Ordinary Differential Equations

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Lecture - 4
Examples Continued

Welcome again back to the lecture, especially the lecture on the first module. In the last lecture we have seen few interesting examples starting with the population growth model, both its linear and non-linear versions. And then we have seen the atomic waste disposal problem. And finally, 2 problems from fundamentals of mechanical engineering they use it extensively, the spring mass system and the electrical circuit, simple electrical circuit.

What surprisingly we have seen is that both is a model leads to the same second order equation with constant coefficients. The another interesting observation from our first 2 examples is that even after getting solution, especially the population model, getting solutions whether it is an implicit or explicit form is not sufficient; you have to do more analysis on it because the solutions in it explicit or implicit form will not reveal much about the physical model.

And, in the second example, atomic waste disposal problem, we have first modeled it as a linear problem, we have obtained the solution, but it did not resolved the exactly what we wanted to. And then we remodeled it as a non-linear problem, but the non-linear problem was difficult; we could solve it and get the equation in a implicit relation, we could not make it an explicit solution. But, the, that was enough for us to conclude the, that despite of the weather we can do it, the question raised by the environmentalist.

So, this, you see that, how you model, how you solve it; and even after solving, doing it in the analysis are very important. And that is the purpose of a, said in the beginning, the purpose of the course is not just to solve it which normally is difficult, but very quite often explicit or implicit solutions are not the important aspects of the theory. It is, how do you understand its trajectory, and how its behavior, and what is a physical demands, in that what we want to do it.

Now, what I am going to do it, I continue with the last 2 problems, the electrical circuit problem in mechanical thing; and we have obtained the equation; and we want to interpret the, or rather analyze the equation.

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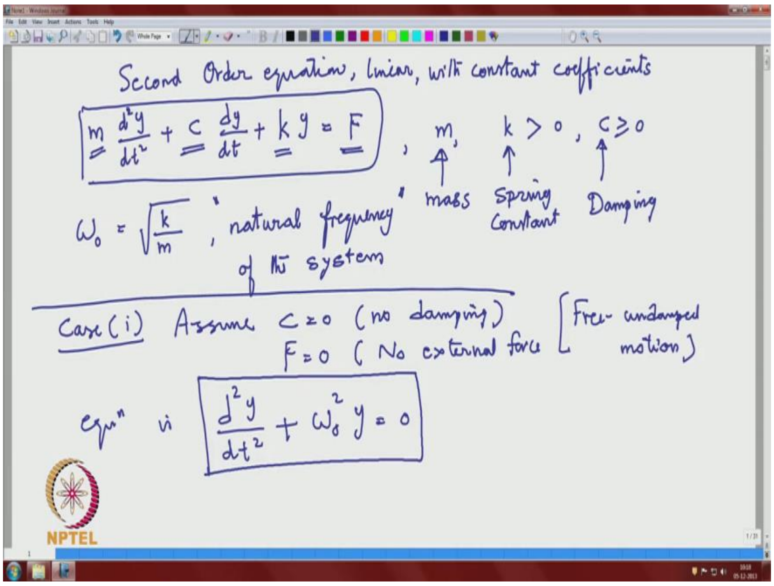

So, let me recall the second order equation which we have derived. Second order equation, what was the second order equation, m into d square y by d t square plus, c into d y by d t plus, k into y, is equal to F. So, this was the second order very simple equation, linear. And what, the other interesting thing is that with constant coefficients; so you will learn more about this first order, second order equation later. But, so I am not introducing to you how to solve this equation in this particular lecture.

I will write down the solution, and my aim is to give you the interpretation of this equation where the thing is that where m, c, k, are greater than; in fact the let me tell you m and k are greater than 0, then only you will have non-trivial mass I think. And c can also be, so this is c is the, this is the mass which you have, let me recall mass, this is the spring constant spring and this is the damping. So, I want to analyze this equation extensively now, and give you an interpretation.

So, I will be introducing few notations like that. You will understand why this is, a call it omega naught is equal to square root of k by m. This is called the natural frequency of the system. You will understand why it is called a natural frequency when you will analyze it. This is called the natural frequency. So, when you develop a mechanical system it has natural frequency, natural frequency of the system, use it. So, this, a, we are introducing more and more notation. So, I am going to analyze it case by case.

So, we will start with case 1. And this, in this case assume, so I am going to do it more and more, assume c equal to 0; that means, no damping. And then I am also assuming that F equal to 0; that means, this is a no force, no external force. So, this is the situation free undamped motion. Then you knowing, made it this equation; in this case equation is, what is the equation, m, I will, c is equal to 0, F is equal to 0; and d by d k by m, this equation.

I can write this equation as, d square y by d t square, is equal, plus, omega naught square y, equal to 0. This is the simple harmonic type motion using equation; it is a nice equation. There is no external force, no damping, then you know that the second law, the first law of motion, Newton's law of motions should obey; let us see, we can interpretate that equation.

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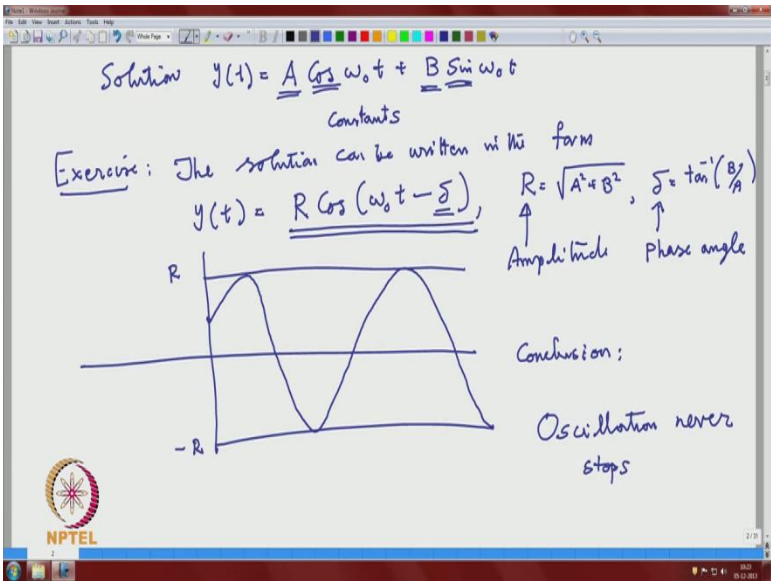

So, let me write down the solution. I am not going to tell you, you will learn this when we, as I said, when we study the second order equation. I will just write the solution y (t). This was learnt of course, a preliminary course on ordinary differential equation, know how to get the solution, those who do not know will learn soon. But, let me write the solution.

