Ordinary Differential Equations
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Module - 6
Lecture - 33
Stability Equilibrium Points Continued

Recall that in the last class we were discussing linearization process and linear stability analysis of non-linear systems.

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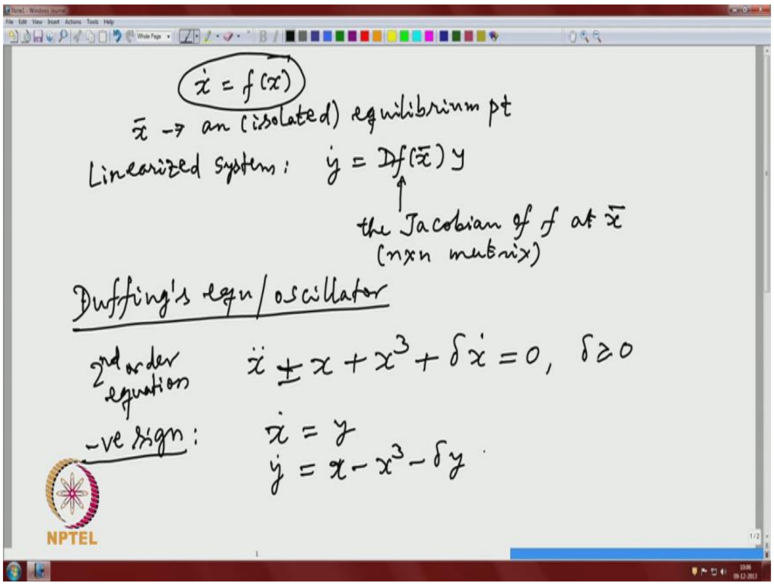

So, let me just recall what we are doing. So, this was our autonomous system and x bar an isolated, so that is standing assumptions equilibrium point. And then, the linearized system is given by y dot equal to D f x bar y. And, this is the Jacobian of the vector f at x bar. So, it is an n by n matrix when we analysis this one, we call it linear stability analysis of this non-linear part, so doing examples.

So, let me again recall that Duffing's equation or its also called Duffing's oscillator. So, after some simplification o e reduces the equation with any parameters to only one parameter. And, this is second order equation x double dot. So, second order equation. So, x is a real valued function x dot plus or minus x plus x cube plus delta x dot is equal to 0 delta is positive.

So, let me just concentrate on the negative sign. And, the positive sign is similar in fact, there is 1 equilibrium point in that case. So, let me write that negative sign as a system. So, this is dot equal to y and y dot equal to if I take negative sign, so this is x minus x cube minus delta y.

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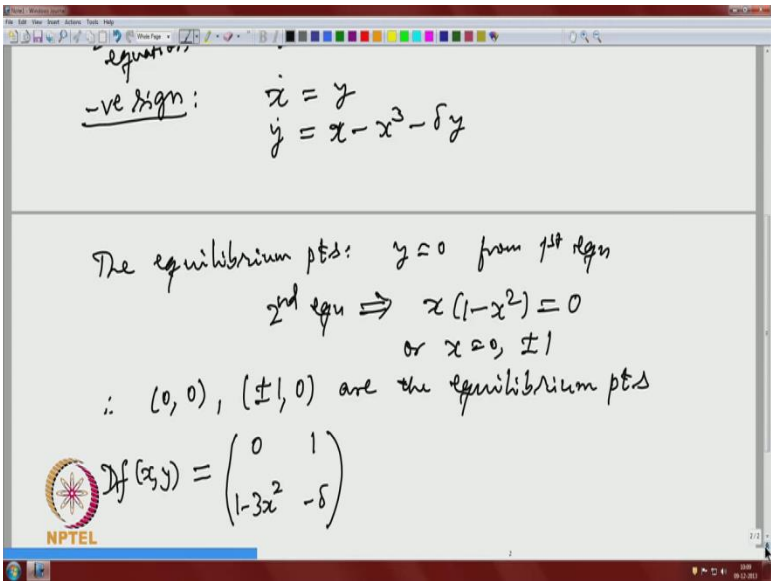

So, the equilibrium points, so if you compute. So, the first equation gives you y equal to 0 and this is from first equation. And, second equation then implies that x into 1 or minus x square equal to 0 or x equal to 0 plus or minus 1. So, therefore, in this case we have 3 equilibrium points 0 0 plus or minus 1 0 are the equilibrium points.

And now let us, calculate Jacobian at this three points. So, let me just calculate the general the Jacobian at a general point x y. So, if you this is now two by two matrixes. So, we are just two dimensional systems. So, if you take the first function. So, I differentiate with respect to so first again go back. So, if you go back, ((Refer Time: 00:43)) so you see that the first equation contains only y and second equation is x minus x cube minus delta y. So, this will be 0 1 1 minus 3 x square minus delta.

