Ordinary Differential Equations
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Module - 6
Lecture - 37
Periodic Orbits and Poincare Bendixon Theorem Continued

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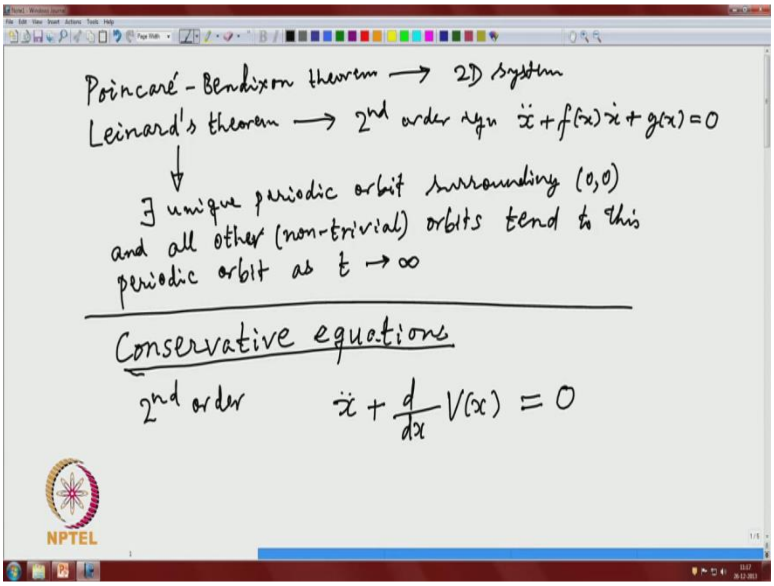

Recall in the previous class, we were discussing two important theorems. One is Poincare Bendixon theorem and another one is Leinards theorem. So, both give sufficient conditions for the existence of the periodic orbits in 2D. Let me stress that again, these are very 2D specific theorems. And this is applicable for 2D systems of first order equations and this one second order equation, x double dot plus f x x dot plus g x equal to 0 under certain conditions on f and g which we stated in the previous class.

So, Leinards theorem proves that there exists unique periodic orbit surrounding. The only equilibrium points 0, 0 and all other non-trivial orbits. So, here the trivial orbit is only the equilibrium solution. All other non-trivial orbits tend to this periodic orbit as t tends to infinity. So, that is important conclusion of this Leinards theorem.

Today, we will discuss another class of equations, where the phase plane analysis can be done somewhat easily. And we obtain a complete phase portrait of 2D systems and these are termed as conservative equations. We will shortly see why this name conservative equations and these are again second order equations of the part x double dot plus d by d x of V x equal to 0. So, this function V is assumed to be smooth.

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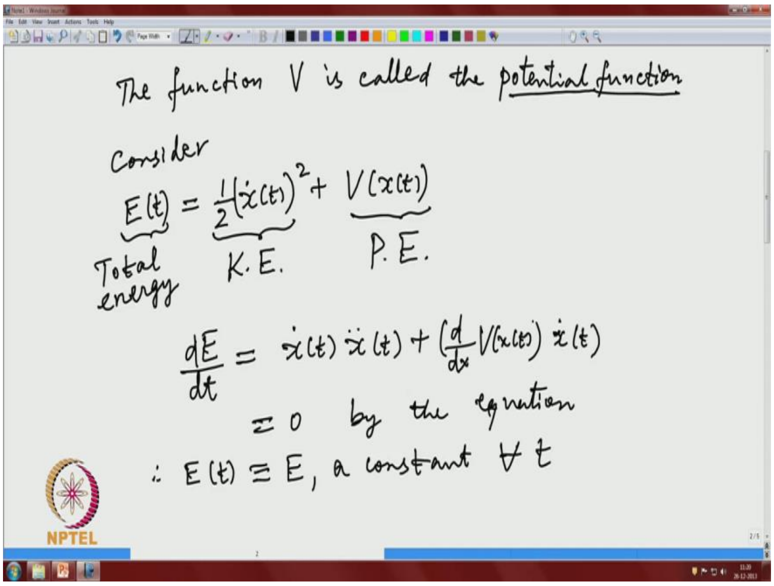

The function V is called the potential function. The terminology again comes from classical mechanics. So, consider this quantity. So, if x is a solution of that conservative second order equation. Consider this quantity x dot t square plus V x of the, and using, so if. So, let me call this as E t, E of t. So, x t is a solution of that equation. So, this represents kinetic energy and this represents potential energy. So, this is the total energy at time t, total energy of the particle.

