Ordinary Differential Equations
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Module - 6
Lecture - 30
Stability Equilibrium Points

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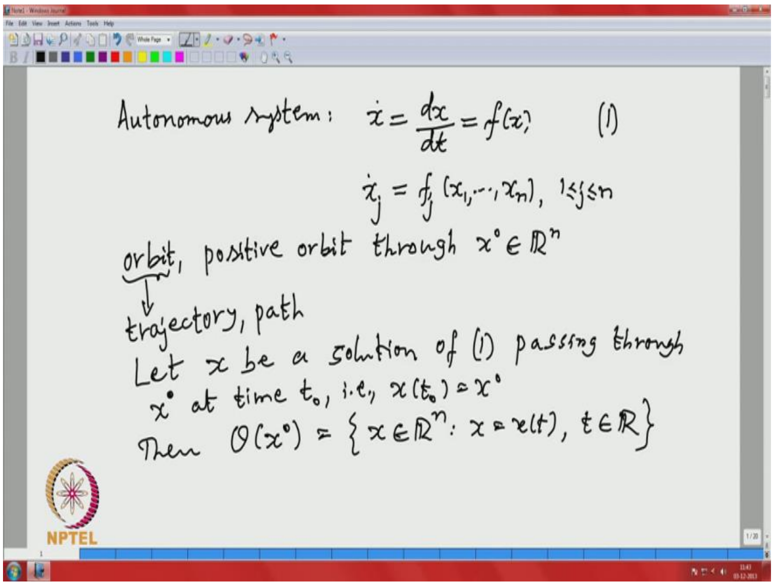

Welcome back, let us recall what we were trying to do in the previous lecture. So, we are considering this autonomous system f of x. So, let us call this equation 1. So, we are going to refer to this for the whole lecture. So, or in the expanded form we have this x j dot is equal to f j x 1 etcetera, x n and these are n equations for each j. And where defining orbit, positive orbit through a given point x 0 in R n. The other terminology for orbit the synonyms, they are also called trajectory path, this terminology also used.

So, recall how did you define for example, the orbit, similar remarks holds for positive orbit. So, we started with a solution. So, let x be a solution of 1 passing through x 0 at time t 0, what does this mean that is x t 0 is x 0. And x satisfy the differential system x dot equal f of x are in expanded form is component of x satisfies these equations; and then the orbit was define. So, then the orbit of x 0 is just set of all x in R n such that, x equal to x t, so using all x, x, x, but it should be clear from the context.

So, what I am just taking is all this. I am varying t for every t I just pick up that x of t coming from the solution. So natural question arises, natural doubt arises. What if I have another solution of 1 passing through x naught at a different time. And then, I use that solution to define the orbit, do I get a differentiate. And obviously, I should not get. And let me state this as a lemma and it does not depend on the particular solution used for defining the orbit.

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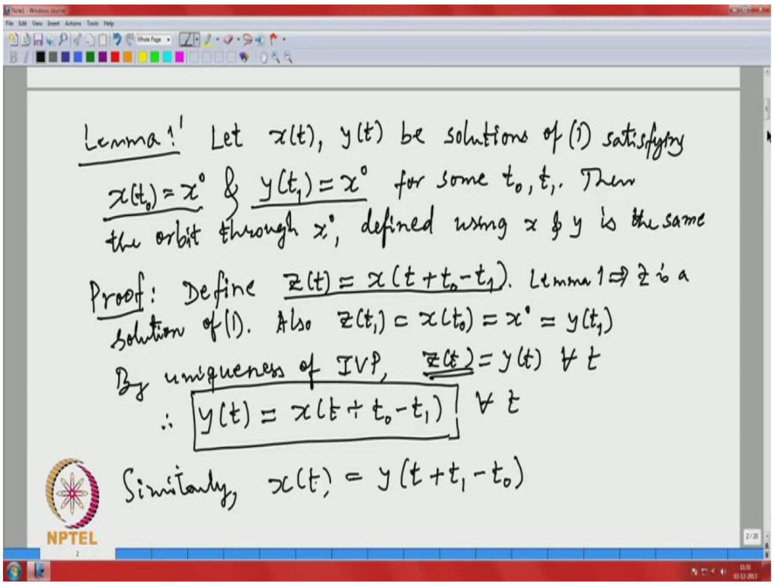