So, second order equation, there will be 2 equations. So, you can, the solution will be, A cos omega naught t plus, B sine omega naught t. So, that is where there will be, A and B are constants, arbitrary constants we call it. So, it is a general solution. So, I want to give you an exercise here, this nothing to do with a, p d e.

So, I will keep on giving the, you should do all, when you learn this course, the solution can be written in the form, we just do a bit of computations, you write the solution in the form; this is a better understanding in physics. You will get a better understanding with the, this way of y (t). You can write, R cos omega naught t minus delta. You do the, a bit of a computation in the trigonometric.

And, what is R? R is nothing but square root of, A square plus B square. And your delta is equal to, tan inverse of B by A. This is called, this R is called the amplitude, and this is called the phase angle, you will see by its important, let us see. You, whenever there is appears a motions, and you will see; and it is also cos is a sign of periodic function. So, you see that solution is periodic.

And, the, if you see that, if you look at this case, you see, cos will take values between minus 1 and plus, and hence this y (t) oscillates between minus R and plus R. And you see the Newton's first law obeys. It will never stop. The only thing is the, this gives you phase angle. So, when t equal to 0, it starts fro; so if there is not damping you will have a problem with the system, you see.

So, if you plot this curve y (t), and it will oscillate, so if you mark here minus R and plus R, it may start with a something, somewhere, so that is what it is a phase; it shows it, you know, start with 1 or something. So, you will have phase angle, it will go. So, you see, it is a periodic motion. And the conclusion, oscillation never stops. And that is what except, oscillation never stops, and as excepted, that is what it says, because there is no damping and no external force. So, the moment if there is no external force acting on a system, the motion has to continue, and you have your complete thing.

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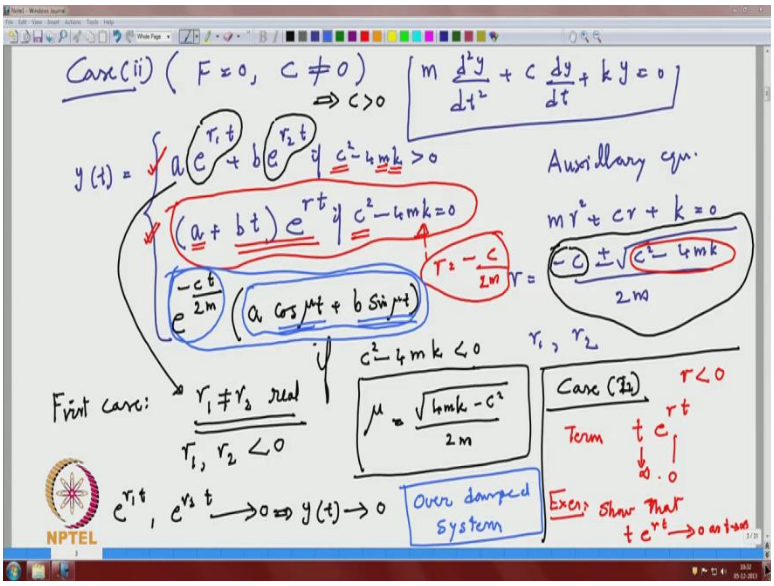

So, let me do to the next one case 2, you from this solution you understood. Case 2, you are still taking the equation with F is equal to 0, but c not equal to 0. So, how does that equation look like? There is no external force; so you have a system, no external force coming to the system, but then you are putting the damping; you are putting it in the void for that particular example; or, in other mechanical system, you add damping.

So, you will have this equation, m into d square y by d t square plus, c into d y by d t plus, k y equal to 0. So, this is your equation. And let me write down again the solution I told you; you will know how to find the solution. So, let me write down the solution in different cases. It depends on, depending on the there is what is called an auxiliary equation, and you look at the roots of this auxiliary. So, what is auxiliary equation here? Auxiliary equation that is equal to, m into r square plus, c r plus, k equal to 0.

And, you have your solution, r is equal to minus c, plus or minus, square root of c square minus 4 m k by, 2 m. So, you have 2 roots are, r 1 and r 2. So, if c square minus 4 m k is positive, you have 2 real roots; and when it is 0, there is a double root, and all of them you know. Accordingly, you can write down the solutions we know, a e power r 1 t plus, b e power r 2 t, if c square minus 4 m k is positive.

And, if it is a double root, again you, how to find solutions which we will see later; a and b are arbitrary constants, b t e power r t, because there is a, if c square minus 4 m k is equal to 0. When c square minus four m k is equal to 0, you have this r is nothing but in this case, r is equal to nothing but minus c by 2 m; so you have that, you see. And then the last case, when c square minus 4 m k is negative, in that is a complex roots. In that case, you can write down, c is equal to your solution minus, c t by 2 m into, a cos mu t plus, b sine mu t; this is the case if c square minus 4 m k is negative.

And, what is mu? If this is negative, you define your mu is equal to, in this case the mu is equal to square root of 4 m k, so that is positive, minus c square by, 2 m, that is what; you can write down that solution, c square minus 4 m k less than 0. So, now, if a situation is different, we can see if c is always not 0, in fact, c is positive; that means, this implies c is positive because damping is always non negative. So, you have the case c is positive.

