Ordinary Differential Equations
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Lecture - 29
Basic Definitions and Examples

Hello everyone, I am P. S. Datti from TIFR Centre for Applicable Mathematics Bangalore. I will be covering some portion of this ODE course along with my colleagues Nandakumaran and Raju k George. Today I will discuss this qualitative theory of differential equations.

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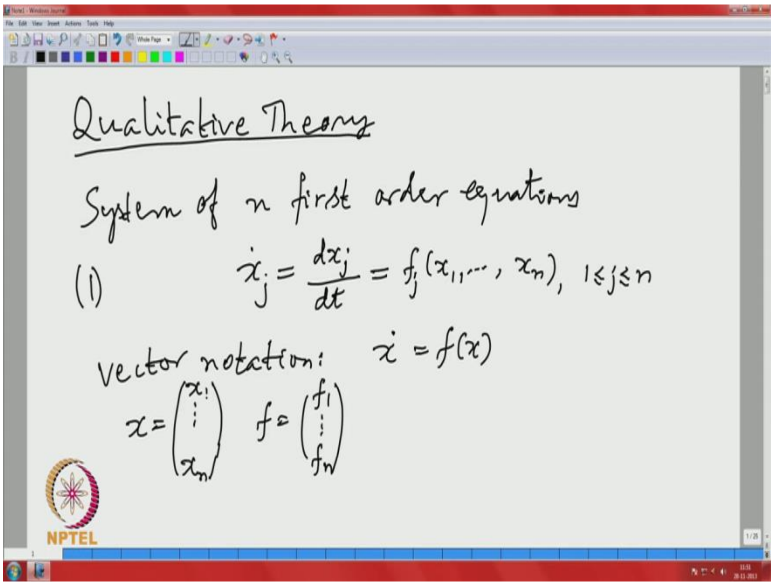

So, let me first explain my notation and some motions. And what is the main objective of this part of the theory of differential equations. So, we will be studying a system of n first order equations. So, I will write this system in the following form x j dot. So, dot for me d by d t. So, let me write that, so this is same as d x j by d t. And that is equal to f j of x 1 x to x n.

So, there are n equations this is i j 1 less than equal to j less than equal to 1. So, let me number this equation for this class. So, I call this... So, can you this shortened vector notation, so the same thing. So, vector notation, which will be using more often. So, this whole system just written as x dot equal to f of x. So, this x is for me this n vector and each is a function of t. And similarly this f is a vector of this n functions f 1, f 2, f n.

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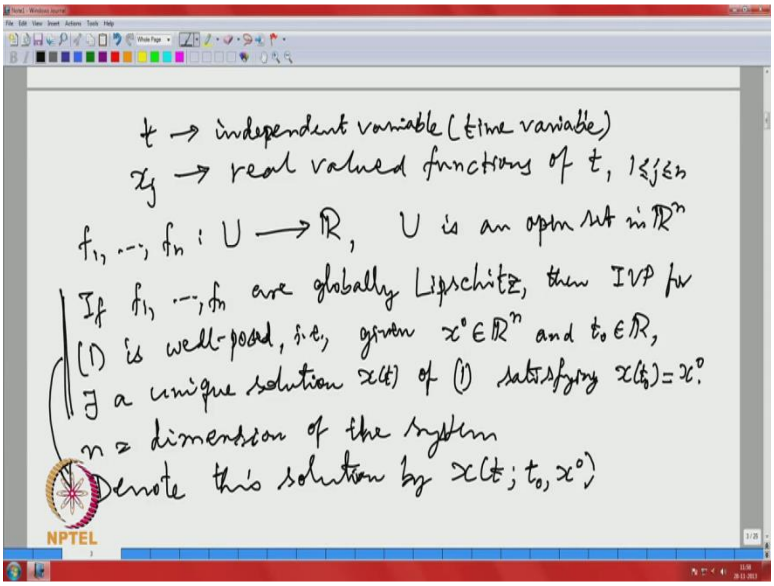

So, this t is the independent variable. So, usually we call it time variable. So, we would like to think the system one as a dynamical system. And this x 1, x 2, to x n are positions of the particles. And you want to determine given some initial position, how the system evolves given that law of the differential equations system of equations.

