Ordinary Differential Equations
Prof. P S Datti
Department of Mathematics
Indian Institute of Science, Bangalore

Module - 6
Lecture - 36
Periodic Orbits and Poincare Bendixon Theory

Welcome back again. So, in today's class we will be discussing Periodic Orbits, especially their existence.

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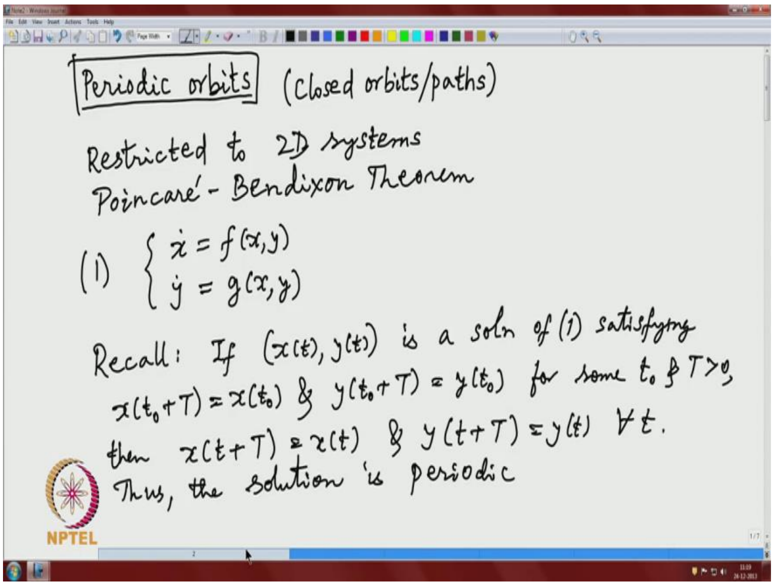

So, these are also referred to as closed orbits or paths. So, the existence of periodic orbits is an important part of qualitative theory, and quite useful with respect to many physical and other applications. I am not going to prove any result in this discussion, mainly because the proofs are quite involved. And they also require new concepts and tools. I will try to explain the results through some examples and through some pictures.

The discussion is restricted to 2D systems. And with regard to 2D system there are fairly general results. So, those things we are going to discuss, the main the important theorem with regard to t to 2D autonomous system is the celebrated Poincare Bendixson theorem, which will be coming more or less at the end of this hour. So, since we are discussing only 2D systems.

So, let me change a notation little bit and write our system as x dot equal to f of x y autonomous. So, there is no t dependence on the right hand functions. So, g x y, so x and y are the unknown functions of t and this is our 2D system. So, we would like to discuss the existence of periodic orbits for this system one. So, what are the necessary conditions, what are the sufficient conditions? There are only sufficient conditions necessary conditions are too hard to come by there are some that we will discuss them.

So, recall again from the general lemma's we have learnt about autonomous systems. So, let me recall that, so in this 2D setup if x t. So, let me write that y t is a solution of 1, satisfying x of t 0 plus t is equal to x t 0 and y of t 0 plus t equal to y t 0 for some t 0 and t positive. Then x t plus t equal to x t and y of t plus t is equal to y t for all t. So, thus the solution is periodic. So, this we have already seen in the general context, so in particular, this true for this 2D system.

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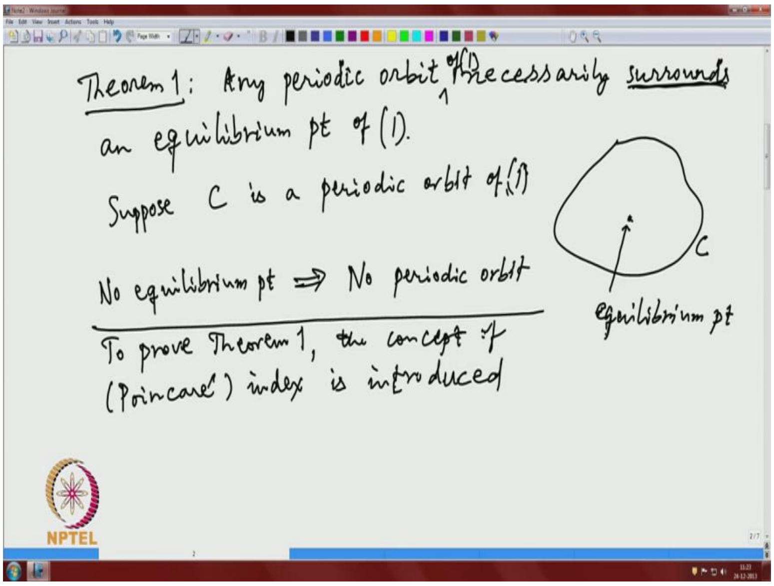

So, our first result let me state it as theorem 1. So, any periodic orbit, so orbit that is related to a periodic solution that is periodic orbit. Any periodic orbit necessarily surrounds an at least one an equilibrium point of 1 here of 1. So, let me just what does this mean.