So, if we differentiate this E with respect to t. We get is x dot t x double dot t plus using chain rule. So, we have this d y by d x V of x t times x dot and now, x dot is common and if, we look at the other terms x double dot t plus d by d x of V x of t and that is 0 by the equation. So, what we are see by this simple calculation that total energy is conserved. So, it is the same per all t. So, therefore, E t is identically equal to E a constant for all t. So, this is the region why that second order equation is called a conservative equation.

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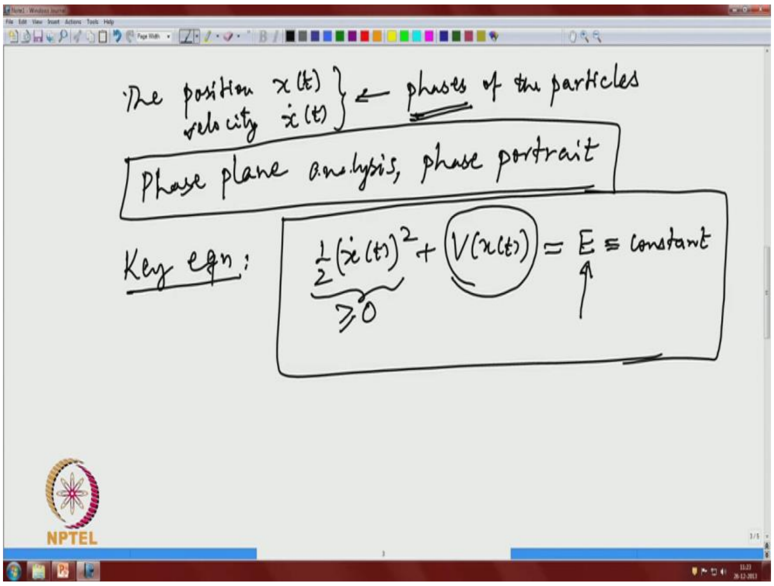

So, and the position x t and velocity x dot t, these are called as phases of the particle. Again, terminology comes from mechanics. So, we will do analysis involving phases that is phase plane analysis. And when you describe the position and velocity for all time and that is described as phase portrait. So, that is phase plane analysis, these are the terminology coming from this. This plane because there is only position and velocity.

So, its two dimensional phase plane analysis, phase portrait, refer to this analysis and description of all the orbits. And that is what we are going to do for some simple examples. So, again the key equation is the following. So, this is the key equation, this half x dot t square plus V of x of t is equal to E and this is remember a constant, and this, in this term is always non-negative.

This V has to be less than or equal to E, so for different levels of this will see, how this phase portrait will change. And so once, we are given this potential function and this different energy levels E. You take and that will restrict the values of x, and then you use this key equation to analyze the behavior of x t and x dot t. That is the position and velocity of the particle. So, we will explain with through simple examples. How this, we done examples, the first example is pendulum equation.