So, let me state it as a lemma, lemma 1 prime, so let x t. Let me write that t, y t be solutions of 1 satisfying x of t 0 for that t's of 0 x 0 and y of t 1 is equal to x 0. So, both the solutions are passing through the given point x 0. May be at different times if they pass through x naught at the same time, then by uniqueness they must be same. So, they might pass at different times for some t 0, t 1. Then the orbit through x naught that is our purpose defined. So, this is important using the solutions x and y is the same.

So, I will not get a different set, so that is important, that remove the ambiguity I can just use any solution passing through x naught, the time is not important. So, only the solution has to pass through that given point that is important, so this will remove that ambiguity. So, no special attention is given to that time t naught only the solution, this proof is very quick, so define z t is x of t plus t 0 minus t 1.

So, recall that for the autonomous system, earlier we saw lemma 1 that if x t is a solution then x of t plus c for any constant is also a solution. So, that defines that prove that, this z t is also a solution, so lemma 1 implies z is a solution of 1. So, more over now you compute z of t 1 also z of t 1, so you substitute t equal to t 1 in this expression. So, you get x of t 0 and that is x 0, but x 0 is also equal to y of t, so that is the hypothesis you see that is the hypothesis.

So, now by uniqueness, so this is important, so all the time you use this thing, so by uniqueness of the initial value problem, uniqueness of I v p, z t is equal to y t for all t. So, the solutions z and y are the same because they have the same initial value at t equal to t 1, so this condition imply that, so uniqueness is an important thing. So, all the time we use that, now look at z, look at y. So, z t is given by this expression, so therefore, so this is an important thing for autonomous systems and it is not true in general for non-autonomous systems x of t plus t 0 for that t 0 minus t 1 for all t.

So, though x and y are 2 different solutions, the only common thing they have they pass through the same given point x 0 at different timings, they are related by this equation. So, they are not really different, so similarly you can use y or you can just see here. So, similarly you get x of t is equal to y of t, now you have you have to just change the arguments carefully. So, have t plus t 1 minus t.

So, therefore, so if you now go back to the definition of the orbit, so what is orbit? So, I use that orbit through x naught, I just use that solution passing through x naught and collect all those points.

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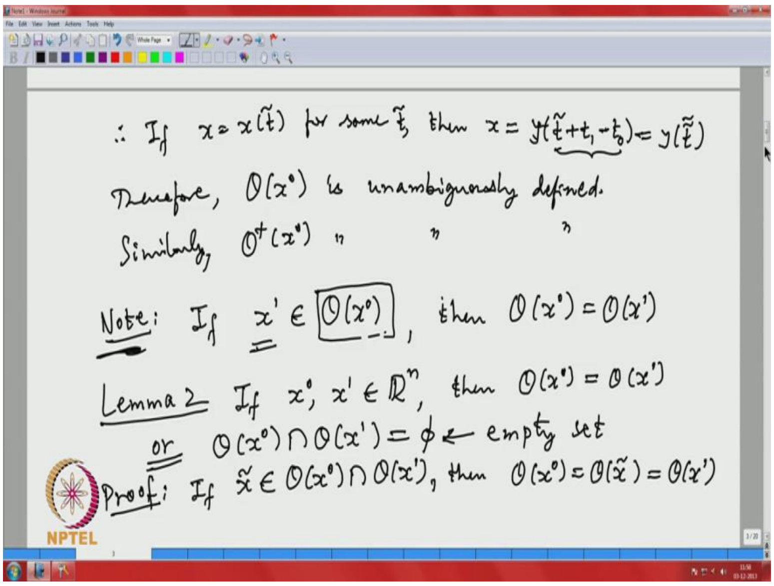

So, therefore if x is equal to x t are tilde. Let me say for some t tilde on then x is also equal to just look at the relation between x tilde and x of t tilde and y of t tilde you get that y of t tilde plus t 1 minus t 0. And this entire thing I may just call it y t tilde and similarly if h is equal to y of some t tilde, I can get x is equal to x of t tilde. So, therefore, this completes o of x naught is unambiguously define. So, let me stress what does that mean again so that means, I just take any solution passing through x naught, any solution of 1 is system 1 through x naught and time is not important at all I get the same set.

