Ordinary Differential Equations
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Module - 7
Lecture - 38
Linear Second Order Equations

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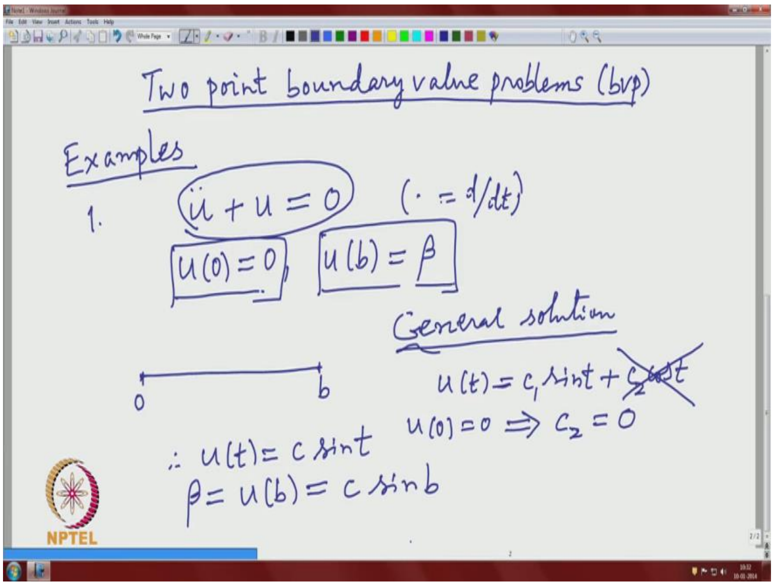

Welcome back, today we will start a new topic. So, in this chapter we study two point Boundary Value Problems. So far short will denote it by BVP, so in many important applications concerning differential equations. The data on the unknown function is given not exact one point, but many points. And we now discuss a restrict our discussions to two point boundary value problems. So, in contrast with initial value problems, which we have already studied. So, the data on the unknown function governing a differential equation or given just at one point that is initial value problem.

And this is when it is more than one point, the data is given at more than one point. Then it is referred to as two points or more points boundary value problems. So, our discussion is restricted to two point boundary value problems. And let me start with some more problems examples. So, just to see what are the difficulties? What are the issues? The first example is again u double dot plus u equal to 0. And again recall dot represents d by d t differentiation with respect to e variable that is time variable and u is the unknown function.

So, this is second order equation and for the initial value problem we give data of u and u dot at one single point. Now, we are interested in two point boundary value problems. So, data is given a two points. So, for example, say u 0 is equal to 0 and u b, b is another point say it is given as beta. So, we have to find a solution to the differential equation in this interval 0 b. We satisfy the differential equation u double dot plus equal to 0 and in addition to that it also satisfy these two boundary conditions at 0 and b.

So, since this is a linear problem we know that the general solution is given by… So, that we have already learnt. So, u of t is c 1 sine t plus c 2 cost for this linear equation this is general solution. And if you want this satisfy these two boundary conditions. So, let us start with at u equal to 0. So, u 0 equal to 0 imply just you put the value substitute the value t equal to 0 in this general solution and require that u 0 equal to 0, then you see that c 2 is 0.

So, with then this cost term is not there. So, we have just therefore, u t is equal to, I just now replace c 1 by just see sine t. So, c is a constant and now we have to determine that constant c if possible in order to satisfies the second boundary condition. So, let us calculate that thing. So, beta is equal to u b, this is equal to c sine b, now you see already difficulties here.

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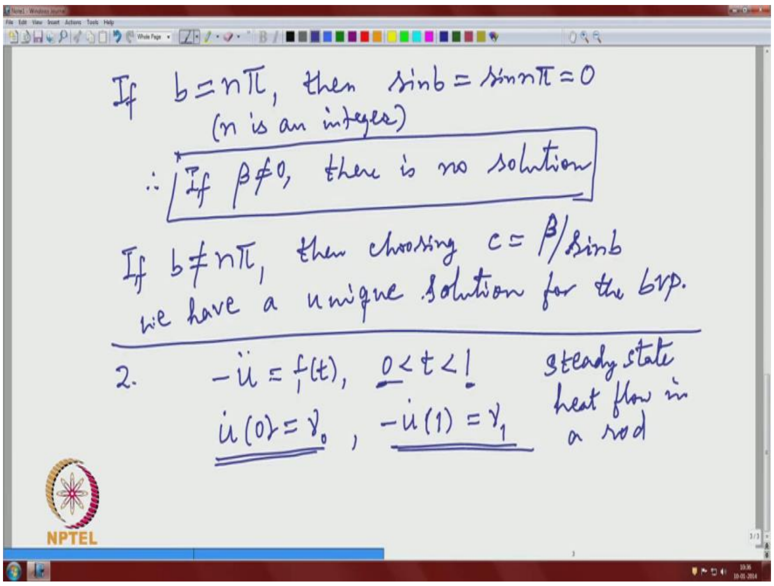