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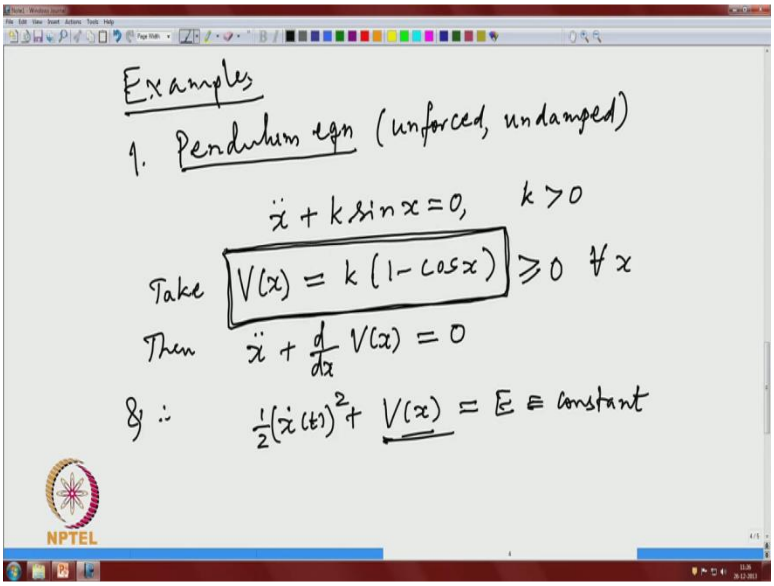

So, we are not adding any force. So, this is unforced and there is no damping. So, this is un-damped. So, this is the equation x double dot plus k sin x is equal to 0. So, k is some fixed constant, k is a positive constant. So, this can be written as a conservative equation so by. So, take there are many choices and I will explain by this particular choice. So, this you take V x is equal to k into 1 minus.

So, you can always add a constant to the potential function without changing the equation. And that constant can be added in such way to achieve some simplification for the potential function. So, here you see that since it is 1 minus Cos x. So, I added this constant. Otherwise it is just minus k Cos x. I, we can also take that 1. I have added k here just to make this potential function non-negative for all x.

So, then the equation becomes x double dot plus d by d x of V x equal to 0 and therefore, the key equation in this situation is. So, half x dot t square plus V x is equal to constant. So, remember this is constant and our V is given by this. So, remember this equation. So, just what we are going to do now. So, let me draw this potential function as a function of x and let us see how it looks like.

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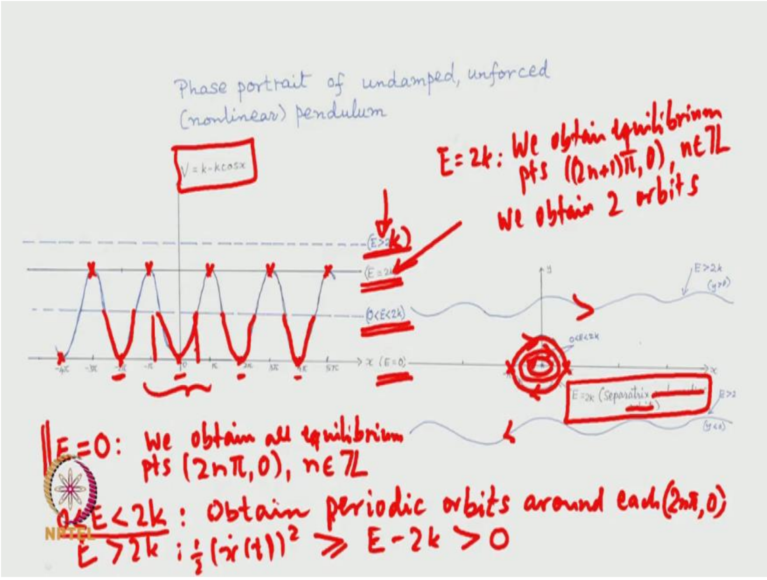

So, this is the function V of x, V of x is given here. So, this is remember, this is V of x and the graph of V of x suggest that, we take these different energy levels E equal to 0, and then E between 0 and 2 k, and then E equal to 2 k. And, then finally, E bigger than 2 k. This graph itself suggests and so let us take one by one. So, just observe carefully. So, let us take with E equal to 0. E equal to 0, you know that again the key equation.

You remember that half of x dot t square plus V of x t equal to E and now, we are taking the E equal to 0, and so, and V will be 0, only at this points. These are all even multiplies of pi and x dot will be 0, because of that conservative key equation. So, here we obtain all equilibrium points. 2 n pi 0 an n integer. So, all even multiples of pi, we obtained at this energy level, important to remember that E equal to 0, we are taking now. So, x dot is 0 there and we obtain these equilibrium points of the system.