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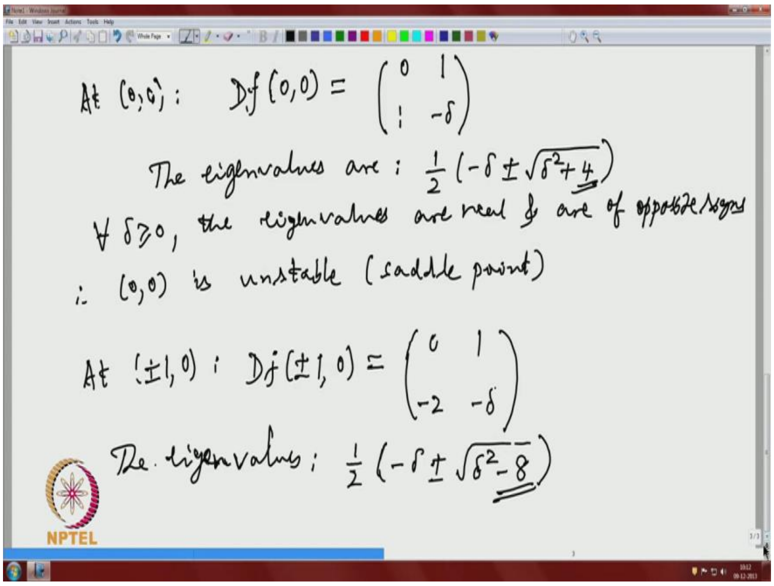

So, at 0 0 let us calculate the Jacobian. So, D f 0 0, so this is just 0 1 1 minus delta. So, the Eigen values, you can compute them are half minus delta plus or minus square root of delta square plus 4. For all deltas nonnegative, the Eigen values are real and are of opposite signs. So, 1 is positive and 1 is negative. So, we conclude that from the linear stability analysis. So, we invoke all the linear theory here, so that 0 0. So, is unstable in this case we also call it saddle point.

When the Eigen values are real and of opposite sign the unstable equilibrium point is called a saddle point. And at plus or minus 1, since the Jacobian as x square term, so it is the same, so D f plus or minus 1 0 it just 0 1. Now, it is minus 2 minus delta 1 minus 3 x square. So, that gives us minus 2. So the Eigen values, here is computable 2 by 2 matrix. So, no problems Eigen values are half again minus delta plus or minus delta square minus 8. So, this in case of 0 0, it is plus 4 and here it is minus eight and; that means, it difference.

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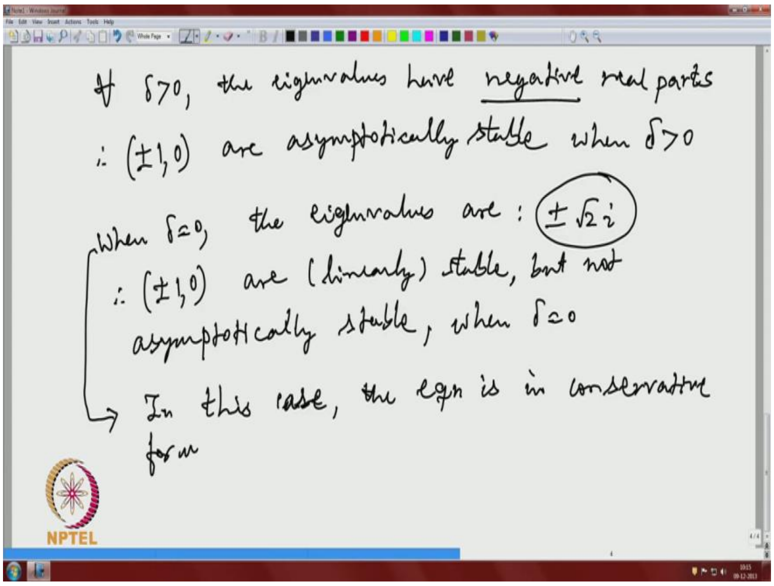

So, for all delta positive, the Eigen values have negative real part either themselves are negative real numbers or certainly they have negative real parts. So, therefore, plus or minus 1 0 are asymptotically stable, linearly asymptotically stable. When delta is positive, what happens when delta is 0, when delta is 0, the Eigen values are plus or minus root 2. So, therefore, plus or minus in that case is both are same, are linearly stable, but not asymptotically same.

Because here, the real parts are 0 they are purely imaginary. So, this is again from the linear when delta is 0. So, delta equal to 0 also falls this case, in this case the equation itself. Equation is in conservative form and we will study these things little it in detail. So, at that time again I will recall this. So, when delta equal to 0, the equation is referred to as undamped unforced Duffing equation.