So, for example, so, if b is equal to n pi, then sine b is sine n pi. And that is 0, n is an integer in this case positive integer does not matter. So, therefore, if beta is not 0, there is no solution. So, we just one observation, when equal to n pi n is a positive integer. Then will not have a solution beta is not equal to 0. So, that the other the boundary condition at t equal to b is not satisfied in this case. On the other hand if b is not n pi, then sine b is not 0.

And then choosing c is equal to beta by sine b. You have a unique solution for the boundary value problem. So, even in this simplest case, the second order equation with constant coefficients, we may have a solution or we may not have a solution. So, let us continue with one more example. So, this minus u double dot equal to f t, 0 less than t less than 1 and now the again we have two point 0 and 1.

So, instead of u give the data u double dot u dot 0 is equal to gamma 0 minus u dot 1 you see why I am putting minus 1 in a minute gamma 1. So, this represents steady state heat conduction heat flow in a rod physically that is the. So, rod is placed at the interval 0 1 these two end the points and this is the these represent the heat flux. So, that is one physical interpretation. Now, let us and f is the first theta, f is given to us and now let us integrate the differential equation.

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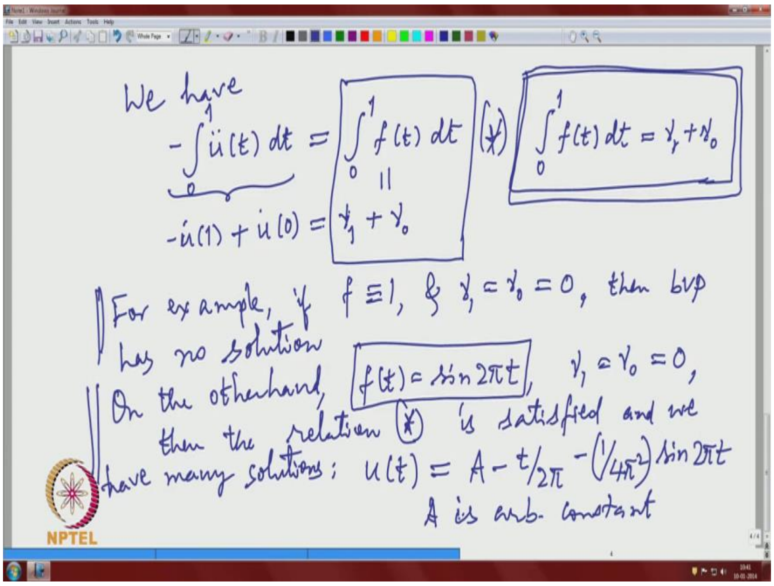

So, we have integral 0 to 1 minus u double dot t d t is equal to 0 to 1 f t d t this is from the differential equation. And the left hand side just to we can integrate it and in just you find that this is minus u dot 1 plus u dot 0 and that is gamma 1 plus gamma 0. So, hence we have this equality between the forcing term and the boundary conditions. So, necessarily the forcing term and the boundary condition should satisfy this. Let me right it. So, 0 to 1 f t and d t equal to gamma 1 plus gamma 0.

So, now this relation will tell us, which boundary conditions and which forcing terms are possible and which are not to have a solution to the BVP. So, for example, if f is identically equal to 1 and gamma 1 equal to gamma 0 is 0. So, more than gamma 1 plus gamma 0 is 0. Then this condition I s not satisfied, then BVP has no solution, because this condition is violated and the left side you get 1 and the right hand side you get 0. So, that is violated. So, there are no solutions. So, on the other hand... So, if I take I keep the boundary conditions same.

Now, I take f t is equal to sine 2 pi x gamma 1 equal to gamma 0 is 0. Then certainly that relation is satisfy, then the relation call it star is satisfy and we have many solutions. So, write the solution again you can just with take this f t equal to sine 2 pi t and solve the differential equation. And you see that you have this minus 1 by 4 pi square sine 2 pi t. So, you can check that this is satisfy the equation in and so, satisfies the boundary conditions.

So, then no problem there and a is arbitrary constant. So, again you see the solution is not unique in one case there is no solution. And in another case we have infinitely many solutions. So, these are the typical problems, we face in the study of the bound two point boundary value problems.