So, what about 0 next level, the next level 0 E less than 2 k and if you see observe the graph of E. The values of x will be restricted to these portions. So, these are symmetrical intervals around each of this equilibrium points. 2 n pi 0. They are around that and now x value is restricted only to that interval. So, you can explicitly calculate what this point is, that is not difficult we can. So, it is strictly in the interval here minus pi pi and similarly it will be in different intervals.

So, since all are have similar structure. So, let us concentrate around this interval g get 0. The equilibrium point 0 and corresponding to that value of x. We obtain periodic orbits which are exhibited here and arrow is this. So, we obtain the periodic orbits, when 0 E less than 2 k. So, obtain. So, let me write here periodic orbits around each 2 n pi 0 around each equilibrium point 2 n pi 0. So, here I have just shown it around 0, 0 is same thing happens at all other points.

And next, when you go to this energy level E equal to 2 k, that is the next level and here, we do not have any restriction on x. Just to the point that, see now, in this case first, we obtain the equilibrium points 2 n plus 1 pi 0. So, n is again an integer, which we did not obtain in the energy level 0. Now, with energy level E equal to 2 k we obtain all this. These are the equilibrium points, odd multiplies of 2 pi. And now, if we start an orbit at a point different from this the equilibrium points and this is what we get. So, this is the, and this just this is the one.

Actually, there are two orbits, both approaching two different equilibrium points. The upper one is as t increases the arrow is mark like that and the bottom one like this because in the upper portion and x dot is positive. So, x is increasing and in the bottom orbit x dot is negative. So, it is approaching it going to minus pi as t tends to infinity. So, we obtain addition to this thing. We obtain two orbits and together they are referred to as so, here high-lighted it.

So, Homoclinic is not there. Let me, it is a separaterix. As it is going separate two kinds of orbits and the final one, E bigger than 2 k and there is absolutely no restrictions on the values of x and from the key equations. You also see that, in this case, E bigger than 2 k. Let me write that, you also see that from the key equation. x dot t square is always bigger than or equal to, there is half. Let me write that half E minus 2k and now, E is bigger than 2 k. So, this is always positive. So, the x dot t is always bounded away from 0. So, either it is always positive or again always negative.

So, that is why we get two orbits. So, one here it is since x dot is positive. So, x is going that direction and what a one is in this direction. So, In this case, for the un-damped, unforced, Pendulum equation. We obtain a complete phase portrait of this orbit; several orbits; you see that. So, there are again, let me repeat it, so for there are four energy levels, that to be considered in this case. So, E equal to 0, we obtain all equilibrium points, which are even multiples of pi and 0. And for the energy level 0 less than E less than 2 k.

We obtain periodic orbits around each of the equilibrium points 2 n pi 0 and for E equal to 2 k. We obtain the equilibrium points, which are odd multiples of pi and we are in addition to that, we also obtain two orbits and there are around each this importance. So, the same pictures happen at all other points. So, that is important, it is not just restricted to this zeros, the same thing you repeat it to other parts, other 2 n multiples of, even multiplies of pi and E bigger than 2 k. We obtain unbounded solutions and either the solution is always increasing are always decreasing and as shown in the picture, second example.

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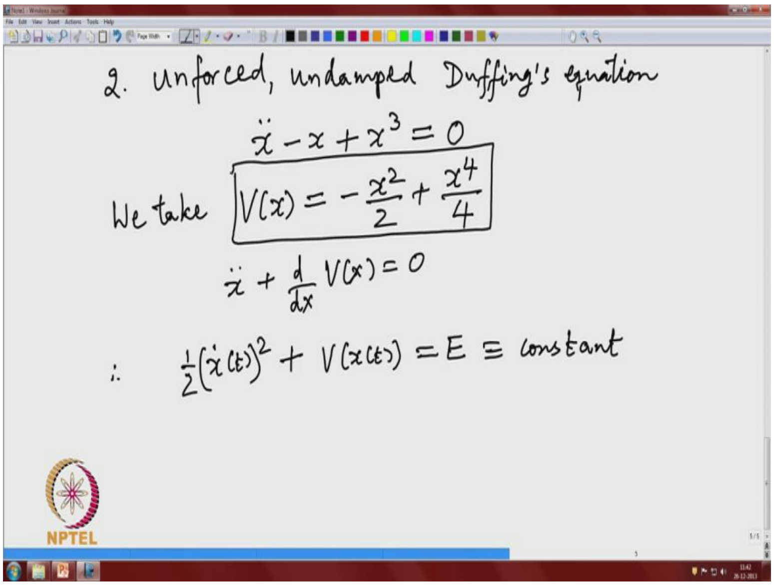

