Ordinary Differential Equations
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Lecture - 39
General Second Order Equations

Welcome back. Recall that in the previous lecture we were constructing Green's function for linear second order boundary value problems. So, let me recall what we were trying to do.

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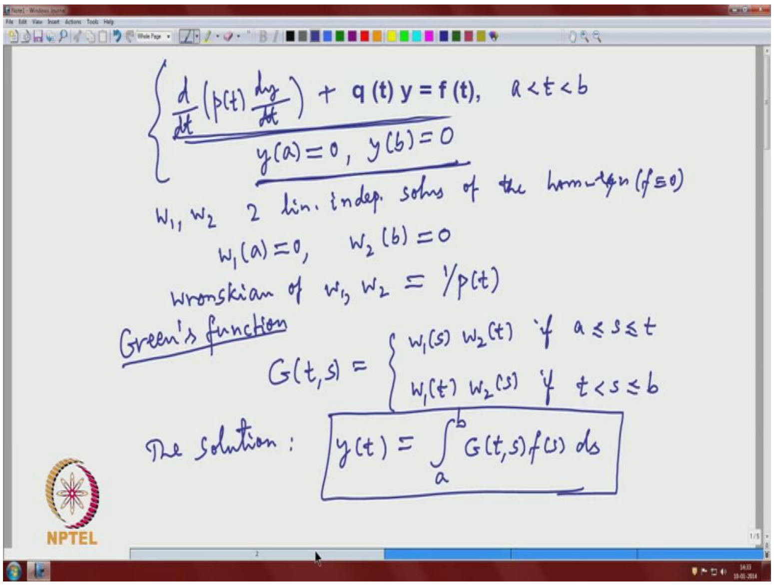

So, this second order equation, d by dt how p t dy by dt plus q t equal to f of t in the interval a b. And we have taken the boundary conditions y a equal to y b equal to 0. So, p is a continuously differentiable function and a positive function in interval a b. q and f are continuous function.

So, let me recall the order we adopted to construct the Green function. So, we started with w 1, w 2, two linearly independent solutions of the homogenous equation. So, that is, f is identically 0 and w 1 a equal to 0. So, one of them satisfies the boundary condition at one end and the other one at other point b. ok. So, we also normalized w 1 and w 2, so that the Wronskian of w 1, w 2 is equal to 1 by p t, ok.

Under these conditions, the Green function of the boundary value problem was defined by this expression w 1 s w 2 t, if a is less than equal to s less than equal to t. And other way round, if s, if t is less than s less than equal to… And this solution, required solution for the boundary value problem, the solution y t is given by the integral a to b G t s f s ds. And we verify that this while given by this integral, satisfies both the boundary conditions as well as the difference in equation, ok. So, we will illustrate this one through an example, simple example.

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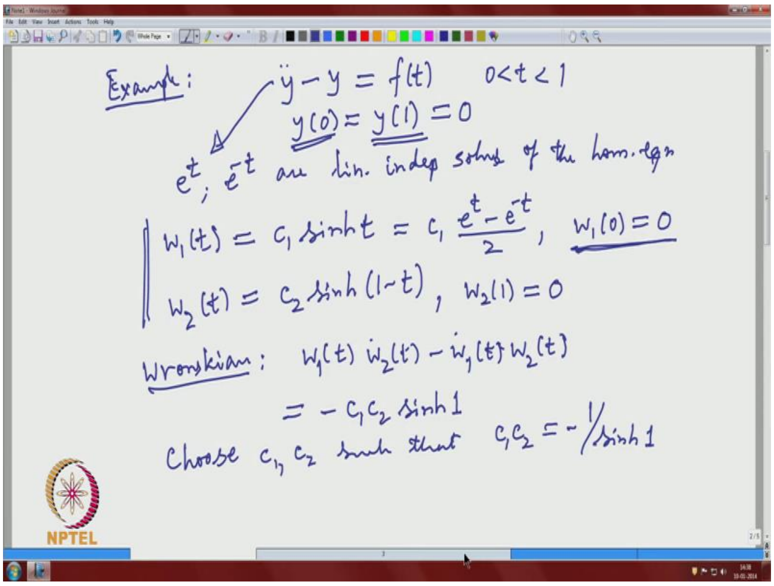

So, an example very simple one. So, y double dot minus y is equal to f t. So, in the interval 0 less than t less than 1 and again, I take the boundary conditions as… So, while vanishing at both the end points 0 and 1. So, in this, so this is a second order equation with constant coefficients. So, if you study the homogenous equation, so e exponential t, exponential minus t are linearly independent solutions of the homogenous equation.