So, this x j's are real valued functions of t. So, what about the functions f 1, f 2, f n? So, we need some assumptions on that. And to start with at this stage, the assumptions on f 1, f 2, f n will be that the system one, the corresponding initial value problem as a unique solution given an initial vector in R a. So, that is those are the assumptions. So, f 1, f 2, f n are functions either from an open set in R n.

So, U is an open set in R n it could be R n, at this state again. So for example, if f 1, f 2, f n are globally Lipchitz. So, if f 1, f 2, f n are globally Lipchitz, that you already learned in the existence theory of differential equation, then IVP for 1. So, initial value problem IVP for 1 is well posed. So, what does that mean that is let me...

So, given a vector x 0 in R n and t 0 is in R, there exist a unique solution x t of 1 satisfying x t 0 is x j 0. So, the for the time being these assumptions on the functions f 1, f 2 is sufficient in order to introduce some motions. And later on in fact, to need more and will add those hypothesis on these functions as we go along.

So, we denote the solution this. So, this is guaranteed for us from the existence theory. Once, we assume this global Lipchitz conditions on this functions f 1, f 2, f n. And most of our examples though I m stating it for a general n most of our examples are n equal to 1 that is 1D n equal to 2D and n equal to 3, that is 3, 1, 2, 3 the dimensions.

So, n is referred to as dimension of the system. So, before I introduce some notations. So, let me also just... So, this solution we denote this solution by x of t, t 0 x 0. So, this I am emphasizing that initial time and the initial data. And when they are not changing in a discussion, then we might as will ignore those t 0 and x 0 just simply write x t. But, when they are important in a discussion, then we will emphasize the dependence on t 0 and x 0. And usually we take the t 0 equal to 0.

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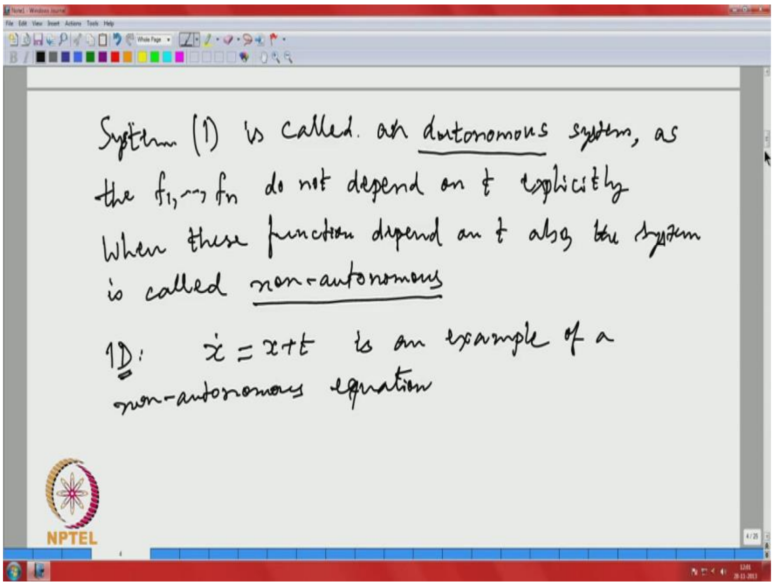

System 1 is called an autonomous system. For the region as the functions the right hand functions f 1, f 2, f n do not depend on t explicitly. So, when they do depend on t explicitly, when these functions depend on t also the system is called non autonomous a simple example in 1D. So, D this capital D refers to dimension. So, this is just a single equation, so this x plus t. So, is an example of a non autonomous equation there is only one.