So, this is again unforced un-damped Duffings equation. So, earlier we have discussed about this thing and now, we are taking the un-damped Duffings equation and this is again second order equation given by. So, there is x dot term is missing, that is the damp one. So, we are taking delta equal to 0. So, here we take looking at the equation. We take V x. So, here we are not going to add any constant. So, just to keep it like that is square by 2 plus x 4 by 4. So, with this potential function the given equation is written as x dot plus d by d x of V x equal to 0.

So, therefore, we have the key equation again, x dot t square. So, any solution will satisfy this equal to, this is a constant, so before deriving the phase portrait. So, let me just spend a few minutes on this potential function. So that, the explanation of x portrait will be very clear. So, here the V x is minus x square by 2 plus x 4 by 4. So, this is symmetric V x, V of x is equal to V of minus x and quickly draw, I am showing this more in the next picture.

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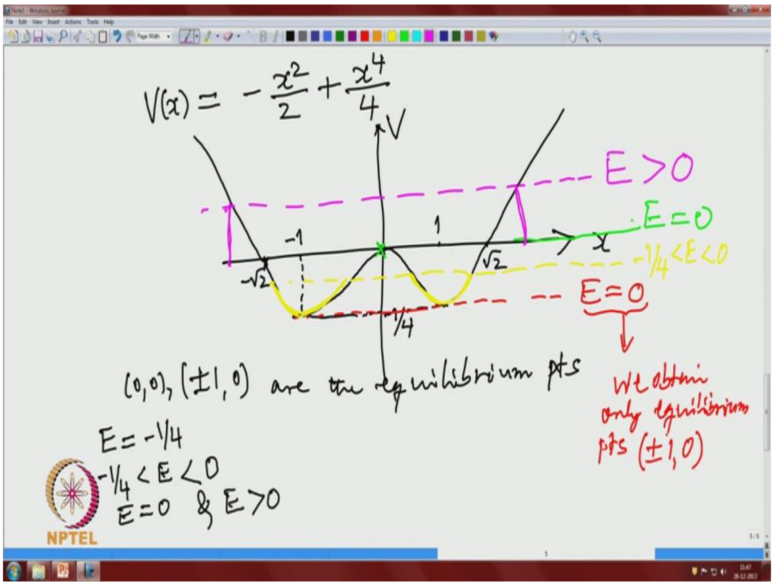

So, this is just V here. So, this is a 0 here, and then if you solve this thing you have two more zeros minus root 2 and plus root 2 and this is negative here, 0 here. So, it is symmetric varies more, because it is a polynomial, just you can do that yourself. So, this minimum occurs. So, this is level minus quarter. So, this occurs at plus 1 and this is minus 1 and recall that. In this case, we have 0, 0 plus or minus 1, 0 are the equilibrium points and you may recall the linear analysis. We have done at the test to understand the behavior of orbits very clearly, this we have already done it.

So, this 0, 0 is always unstable and plus or minus 1, 0 in this un-damped case. They are stable, but not assume to decrease table. So, we can recall that thing and the graph of V suggest that the energy levels. We are to consider E is equal to minus quarter, and then we have x minus quarter less than E less than 0, and then we have E equal to 0 and E is bigger than. So, let me just for a reference. So, this is the, E equal to 0 levels and this one in any varying between. So, there are many. So, this is minus quarter less than E less than 0, and then will have this x axis is also energy level 0 and the last one.