And we have to construct two linearly independent solutions satisfying boundary conditions at different points. And if you just look at the boundary condition here, so you just w 1 t, which is a linear combination of e to the t and e to the minus t and we want that to vanish at 0. So, one that easily found is sinh t. So, this is sine hyperbolic t. So, let me just, you all know this thing, but let me just write that, ok. And you see, that w 1 0 is 0, ok. So, that is the solution of the homogenous equation and it satisfies this boundary condition.

And similarly, at t equal to 1 you just see, that this fellow into 1 minus. So, again when you expand this, this is a linear combination of e to the t and, and e to the minus t. Hence, it is a solution of the homogenous equation. And at t equal to 1, this is 0, t equal to 1, so now let us check whether that is normalized or not. So, let us compute the Wronskian.

So, Wronskian, so we just w 1 t w 2 dot t minus w 1 dot t and w 2 t. So, if you use this expressions and write the derivative this is simply minus c 1 c 2 and you use addition formula for the hyperbolic sign function. What you find is sinh 1. So, that is the constant. So, choose because we want this to be 1, Wronskian to be 1 by p t and p is 1, identically 1 here. So, choose c 1 c 2 such that c 1 c 2 is equal to minus 1 divided by sinh 1 sine hyperbolic 1.

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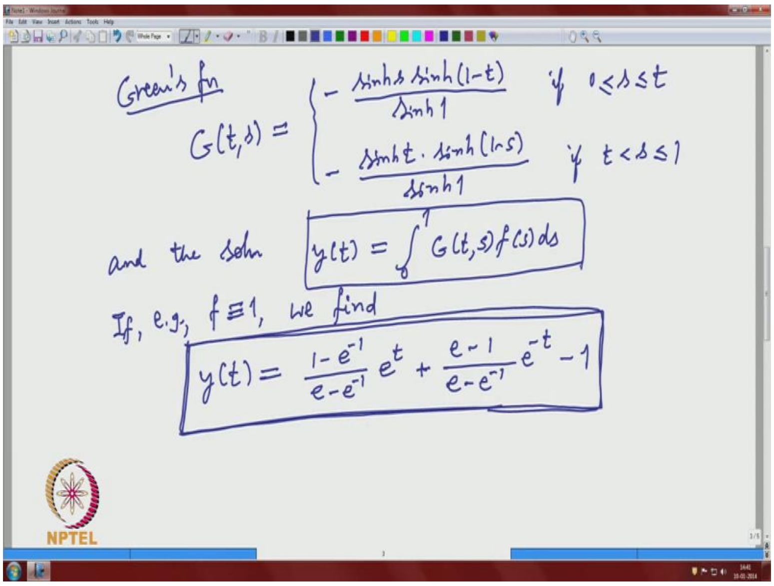

And then, the Green function in this case. So, G (t, s) is minus sinh s sinh 1 minus t by this normalizing thing sinh 1 if 0 less than s less than equal to t and minus sinh t into sinh hyperbolic 1 minus s divided by, if t is less than s less than equal to 1. And the solution, solution is just simply y t equal to 0 to t, 0 to 1 G of t s f of s ds.

So, if you take a particular f for example, if for example, f is identically equal to 1, so we can integrate this easily we find, and so we can easily check that y t is equal to… So, let me write that 1 minus e to the minus 1 divided by e minus e to the minus 1. This is coming from sinh 1. So, e to the t plus e minus 1 by e e minus 1 e to the minus t and there is a minus 1. So, you can verify this. So, this satisfies the equation y double dot minus y equal to 1 with boundary conditions y 0 equal to y 1 equal to 0.

So, as I remarked earlier, so if we again change the boundary condition, the solution w 1 w 2, they change and accordingly the Green function changes. So, Green function is very much dependent not only on the equation, but also on the boundary conditions. So, that you should keep in mind. And we have a nice recipe in this case to construct the solution of boundary value problem via the Green function. So, we will discuss the uniqueness little later. So, we have found a solution.