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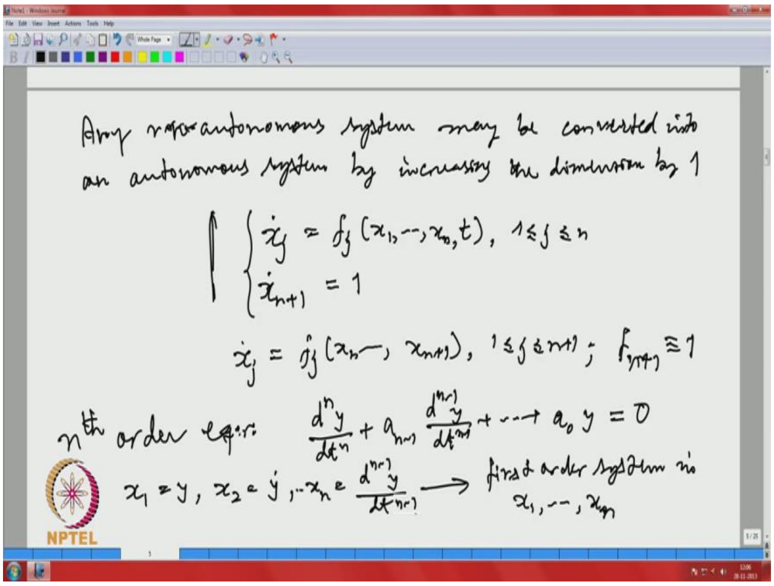

So, let us call it equation not system and any non autonomous system just as a remark I will any non autonomous system may be converted into an autonomous system. But, then you have to pay a price for this by increasing the dimension by 1. So, I quickly explain this. So, for example, even if you have a 2D non autonomous system and if I want to convert that to autonomous system, my system will now has dimension 3.

And this is quite significant, as we will see as you go along. So, how it is done? So, let me just quickly mention that, so x. So, we have a non autonomous system. So, x j dot f j x 1, x 2, x n and now these f j's also depend on t, so this is... So, I have again system of n equations, now the right hand side explicitly depend on t. So, I will make convert this into a autonomous system by introducing another.

So, these are only n equations, now I introduce another equation x n plus 1 dot equal to 1 and whole thing now I can rewrite as an autonomous system x j dot equal to let me write f j x 1, x 2, x n plus 1 now, so 1 j less than n plus 1, and if you look at this system here. So, f n plus 1 is identically 1 that is constant function. So, we restrict ourselves to study of autonomous systems and they enjoy certain properties.

So, that we are going to list now, if you nice properties. And one more think I would like to mention at the stage. So, if you have a nth order equation. So, this also we have already learnt nth order equation of the form d n y by d t n plus a n minus 1 plus a 0 y equal to 0. For example, where this coefficients a n minus 1 a and minus to a 0 can be functions of y and if derivatives of to be nth minus 1 order.

So, in that sense it is Cauchy linear we call it Cauchy linear. This can be converted into a system of first order equations, autonomous first order equations by simply introducing the new variables. So, x 1 equal to y, so you have already learned it. So, just mention it x 2 is equal to y dot etcetera. So, x n is equal to... So, this converts into first order system in x 1, x 2, x n.

So, the consideration of first order systems in essence more general than even considering any nth order equation. So, that is also makes it important. So, just we have to concentrates just on the first order systems. So, that will also cover. In fact, most of our examples are second order equations and will be writing a first order system. So, that just concentrate the study on first order systems.

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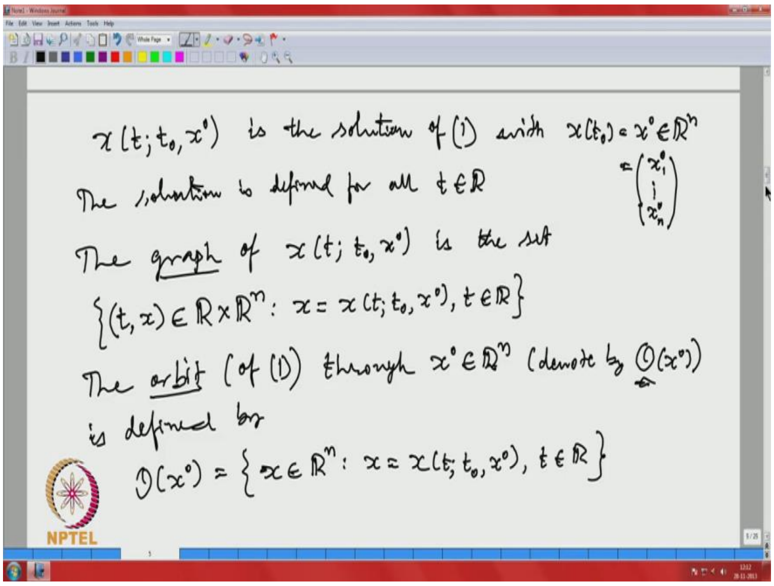

So, recall that we have denoted by this x t, t 0, x 0 is the solution of the system 1 with x of t 0 equal to x 0. So, x 0 again remember it is a given vector in R n. So, you can also write it as x 0 1 x by x 0 n, so n in vector. So, since we assume nice conditions on f 1, f 2, f n. So, the solution is defined for all t the solution is defined for all t in R.