So, anything above E bigger than 0; so again in the like in the Pendulum case, we also have four energy levels and at E equal to 0, you only get. So, let me again see it here. So, this level, we obtain only equilibrium solutions, equilibrium points, plus or minus 1, 0 and when energy level is between minus quarter and 0, the values of x is restricted to this portion is called potential bell. Again physics language so around each of this equilibrium parts. So, they are around this plus or minus 1, 0 and again, we obtain periodic solutions around that, and then at energy level 0.

We do obtain this equilibrium point 0, 0, and then when we start the orbits at it different point. We do obtain again separatrix and when E is bigger than 0. Again, you see this value of x is restricted to this interval. This interval and that level is restricted to that. So, again, we obtain periodic solutions and these are shown in the next picture. So, that, that is what I am going to show you. So, just remember this diagram is more elaborately done here.

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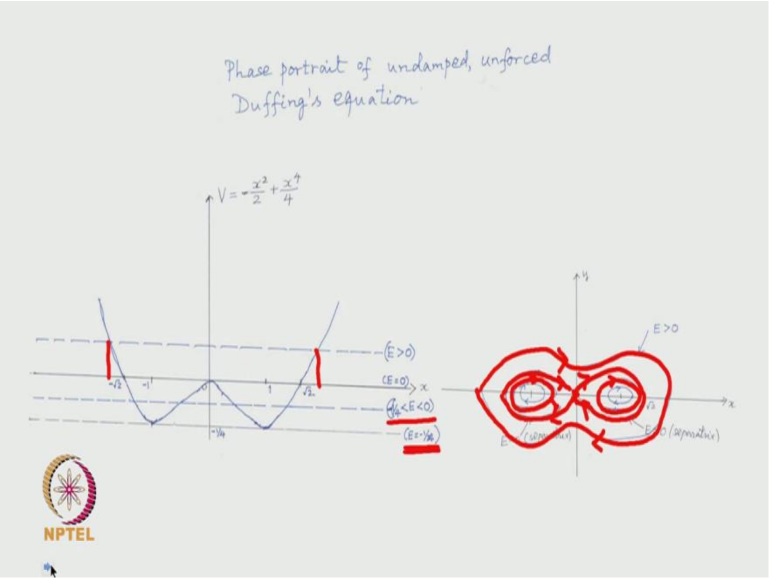

So, what we saw in the previous picture same thing is done here and see at energy level E equal to minus quarter. We obtain only the equilibrium point, plus or minus 1, 0, and then when the energy level is between minus quarter and 0. We obtain this periodic orbits, I am just showing one around each of this equilibrium point plus or minus 1, 0. So, this is minus 1, 0 and that is, this is minus 1, 0 and that is plus 1, 0 and when energy level is exactly 0 apart from this equilibrium point 0, 0.

We do obtain the orbits one on the right side and another one. So, this is single orbit. Let me just explain that, it is not periodic orbit that, you should keep in mind because, this is an equilibrium point. So, another one and the nature of these orbits clearly indicate that the origin is unstable and when again E is greater than 0 as I said. So, it is restricted to x is restricted to this interval and we obtain this red color one. It is going in this direction. So, that is the periodic orbit. So, we have in the case of un-damped unforced Duffings Equation.

We do have again several different cases of orbits at one energy level. We get only the equilibrium points. Then, we get the periodic orbits then we get separatrix, and then again periodic orbits. So, in the, in the previous example, in the final thing, we got only unbounded solutions, but here at all levels, we get only periodic orbits. So, that is the big difference. So, let me just make few remarks about this duffings equation again. So, again here; so, let me just concentrate on this level energy level 0. So, we obtain. So, let me write that thing lets go to the next one.