So, this suppose c is a periodic orbit of 1. So, in the phase space, so it will be like this. So, that will be c. So, displayed by a periodic solution of this system 1, so the interior of c. So, that is what is meant by this surrounding. So, there is at least 1 equilibrium point.

So, this equilibrium point. So, theorem 1 is in some sense a negative result. So, theorem 1 implies that no equilibrium points, point implies no periodic orbit. So, in order to have periodic orbits necessarily the system 1 has to have some equilibrium points.

So, let me just briefly describe how this theorem 1 is proved. So, that is where the new tools new concepts new ideas come in and they are bit advance. So, just let me restrict myself to a discussion of that. So, let me just discus that. So, to prove theorem 1 the concept of an index this is also called Poincare index. So, let me just Poincare index is introduced. So, let me just briefly discuss what does that mean.

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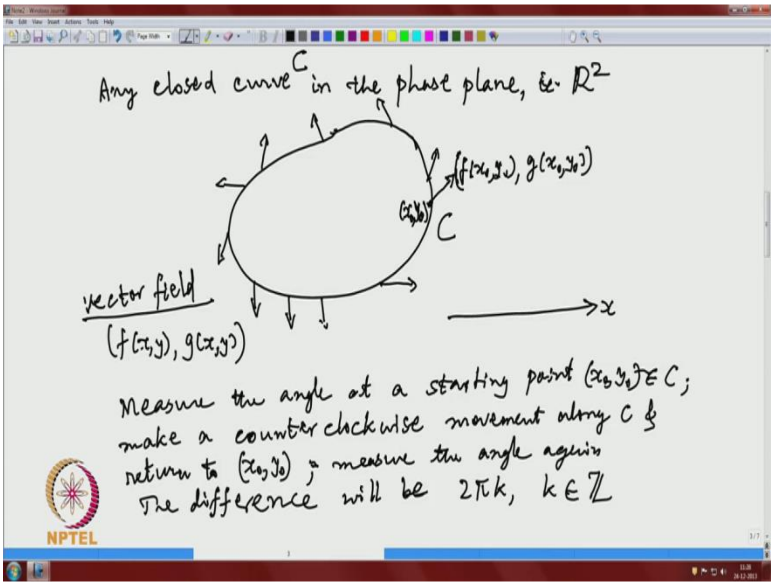

So, this is again very specific to 2 D. So, this even this concept of index. So, you consider any closed curve not necessarily an orbit or curve in the phase space in the phase plane we are in r two. So, that is r two. So, let me describe how the index is defined. So, this is ((Refer Time: 10:59)) suppose this. So, at every point on the curve you just compute this f x y g x y this is vector. So, this in r 2 now. So, that is the reason why you call system 1 as a vector field.

So, vector field suppose this vector is nonzero vector at all points of c. Then it defines a definite direction. So, let me just draw here. So, this is. So, this say x 0 y 0 a specific thing and this is f of x 0 y 0 g of x 0 y 0. So, we say that this vector field is nonvanishing on this closed curve c. So, then a tau points on the curve this vector has a unique direction.

So, we can measure its angle from some fixed direction say x axis. So, you can just keep it here x axis. So, we measure the angle of this vector with respect to this axis and then you start moving along the curve. So you just move along this thing. So, they may just depending on f. So, you do that.

So, you draw all the vectors along this and then you come back. And again you measure the angle at the point we started with, and so measure the angle at a starting point x 0 y 0 on c then make a round. So, you keep on measuring the angles at all points make a counter clockwise around clockwise movement along c and return to that starting point again.

And then measure the angle again. So, the difference will be 2 pi k. So, k is an integer. So, it could be positive it could be negative or it could only zero. So, that is what we do? So, this k is called, so take some simple examples of closed curve like circles ellipses and you take a vector field which is nonvanishing on that closed curve. And then try to measure this angle. So, there is an analytical formula for this angle measurement and describe that in a minute.