So, we already seen examples that this may not be the case for all differential equations. So, there is a for certain differential equations, the solution does not exist for all t in R.

So, in that case one can consider the interval maximal interval of existence. But, for simplicity I just assume that, will see those examples again and that will not affect the qualitative analysis we are going to do.

So, next we define two notions; these are important notions as far as the autonomous systems are concerned. The first one I will not give any notation, that is just passing I will mention that the graph of x t, t 0 x 0 is the set. So, this is a subset of R n plus 1, so I write the coordinates as t x. So, t belongs to r and x belongs to R n and what is the x to do with t and this solution.

So, I start with a solution, the initial value problem of one and now I define it is graph. So, x is nothing but, x of t, t 0 x 0 and t belongs to R. So, we will have a some simple examples later. So, this is nothing but, suppose n equal to 1. So, this is a just I am instead of array drawing the graph, I am just writing it as a set. So, that is...

And the next notion is very important, that is where we are going to... So, this graph is also important, but more than graph, you want to suppress t. So, you want to project this graph on to R n and that is the name, we give for that. That, what is that projected set is, so the orbit of 1. So, that always you remember that system orbit of 1 through x 0. So, this is a given vector in R n. So, this we denote by o x 0, so this o for orbit.

And x 0 is the point through with the orbit is passing through is defined by... And now this is a just subset of R n, this is just x belongs to R n such that x is t in R. You see, if you compare this definition of the graph, that is the set. And now I have just suppressed that t. So; that means, I am projecting this graph on to R n. And I m calling that has orbit through x n. And if you do not restrict this t belongs.

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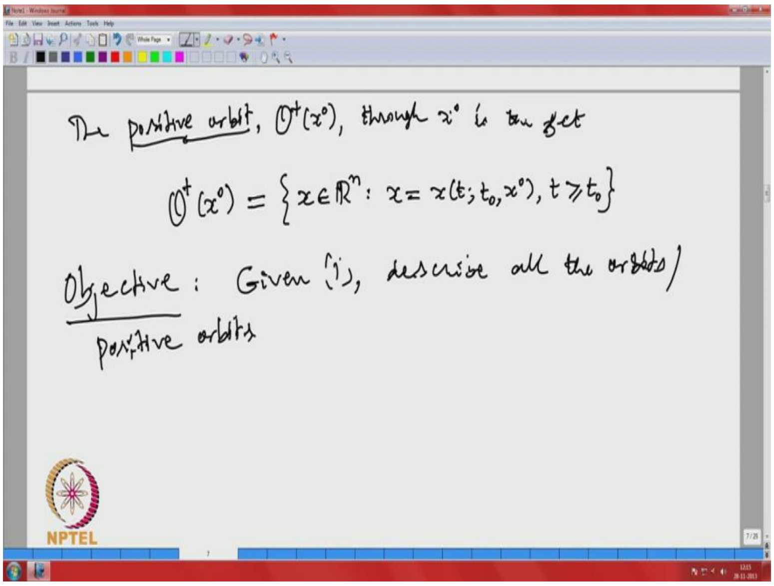

If you just restrict it to only certain set, then the positive orbit by denoted by o plus x 0 again through x 0 is the set. So, it is a subset of that full orbit. So, again x is in R n, x is a equal to x of t, t 0, x 0 now I restrict only. So, what happens in the feature I am not interested in… Suppose, I not interested to know what happens before I start I am interested only in the feature.

So, I start at time t equal to t 0 and just would like to know, what happens after that. So, I just study this positive R n. So, the objective of this qualitative theory. So, the main objective, now I can is after introduction of these notions, we can simply say that. So, given 1, so that is describe all the orbits or just positive orbits.

So, at least objective is very easy to state. So, I we are given a system 1 and just describe the orbits and positive orbits. So, easier to say but very difficult to do. So, we will be trying to do in some situations, especially in 1, 2 and 3 dimensions. So, before I go further to describe the properties of these orbits and other things. So, let me just take some simple examples.