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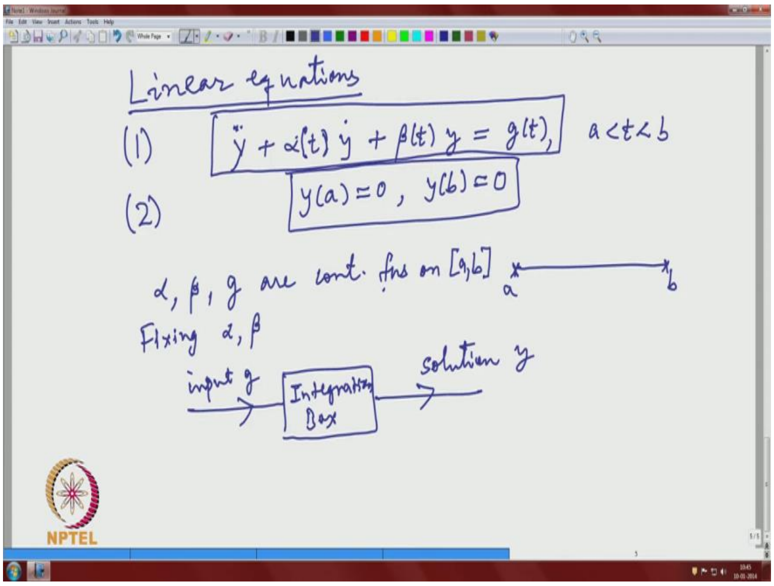

So, with this let me now discuss a general theory, first for linear equations. So, how to obtain a solution for a linear second order equation, satisfying boundary conditions. So, here. So, for simplicity. So, let me just consider y double dot plus alpha t y dot plus beta t y is equal to g of t. So, this is in a less than t less than b and for simplicity let me take the boundary conditions as y a is equal to 0 y b equal to 0. So, this is the problem here. So, we have this interval a b.

So, we are to find a solution for this equation satisfying these two boundary condition. So, you want a c 2 solution y is a c 2 function it is twice continuously differential function satisfying this differential equation and these boundary conditions at a and b. So, while dealing with linear equation we have lots of tools already developed. So, we are going to make use of that. So, the theory for the linear equation is somewhat easier. So, let me start with this. So, let me put some numbers.

So, this is equation 1 this is equation 2. So, this alpha beta g, are continuous functions. So, some minimum hypothesis defined on this close interval a b. So, what would like to view this 1 as. So, if alpha beta are fixed. So, fix fixing alpha beta. So, what would like to do in an analogy with first equation, see first order equation it will just one integration was involved and similar thing we expect even for the second order equation.

So, there is this integration box. So, we will see what that integration is and here the input g and comes out solution y that is would like to prove this boundary value problem one into.

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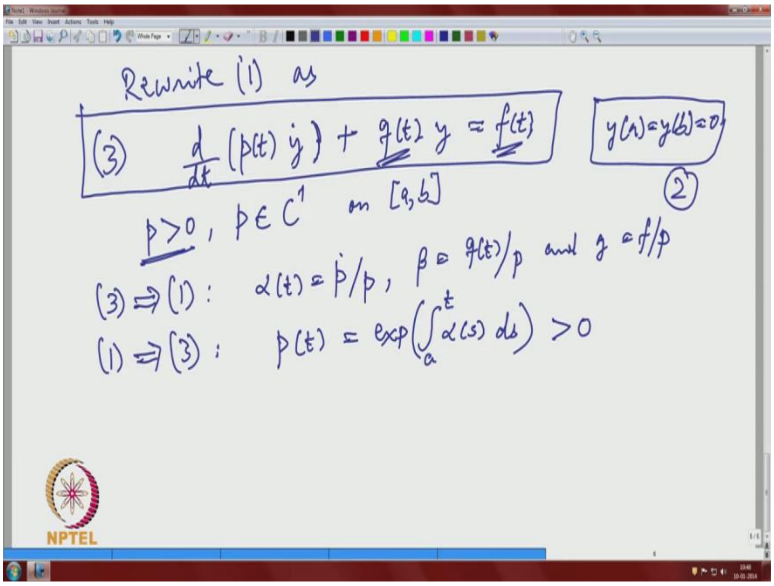

So, will simply little bit, we will rewrite one. So, this is for some simplification rewrite one as. So, this b by t of some p t by dot plus f t y is equal to f t. So, there is some advantages in this thing. So, let us exploit that thing. So, this is I call it 3. So, what is p? P is a positive function and p is also c 1 on a b. So, it is continuously differential function which is positive that is important. So, a very easy to see that one implies. So, first 3 implies 1 that is easier part. So, 3 implies 1.