And similar you can also, now work out and just look at the definition of the positive orbit, so you also get the same similarly, so this positive orbit is unambiguously. So, what to all lead is a solution passing through x naught, so that simple clarification. And now immediately you see that, so this is a note remark. So, if x 1 belongs to an orbit passing through x 0. So, that mean there is a solution, I am using here in the definition of this orbit.

So, there is a solution passing through x naught, at some time that is not important and the same solution will also been now passing through x 1. So, I can use that same solution to define orbit passing through x 1. Since, it is the same solution, so I immediately get that x 0 is o of x 1. So, in lemma 1 prime we have seen that it is only a solution passing through a given point that is important to define the orbit through that point. So, since the same solution that is passes through x naught also passes through x 1, when x 1 is in the orbit of x naught.

So, I can use the same solution and that implies immediately that orbit passing through x naught and orbit passing through x 1 are the same. Whenever, x 1 lies on the orbit of x naught and that also prove that x naught lies on the orbit of x is passing through x 1, so this implies. So, is a simpler proof last time I stated that so lemma 2. Let me again recall that, so if we take two points in R n and then x 0, x 1 in R n and then consider the orbits passing through x naught and x 1, so then is equal to x 1 or empty.

So, this is empty set, so it says that any two orbits of the autonomous system 1 again, so remember that everything is reference to the autonomous system 1, x dot equal to f of x. So, that is important, so everything, so if I have 2 orbits either they are same or they are disjoint. So, there is nothing like, no they meet at one point and then they are not equal.

So, if they intersect at one point then they must be equal. So, and this in now is a simple proof, so let me just.

So, if x tilde belongs to o x 0, suppose there intersection is non-empty, so if there intersection empty than you do not have to prove anything. Suppose, there is common point between them then you look at this note. So, then since x tilde belongs to the orbit passing through x naught. So, I have this is equal to x tilde and x tilde also lies on the orbit passing through x 1 and that is equal to x 1. So, if they have a common point then they must be equal, so that is, so now the question is…

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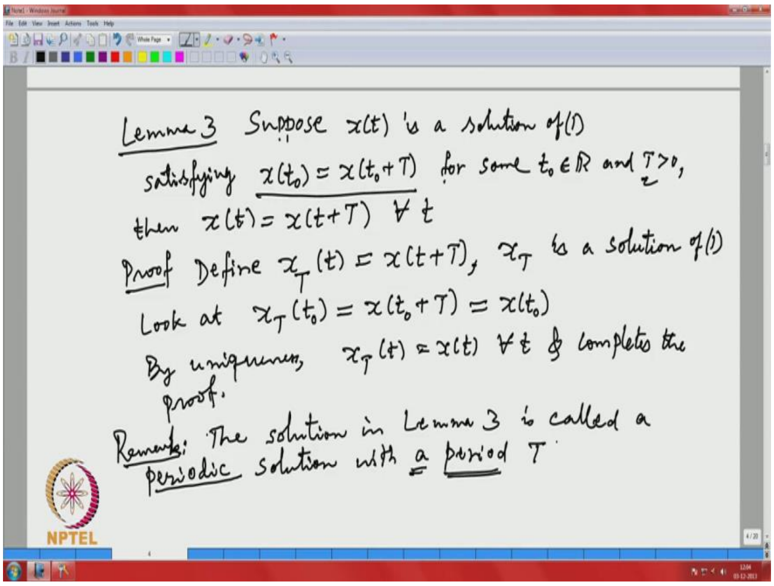

The previous lemma says about 2 orbits; what about the same orbit can it intersect itself so this is the content of the next lemma, so let me write it. So, if suppose x t is a solution of 1 satisfying. So, this is the condition I am, so suppose that it the orbit defined by the solution x intersects itself, satisfying x t 0 equal to x t 0 plus t for some t 0 in R and t positive.