Now, look at the first 2 cases, when the roots are 0, in these 2 cases, first 2 cases, first case within these; first case means this equation; that means, r 1 is different from r 2, and real, that is a situation. In fact, case it is a real, you can analyze this equation, if you analyze this equation, c square minus 4 m k is positive and hence this is, this smaller than c. So, it will retain the sign of this one, means in this case is actually r 1 and r 2 are negative, you can see that.

Because, this is a, even though if you take plus, if this minus, no problem; both are minus, even if you take a plus, it will be a quantity less than c, and absolute value. And hence, this will retain the sign. When r 1 and r 2 are negative, look at these terms, e power r 1 t and e power r 2 t, r 1 and r 2 are negative; that means, this will go to 0. So, you will have, e power r 1 t, e power r 2 t, goes to 0; that implies y (t) goes t 0, tends to 0, if there is no problem.

So, the case 2, this you know how to plot it. So, it is a, it is all vanishes the solutions. So, the moment you add some damping the, earlier when there is no damping the solution were oscillating between minus r and plus r. The moment you add damping, the solution goes to 0. It may work, in whatever depending on this sign of a and b; whatever you take it, initially it behave the, whatever way it is; but eventually, every solution, d k in an exponential rate to the 0 level.

And what about the case 2? Case 2 here, within this case 2, this is the situation. So, you have the case 2 situation, this is the thing I am analyzing it. In this case, c square minus four m k is equal to 0. So, your y (t) is, has a term; the first term, no problem, a into this one again will go to 0; the second, there is a term, e power r t; r is negative, there is no problem; even here, r is negative because r is nothing but minus 2 by m. But, here is a term which goes to infinity, and here is a term which goes to, there is a term which goes to infinity; here is term which goes to 0. So, there is an in determinant form.

But, let me give an another exercise. You have to keep on doing this exercise. This is a nice exercise, a small exercise from your analysis, show that, t e power r t goes to 0, as t tends to infinity, you see. So, you a have nice exercise, again so solution d k. So, both these cases, case 1 and case 2, the solution d k's. And there is no oscillation, so when you add damping, damping is the when this will become a 0, suppose m and r fixed, for a particular system, m is fixed, k is fixed.

And, this sign of this determined based on how much damping you add. So, if you add sufficient damping, for in case a positive quantity, and if you add sufficient damping, in such a way that c square is greater than or equal to 4 m k, there is no oscillation, there is no oscillation phenomena, and those selection goes to 0. So, these 2 cases are called over damped system. So, you are giving so much damping, so that the entire, over damped system. So, you have add a so much damping, so that complete oscillation has vanished and the solution goes to 0, exponentially.

But, now, to the third case, third case is this situation, when you look at that term this is a cosine term, this is a sine term, both are bounded. So, the cos, and hence this entire term is a bounded quantity. But, this term is a quantity which, it is 0. So, let me take to the third case.

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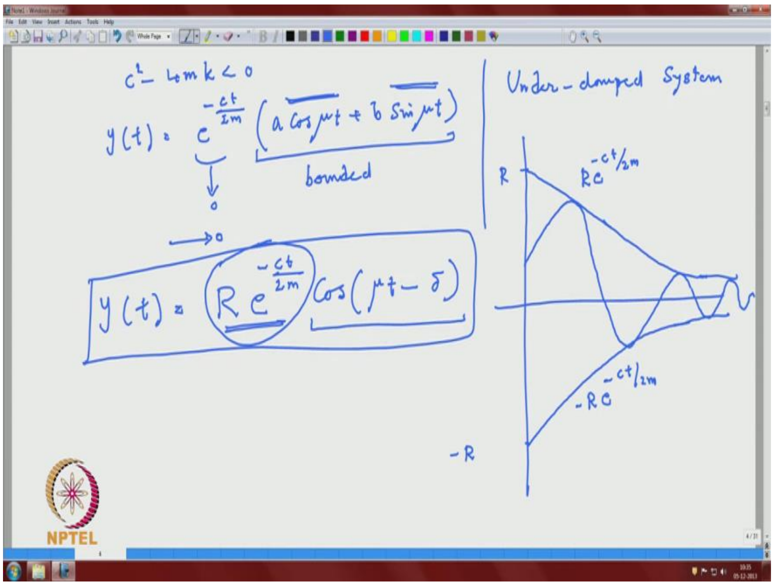

So, case, c square minus 4 m k is negative, in this case your solution y (t) is equal to e power minus e t by 2 m into, a cos mu t plus, b sine mu t. What I said is that, this is a bounded quantity, bounded, and this terms goes to 0. So, why it is still goes to 0? And this is a situation you have damping, but then damping is not too much. So, this is called the under damped system; damping is there, but under damped system.

So, you see, all this we have; when there is no damping, complete oscillations, oscillation does not decay. So, these 2 quantities show that there are still oscillations. So, if this will oscillate between some quantity; so let me write down the solution in the another form; the other form, you can understand this oscillation much better way. So, that is what I say, you can rewrite this equation in another form, minus c t by 2 m into, cos of mu t minus delta. So, you see, so solution.

So, this shows, the oscillation is not removed completely, but then there is an amplitude, and that amplitude basically goes to 0. So, it oscillates, but eventually goes to 0. So, if you plot the graph of this function, it oscillate cos takes value between minus 1 and plus 1; and hence it will also takes the value between is the. So, if you have this, if you plot this curve, this is an exponential decaying curve with r.

So, if you plot that curve, you will have a curve which goes to here; this is nothing but minus r e power, minus r I think; and these yeah, minus r; and here you have a plus r, and you have a this curve, this is a symmetric curve. So, it does not look symmetric, that is a, you can plot it. So, this is the curve, minus R, e power minus e t by 2 m; and this the curve, R, e power c t by, minus c t by 2 m. And it oscillates, so it starts here. It will go, and maximum it will go, so it will oscillate something like that.

So, you see, like that it will go. So, you have these complete analysis in the, this is the case 2. So, what you, so the what is so far we have concluded? You have a system, there is no external force. We have considered only 2 cases; the case 1, we can, no external force, no damping, full oscillations. So, oscillation continues in the same pattern, then there are no external force.