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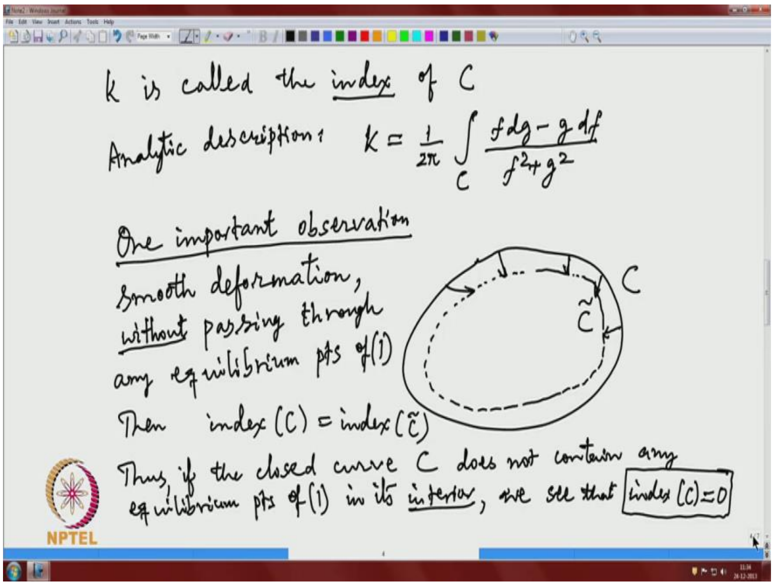

So, first the definition. So, this k is called the index of c. So, here the vector field is fixed that f and c that is coming from our system one. So, that is why you are not otherwise that is also part of the definition, but we have fixed that vector field f g. So, f and g are assumed to be smooth enough. So, the analytical formula for this k is analytic description. So, k is equal to 1 by 2 pi.

So, this is a contour integral over c. So, f d g minus g d f this is just by taking the angle is in terms of r tan of g by f. And then you work out and this will be the definition. And since we are dividing by f square plus g square you see that assumption is necessary that this f vector field is nonvanishing on c, otherwise we will have trouble there. So, this is a line integral. So, when you have a parametric representation of this closed curve c.

Then this d g and d f will be expressed in that thing. So, it will be a 1 dimensional integral. So, I am not going to that details, so one important observation about this index. And that is very much used in proof of theorem 1. And this is the key observation. So, you start with c again whatever c is you determine its index. So, if you deform this thing this closed curve smoothly to another closed curve c tilde, so this smooth deformation.

So, there are again very precise description of this thing in terms of mappings. So, again I am not going do that this is the thing. The only requirement is this deformation should happen without passing through any without that is important any equilibrium points of system 1. So, when we deform this c into c tilde. So, now here we should come across the equilibrium points of one.

So, that is that we should keep in mind. Then one proves this is the important observation. The index of c is same as index of c tilde. So, that is one important observation. So, in particular if a closed curve is c such that in the interior of that c if there are no equilibrium points of 1 then I can keep on deforming. And make this length of this closed curve very small. And eventually going to zero, so what we get is so… So, thus if the closed curve c does not contain any equilibrium points of 1 in its interior. So, I can keep on doing this deformation we see that index of c is zero.

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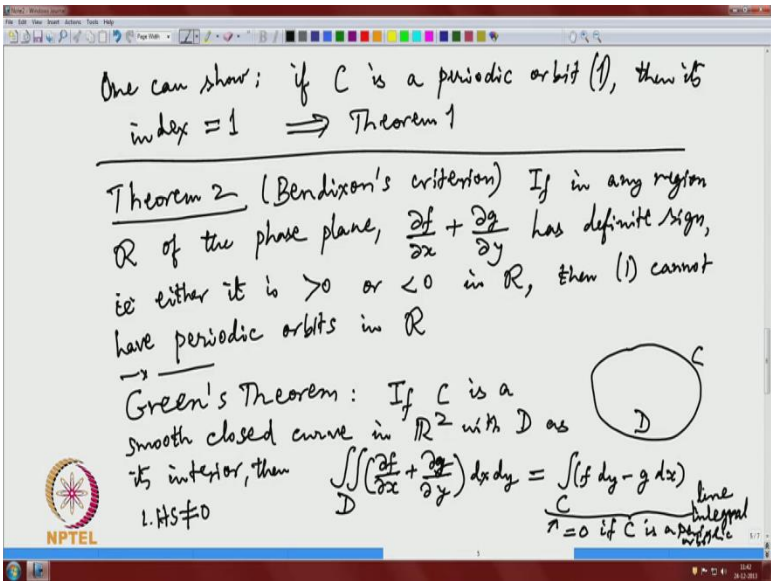

So, whereas, one can show there are many results in computation of indices. So, it is one of them. If c is a periodic orbit of 1, then of this is the system 1. Then its index is equal to 1. So, deep result they are not at all trivial results.