So, there t 0, t 0 plus t are two distinct points and suppose that intersects then x t is equal to, so this is again a remarkable property. So, if intersects one point than it has to do that thing for all time, not just for that for one point, so that is. So, again let me stress that 1 is an autonomous system and these properties heavily use that fact. And so in general these properties of lemma 1, 2 etcetera are not satisfied by non-autonomous system. So, this you have to bear it in mind, again proof is very simple more than proof the statement of the lemma's are important.

So, again define, so you can see the repeated use of the same set of ideas, so x sub t of t is equal to x of t plus T and again by lemma 1. So, this is just a fix translation t is fixed, capital T is some fixed number, this is just a translation. So, x T is a solution of 1 and look at now x T. So, look at x T at t naught and this by definition is t 0 plus T and by hypothesis.

Now, look at the hypothesis, so this is x of t 0, so here we have two solution of 1, x t and x and they have same initial data at t equal to t 0 again by uniqueness, by uniqueness x T is of t is equal to x of t for all t and this is same I m saying a that x of t. So, that is and completes the proof that is fine. So, the solution remark, the solution in lemma 3 is called a periodic solution with a period, so let me stress that a period this T.

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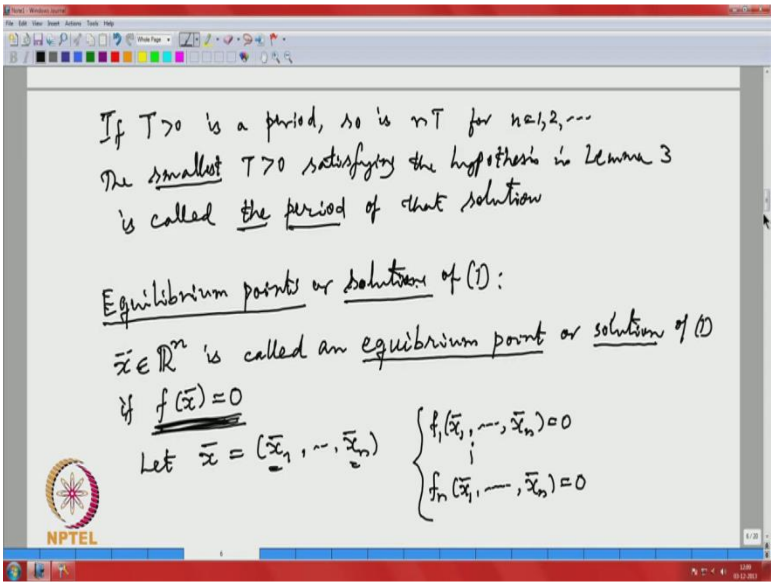

So, if T is a period, if T positive is a period, so is n T for n equal to 1, 2 etcetera. So, we will pick up one which is least, the smallest T positive satisfying the hypothesis in lemma 3 is called is called the period of that solution, these are some terminologies. So, we will see the importance of this periodic solution little later as only go along, so another concept, now will introduce another concept. So, this is the equilibrium points or solutions of 1, so this is our next concept, so definition. So, you pick a point let no I generally use this x bar, x bar is called an equilibrium point or solution of 1.

So, whatever the concept we are doing all refers to the 1 remember that, if f of x bar is 0, so let me explain, so in case of when n is bigger than 1. So, this consists of, so this let me explain this what, so let x bar be x 1 bar x n bar, so this is a, r n and let me expand the system of equation so; that means, I have f 1 of x 1 bar, x n bar is 0 etcetera f n. So; that means, the components of x bar, this x 1 bar, x 2 bar, x n bar are the solutions of the simultaneous equations f 1 bar x 1 bar etcetera 0.

So, in case of f 1 is linear, so this is just a system of algebraic equations, if f 1 are any of the linear, so many examples, so it consider them as polynomials then we have a set of non-linear equations and we are seeking a common 0 there may be 1, there may be many, so we will see an examples. So, these you should remember this f x bar means, this components x 1 bar, x 2 bar, x n bar they satisfy this system of equations that you should remember that.