But then there is a damping, it depends on how much damping you give, either you can kill entire oscillations, or the oscillations will be, if the damping is less, the oscillations retains. And then you have, but is still goes to 0, and that is the. So, the moment you have damping, eventually the solution goes to 0.

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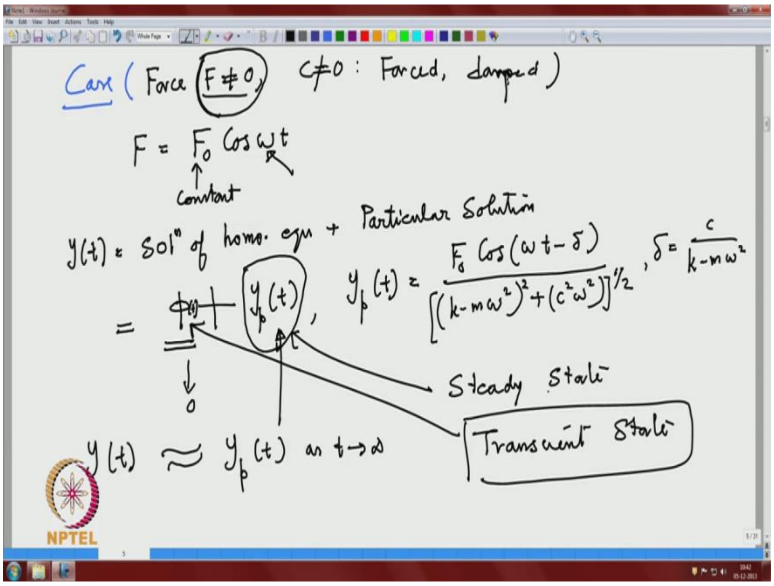

Now, let me go to the next case, where you have the situation, you have the case where you have force F not equal 0, but you take c is equal to; so the, we analyze the case when F equal to 0, both c is equal to 0, c naught equal to 0. And we are trying to analyze the case when there is an external trouble or external advantage. We do not know the external force is advantageous or external force is. In mechanical system, quite often the external disturbance is advantageous.

But, we will remark that the same external force is advantageous in the electrical circuits. So, the interpretations will be different. So, we are trying to do think in the set of mechanical system. So, this is nothing. So, it is a forced, but undamped system. So, there is no damping, what will happen is that one. So, this is; so maybe I will, this is a more difficult case. So, let me do the difficult case little later.

So, let me do first, because when, already seen that when c is equal to 0, when F equal to 0. there is vibrations. But, now there is an external force coming into picture and still there is no damping, you except more trouble in the mechanical systems. So, let me do the easier case first, and then we will go to the other case. This is the standard case. So, when it F equal to 0, how do you write them in the solution. So, you may not be able to write down the solutions with general F because it is a non-homogenous second order equation.

So, first you have to solve the homogenous linear equation, and then with certain right hand side you will be able to write down the explicit solution. So, you have to find what are called particular solutions, which we will study later. So, but certain F, you will be able to study, obtain the explicit solution. So, we will assume F is of the form with a single frequency coming, at external force coming, F naught into cos omega t. So, this is F t, where F naught is a constant; this is a constant. And then omega is the frequency coming from the external one. Omega naught is the corresponding, frequency corresponding to your mechanical system.

So, I can write down; so already as a told you, write down the solutions y (t). You need a solution of homogenous equation which you know already; homogenous equation plus, 1 particular solution. You will understand this more when you study the second order equation particular solution. And let me write down the solution completely; how to do it, you will learn it later.

So, let me write down the homogenous solution which you already seen. Whatever I have written down is the homogenous solution plus, you will have a particular solution. The particular solution, let me denoted by y p (t). And what is particular solution, y p (t)? Let can write down the particular solution I have something, F naught; this is, cos of omega t minus delta by, is a complicated, little complicated, but you can see that omega square whole square plus, c square omega square power whole power half.

I hope the computation is correct. Anyway, you should do that; where delta you can also write down delta here is, c by k m omega square. So, see, you can come exactly calculate your phase coming, using your external force here. So, there is nothing specific in this case. The moment you have damping, it should work; when there is no damping, this term was not, you know, when there is no external force, this term was not there, due to the, but this was there is without a homogenous solution.

That means, it is a solution with a F equal to 0, you understand the behavior. So, that behavior will be retained. And plus, you have an extra term here, and the extra term also have, so there is no additional feature here. The only additional thing is the; so the behavior everything more or less is the similar thing. And we have seen that in all cases with F equal to 0, you have seen that the solution goes to 0.

So, this the term phi t which goes to 0, this will oscillate. So, what will happen is that y (t) eventually will go to this particular solution y p (t); y p, you know, let me not write, it will go to 0, it will behave like y p (t). So, you can understand the behavior of the y p (t), and so the, as t tends to infinity. So, the behavior at large time, y p will be of y p (t) which is coming due to the external force, and nothing more interesting. So, this is called the steady state, this y p, a physicist call this is the steady state solution. You can understand, why it is called a steady state?

Because eventually steady state and this phi solution to the homogenous system is called the transient state. You can understand this term terminologies very well, it matches. Because, in the beginning of the time, initial time, you will have the contribution of transient state and eventually that transient state will go to 0, and the n t r solution will behave like a steady state solution.