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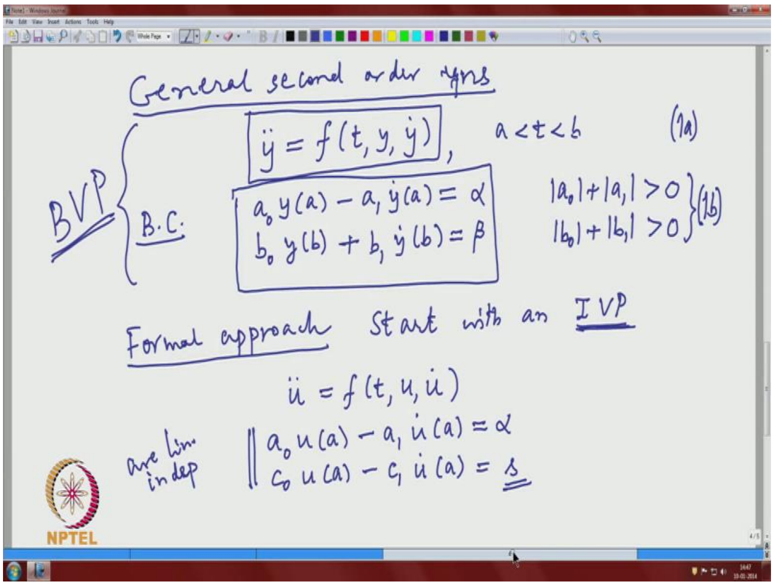

The next question, obviously, is that weather it is unique or not, ok. So, now we go to general second order equations, order equations and see how far we can carry whatever we have done in linear case. So, second order linear equation. So, let the second order general equation. So, I consider this non-linear equation. So, y double dot equal to f of t y y dot. So, again this is in the interval a less than t less than b.

And now, I take general linear boundary conditions. So, we are not able to do much with general boundary conditions. Consider, so boundary conditions, B.C, so a 0 y a minus a 1. So, there is no particular reason why I take minus, but can also put plus, no problem. So, a 1 y dot a is equal to alpha and b 0 y b plus b 1 y dot b is equal to beta. So, in the previous case what we had as a 0 equal to 1 and a 1 equal to 0 and alpha equal to 0. And similarly, b 0 equal to 1, b 1 equal to 0 and beta is 0. So, only the condition is that this a

0, a 1 should not vanish simultaneously, so that we will put here. And similarly, the other condition, so they do not vanish simultaneously, one of them can be 0, fine. And alpha, beta are again given real numbers. So, now we will discuss this.

So, let me again put some numbers here for this this is 1a, and the boundary conditions 1b and the whole thing together called problem 1. So, would like to discuss the existence of a solution to this problem 1, that means, we want a function y to be determined, which is 2 times continuously differentiable and satisfies this given differential equation and these boundary conditions at the end points a and b.

So, unlike in the linear case we do not even know how to start because here there is no concept of the homogeneous equation because everything is clubbed here. So, I cannot separate the linear part and inhomogeneous part, homogeneous part and inhomogeneous part and how to start. So, that is, so we are already facing a problem even to start. Where to start a formal approach is the following.

So, we start with an initial value problem, start with an IVP, so this is BVP. So, this is one and we start with an, an initial value problem. What is the initial value problem? Equation is the same, so now I use different unknown function. So, u double dot equal to f of t, u, u dot. So, this is second order equation. And now, I impose conditions only at a, so I will not bring in b because we are studying just some initial value problem.

So, at one point we give the data, what is that data? So, we just a 0 u a, which is same as the condition at a in the BVP. And now, I will have another independent condition at a only, c 1 u dot a, I put an s here, s and this s is at my choice. So, this, so s is any real number for the time being and these two conditions are linearly independent, linearly independent. What does that mean? You cannot obtain one condition from the other.