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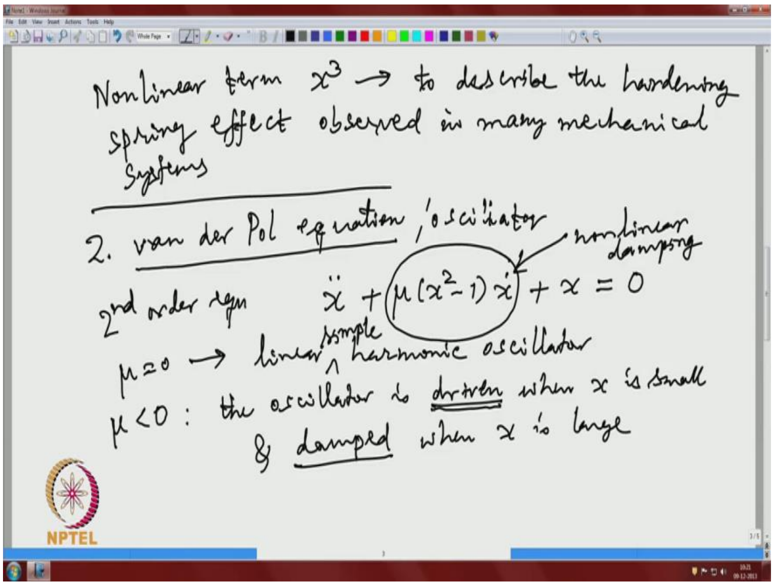

Duffing is equation also refereed that if you recall that non-linear term x cube is referred to as cubic stiffness term and that is used to describe the hardening spring effect observed in many mechanical systems. So this is to describe, the hardening spring effect observed in mechanical systems.

So, with this we move to the next example. So, this is another important equation that is studied extensively and it has also given arise to move mathematics in the theory of the non-linear dynamics. So, this is called van der pol equation are again oscillator. So, van der pol in the years around 1927, when he was working for the Philips Company in Netherlands extensively studied this equation both theoretically as well as the experimentally using electrical circuits.

So, this is one of the, again this is a second order equation. So, given by x double dot plus mu x square minus 1 x dot plus x is equal to 0, so this is unforced. So, there are having been many studies with one periodic forcing term as give some reference. So, when mu is 0 this term vanishes and you get back our simple harmonic oscillators. So, when mu is equal to 0 this leads to linear harmonic oscillator simple harmonic oscillator.

So, this non-linear term is added to that and see whether you still get periodic solutions when mu equal to 0 certainly you have periodic solutions and would like to see whether mu non equal to 0 also produces periodic solutions. So, you can imagine so when mu is negative, so that is the interesting case. So, do the linear stability analysis for this also when mu is less than 0 and x is small. So, this if you compare with the spring mass dash part system. So, this is the damping term, but this is non-linear damping.

So, in spring mass dash part system this was a constant. But here, we have this non-linear this coefficient depends on the solution itself. So, in the engineering parlance, this is called the oscillator is driven. When x is small, we will see what that means, a mathematically and damp are slow down, when x is large. So, when x is small means this x square minus 1 is negative and I have mu negative.

So, this whole thing will become positive and that will produce exponential terms with positive terms. So, that is so there are large oscillations. But, when x square is bigger than one, then with mu negative this also becomes negative and that acts as damping and the oscillations we will be slow down.

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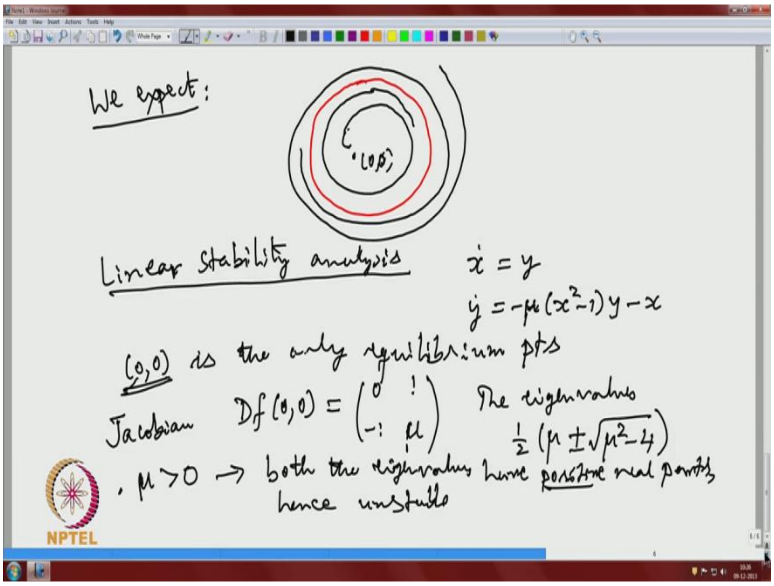

So, effectively what you expect is in this case, so we expect this figure. So, we see that we will check that 0 0, if the only equilibrium point in this case, whatever may be new. So this when start the solution mu is 0 So, it just moves around and try to approach a periodic solution.