So, if we expand this by this differentiating this product, you see that alpha is p dot by p and beta is q t by p and g is f p. And for the converse you have to just find what p is and p comes from this. So, put p t is equal to exponential a to t alpha s d s.

And since this is an exponential. So, p is always positive and it is differentiable, because alpha is continuous and we are integrated it. So, that will produce a c 1 function. So, you can verify and with this p you can calculate what q is and what f is. So, now, we restrict this our attention to this 3 and again recall the boundary conditions y a they are not changed y b equal to 0, this is remember that.

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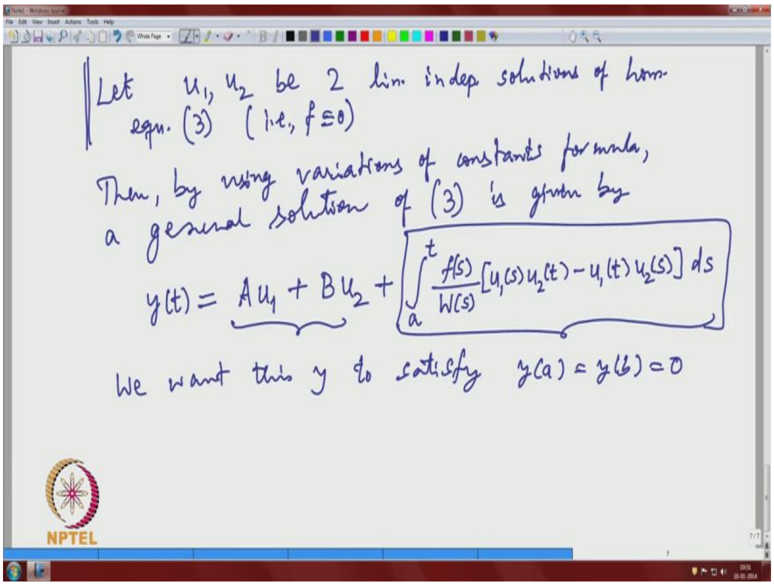

So, let. So, a procedure. So, let u 1 u 2 be 2, linearly independent solutions of homogeneous equation 3. So, that is in 3 you take f j and then you and this existence of these 2 linear independent solutions come from general theory of linear equations. And we are sure that they are there they exists. Then by using variation of constants formula a general solution of 3 is given by. So, one part consist of solution of the homogeneous equation. And that is a linear combination of u 1 and u 2; the other one is coming from the particular integral and let me just writes that.

So, this is a to t f s w s. So, check this one from bit complicated, but straightforward, so u 1 s u 2 t minus u 1 t u 2 s d s. So, this is for the homogenous equation and this is for the in homogeneous part. So, this is general solution of 3. Now, we want this y to satisfy the boundary conditions y a equal to y b equal to 0 and when t is equal to a this integral is no more there and one that integral is there when t equal to b. So, let us write those 2 equations.

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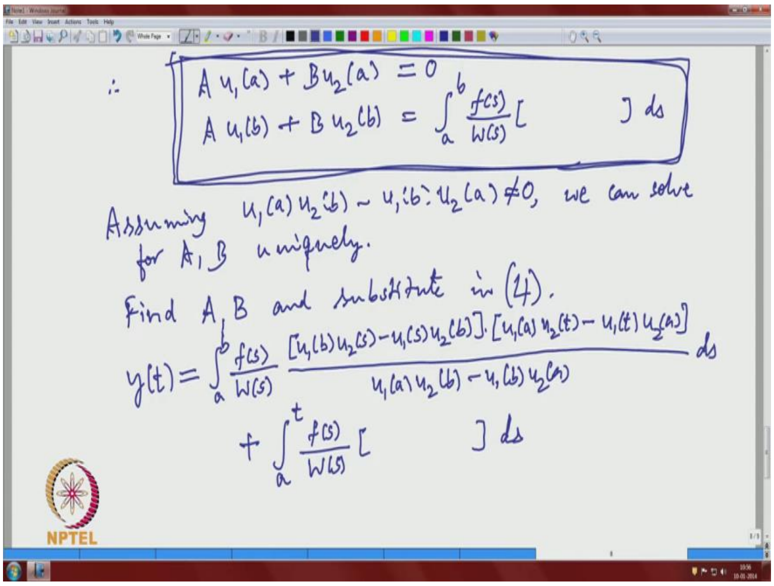

So, therefore, A u 1 a plus B u 2 a equal to 0 and A u 1 b plus B u 2 b I s equal to a to b. So, let me not write the whole thing. So, you just copy from the previous line, only thing is I am taking the other side. So, there will be a sign change. So, a b are constants a and b satisfies these 2 equations linear equations. And you have to solve them provided, this determinant is non-zero the determinant of the coefficient matrix is non zero. So, assuming. So, here the determinant is just this one u 1 a u 2 b minus u 1 b u 2 a non zero, we can solve for a b uniquely.