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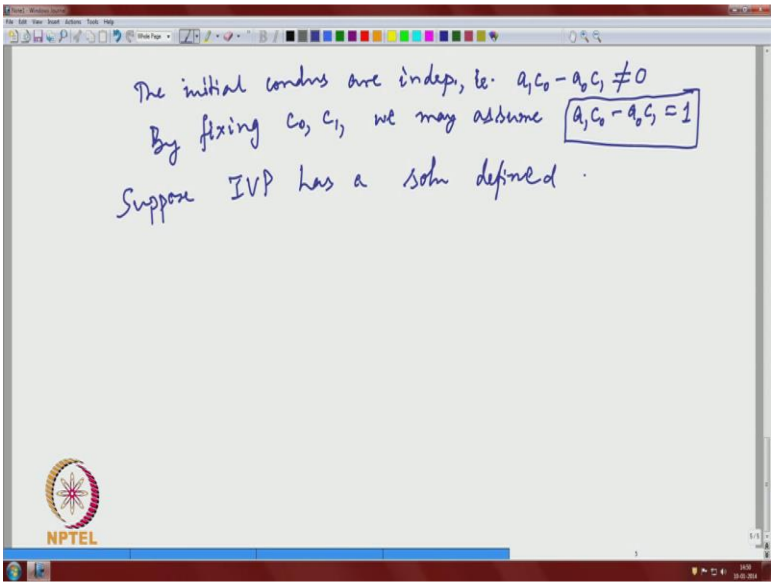

And in terms of the coefficients this means, so that means, the initial conditions are independent, that is, this matrix we have a 0, a 1, a 0 minus a 1 c 0 minus c 1 is non-singular. That means, we have this a 1 c 0 minus a 0 c 1 not 0, ok. That is the meaning of that linear, the conditions are linearly independent.

So, by fixing, so since the second condition is at our choice, by fixing c 0 c 1 we may assume, so this is kind of normalization, that a 1 c 0 minus a 0 c 1 is equal to 1. So, then you just forget it and we have that initial value problem. So, let me just again, so this IVP. So, this let me call it, so this is 2 a and 2 a, 2 b.

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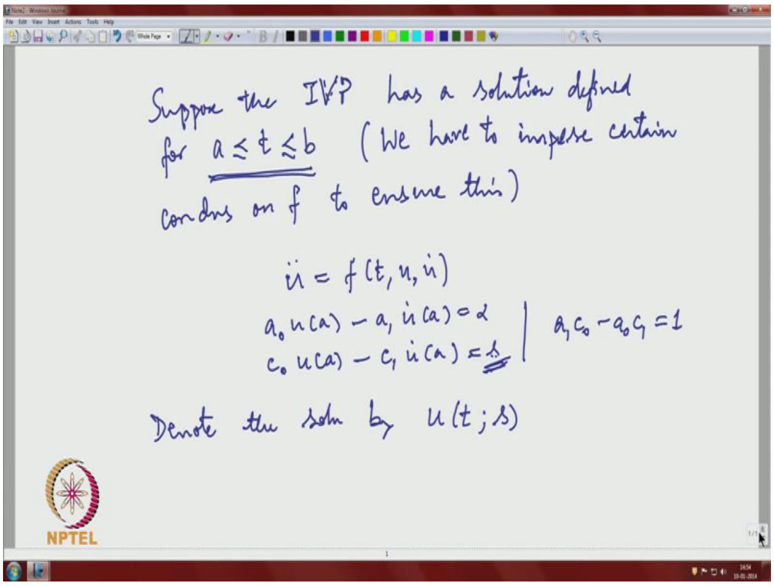

And suppose, the IVP has a solution defined for a less than or equal to t less than or equal to b. So, we want the solution to exist all the way up to b and perhaps one beyond b, but at least we want up to b. So, here we have to, so we have to include, we have to impose certain conditions on f to ensure this, like global Lipschitz and other things.

So, you know, now the solution for the, so let me just write again, u double dot equal to f of t, u, u dot and we have this a 0 u a minus a 1 u dot a equal to alpha, and c 0 u a minus c 1 u dot a equal to s. s is at our freedom and we have normalized, so that this a 1 c 0 minus a 0 c 1 is 1.

So, these two conditions are linearly independent and we have a solution, which exists for the entire interval a b. So, to emphasize the dependence on s, so denote the solution by u t and you stress this s and s is coming from the second condition we are imposing at a, and now you solve this.