So, let me just take, so this is the periodic solution and again when you start from for a half that also try to come and approach this. And we see later that this is the case all solutions starting at the different from different point. Then the equilibrium point all approach a periodic solution and that is indicated as a red circle here not circle, but some close orbit. So, we will see that later.

So, let us do the linear stability analysis very simple here. No I am not restricting mu to be negative analysis. So, these two examples are important for us Duffing's equation and van der pol equation. So, again write as a system, so x dot equal to y and y dot equal to minus mu x square minus 1 x dot is y and I have minus x. So, 0 0, so if you solve again the right hand side is equal to 0. So, first equation gives me y equal to 0 and if I put that in the second equation that gives me x equal to 0, whatever may be mu 0 0 is the only equilibrium point, so hence isolated.

So, even in the previous case are all isolated equilibrium points. So, 0 0 with the only and if I calculate the Jacobian that is again simple here d f 0 0 and this is again 0 1 minus 1 mu. So, the Eigen values are given by 1 by 2 mu plus or minus mu square minus 4. So, when mu is positive. So, this always dominates and so you will have so both the Eigen values have positive real point, so hence unstable.

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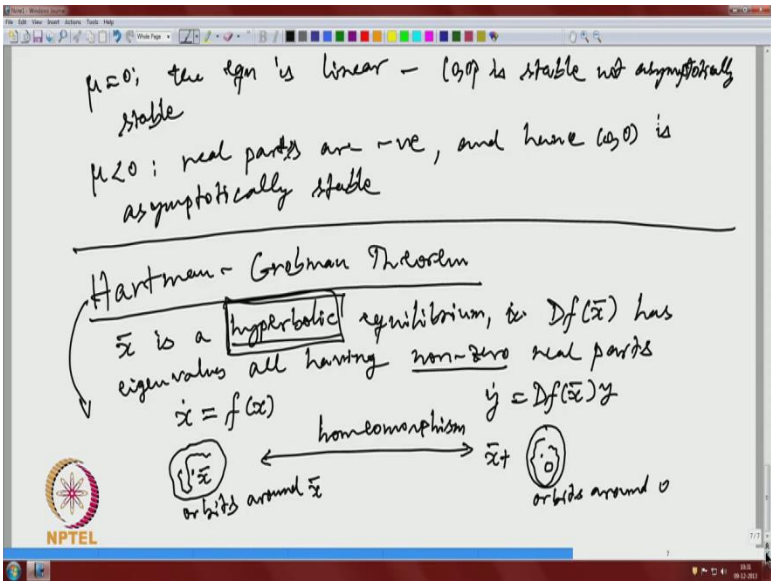

So, mu equal to 0, the equation itself is linear and it is 0 0 is stable, but not asymptotically stable. And, mu less than 0, so the real parts real parts are negative and hence 0 0 is asymptotically stable. So, as I said in the beginning mu is positive that is interesting case and we will see that little later.

So, let me now describe, a situation when the linear stability analysis imply the non-linear stability. So that means, if linear system is stable then the non-linear system is also stable and if linear system is unstable then the non-linear system is also unstable and this goes by the name Hartman Grobman Theorem. I just explain this only in English. I will not even write down the precise terminology, I just explain with only words. So, with the situation is x bar is a hyperbolic equilibrium point.

So, last time we defined this. So, let me again recall what; that means, is this D f x the Jacobian of f at x bar. So, this is a matrix has Eigen values with as Eigen values all having non zero real parts. So, this is important non zero real parts. So, in this context when x is a hyperbolic equilibrium point the Hartman Grobman theorem says that you have this non-linear system and you have the linearize system.

So, you just work here orbits around x bar the orbits and here also you this is 0 and you take orbits of this linear system here. And, then two compare these two so you just add x bar here. And, the orbits here, so orbits around x bar and orbits around 0 of this linear linearize system. Add x bar to that, so they are link by a homomorphism. It is an interesting and important theorem.

Locally describes around a hyperbolic equilibrium point. So, the orbits of the non-linear system can be gotten from the linear system and vice versa. So, this is an important result and this happens only around a hyperbolic.