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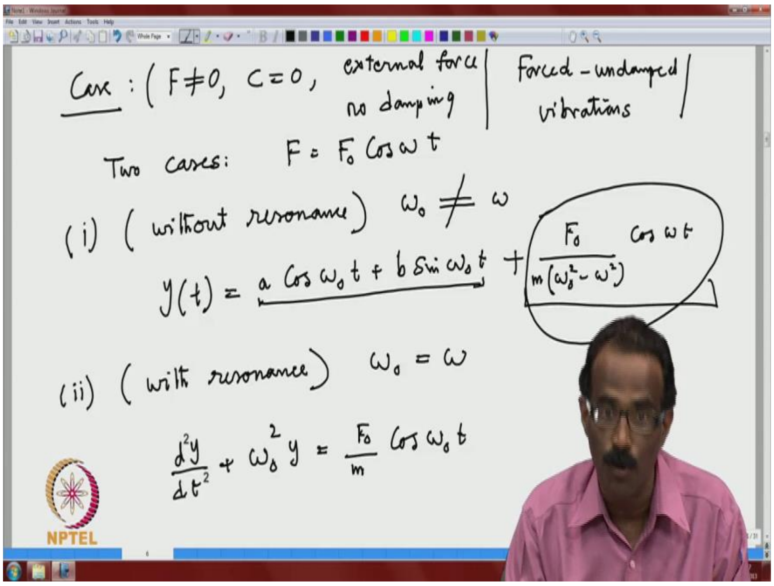

So, now we come to the last part, a next case; the more interesting and difficult case. This is the case where F not equal to 0, but c equal to 0, you see. This is a no damping, external force is there, external force, but no damping, you see. Why damping is important, you will see here, well in any system. So, that it is the forced undamped vibrations, this is the situation. So, let me, here itself there are 2 cases. So, I have to study separately 2 cases now, you can see.

So, you have to split because it is a complicated thing, 2 cases behaves differently. So, what is your F? F, I may give, assume two of the form, F naught into something like cos omega t; I have writing this one. So, what are the, in this case, first let me tell you the subcase with that one, this is called, you understand why without resonance. I am going to introduce a new terminology without resonance; that means, you have a natural frequency, omega naught, coming from the system, and you have an external frequency coming from the external force.

So, the without resonance is the case, the external frequency is different from the natural frequency. So, omega naught is different from the frequency coming here. Then, this is not the most interesting case. You can just write down the solution, y (t). You have a solution of the homogenous system. This will look like a into cos omega naught t plus, b sine omega naught t, you see. So, this is the solution corresponding to your homogenous part.

And then there will be a particular solution; it may look like, m, F naught into, m into omega naught square minus omega square into, cos omega t. So, you see that when omega equal to, not equal to omega, this is not the particular solution. The particular solution can have problem, and why that problems? Mathematically you will understand when you solve second order linear equations with constant coefficient, which we are going do it any way in future.

But, you see, the immediately a trouble we cannot write, this equation is clear from here. So, this is again a natural thing. So, when you have c equal to 0, you have the your normal vibrations it is happening because then there is no damping naturally that full vibrations will be there plus, you have a term here; and it will vibrate as it is. So, this is the case which you have. There is nothing new whatever we have discovered.

The interesting, another interesting case is with resonance. Why call it with resonance? This is the case when omega naught equal to omega. This means, you have a natural frequency to your mechanical system, an external trouble, external force is coming with a same frequency; when these things are there to solve this equation, then finding the solution and writing the solution is in a different way. And let me again write down the solution here.

Just write down the solution. Here the equation in this case looks like this, d y by d t plus, omega naught square y, is equal to F naught by m, cos omega naught t. You already seen that cos omega naught t is a solution to this one. So, whenever you want to solve an equation, if you want to find a solution to this one, anything like cos omega t were not work like a particular solution; cos omega naught t, is a solution to your homogenous equation. So, the solution is slightly you have to write it in a entirely different way.

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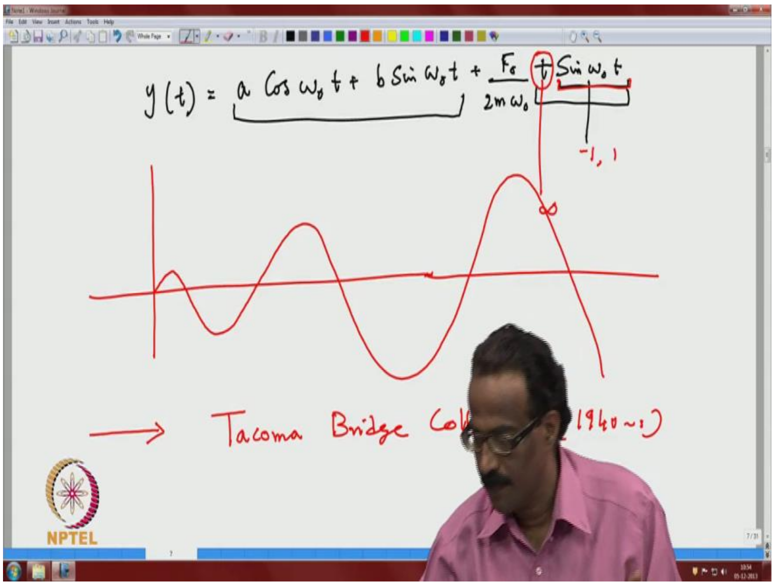

The solution, let me write again here. The solution will look like this, y (t) is equal to, a cos omega naught t plus, b sine omega naught t plus; this is the difference here; how things behave, 2 m omega naught into, t sine omega naught t. So, the behavior of this term is fine. It oscillates between a and b. If you write it, r e power, you already seen that, it will oscillate between full wave. But, look at this term, this oscillates, the sine part oscillates between, minus 1 and 1.

But, you see, as t increases, these increases to infinity. So, what happens is that, even though this oscillates, if this term, this such a term, such an unbounded term, as time goes the amplitude is increasing; that is what happens when t becomes bigger and bigger, the amplitude is increasing like that. So, it will start from here, the amplitude increases, that means, the vibrations become unbounded.

So, as you plot this curve here, the curve may, you can look like the, as time goes, you see, it goes to become bigger and bigger. So, you have an mechanical system. You are meeting with an accident or something, some vibrations coming in your motor bike or something, and the vibrations comes into you that, and the person who is riding the thing you do not want the vibrations to on you; but here, if this is the situation, an external frequency comes with that one and this vibrations becomes larger and larger, these are going to have meant catastrophes.

And, this is what when you design any mechanical system like a bridge or a anything, you have to take into account in such a way that, you have to put enough damping in the system is such a way that, the external frequency, it may be or may not be, the external frequency may not be due to your, in, under your control. For example, there will be lot of vehicles travelling through your bridge. So, when your vehicles travelling through the bridge it will create automatically, it is on external frequency. And your system will have certain frequency.