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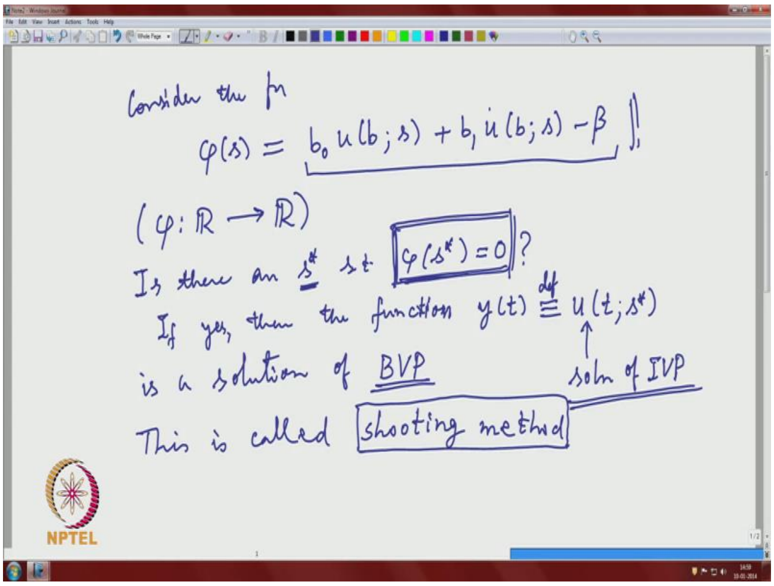

So, consider the function phi of s. Now, I define this function, so which is b 0 u b. So, u is the solution of the initial value problem. So, I remember that s plus b 1 u dot b s minus beta. So, if you remember the second condition boundary condition, so it is of that form. So, I am just evaluating the solution of the initial value problem at t equal to b. So, this is nothing but the evaluation of the initial value, solution of the initial value problem at t equal to b. And now, this is purely a function of s because every, everything, b 0, b and beta and u, now everything is fixed. So, this is just a function phi from R to R.

So, now, that we have got a function defined from R to R, so for every s I will have solution to the initial value problem. And once I obtain that solution of the initial value problem, I will compute this thing and call it phi of s.

So, now ask the question. So, is there an s star such that phi of s star is 0, that is the question. So, remember this, phi is a function from R to R. So, this is no guarantee, that there is a root of that thing. So, in case there is, if s, then the solution, then the function y t. So, this is the definition. So, u t s star is a solution of BVP. So, this, you remember this is solution of IVP. So, this we already know, and certain conditions on the right hand side, function f, we always have a solution at this point, ok; uniqueness and other thing we will not discuss.

So, just there exists a solution and in addition to that if there is an s star satisfying this condition, which is a root of the function phi, then using that solution of the initial value problem we obtain a solution of the boundary value problem. So, in the literature this is called, this is called shooting method and this is widely used in obtaining numerical solutions of the BVP. So, this is very, very important method and same thing can also be used for theoretical purposes as we are studying, ok. And so if there are more solutions to this equation, if there are more roots to this phi, then we obtain as many as solutions for the boundary value problem.

So, in general we cannot say whether there is, there could be no solution. There could be only one solution, there could be multiple solution and depending on that the boundary value problem will have a solution, a unique solution or multiple solutions or no solutions, so in order to sum up, so let just me state a theorem.

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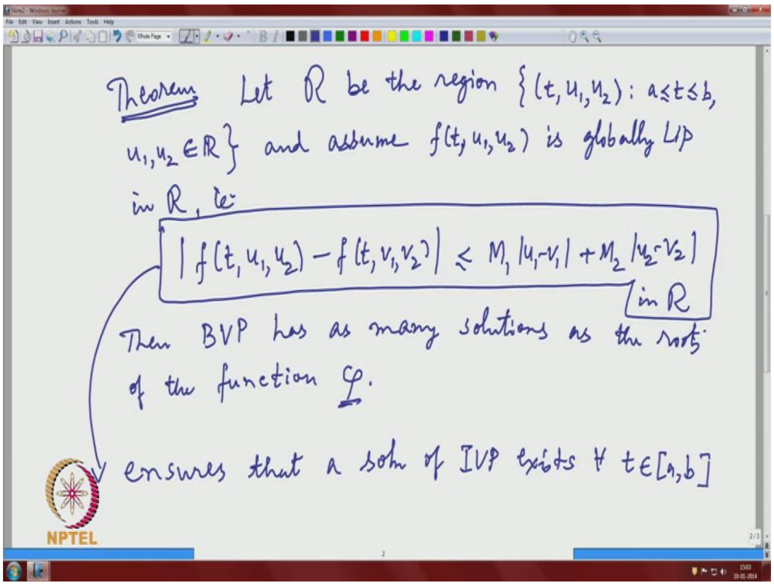

So, let me just put that theorem, ok. So, let R be the region consisting of all triplets t, u 1, u 2. So, a is less than equal to t less than equal to b and u 1, u 2 in R. So, this is the region. And assume, so this is our right hand side M, f, so this is a function of t, u 1, u 2 is globally Lipschitz in R, that is, if you take f of t, u 1, u 2 minus f of t, v 1, v 2. This is less than or equal to some M 1 u 1 minus v 1, that is, M 2 u 2 minus v 2 for all t u 1, u 2 t v 1, v 2 in the region. So, let me not write that for all, ok. Let me write in R in R, ok.