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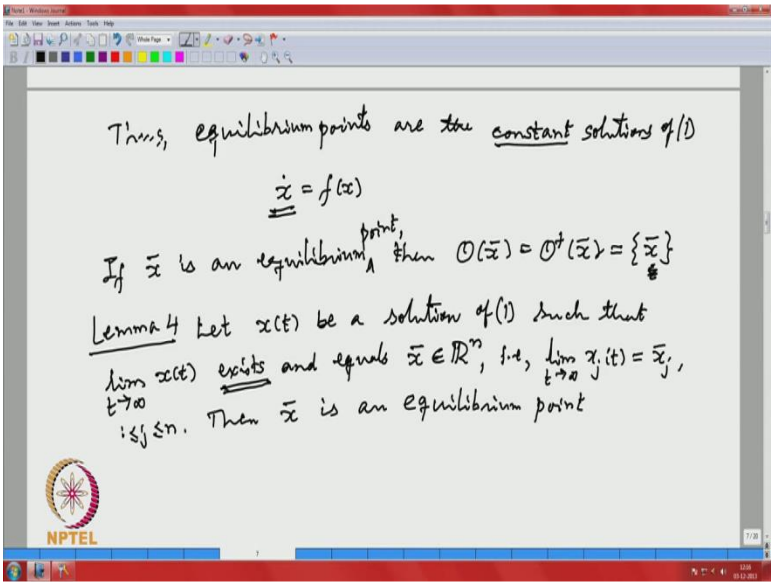

So, another interpretation, so of this equilibrium points thus equilibrium points or the constant solutions, let stress that constant solutions of 1. So, constant solution means it does not change with time, so if x does not change with time then the derivative 0; that means, f of x 0. So, just look at the condition here x dot equal to f x in either one equation or more equation. So, when x is a constant solution that d x by d t is 0, so that means, f of x is 0 and conversely, if it is equilibrium point then f of x is 0 there and then you get that constant solution.

So, equilibrium points are nothing, but the constants solutions of y, they are solutions that is important, so now you immediately see that if x bar is an equilibrium point and now I want to look at the orbit of that x bar, since that itself is a solution. So, immediately obtain, that x bar is positive orbit of course, a why only positive orbit I am also consider negative orbit, but we are interested in the future.

So, that is why we consider the positive orbit, so here is a situation, so where you can immediately determine the entire orbit the positive orbit, they are all equal to this single term set x bar, so that is the beauty of this equilibrium points. So, in this case you can immediately determine its orbits. So, now another important in that is connected with equilibrium point is this following lemma, so let me do this thing and then you go to examples.

So, we also see the significance of the equilibrium points and what happens if the system does not have equilibrium points see if this for example, this f may never be 0, then that is particular system will not have any equilibrium points. So, will see through some examples, so what is that significance and when there are equilibrium points, what is there significance will see all these things in detail.

So, before that, so if x bar is an equilibrium point then we know it is orbit precisely. So, if I have a non trivial orbit what does that mean, so it is an orbit not passing through the equilibrium point, then that cannot intersect this orbit. So, that is because we have seen in lemma 2 that either 2 orbits or the same or they are the disjoint all the time. So, if I have solution not pass, cannot pass through this thing because again uniqueness because that is the only solution passing through the given equilibrium point.

So, if I start with another solution that orbit will never intersect this equilibrium point, but it can approach, so that is what the lemma says. So, let x t be a solution of 1 such that limit x t as t tends to infinity exists, when we say exists, so you always mean a finite number and equals x bar, so this is in R n, so let me again write it component wise. So, that is, so if you take the j'th component of the solution this is x bar j, so this is bar.

So, that I am assuming this solution x t has this limit, so then necessarily then x bar is equilibrium points. So, if a solution has a finite limit s t goes to infinity, so this also you can do it for t tending to minus infinity, so necessarily the limit is an equilibrium point.

Now, just for curiosity we can just argue that suppose the system does not have any equilibrium point so; that means, no solution can have this finite limit.

In most of the cases, in the options of equilibrium points the all the solutions of the given system will be unbounded will see through some examples, so that is one significance of the equilibrium points.