And if you compare the previous remark and this result you will see that theorem 1 is proved. So, the next result… So, let me state it as theorem 2. So, this proves this implies theorem 1. Theorem 2 this is called Bendixson's criteria. Again in some sense it is a negative result. If in any region I will write script R of the phase plane. This quantity divergence of the vector field, del f by del x plus del g by del y. f and g are smooth functions. So, this is in 2 D this is referred to a divergence of the vector field f g has definite sign. So, that is either it is greater than 0 or in R at all points in R.

So, either it is positive or negative. Then 1 cannot have periodic orbits in R. In that region the system 1 cannot have periodic orbits. So, this is a simple consequence of Green's theorem. So, let me just describe that, so Green's theorem in 2 D. So, it is something like integration by parts. So, there are of course, some sudden smoothness assumptions on the curve in the domain. I am not explicitly saying them. So, if again in just in terms of the f and g let me state that.

So if, so let better write this C. This is D. If c is a smooth closed curve in R 2 with D as its interior as in the figure, then this integral D of the divergence del f by del x plus del g by del y del g by del y. So, this is double integral d x d y. And this is given by line integral c this f d y minus g d x. So, when you parameterize the curve c. So, d y and d x will be expressed in that terms of parameterizations. So, this will be a 1 dimensional integral that is line integral contour integral line integral.

So, now if this region does have a periodic orbit. Then I can take that as c with this given region has a closed orbit to get a contradiction. And when c is a periodic orbit you see that using the system 1 the right hand side is zero. If c is a periodic orbit and whereas, by hypothesis the left hand side is nonzero. Since this del f the divergent del f by del x del g by del y as a definite sign. So, the left hand side will be a nonzero. One H s will be nonzero and that contradiction proves that the region will not have any periodic orbits.

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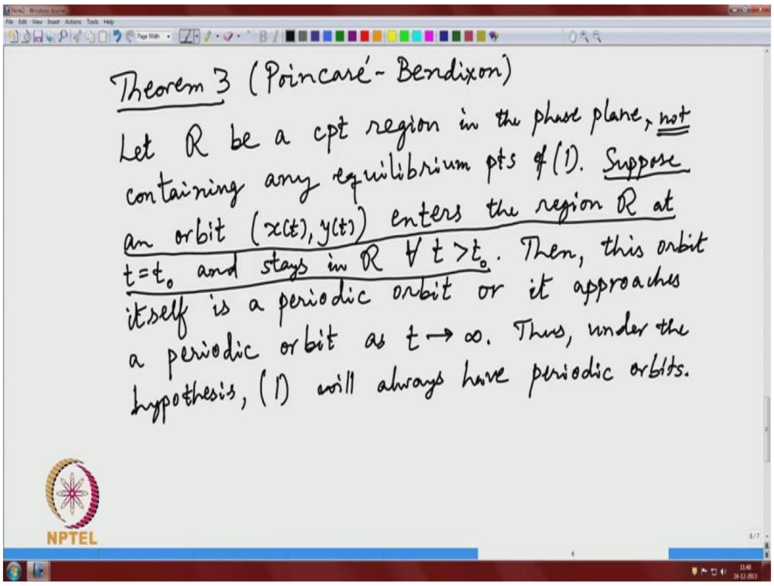

So, now we go to the celebrated Poincare Bendixson theorem. So, let R be a compact region in the phase plane. So, that means it is bounded and ((Refer Time: 29:30)) bounded not containing any equilibrium points of 1. So, let me stress that not containing suppose an orbit. So, let me write it since we are only in 2 dimension. So, let me just write explicitly x t y t. So, that is a solution of system 1 enters the region.

For example, it can start in region R. Region R or it can even come from somewhere else enters the region R at time t equal to t 0. And stays in R for all t bigger than t 0.So, this is the assumption on the orbit. So, either it can start in the region R at some time t equal to t 0. And then never leaves that region R for all future times. Or it can enter the region R at t equal to t 0. And then it is does not leave the region for all future times. So, this is hypothesis on the orbit. Then the conclusion this orbit itself is a periodic orbit.

If not, so there are only 2 alternatives or it approaches a periodic orbit as t tends to infinity. So, thus under the hypothesis 1 will always have periodic orbits. So, this is to some extent a satisfactory result. So, it under these assumptions on the orbit it give the existence of periodic solutions. But however, if you look at the hypothesis its somewhat difficult to verify in a general situation.

This can be done in particular situations, but to say that an orbit entering the region R will stay there for all time that requires again some analysis. And it is not easy to verify in a general context. So, there are some simplified theorems we will state 1 of them which are specific to second order equations. So, this we remember this is valid for first order systems. And this is more general than that, but whereas in that other theorem of lie nard. We are going to state that, the verification it directly in terms of the functions involved in the equation, and not in terms of the orbits of the equation. So, that is there is 1 difference. So, let me just briefly discuss the hypothesis the contents and the conclusions of the Poincare Bendixson theorem.