So, then BVP has, has many solutions, has as many solutions as the roots of the function phi. So, for every root of this function phi you get a solution of the boundary value problem. So, remember that what phi is. So, this one, somewhat very strong condition and that ensures, so this is just for ensures, ensure that a solution of IVP exists for all t in a b. So, if you have some other means of ensuring the same thing, we do not meet the strong condition.

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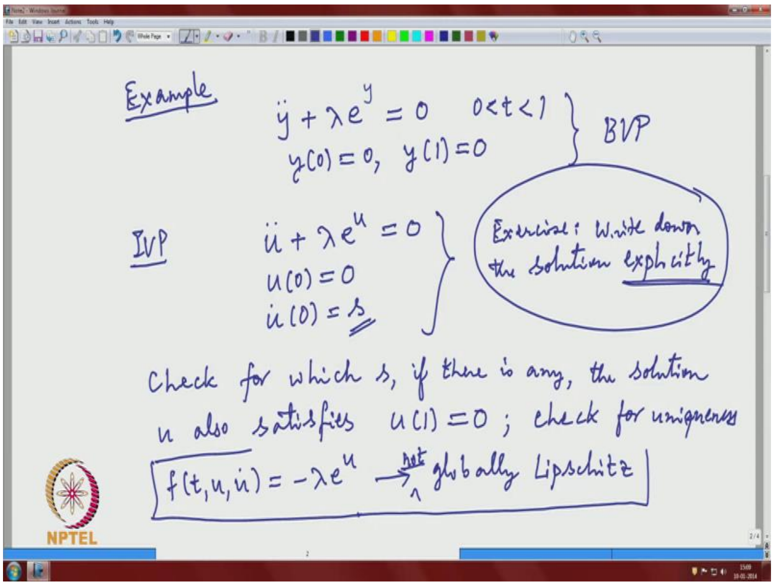

So, in fact, let me give an example. So, an example somewhat difficult integration, but it is doable. So, this is again second order equation. So, you let me, let me just write y, y double dot plus lambda e to the u is equal to 0 in 0 less than t less than 1, and y 0 equal to 0 y 1 equal to 0. Let us consider this BVP. So, if you follow this shooting method. So, what we should do is this. Consider the IVP, u double dot plus lambda e u equal to 0 and 1. Initial condition is same as the one coming from the BVP and second one we want linearly independent. So, it has to be u dot equal to s. s is at our choice, as an exercise write down the solution in explicitly. So, this can be done. So, there is no problem with that. So, in fact, you can use the conservative nature of the equation. So, you can easily integrate once and you convert that into a first order equation and then, using method of u separate the variables and integrate it. So, it is bit complicated, but doable. So, you can explicitly solve it and in this solution you see, that the dependence of s very clearly in that, ok.

Now, you please check using that solution, using that solution check for which s, for which s, if there is any, it is not guaranteed. If there is any, if there is any, the solution u also satisfies the other boundary condition, namely u of 1 equal to 0. And if, and also you check, check for uniqueness, whether there is one s or more s, check for uniqueness. So, it is a good exercise. So, we will see how this shooting method is at work.

So, why this example? So, why I want to stress this example is this, in this case, this f of t, u, u dot, very very simple it is, just minus lambda e u, if you want to write it. And this is not globally Lipscitz, not globally. But yet, the solution exists for all t in the interval 0 1. So, that is important. So, there are and that is one, one wants to state as a theorem, one has to be very stringent. So, may be you put lots of conditions, but in practice such stringent conditions may not be required. So, for example, this one. So, even without the assumption of globally Lipschitz you can write down the solution explicitly and you see, that the solution exists for the entire interval 0 1, ok. So, this is one example on that.