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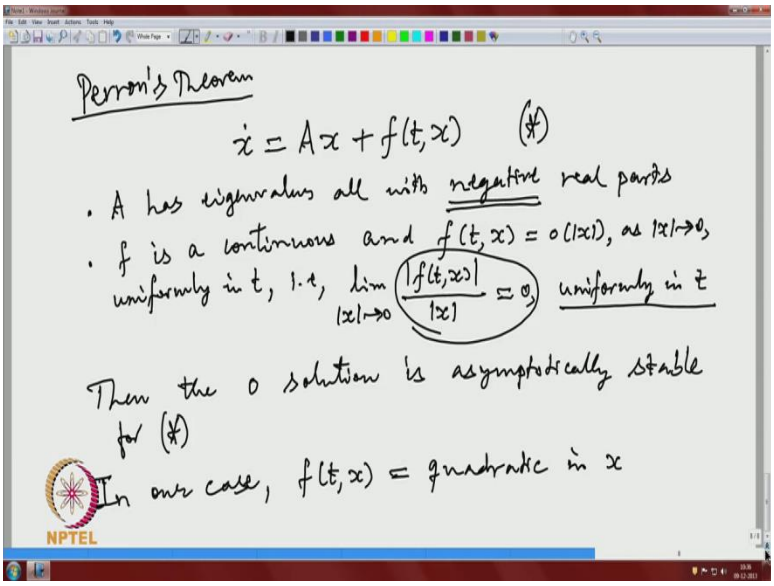

A somewhat simpler 1 and older than this Hartman Grobman theorem referred to as Perron’s theorem though, it is not as precise as Hartman Grobman theorem, it gives us some sufficient conditions gain for comparison of non-linear systems orbits of non-linear systems and linearize systems.

So, let me state this and give you a proof though it is technical. Since, we have developed all the technique technicalities in the linear algebra portion. It is very east to give a proof of this. So, let me just take this. So, we have the system linear systems perturb by a non-linear, so this f s different. So, this is a system. So, the hypothesis on A. So, A has Eigen values all with negative real parts.

So, Hartman Grobman theorem requires the Eigen values all to have non zero real parts, but here the Perron’s theorem it only concerns about the Eigen values, when they all have negative real parts. And, f is a continuous function. Let me not bother about where it is defined, but it will be defined in a neighborhood of 0. And, this is important hypothesis and this indicates small x. So, this is little o of mod x as mod x tends to 0.

So, uniformly in t, I will clarify this little later. So, what does it means is this limit mod f t x mod x mod x tends to 0, this 0 uniformly in t. So; that means, this limit process does not depend on t that is uniform t. So, the usual epsilon delta that appear in the definition of this limit. The do not depend on t. So, that is what meant by this uniformly t.

Then, the 0 solution 0 solution is asymptotically state. So, again read the definition of asymptotic stability carefully stable for star, so input this star. So that means, if I start a solution of start star we at the origin, but at the origin, because origin is always a solution. I would like to show that the solution is exists for all time. And, the solution tends to 0 as t goes to infinity there are two steps. So, proof let me just indicate a proof of this.

And in our case, when we want to apply to the linear stability analysis f t x does not depend on this is just a quadratic in x. So, it does not depend on t. So, automatically this condition is satisfies. So, there is no problem. So, this Perron’s theorem implies that 0 is asymptotically stable. When the linearize problem as the matrix the Jacobian matrix in the linearized problem has Eigen values all with negative real part.

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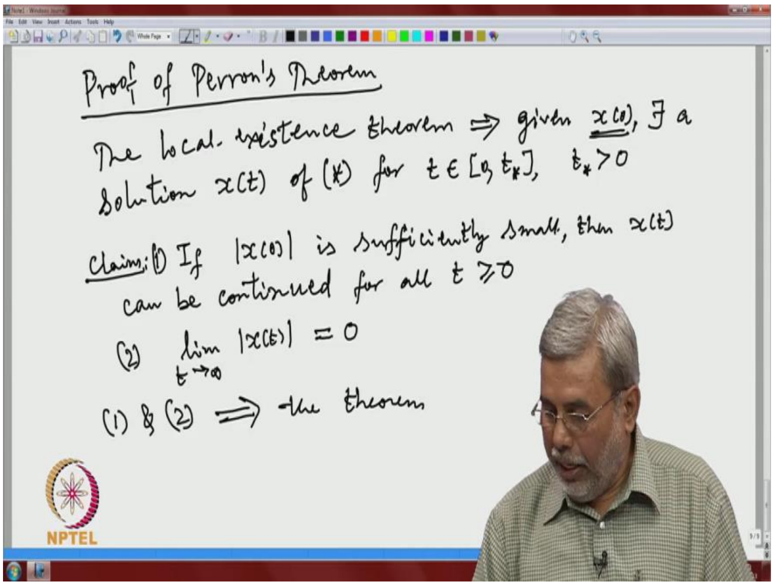

So, let me just give a proof of this. So, some inter play between analysis and linear algebra and differential equations of course. So, proof of Perron's theorem. So, the local existence theorem local in time theorem implies. So, whatever may be my x 0. So, given x 0, there exists a solution x t.