So, if your bridge or whatever it is, do not have the frequency with a damping enough; and it is a possible that the frequency of your bridge may coincide with the external frequency coming, and the via, the bridge will start oscillating, and bridge will collapse. This is the famous, you should go and read the famous Tacoma bridge collapse. This is a probably happened in 1940s or something.

This is a famous bridge in USA where what happens is that in the early morning the bridge starts vibrating with smaller and smaller vibrations, by the time it, and the vibration became bigger and bigger, and by the time at 11 O' clock or something, the vibrations are so big. So, the amplitude was so huge, and the bridge got collapsed. The interesting tale of that one is that nobody died in that huge disaster except a dog.

And, this is also the reason, probably you have seen that in the military or army marching in the cadence marching through the bridge, the army march not like what we create different types of vibration, but army marching will be with a particular frequency and it will be very high. That is why, when the army marches through the bridge they would not allow the continuous marching of the army; they will break into small, small pieces; a small set of people will cross the bridge and then only the other seat will.

So, they will break because they do not want to create a frequency from the external side which marches with this one. So, as this example, before going to, quickly I will not go into the quickly the other examples, I will just explain few examples which you will see details in future other lectures, in the later part of this course. But, I will explain few important examples.

And, before going to that one, when you analyze this same system in the electrical circuit the analysis is the same you look at it, but the interpretations are now different where here if the exactly what is undesirable in a mechanical system is used to good use in electric circuits. Because, what is all happen is that, when you want to tune a radio or t v, what you are doing is that you are setting a frequency, natural frequency to the system. So, when you are saying changing your channels or changing the radios, when you are changing the frequency means, you are changing the frequency of natural frequency of the system; and externally, all that telecasting all the channels with different frequencies.

What you are doing, trying to do is that you are trying to put a resonance situation in such a way that when you are setting a particular frequency to your natural frequency to your system, there will be all the frequencies around your radio or t v, it will only match 1 of the frequencies of the particular channel, or a particular broadcasting thing, when these 2 matches, these 2 will create a resonance. And hence, it is, that is what is called an amplifying.

So, basically a situation of resonance happening, even though whatever is coming is very little, it matches, and then it amplifies. And that is why, you get a particular channel. So, you see, the same thing, the interpretations are different, but you have to, is coming from the same equation, entirely 2 different fundamental problems, from 2 fundamentals areas of engineering.

So, this much details I will not do it any more in a next 10 minutes or something. I do not know, how many examples I can do it. Anyway, I am not going to do too many examples. I will be writing down few interesting, I already wrote down 1 equation, is called the predator model called a Lotka Volterra model.

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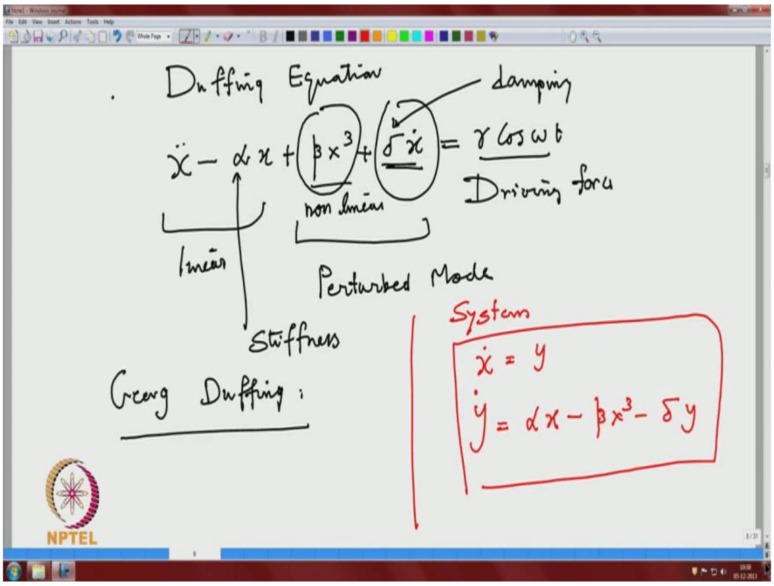

And, another equation you will be studying here, in later, is called the duffing equation. The equation looks like this, x double dot minus alpha x plus, beta x cube plus, delta x dot. And you can also have the driving force if you want. So, that you have a homogenous equation, this is non-linear, due to this is non-linear, and you have the, you can put something like gamma cos omega t, if you want a driving force, external driving force. Of course, if this beta and delta are 0, is nothing but a, this part is a linear part; of course, this is also linear; but the linear, simple linear oscillator.

So, you can think it as the perturbed mode of the linear, perturbed mode, if you want to see; and alpha is a, if you want something, this alpha is nothing but your stiffness if you want it, in mechanical model; stiffness, you can interpret that one; and beta be the amplitude, and delta whatever it is the damping. So, this is something the similar thing. If you look at our mechanical model, it is a similar thing. So, you have this term, this term, similar terms; only this is more term; this is the damping, you can see that one.

This model was probably introduced by George Duffing, as I said, if you want to understand more, if you wanted linear system you will be restricted to only thing if you want to more general understanding of your system, you have to have your components, then linear perturbations and perturbations, and this the thing. So, you may see, so for example, your stiffness of the spring may not obey the Hook's law exactly. And such kind of things, it will not follow precisely the things like that.

So, you will see this equation in probably more about the derivation, not too much, but we are going to study this equation in detail in our future lectures. You can write down these as a system. You will see that also. You can write down this as a system with, x dot is equal to y, easy to write, when you have an x dot is equal to y. You have your, y dot is equal to, you put that all there, alpha x minus beta x cube plus minus delta y. So, you see, you can write this as a system.