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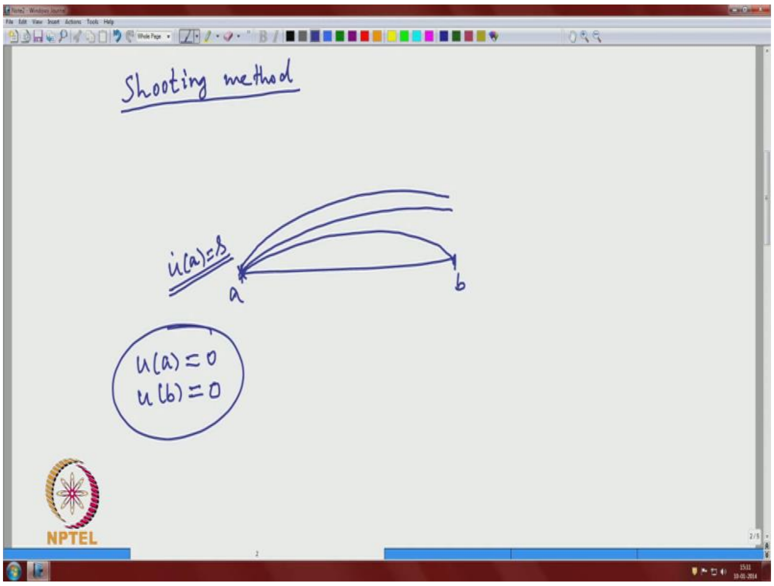

And so let me again continue, why, why this shooting method. So, what is happening, shooting method, so geometric picture. So, here, so this is a, this is b. So, for, let me just take for example simple thing, u a equal to 0, u b equal to 0, just for the illustration. So, what we are doing is, so starting here, but with a velocity. So, u dot a is equal to s u dot a is equal to s. So, we are releasing, we are shooting some particle from here and it may just go somewhere depending on s at b. And if I choose another s, it may go somewhere here. So, we would like to determine an s for which we will eventually go there. So, that is why it is called shooting method.

So, by changing the initial velocity in this particular set of boundary conditions, we are looking for a particular velocity with which if we shoot, we are reaching the destination namely, u b equal to 0. So, that is the idea. So, that is why it is called shooting method.

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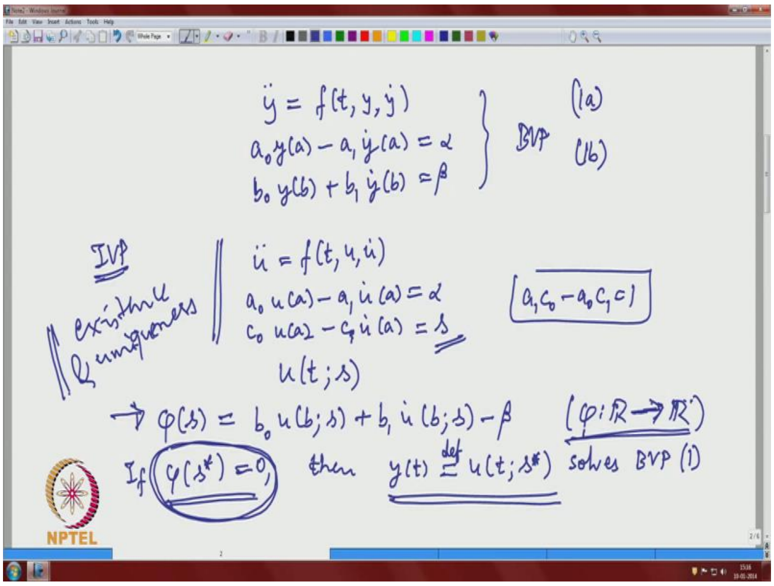

So, again let me go back to the analytical description. So, we started with, let me again go back to y dot equal to f t, y, y dot and a 0 y a… b 0 y b plus b 1 y dot b is equal to beta. So, this is the BVP, call it 1a, 1b.

And the corresponding initial value problem, so u double dot will go to f of t, u, u dot and a 0 u a minus a 1 u dot a is equal to alpha. And c 0 u a minus c 1 u dot a equal to s. And we are normalizing, so a 1 c 0 minus a 0 c 1 is 1. So, normalizing that and we are calling the solution u, stressing the importance on the dependence of s. And then, we form this function coming from the second boundary condition, b 0 u b s plus b 1 u dot b s minus beta. And whenever there is a root of, so if phi s star is 0, then y t defined by… So, this is definition, u of t s star solves BVP 1.