So, let me indicate that x t of star with x 0 that is a for some time t in 0 t star. So, t star is positive. So, that that is always guarantee it what would like to do. Now is, we want to show that when this x 0 is small. That the solution can be extended to all t positive that is the first step. And once, we show that thing we are not interested uniqueness here that is why the there are no hypothesis on little f in star. So any kind, so this is by Peano existence theorem that there is always a solution. So, there are two steps.

So claim, if x 0 is small, so remember this is the standard Euclidean norms. So, this x 0 is a vector in r n. So, if this is sufficiently small than x t can be continue. So, this is also part of the existence theory, we have done earlier when it possible to continue a solution which exist for a short time to all times continued for all t positive. And then, so this is 1 and then limit x t a t tends to infinity equal to 0. So, both these things prove the result. So 1 and 2, we will prove imply the theorem.

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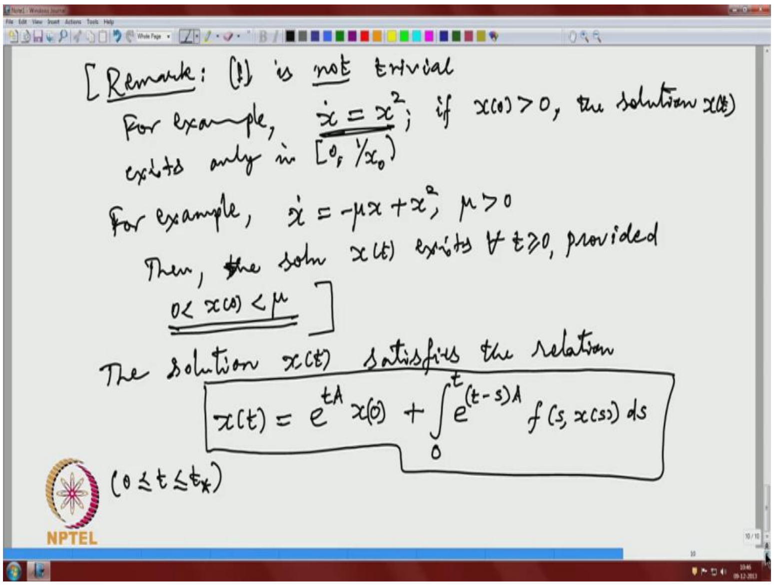

And let me just stress as a remark this is not trivial. So, Remark 1 is not trivial. So, it does not follow automatically. So, we already had seen an example, where this solution does not exist for all time. So for example, x dot equal to x square, so no matter what x 0 is if x 0 is positive; however, small it is that does not matter. The solution exists solution x t exists only in 0 1 by x 0.

We have already seen that of course this does not. This equation does not fall under star, because there is a linear term. So, the linear term is 0 here. So, that is that a part the hypothesis on the Eigen values of a plays a crucial role in order to prove claim 1. For example, so if I want to compare this example with star. So, I take x dot equal to minus mu x plus x square and I take now mu positive.

So this falls, this is similar to star equation event to. Then, you can as an exercise check that x the solution x t exists for all t nonnegative provided. So, this is the smallest as in, what we say in the theorem is provided 0, x 0 is less than mu. When once you exceed x 0 bigger than mu, we can also check that the solution neither does nor exists for all time. So, that is crucial. So Remark 1, this claim 1 is not trivial.

So, this remark explicit that now want to explode the hypothesis on a namely the Eigen values of all have negative real part in order to show that the solution in exist for all time. The local solution the solution which I know exists for short time that 0 t star x t satisfies. So this again you recall, we have prove the existence by converting the given differential equation into an integral equation and I am going to write the same thing, here x t satisfies. The relation equation x t is equal to e to the t a x 0 this comes on the homogenous part x dot equal to a x.

And then, for the in homogenous part we have this non-linear integral e t minus s A f of s x s ds. So, again remember the just t is only part star. So, whenever that x t is the solution of differential equation, it can always be written in this. And, one more remark here, it is a remarkable that the same a technique works even for infinite dimensional systems where the matrix a will be replaced by a differential operator with good spectral properties, what are the good spectral properties, like here the matrix A has all the eigenvalues with negative real part.

And similar to that, if you assume the same process without any change even works for infinite dimensional systems. So, this is a powerful technique the methodology in Perron's theorem. The proof of Perron's theorem is a powerful technique which used even in other situations.