So, now you write a and b and uniquely. So, find a b and substitute in. So, let us go back and see what this one ((Refer Time: 27:16)). So, let us call this some equation and put 3 there only. So, let me put 4. So, after solving for a and b, from those equations, you are solving now a b from these two linear equations in 4. So, let me just write the end results. So, it is bit involve,, but very straight forward.

So, let me just write that a b f s I forgot to tell what w s is let me complete that thing u 1 b u 2 s minus u 1 s u 2 b you also check that I might have done dome mistake. But, you can verify yourself these are just linear equations. So, there is no problem in verification u of b u a and this is, where that determinant comes u 1 a into b minus u 1 b u 2 a d s and as usual there is other part there is no change there. So, let me just a t and let me just put as it is…

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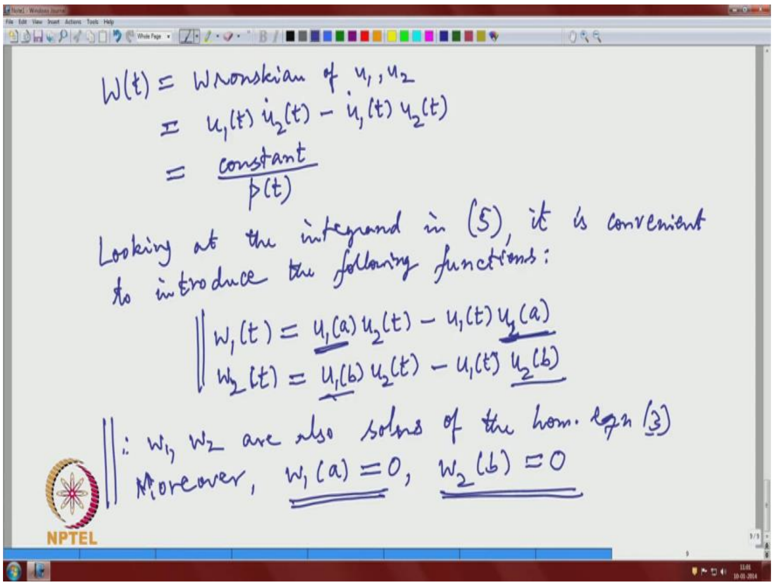

So, what is w. So, let me just what w s is. So, w s at w t in the Wronskian of u 1 u 2 at t. And since both these they are linearly independent solutions of the homogeneous equation. So, this is by definition u 1 t u 2 dot t and u 1 dot t u 2 t. And since they satisfy the same homogeneous equation you can easily check that it, this is actually given by. So, this we already done earlier for general linear systems. So, it is constants by p t and remember we have assumed p t is bigger than 0.

And looking at this expression lets go back again this complicated let me call it 5. So, this complicated expression. So, the first integral is a to b and second one is only from a to t. So, we can split this integral into two parts. So, y t is something a to t plus t to b. So, remember this because we are going to imitate this. So, looking at these expressions, in the integrand. So, it is convenient to. So, looking at the integrand… So, in 5 it is convenient to introduce the following functions.

So, namely w 1 t is equal to u 1 a on u 2 t minus u 1 t u 1 u 2 a and w 2 t at the other end this is at one end t equal to e and this 1 is u 1 b u 2 t minus u 1 t u 2 b. And w 1 and w 2 being linear combinations of u 1 and u 2 these are just constants, u 1 a u 2 a they are just constants here. So, they are linear combinations of u 1 and u 2. So, therefore, w 1 w 2 are also solutions of the homogeneous equation homogeneous equation. So, right hand side is 0.

And more over if you look at these expressions. So, more over and this is the advantage we will take now more over w 1 a is equal to 0 and w 2 b 0. So, they are the… So, when solutions of the homogenous equation and each one of satisfying one boundary condition. So, w 1 satisfy the boundary condition at a and w 2 satisfy the boundary condition at b. And now you just forget whatever calculations we did. And start a fresh from this w 1 and w 2 that is just to get a motivation, how to obtain this w 1 and w 2.