So, you have a very beautiful equation, as I said that a reasonable extensive study will be done here at the end of it. And this is one of the models which we want to understand it in the qualitative analysis. So, to understand, or if you exact, if you want to know how this equations are derived, you can get into the references which we have given, or you can also search through internet, if you want to understand more, very one of the, all right.

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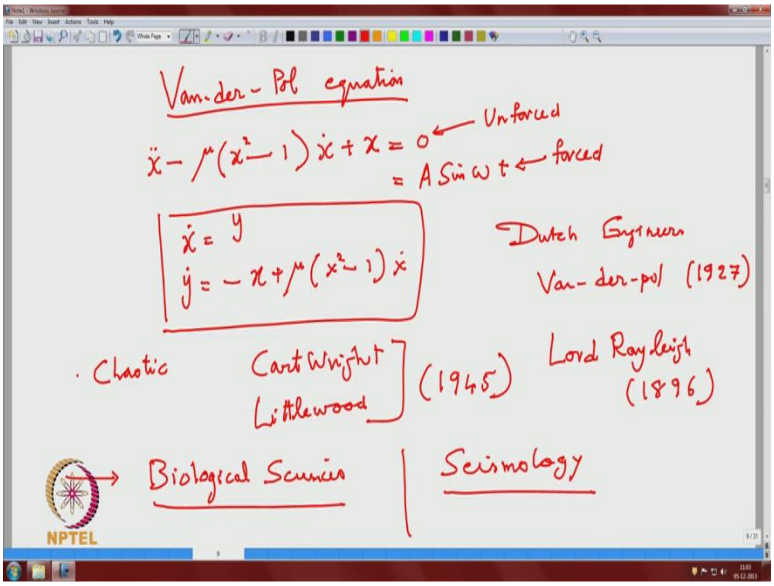

So, another model which you may see is called the Van der- Pol equation. So, it is a Van der-Pol equation, these are all oscillator equations; earlier it was an electric circuit mechanical model; this is also an interesting model coming from electrical circuits, is again we call it oscillator; you can also call it oscillator; everything is a that form, x square minus 1 into x dot, plus x. Here also, if you want, can put it 0, or you can also put it something an external term, A sine omega t, if you want a force. So, this is called the unforced Van der-Pol equation.

So, this you can call it forced, you are putting a force, external force. You can write down this again as a, there are different ways possible you can, so when I am writing it as a system it does not mean that it is a unique way of writing it. You can write down the systems in different ways. So, what I have done here is the easiest way of writing it. And different ways of writing it, will have different F x plus mu into x square minus 1 into x dot.

This is known by the Dutch, Van der pol was a Dutch engineer, Dutch engineer, Van der pol, and introduced in 1927, when he was working in the philips company, that is the time he introduced. But, it is also seems that this equation is derived little earlier than Van der pol due to Rayleigh, lord Rayleigh, but somehow it is not in that name. Say, introduced probably 1896 itself, 30 years before that one, but it is a he is the one, atleast Van der pol is the one who extends, he only studied this thing.

So, you will see the behavior, probably in this study in our course, you will see the chaotic behavior, probably, periodic solutions and chaotic behavior; all that analysis probably you will see that. There is also further equation. Other people like CartWright, I think I am right, CartWright and Littlewood, they are further whole mathematical study later, more detailed mathematical study. Here, it is engineer, he is more like a done like, some mathematic, theoretic and experimentally a complete study, more detailed mathematic and analysis, probably a little later by this people.

So, you will see about this type of equations also in our, this course you are going to see that. Of course, one of the easiest; this equation; the one more thing. This equations have not only application this kind of thing, it also has applications in biological science that is the, in, so as I told you, as you already seen, the same equation may have different effects in different thing, just like the second order equation which we have derived and analyzed, comes from a mechanical system, it also comes from the thing.

So, here, in the study of the neurons, the action potential of neurons, the same equation is have extensively studied, and its chaotic behavior; it is also seismology is used where you want to understand the georgia plot 2 plates. So, this same equations has applications in other various also. So, when you study, so what you will be seeing here, our concentration is not to a particular problem, but our concentration will be how this equations behave. So, our analysis is going to be the qualitative analysis of this general equation; interpretations has to be done according to our problem, according to physical problem.

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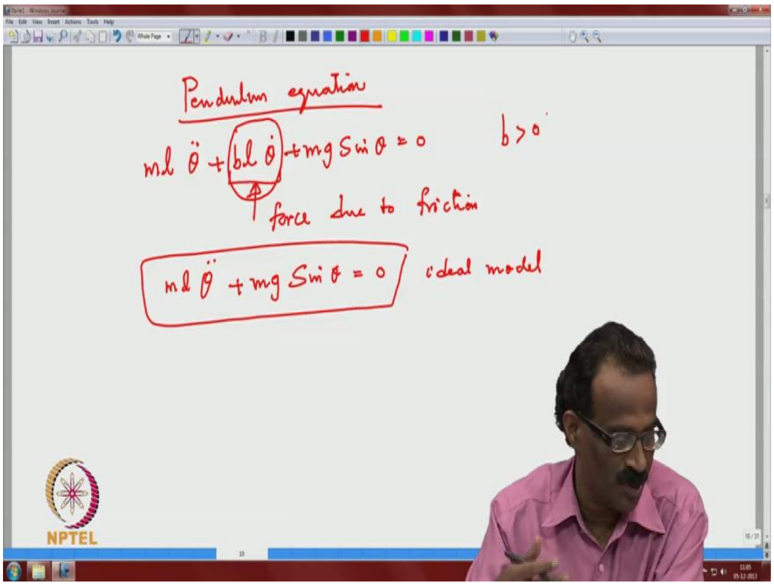

The other equation, of course, this would have been come here; mainly, the pendulum equation, this you also may see a pendulum equation; and this looks like this more complicated, let me write it in slightly general way, plus b l theta dot plus, m g sine theta, is a standard pendulum, you have a string, a list length of string and mass is attached, all of you know that one, those who gone through your plus 1, plus 2 class, and how the pendulum swings.