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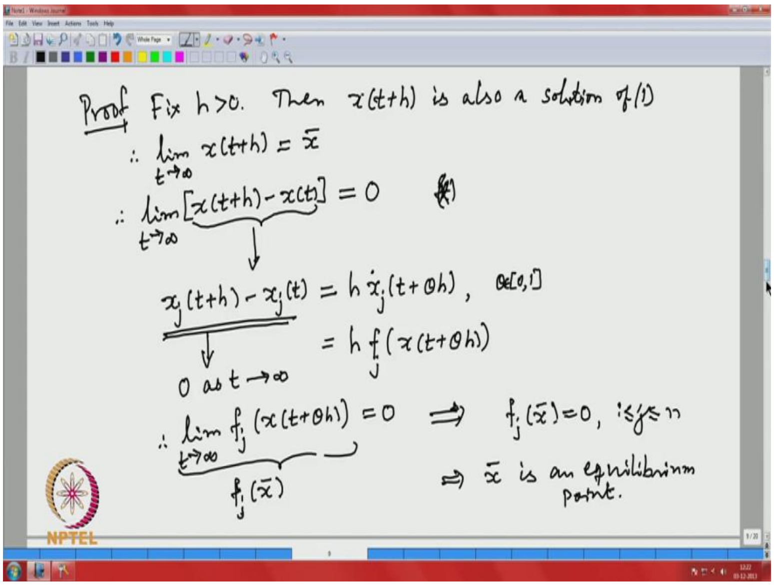

Let me just give a quick proof of this, so let me a write again component wise fix h greater than 0 then x of t plus h is also a solution. So, repeatedly using that lemma 1 you see that, so that is an important thing, this also a solution and as t tends to infinity, t plus h also tends to infinity. So, therefore, I have limit x t plus h is t tends to infinity is also x bar and since x bar is finite the difference if I take the difference.

So, therefore, if this not of the infinity minus infinity forms, so you can separate the limit, so this is calculus, since both are x bar and they are finite, so this is 0. So, this is not of the, so I can write this as limit of x t plus h minus limit x t because both are finite and now look at this component wise. Let me write this, component wise for each is j equal to 1 to n.

And use the mean value theorem, each x j is a differentiable function, so by mean value theorem, so this is just h x j dot at some point theta h, so this is theta is 0 1. But, x is not an arbitraries function it is a solution of 1, x is solution of 1, so by using the equation, so this is f of now let me write it has x of t plus theta h, because x satisfies equation 1, system 1.

And now you see look at this limit, so I am using this limit star. So, the left hand this one goes to 0 as t goes to infinity, that is from star that we are seen it and therefore, since h is positive. So, I conclude that therefore, limit f of f j for each j, x of t plus theta h, as t goes to infinity is 0 and now the continuity, so this is again part of the preliminaries in analysis. So, this is f j is continuous function x is a continuous function.

So, I can push this limit inside here and this limit let me just write that f j of x bar so; that means, so this implies f j x bar is 0 and this true for all j, so that implies x bar, so with these few preliminaries.

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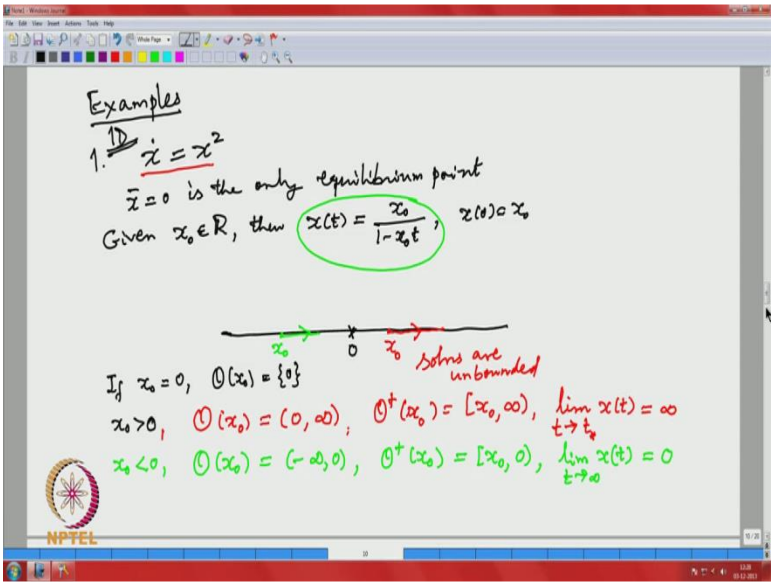

Now, we consider some examples, so first the simplest thing just consider this 1 d, so most of the examples are 1 d examples, so except perhaps the last one, if I am able to come to that thing that will be 2 d examples, so this is 1 d; that means, just a single equation and. So, f of x is x square here and so the 0's of that function is just is in this case it is only 0. So, x bar equal to 0 is the only equilibrium point, so given you can also write the explicit solution.