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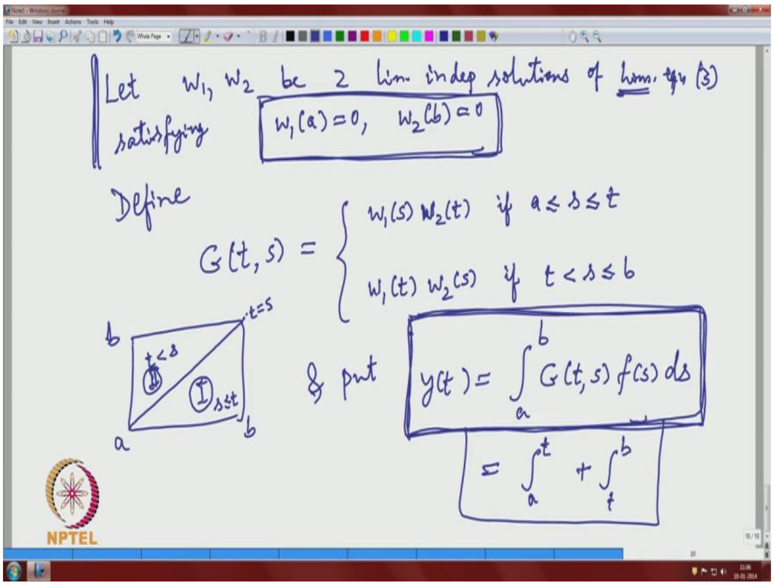

So, now let us start with, so a fresh. So, let w 1 w 2 be 2 linearly independent solutions of homogeneous equation 3 you have to remember that homogeneous equation 3 satisfying w 1 a equal to 0 and w 2 b equal to 0. So, if we have a more general condition this is what we have to do. You have to find two linearly independent solutions of the homogenous equation 3. Satisfying one of them is satisfy is boundary condition at one point and another one at another point. So, for simplicity you have taken this simple boundary condition here, that namely y a equal to y b equal to 0.

And w 1 satisfy the boundary condition at one point a and w 2 satisfy the boundary condition at b now define. So, g of t s it has two parts. So, let me draw a picture little later. So, this is w 1 s and w 2 t if a is less than or equal to s less than or equal to t. And other we write w 1 t w 2 s if t is less than s less than or equal to. So, if you take this square a b. So, this is the line t equal to s in one portion this is less than or equal to t and here t is less than that is other portion.

So, this is the first portion and that is what is happening and defined and put y of t is equal to, and little comment later, just look at the definition of g. So, g has 2 parts, one in a t. So, this integrals splits into two parts a to t and t to p similar to given, we did for p and we except this y two satisfies the differential equation and also the boundary condition. And this is the integral box we were talking at the beginning. And so, see this integral box is just manufactured with the help of two linearly independent solution nothing else. And that is comes from the just homogeneous equation. And now we put this input f the right hand side. And we obtained corresponding solution. So, this is what we have obtained and now we verify that y t indices satisfies the differential equation 3 and also the boundary conditions.

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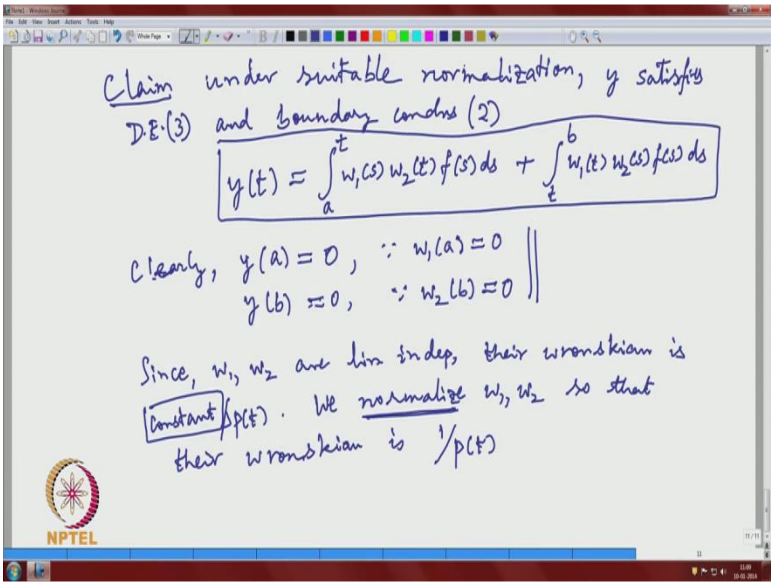

So, this is the in the next 10 minutes we will do that thing. So. So, claim under suitable normalization, will see what this normalization is y satisfies D 3 differential equations 3 and boundary conditions. So, it is very easy to verify. So, remember y t is given by. So, let me just write that. So, a t w 1 s w 2 t f s d s plus t to b w 1 t w 2 s f s d s that remember this every time you have to do that thing.