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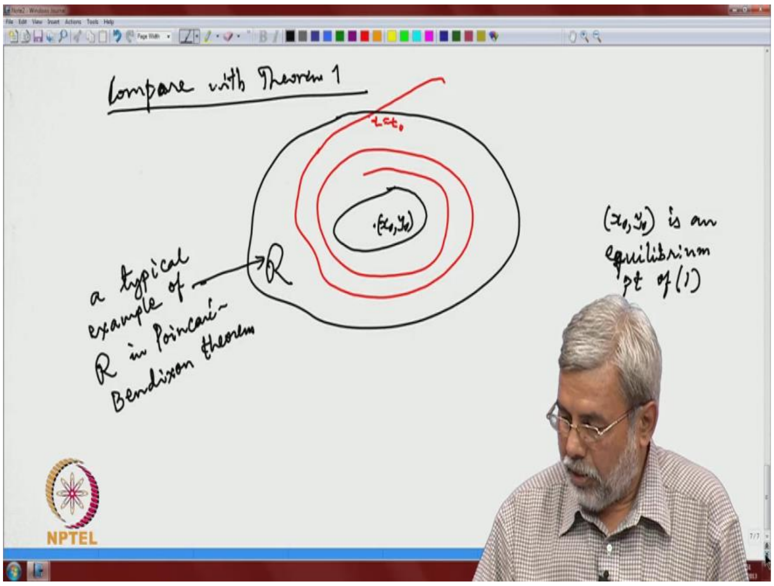

So, if you compare theorem one. So, compare with theorem 1. Theorem 1 says if there are no equilibrium points then there are no periodic orbits, whereas Poincare Bendixson theorem require that compact region not to have any equilibrium points. So, how should that look like that, compact region in the hypothesis of Poincare Bendixson theorem how should look like. So, it should like typically an annulus type. So, this is. So, something like that and there is another 1 here and there is some let me call it x 0 y 0.

So, x 0 y 0 is a equilibrium point of 1 system 1. And this is R. So, this is a typical example of R a compact closed and bounded of R in Poincare Bendixson theorem. So, the requirement there is. So, an orbit of 1 it enters may be at time t equal to 0. Here t equal to t zero. but once it enters it will stay there for all. So, you can just visualize that it may be approaching a periodic orbit. So, this state that it stays in that R forces that orbit to approach a periodic orbit. We will see some examples. Some simple examples we will see.

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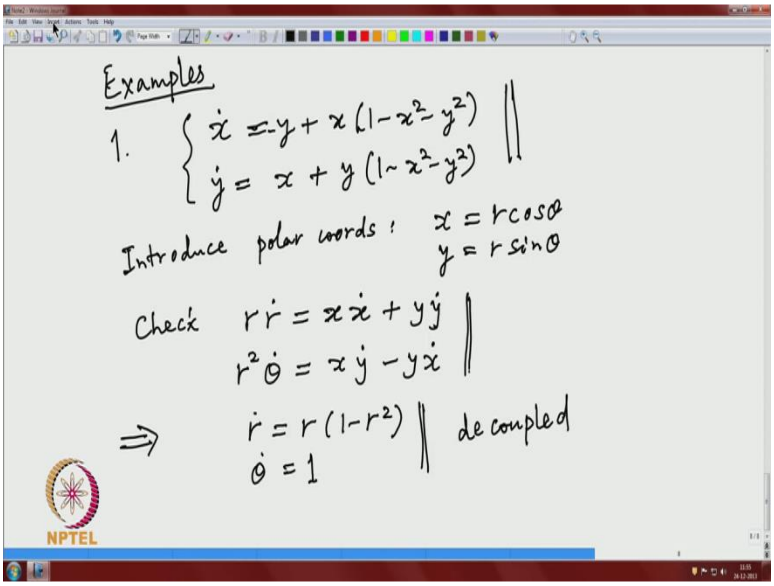

So, examples… So, these illustrate the Poincare Bendixson theorem. So, this is first let me just state that. So, x dot equal to minus y minus y plus x into 1 minus x square minus y square. y dot is equal to x plus y into 1 minus x square minus y square. So, because of the presence of x square plus y square it is convenient to express this equations in polar coordinates, so usual polar coordinates. So, introduce. And these are helpful in many situations in 2 d systems introduce polar coordinates, so in instead of variables x y we have r and theta r cos theta y is equal to r sin theta.