So, you should complete all the steps, then this x t is a solution, x of 0 is for more calculation we just take the t 0 equal to 0, but without this using the solution I would like to calculate the orbits, positive orbits for arbitrary x naught in R. So, for that just draw this line and here is your equilibrium point, so the orbit of that equilibrium point is; obviously, that single term, so any orbit starting here. So, automatically it has to stay there and any orbit starting in the left, it will stay here because it cannot intersect with that orbit.

So, orbit, so this if x 0 is 0, then this orbit just 0, so any other orbit cannot intersect, if x 0 is positive, let see what happens. So, let me use a different color here, so x 0 here and without this knowledge of the solution, let us try to see how the solution moves. So, since some simple calculus would help us in most of the 1 d problems. So, look at the equation, it says d x by d t is equal to x square and x square is always non-negative, so derivative of that solution is non-negative that means the solution is always increasing with time.

So, if I start here the solution has to only move to the right, simple calculus I m not using anything here, so using that fact, let me write here. So, you just see that x 0 is 0 infinity and the positive orbit, just x 0 infinity and since it will stay here and it is keep on increasing, we can just conclude that x t of course, it will not exists in this case particular case, for all there is a t star, so let when this denominator become 0. So, it is infinity the solution is unbounded here.

Because it is the orbit is this entire 0 infinity the solutions in this case are unbounded, so now look at a point here, x naught is again the solution is increasing. So, there is no because x dot is x square again it is increasing, so it will be moving this direction and there is a barrier here, so cannot cross that thing. So, at the most it can read that in infinite time, so in this case we verify that x 0 is minus infinity 0 and 0 and limit of x t and in this case it exists for all time here it is 0, so verify these facts. So, this just follows from this simple analysis, so I do not really require this explicit solution.

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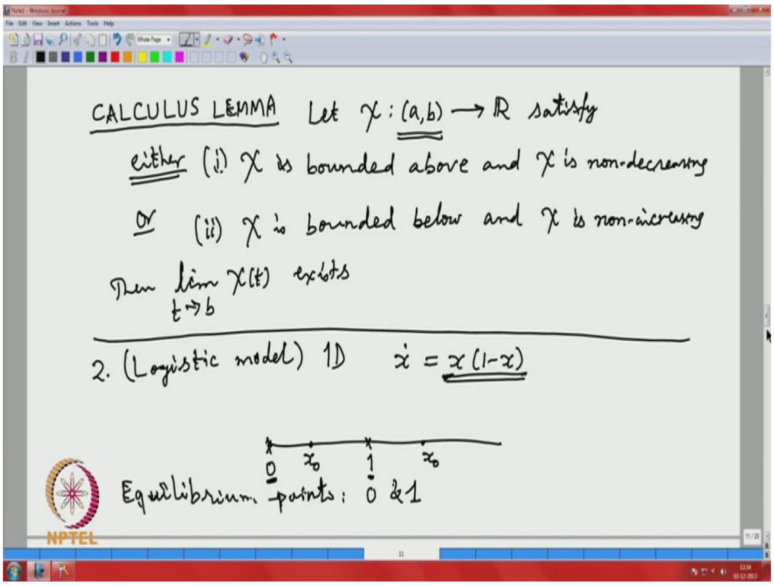

So, before moving to the next example, so let me just in remaining ten minute, let me just write this calculus lemma. So, it is very easy to prove will include in the, we will discuss in the discussion on preliminaries on analysis. So, let chi the real valued function defined on an interval a b and this could be entire R, it could be finite infinite, it does not matter semi infinite, whatever in any arbitrary interval of the real line and chi is a real valued function defined on that interval satisfy either, so one of the two conditions.