And looking at these expressions it is very easy to see that. So. So, clearly y a, y a just 0, y a 0 because w 1 is 0, because a is 0 and y b is 0, because w 2 b is 0. You see the way we have chosen the two linearly independent solutions of the homogeneous equation they are helping us in satisfying the boundary conditions. And since we are assuming w

1 and w 2 are also linearly independent. So, since w 1, w 2 are linearly independent their Wronskian is this, we have already seen is constant divided by p t.

And the normalization we normalize. So, normalize means you just multiply w 1, w 2 by constants normalize w 1, w 2. So, that their Wronskian is 1 by p t we want this constant to be 1. And that can always be achieved by multiplication by some constants. So, there is no problem there. So, this is the normalization, we are talking about.

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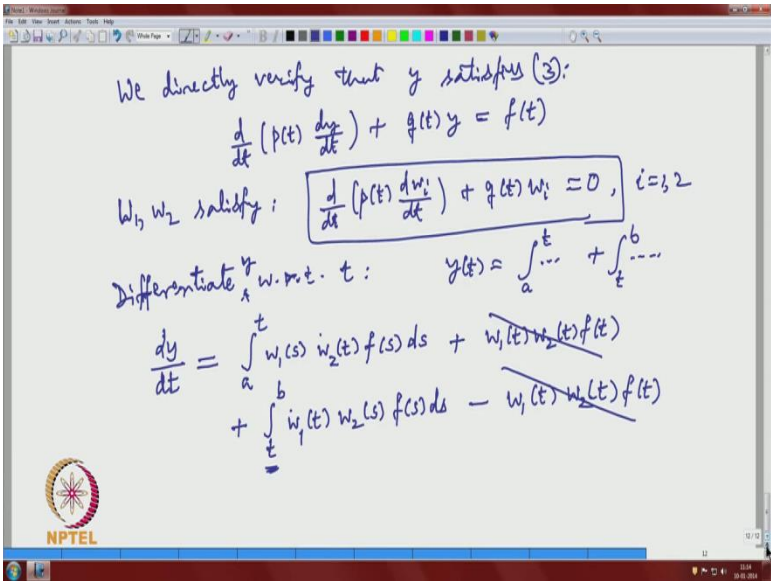

So, now we directly verify that y satisfies 3. So, again let me recall what that 3 is the differential equation. So, it is d by d t of p t d y by d t plus q t y I s equal to f t. And w 1, w 2 satisfies the homogeneous equation so; that means, d d t p t w I dot let me write i t d by d w i by d t plus q t w i equal to 0. So, I equal to 1 2. So, remember that this satisfy this they are linearly independent and their Wronskian is just 1 by p t. So, that also you remember now y is given by that again let me not write it here. So, just go back here. So, y is given by this remember that thing and no we will just make this.

So, differentiate. So, you have to be bit careful there, because the integral limits also involve to I t. So, you have to just bit careful there, which you already done it differentiate y. So, y is given by. So, let me just write it. So, a to t plus t to b recall that. So, d y by d t is equal to. So, let me just write it a to t w 1 s w 2 dot t the t is only here f of s d s, and since this t is there in the limit. So, you have to also differentiate that. So, that you get. So, w 1 t w 2 t f t and I f you do for the other integral. Similarly, there is t in w 1. So, w 1 dot t w 2 s f s d s and now t is in the lower limit. So, you get a minus here. So, w 1 t w 2 t f t and this and this cancels.

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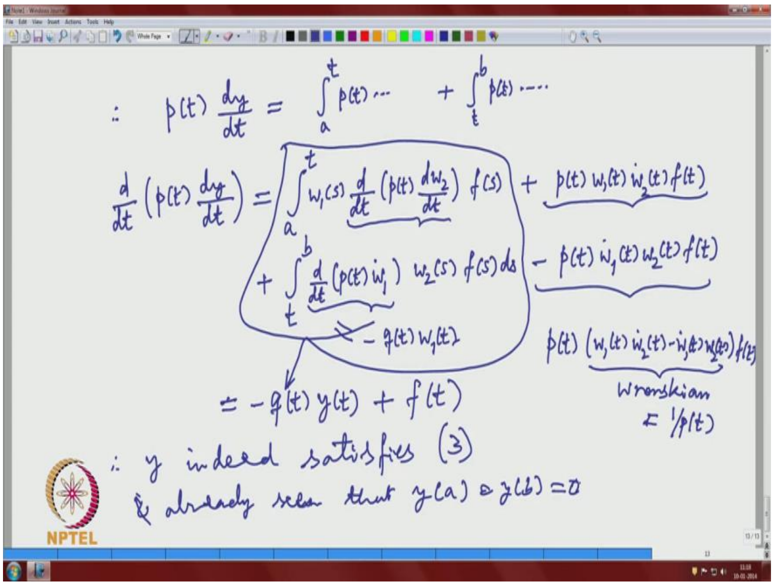