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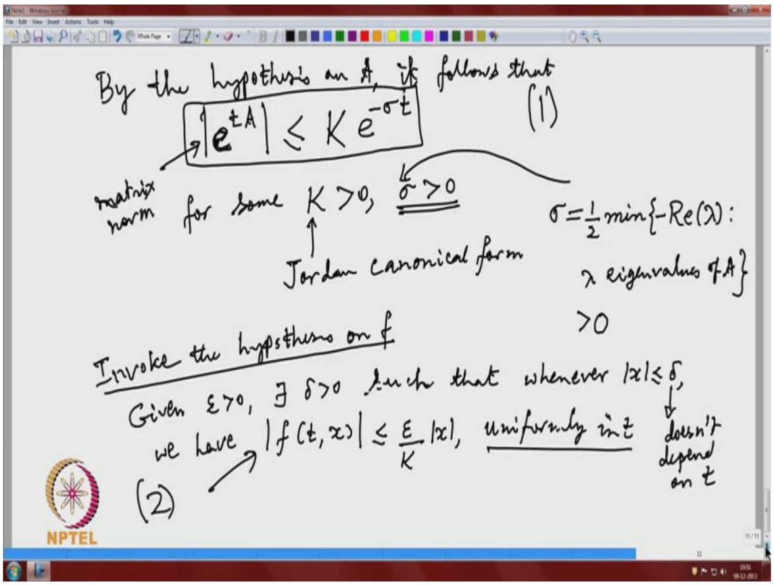

So, now the hypothesis, so let us by hypothesis. So, in the previous thing that e to the t A with the exponential matrix which we have defined in the linear algebra portion and I am going to use some more properties from that linear algebra, so by the hypothesis on A. So, this will show in detail in linear algebra portion. It follows that so e to the t A, so this is matrix norm which we have introduced in the linear algebra part. So, this is less than or equal to some constant positive constant k, e to the minus sigma t for some k positive and sigma points. This is important the sigma positive.

Let me, just briefly as explain where do this two constants come from. So, this essentially comes from the Jordan canonical form which we have already discussed in linear algebra portion. So, when we use the Jordan canonical form of A. So, this constant k is produced and sigma comes from the eigenvalues. So, sigma is essentially half, half you can put any fraction there. So, minus of real of lambda, lambda eigenvalues A, and that is where the, so this is spectral property of A.

So, eigenvalues this can be spectrum. So, this comes from the non use of the eigenvalues of A and by our hypothesis this minus real part of lambda are all positive and I am just taking there are only finitely mini eigenvalues. They are expression all this real part is now minus real part is positive. So, this sigma is positive. So, half you can replace by an fraction. So, that is no problem. So, this is a very crucial step in the proof of Perron's theorem. So, this estimate is very crucial. And, this we have done in linear algebra.

Once you have this thing, now I invoke the hypothesis on invoke the hypothesis. I want to now write it in terms of epsilon and delta. So, given epsilon positive, there exists delta positive such that whenever mod x less than or equal to delta we have f of t x less than or equal to epsilon by k. So, this is for technical reason that k I am under using it mod x uniformly in t.

So, just this means again as I said earlier this delta does not depend on t. So, this does not depend on t. So, n itself is independent of t. So, that uniformly in t does not arise, but when t is there. So, you want this delta to be independent of t, so that we can just concentrate only on the x variable. So, now, let me call this as 1 and this as two. So, these two estimates play a crucial role.

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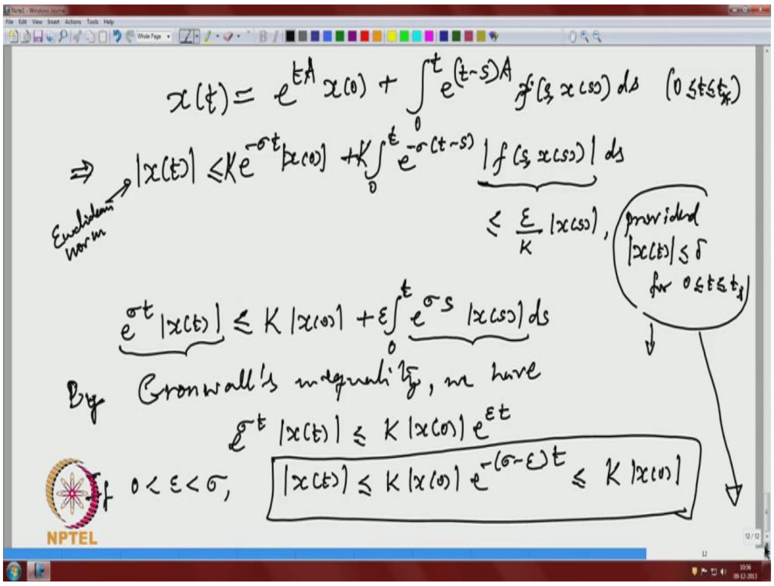

So, again go back to this. So, remember this solution satisfies this integral equation 0 to t e to the e minus A s A and f of s x s ds. So, this implies, so remember this valid only for a short t our aim is to extend this to for all t, so taking norm and using estimate one. So, this gives me minus sigma t x 0. So, this is Euclidean norm plus 0 to t e to the minus sigma t minus s and this f of ds.