And if there are more roots, then we will also have more solutions for the BV problem and this follows from uniqueness of the initial value problem. So, the important thing is this. So, we have stated theorem using globally Lipschitzness of f. So, you have existence and uniqueness. So, later on we also need continuous dependence on the solution. We are going to differentiate with respect to s.

And as of now, sexistence and uniqueness results of IVP are giving us the solutions of the boundary value problem provided, provided this function phi has roots. If this has no roots, then we are, by this method we are not able to get, produce any solutions at the BVP. So, that is the, so you can imagine, if you go to higher order equations or even first order systems. So, this phi, say remember, here the phi is from R to R and soon we will state a result where existence of roots is guaranteed conditions on phi.

So, if you go to higher order, equation are more general, first order, systems of first order equations. Then, this phi will be mapping from some Euclidean space, to some other Euclidean space. And the study of its 0s, existence of its 0s becomes more complicated and that is where you need the tools from non-linear analysis. In this one-dimensional case we can do all that thing just using one-dimensional calculus and that is what we are going to do. So, let me just state a result and we will discuss in more detail next time.

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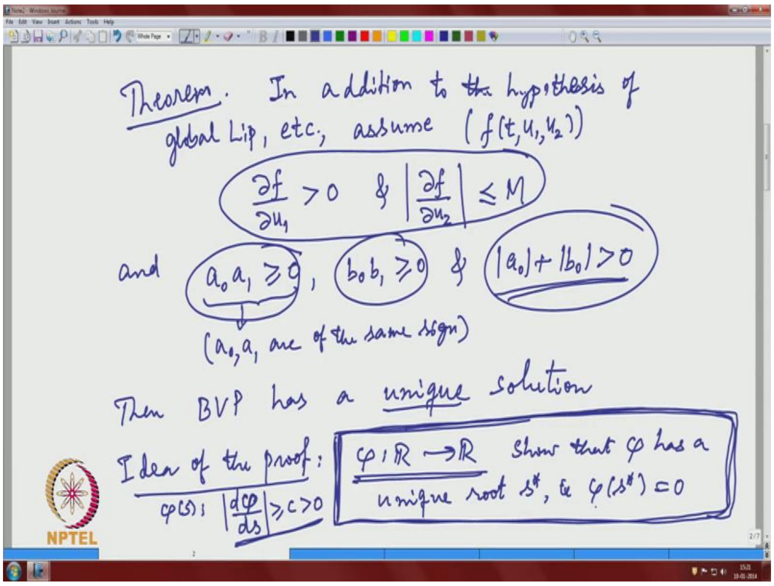

So, just, I will just write a theorem in addition to the hypothesis, hypothesis of global Lipschitz, etcetera. So, we have also several conditions on the coefficients, etcetera. Assume, so this is some extra assumptions on the non-linear right hand side. So, del f by, so remember f of t, u 1, u 2. So, this is the right hand function, so del f 0 and del f by u 2 boundary. And so a 0 a 1 non-negative product, b 0 b 1 nonnegative, and a 0 and b 0 do not vanish simultaneously. So, this is the condition. So, this means, a 0 a 1 are of the same sign and similarly, b 0 b 1 and a 0 b 0 do not vanish simultaneously. So, this is the condition. Then, BVP has a unique solution.

So, we will prove this theorem because it is quite interesting, bit technical, but it uses only one-dimensional calculus. The idea of the proof is, let me just state that idea of the proof. So, look at this phi again from R to R and show, that phi has a unique root s star, that is, phi of s star e 0 and there is no other s for which phi vanishes. So, even in one-dimensional case we have, of course, several conditions. So, one of the conditions we have such a result in one-dimensional case if you show, that this phi is a function of s.

So, you show, that d phi by ds. Next time we will do, that d phi by d s is bounded away from 0. So, this is one sufficient condition under which this is true. And what we are aiming at these conditions, additional conditions on f and also on the coefficients ensure this, that is, what we use one-dimensional calculus using this hypothesis to show, that this d phi by ds is bounded away from 0 and then, the calculus lemma will give us the unique root satisfying this thing and that will prove, that BVP has a unique solution. So, we will see next time.

Thank you.