So, and now, looking at equation 3 now multiply by p t. So, therefore, p t d y by d t is equal to a to t. Now, just p t just goes I n same way that is all. So, there is no problem there ((Refer Time: 47:39)). And now again we will differentiate this, so d by d t of p t d y by d t. And this is similar to what we did in the previous step. So, just we have now a to t. So, w 1 s d by d t of p t d w 2 by d t f as it is.

And now because of the presence of the t in the integral limit. So, we have this p t w 1 t w 2 dot t f t. And similarly from the other integral t to b now it is d by d t of p t w 1 dot w 2 s f s d s. And now I have minus p t w 1 dot t w 2 t and f t. And remember w 1 w 2 they satisfy the homogeneous equation 3. And this is. So, simply this let me write here this is minus q t q 1. And similarly that one is minus q y w 2 t and q t that nothing to do with the integration variable s. So, that q t just simply comes out and what is remaining I s just minus q t.

And I f look at the expression this is just y of t. So, the integrals are taken care of now that person I s coming here and now about what about this 1 you just simplify them. So, you get p t w 1 t w dot t minus w 1 dot t w 2 t f t and this I s nothing but, the Wronskian of w 1 and w 2. And we are normalized, so that the Wronskian is equal to 1 by p t and this p t p t goes away. So, you have just plus f t, so indeed. So, therefore, y indeed satisfies, and equation 3. So, already check that this y has already satisfies, we already seen that y a equal to y b equal to 0.

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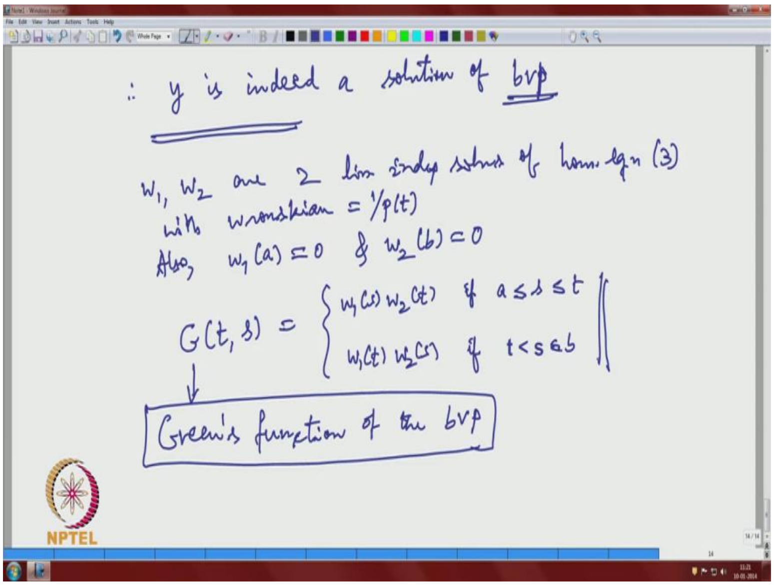

So, therefore, y is indeed a solution of boundary value problem in, we have just in remaining 5 minutes, let me just say what we have done. So, we have started with just let me recall. So, w 1 w 2 w 1 w 2 are two linearly independent solutions of homogeneous equation 3 with. So, we have normalized Wronskian I s equal to 1 by p t also.

So, w 1 a equal to 0 and w 2 b equal to 0. So, if there are more general boundary conditions. So, you have to choose w 1 and w 2 appropriately. And then we define this g. So, it as a name. So, this is w 1 s w 2 t if a is less than or equal to s less than or equal to t. Other way around if. So, g is called green’s function of the boundary value problem. So, it very much independent on the boundary values that is very important.

So, if change the boundary values, if we take more general thing then the green’s function also changes and this g then gives the solution of the boundary value problem. So, next time we will consider some examples of constructions of this green’s functions. And you are seeing that we are taking the advantage of the linear system and the existence of linearity independent solutions. And more generally the fundamental matrix, if we are dealing with a first order linear system. And that is going to help us in construction of the green function and which will eventually give solutions of our boundary value problem.

Thank you.