And now we would like to have differential equations for r and theta from the given equations from x and y and in general you see that. So, compute this thing this is not at all difficult. So, this we can also check simple algebra r r dot is equal to x x dot plus y y dot. And you also for theta you have that r square theta dot is equal to x y dot minus y x dot. This is a more general set if you introduce this polar coordinates x equal to r cos theta y equal to r sine theta. So, then we will have differential equations for r and theta given by this.

And now if you substitute the given expressions for x dot and y dot. So, this implies. So, I have this and let me write that thing and you can easily check. Let r dot is equal to r into 1 minus r square. So, we already see that it is coming from that x square plus y square expressions. Here and theta dot equal to 1. So, in this setup you see the r and theta variable they are decoupled. So, r does not depend on theta. And theta does not depend on r. So, we can separately solve for that whereas, the original system x and y are coupled very much. So, we can solve that thing, but before that. So, we will let us see the applicability of Poincare Bendixson theorem.

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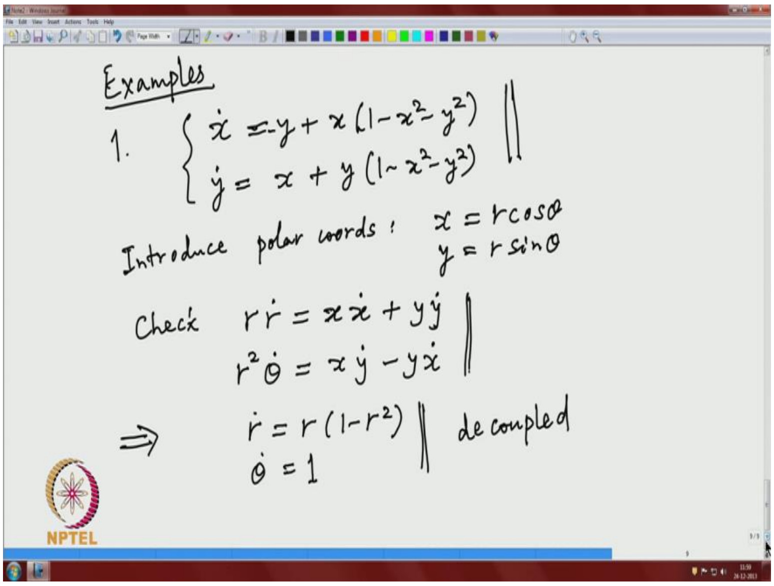

So just so if you consider this annulus let me call it r. So, this is half less than or equal to r less than or equal to two. So, that is the annulus. So, this. So, this is half. And this is 2. And here 0 0 is the only equilibrium point. That is the you can check 0 0 is the only equilibrium point. So, this r does not contain that because we are away from the r.

And now if you look at the equation for r you see that r dot equal to r 1 minus r square. And in the on the internals spheres, so at r equal to half. So, r dot is positive. And at r equal to 2 r. dot is negative. So that means, if an orbit in r try to approach the boundary of inner sphere, it will be again pushed back in the region, because r dot is positive.

So, as soon as it comes here it has to again go back in the region. And when the orbit tries to leave the upper the outer surface of the outer sphere outer circle, then it will pushed in because r dot is negative. So, something comes here again it has to be pushed down. And here if it comes here it has to be pushed inside this annulus.

So, therefore an orbit which enters or which starts. There enters starts in R will remain in R for all future times. So, this is what I was telling. So, in this particular case we are able to prove that the orbit never leaves the region r. Once it enters there or once it starts there. So, this, but in general it may be difficult. So, Poincare Bendixson theorem implies. Bendixson implies existence of a periodic orbit in 1 in R. It will be there. So, in fact in this case we can explicitly construct that.

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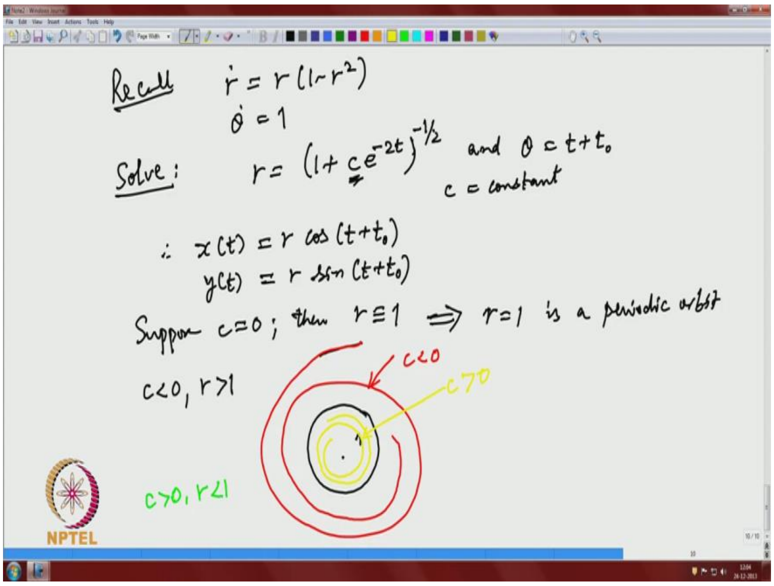