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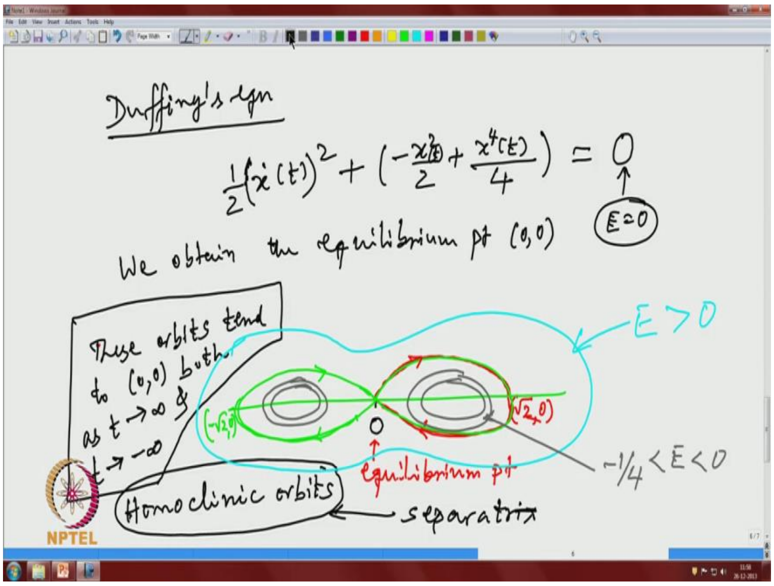

So, Duffings Equation again, so Duffings Equation, let me just call with a remark about that. So, we have again half x dot t square plus. This let me write this x square by 2 plus everything has t x square t by 4 equal to 0. So, energy level 0, E is equal to 0. I am considering that particular energy level 1 to make a remark and that so first. We obtain the equilibrium point 0, 0. So, this just let me draw that. So, this is 0, 0. So, if you start the orbit at a different point, then this one then we got. So, this let me draw that so, for example, if I take x dot positive.

So, we obtain this. So, this will go up to that root 2 root 2 0 and similarly, we obtain the other one. Here, they are supposed to be symmetric of this given and the remark. I want to make is following. So, these are; so, these orbits suppose it is start here. So, it will make an entire round here and come back to the. It can come here only at infinity and you can see that as t goes to minus k. It also goes to 0. So, this is equilibrium point. So, I remember that and similarly, here if an orbits starts here.

Its starts here it goes all the way around and again. Eventually, goes to 0 as t goes to infinity and here also it goes to 0 as t goes to minus infinity. So, let me just draw this. So, both these orbits have the special property. So, these orbit. So, let me now, write these two orbits tend to the equilibrium point 0, 0 both as t tends to infinity and t tends to minus infinity.

So, this is very special here and such orbits are called. When an orbit tends to an equilibrium point both as t goes to infinity, as well as t goes to minus infinity. See in the previous case that does not happen. You can just see that in Pendulum case that does not happen as t goes to infinity. It goes to one equilibrium point and t goes to minus infinity, t goes to another equilibrium point. It does not go to the same equilibrium point and such orbits are called Homoclinic orbits.

In this case, the Duffings Equation exhibits the existence of there are two Homoclinic orbits and they corresponding to energy level 0 and again when you e positive. So, this, these are also now act as separatrix. So, these orbits separate a two kind of periodic orbits one inside this. So, we have here several, we saw that thing there are several periodic orbits here. And here around and above that, we have this another periodic orbits.

So, let me use different color. So, this is you saw that right. So, this corresponds to E positive and this, these two orbits correspond to minus quarter less than E less than 0. So, for every E there we have a periodic orbit. So, that is fine. So, they are not difficult but, you have to just work out patiently in order to see the phase portrait of the orbits. So, with that thing, we come to end of this discussion on Qualitative Theory of Differential Equations.

So, before ending let me make few comments. So, this Qualitative Theory of Differential Equations is an very important and quite deep subject. What we have done in the last few lectures we had. We have been able to scratch this surface of this worst and important subject. So, if you understand this properly, you will be able to read more advanced topics and there are excellent text books providing that material. So, that we have included in the references. So, you will be able to understand much more advance topics, once you grasp this elementary knowledge.

Thank you.