So, let me chi is bonded above and chi is non decrease or chi is bounded below and chi is non-increasing then limit. So, again let me recall that when we say that limit exists it is a finite real number, it is very easy to prove in the first case this limit will be the supreme of chi t. And in the second case it will be in primo of chi t, so will discuss more about this in when you discuss preliminaries.

So, using this thing, so let me now consider second example, so this is logistic model of population growth, so again 1 d, x dot equal to x into 1 minus x, so usually there are some constants here. So, here I have just normalized and assume that the maximum possible population is 1, you can normalized that otherwise you can also put numbers that is no problem, in fact in the discussion on physical models you see actual numbers, but it does not matter.

So, again here one can write down the explicit solution, but without that explicit solution I would like to draw the orbits through arbitrary points, let me just. So, since it is a population model we are interested only in positive solutions and see whether we get positive solutions, if we start initially at positive initial data. So, here there are two equilibrium points, so equilibrium points 0 and 1. So, look at the place where this right hand side is 0 that is x equal to 0 or x equal to 1.

So, then by the discussion on lemma 1, lemma 2 etcetera, etcetera, so if I start a solution here it is orbit cannot go beyond this, because that will intersect the orbit of this equilibrium point 0 and equilibrium point at 1, so it cannot intersects.

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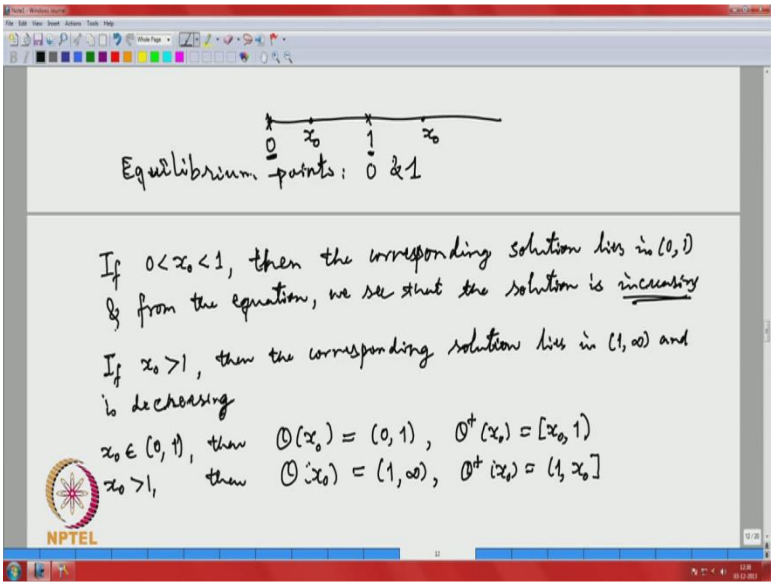

So, if I have an x 0 here, the solution always remains in this interval 0 1 and similarly if I start x 0 here the orbit will always remain there, so let me just let me do that thing. So, if 0 x 0 less than 1 and again look at the equation says x dot is equal to x into 1 minus x. And in this case we have already seen that the solution lies between 0 and 1 again, so the then the corresponding solution lies in 0 1

And then you look at the equation and from the equation, we see that the solution is increasing are at least not decreasing x into 1 minus x, x always lies between 0 and 1, so that product is positive, so x dot is positive, so x is increasing. And in the other case, similarly if x 0 is bigger than 1 then the corresponding solution lies in 1 infinity and is decreasing, now if you look at x into 1 minus x, x is bigger than 1. So, the second factor is negative, so it is decreasing simple calculus we are not using anything else here just simple calculus.

So, if x 0 belongs to 0 1 now you can write down the orbits, so the orbit is just 0 1 and the positive orbit this is more important for us, x 0 1 because it is increasing that is because it is increasing. So, if x 0 is bigger than 1 then the orbit is1 infinity and the positive orbit now it is decreasing, so you see that 1 x j. So, with this thing will continue in the next lecture, so what happens to be limit of this thing, so go and do all this whatever gap is there. So, try to fill in all those gaps and will see in the next lecture.

Thank you.