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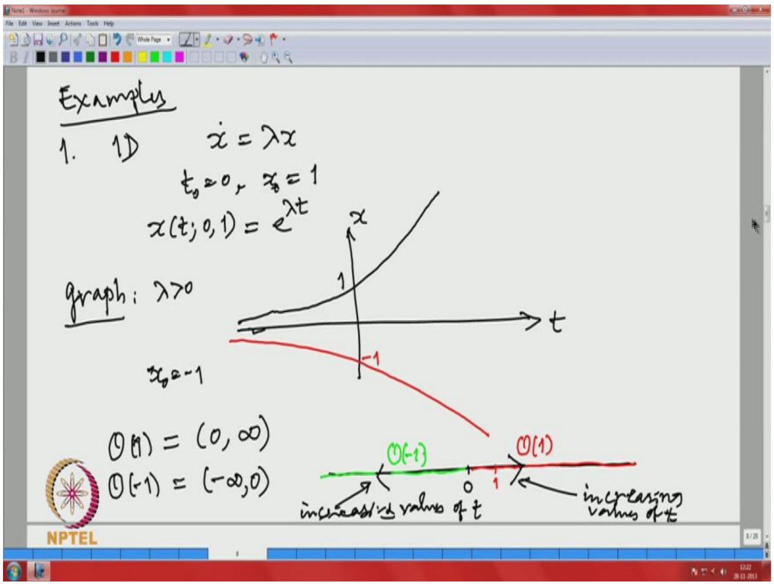

So, simple examples, so the first one is 1D, so one just we already seen this many times. So, just let me write it in a different way that is all, so this is just one equation lambda x. So, I take t 0 equal to 0 and x since it is one equation, so let me write it as x 0 say one. So, the solution is x t 0, 1 in our notation and this is just simply e to the lambda t. So, at time t equal to 0, it takes the value 1. So, whatever may be the lambda.

So, the graph, so let me just the use your graph. So, for example, let me take lambda positive and we will see what happens to lambda negative also. So, this is a let me describe it graphically. So, this is of t and this is x, so I am taking lambda positive. So, it looks something like this, that is one and if I take for example, x 0 minus 1. So, that also I can just do it may be with different color. So, that is still lambda is possible, so this is minus 1.

So, this is the graph of this solution and this is the solution. And these are also refer to a solution curves or even face curves. So, that word face I will explain in a minute, that is not there is no mystery about it. So, face curve, face plane analysis, face line. So, since where in dimension we can call it this is place line analysis, and what about orbit? So, let me just in this case for the same.

So, this orbit passing through one what is this, if you just look at this. So, I have to suppress the... So, this what you see? So, this even lambda is positive, the solution is always positive. So, you see that it is just 0, so this is the orbit. And similarly, minus 1 you see this is minus infinity. So, again graphically will be doing that most of the time.

So, this which you can just represent theta on the line, so this is the origin. So, let me just, so this is o of 1 you can see in fact, it is 1 is most specialty here it is true for all positive values in similarly here this. So, one more thing we have to do with respective orbits here. Since, in case of graph we have the notion of positive t x's and positive x's at least there is direction of t.

So, we see that by looking at the graph, whether t is positive or not. And here, since we are suppressing that t. We have to draw a line to represent the increasing values of t in this orbit. So, as you see here, so sorry for that, so this is important. So, let me just write, so this is increasing values of t. So, the values put arrow in this direction, in indicating that is if you go along this thing.

That means, we are going along the increasing values of t. And similarly, here you see that this is the, so increasing value. So, whenever you draw orbit we will be doing this. So, that will indicate the increasing values of t, so in this simple example. So, similarly if in lambda negative can do that thing, so that is a no problem. So, what about suppose we are interested in only in the positive orbit. Suppose, just we are interested in positive orbit in this case.