So, again recall we have this thing r dot equal to r into 1 minus r square. And theta dot equal to one. So, explicitly we have this. So, solve we get r is equal to 1 plus c e to the minus 2 t to the minus half. And theta is equal to you can just write t plus t here t 0 is some orbit constant. c is also a constant. So, therefore, we have x t is equal to r cos t plus t 0. And y t is equal to r sine t plus t 0. And t 0 and this constant c are fixed by the initial conditions. Suppose c is equal to 0 then r is identically 1 from that equation.

So, this in this case this implies and certainly that is periodic solutions. So, r equal to 1 is a periodic orbit. So, we are explicitly finding in this case that r equal to 1 is a periodic orbit. And it is that orbit is obtained by taking x t equal to cos of t plus t 0 and y t equal to sine of t plus t zero. So, let me just draw that. So, this is 1. So, we put the arrow appropriately in which directions counter clockwise, clockwise direction. So, let me not do that thing, and when c is negative just look at here. If c is negative this quantity is this may be there some c there. And if c is negative then this is less than 1 and we are taking in the denominator. So, r will be bigger than one.

So, this if c is less than 1, less than 0 r will be bigger than one. So, the orbit starts outside this circle of radius 1. And eventually approaches. Let me use different color it will eventually approaches this or it can do it counter clockwise also. So, if c is. So, this is c negative. So, for every c negative you get such an orbit, and when c is positive. So, this is more than one.

So, here in the denominator. So, then r will be less than one. So, c positive r will be less than 1. So, from inside it will be approaching that either clockwise or counter clockwise. It will not go on that thing. So, just it will approach that. So, something like right that is the c equal to c positive in this picture. You can try a similar example. So, let me just say that.

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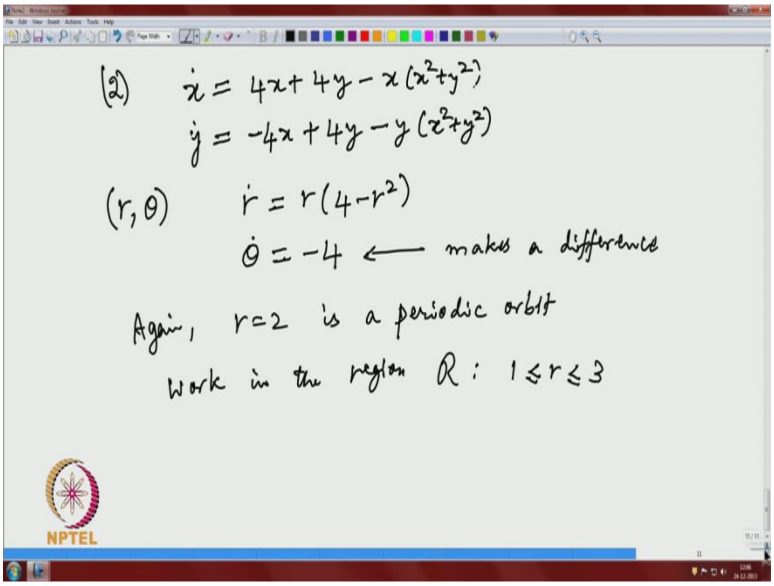

So, example 2 x dot is equal to 4 x plus 4 y minus x square plus y square. Y dot is equal to minus 4 x plus 4 y minus y same thing. So, and again you introduce the polar coordinates r theta. It is similar to the previous example. But there is some certain difference. I would like you to work it out. And see what that difference is. So, here we have r dot is equal to r 4 minus r square. So, again and theta dot is different. Theta dot is minus 4. Suppose and this makes a difference. Work it out from the previous example. So, again this r equal to 2 is a periodic orbit. So, now you work in the region work in the region R say 1 less than or equal to r. So, now next I will describe.