And now, I want to use the hypothesis on f and for that require only say if I want to use that estimate 2 on this thing I require that mod x t is less than or equal to delta and that I assume that for the time being and I will show that. So, this is less than or equal to epsilon by k x s provided norm x t sorry is less than or equal to delta for 0 I will show that.

So, if you now put all these things together in this inequality. So, what I get is e to the sigma t. So, I take that this side this less than or equal to. So, there is a k here there is a k here. So, that is k x 0. So, this k cancels. So, I have that just. So, I take that e to the sigma t other side. So, what I am left is just e to the sigma s there is an epsilon here mod x s ds.

So, provided this again I, so this is valid provided that. But, now look at this inequality. So, concentrate on this function the same function up to here and the left hand side and this integral and this is the situation where we can apply the Gromwell's inequality. So, by, so this is also we have already seen the importance of Gromwell's inequality in proving uniqueness inequality. We have e to the sigma t x t is less than or equal to k x 0 e to the epsilon t. So, that epsilon comes there.

So, if we choose now, if 0 less than epsilon less than sigma. Then, we have x t less than or equal to minus sigma minus epsilon t and now this with a junction on epsilon and this is just k x 0. So, this is the crucial inequality we have obtain provided see all this things you have to remember that. All these things are derived provided x t is less than or equal to delta. So, you have to somehow now, assure that that x t will be remain less than or equal to delta.

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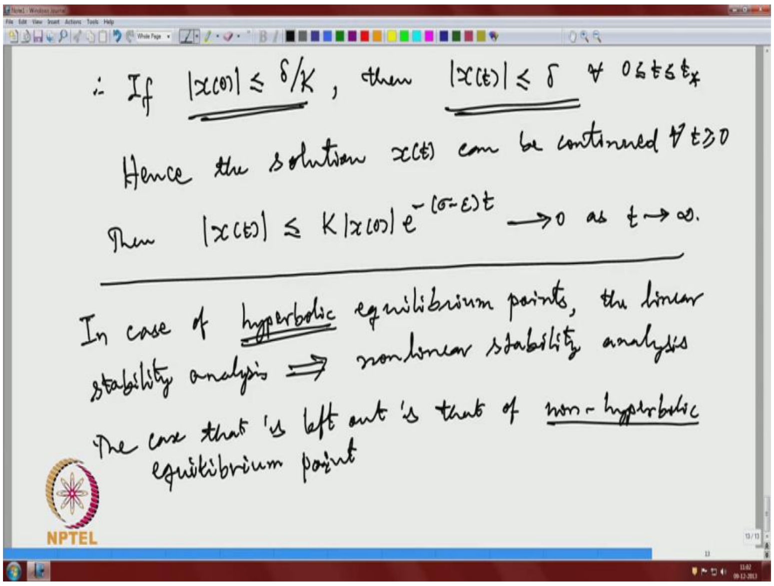

And now that is easy, so now the hypothesis comes into picture. So, therefore, if mod x 0 is less than or equal to delta by k. So, this is the smallest assumptions that is indicated in the theorem then if you are get the previous inequality for all 0.

So indeed, the solution has remained less than or equal to delta. So, all the steps we have derived or let remain and they are valid. So, hence the solution, so this is the bulk the solution x t can be continued. So; that means, it exists continued for all t. And, this is the smallest assumption. So, remember those this is just independent of. So, if they are just come from essentially on the hypothesis of little f and on the matrix A.

And then we also have this k x 0 e to the minus sigma minus epsilon t. So, we have already chosen that epsilon is less than sigma. So, that goes to 0 as t goes to. So thus, we have proved both our claims 1 and 2 and that completes the proof of the Perron's theorem. So, what we have learnt? So, the hypothesis in Perron's theorem is also a hypothesis on the equilibrium point x bar. So, A will be D f x bar. So, it all has negative real part, the eigenvalues have negative real parts.

So, this is the in case of hyperbolic equilibrium points, so this is important. The linear stability implies stability analysis implies non-linear stability. So, if the equilibrium point is hyperbolic, then the linear stability analysis will be sufficient to conclude the non-linear stability analysis and linear stability analysis is much easier there as we because we have explicit formulas and other things. So, the only case that would be left out is the case that is left out is that of non hyperbolic equilibrium points. So; that means, the Jacobian matrix, now will have eigenvalues with 0 real part.

And that we have seen through examples that the linear stability analysis may or may not imply the non-linear stability. You have seen example where in case of non hyperbolic equilibrium points the equilibrium point may be stable in the linear linearization. But unstable or even asymptotically stable in non-linear case. And this one is more effectively handled by the Lyapunov function and that will be the topic of our next discussion next class. So, the Hartman Grobman theorem and the Perron's theorem take care of the case of hyperbolic equilibrium points and the non hyperbolic equilibrium point case will be handled by a Lyapunov functions.

Thank you.