So, this is the force due to friction, the term, the this is something looks like a damping part. So, if this term is not there and if you write it in a simple form say, m l theta dot plus, m g sine theta, this is basically an ideal model, more of an ideal model. So, you will see these things, ideal model, and which you are going to study here, whatever it. And the other case when you study b positive, so there is a, that is a more general case where there is a force due to friction.

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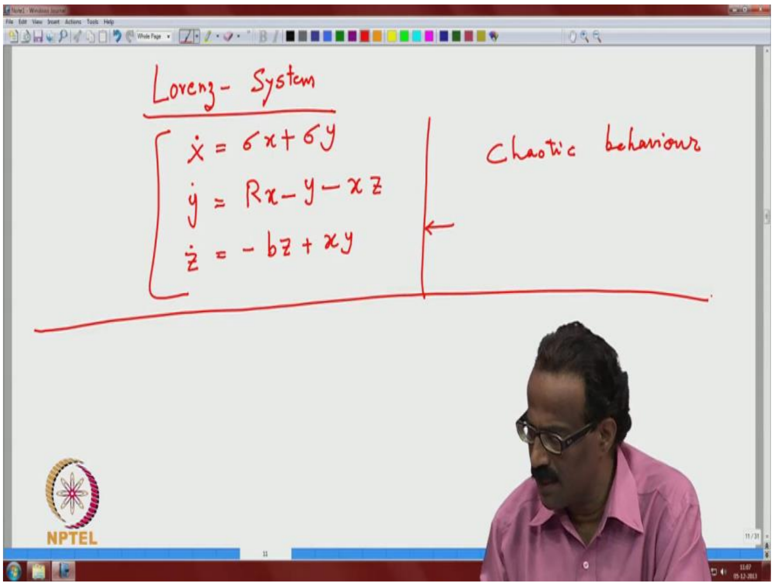

So, let me more or less finish by writing to you one more equation which is the probably the first the chaotic behavior is observed in the Lorenz system; I will write down the system. So, you can, why I am writing this system is that, by the time you come to the end of this course, you should be familiar with a system, so that you will understand its behavior, the qualitative analysis in a much better way. So, if you go through this one, do little more study on this systems, how this systems are coming, and then you understand.

So, let me just write down the thing. I will not again tell much about it. This is minus sigma x plus sigma y; and y dot is equal to R x minus y minus x z; and z dot is equal to minus b z; this is again extensively studied to understand the chaotic behavior. And this is actually developed in the study of meterological problem with of weather production. And you will see again, a little bit analysis of these problems in future.

So, there are, another, some other interesting model. We are not sure about, we will be able to cover all these models because then our 1 semester course may not be good enough to cover that one.

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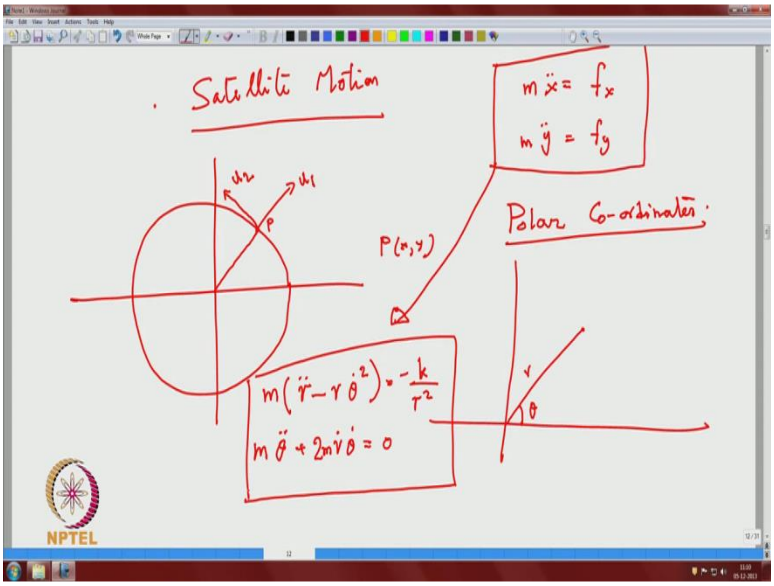

For example, some nice models, if we get time, I am not a, think for satellite motion; So, important, right; satellite motion; examples from satellite motion. So, it is typically you have a satellite. So, you have, a satellite is in this position p, with x, y as the coordinates system, x, y; and then you have, this has the thrusting capacity. So, you can, you have the radial thrust if you want u 1, and you will also have a tangential thrust, u 2. So, you can derive, it is the, modeling is much easier here. So, you can easily model this equations.

So, the modeling is not the important thing. The in the satellite thing, in addition to model, you have to also give a, kind of study the control problems. So, if I want to write down the equation, the equations may look like this: m x dot is equal to f of x, and m y dot is equal to f of y. So, this looks easier model; but then since this is satellite, a better model, better way of writing is not in the cartesian coordinate system, the polar coordinates; writing in polar coordinates is much more better.

So, you can do the very simple analysis. You do not need much things to be done. So, let me just write down the equation. So, it will have something like that radial. So, you have r and theta. So, you can write down the radial. So, you can do very simple analysis, not too much complicated. If we have time we could have done it, but we will not do it, I am not sure, we will be able to do a much better analysis also in this problem.

But, let me write down for your familiarity, this equation can be converted with very simple analysis to the equation of the system of 2 second order equations. So, let me do that one; r theta dot square equal to minus k by some r square, and m theta dot, double dot plus 2, again is a non-linear system, theta dot m, 2 m, is equal to 0. So, you have eventually; so analysis for this kind of things is better to do it in polar coordinates, not in our cartesian coordinate. So, this is a system of 2 equations, second order system that is what it; it is not a first order system. So, when you go, convert into a first order system it will have system of 4 equations in 4 unknowns.

And, there are plenty of other examples where you can study, and it is available. And with this we will complete the module 1 giving examples. In the next module we will be giving the preliminaries. We will give some preliminaries in the case of linear algebra. We will also presents some preliminaries of analysis. We will just do it which we require for our course.

Thank you.