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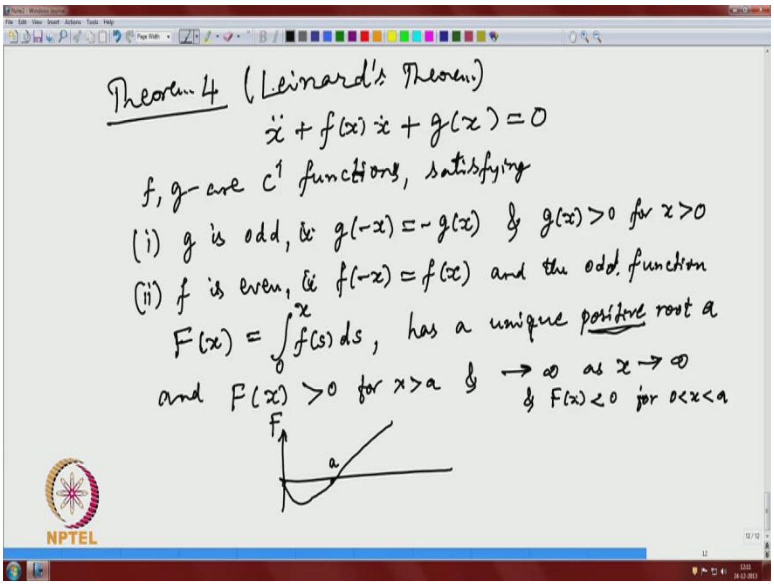

This Lei nard's theorem. Theorem 4, this theorem is very well motivated by the Vander pole equation. So in fact, the hypothesis are well suited to that Vander pole equation. So, this applies to second order equations. So, x double dot plus f x x dot plus g is equal to 0. So, f and g are smooth functions.

So, f are c 1 functions satisfying 1 g is odd that is g of minus x is minus g of x. So, necessarily g of 0 is 0. And g of x is positive for x positive. This 2 f is even. So, that is f of minus x is equal to f of x, and if I integrate now little f that will be an odd function. And the odd function f of x capital F of x which is integral 0 to x f s d s. So, I you integrate this even function. So, the requirement on capital F has a unique positive root. A positive root and f x is positive and for x bigger than a and approaches infinity as x tends to…

So, I said earlier now the hypothesis is purely in terms of the coefficients of the equation. So, no orbits are involved here. It is purely in terms of the coefficients in the equation. So, somewhat easier to verify than the Poincare Bendixson theorem. So, here is the picture of f. So, this is f. Since it is odd it is 0 here. And we require another 0 here. So, it is negative here. And I did not say that and f x is negative for 0 x less than a.

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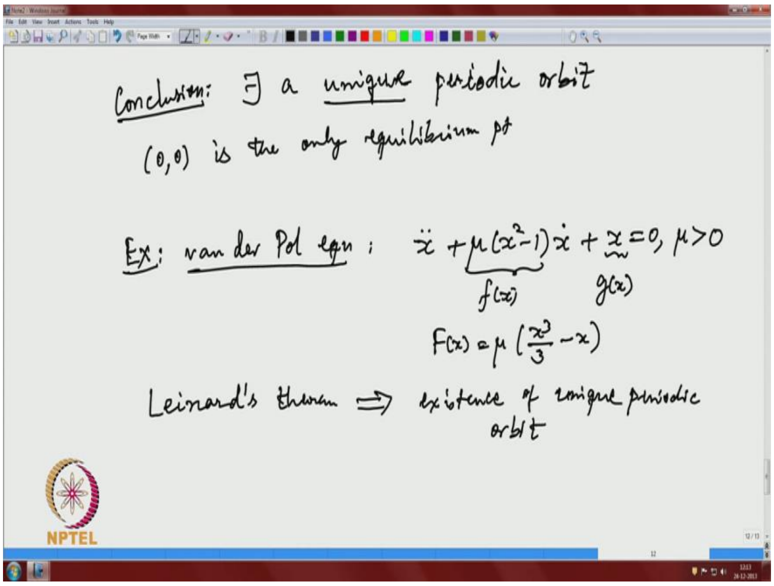

So, this is f and g is that is positive backward. Conclusion there exists a unique. So, there is more unique periodic orbit. And here in this case you can easily check that. So, 0 0 is the only equilibrium point in this case. And there is more to it say any other orbit will spirally approach that periodic orbit. So, the example as I said it is motivated by the Vander pole equation. So, recall this x double dot plus mu x square minus 1 x dot plus x and mu is positive. So, here this is our f x in the Lei nard's theorem. And this is g. So, capital F of x will be just mu x cube by 3 minus x. So, you see that there is a positive root namely root 3. And that will remain positive after that. So, the Lei nard's theorem implies that u existence of makes in periodic solutions. So, with that thing we will come to an end of this discussion on periodic orbits.