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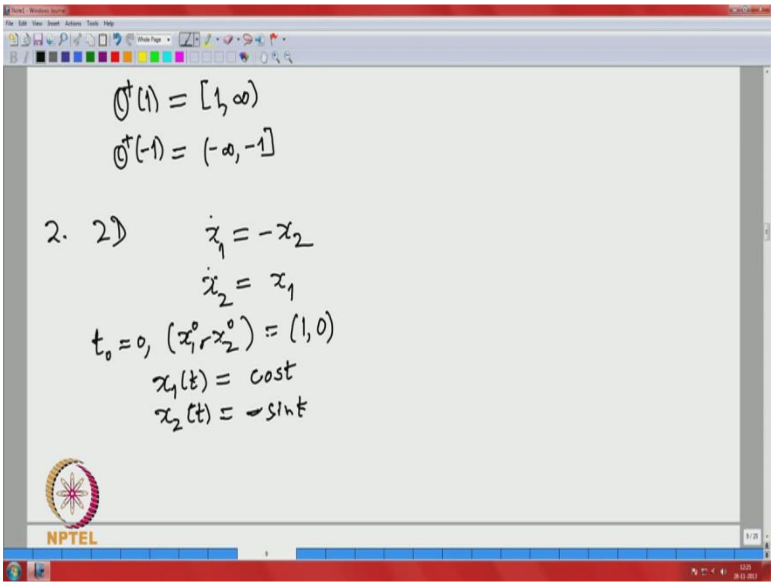

So, this is just, so let me write in the next page, so o plus 1. So, since we have taken t 0 equal to 0. So, we are only interested in t bigger than equal to 0 and that you easily see that it is just one infinity. And similarly here o plus minus 1 it is minus infinity minus 1. So, in the simple examples you if you get used to these notions of orbits, you know that will be useful.

So, whenever you are in doubt you do a simple example and notions will be clear, they are not very difficult once. So, just always try to do some examples, some simple once just like I did. And let me now take an another example, this is a 2D example, again this we have seen many times during the course. So, now 2D, so there are two unknown functions. So, let me just write x dot minus x 2, x 2 dot is equal to x 1.

So, this is linear pendulum equation, this we are seen many times. So, we can write down the explicit solution in this case. So, again let me just, so take t 0 equal to 0. And now this x 1 0 we need the initial point next to 0. Let me take for simplicity just 1 0 passing through y. So, we can immediately write on this solution. So, you have just let me just write x 1 t, so suppress that everything and x 2 t. So, this will be cos t the minus sin t. So, it is very difficult to draw the graph even in this simple case. So, let me try to do that.

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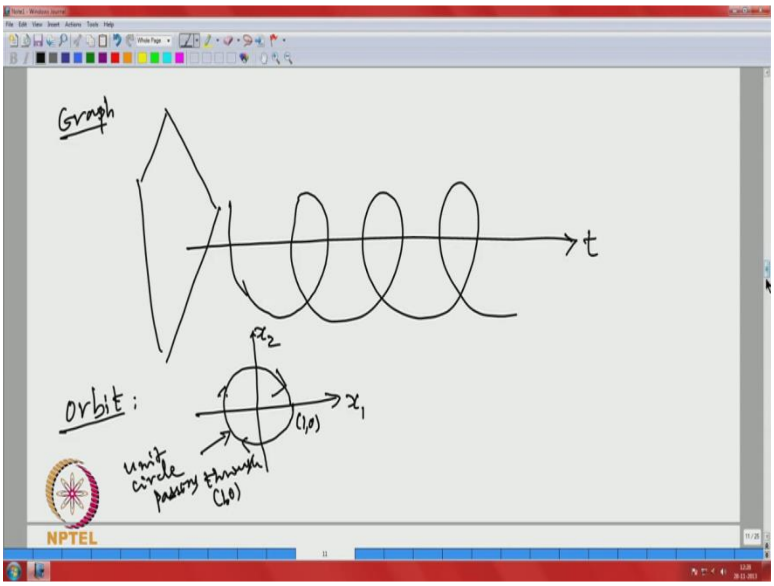

So, graph, so this is the t x is and let me draw this plan here. So, this is, so you will see that pardon me, if I am not drawing it correctly. But, that is, so it will a spiral kind of thing, it goes on like that we will see that write as t moves on. So, if you draw that cos t and minus t, but orbit is very easy to explain. So, if just you project everything on to this plane.

So, orbit is very simple it is just, so let me just find x 1, x 2 and this is just 1, 0 the orbit is passing through 1, 0 and just it is circle. So, unit circle passing through 1, 0 and again want to indicate the increasing values of t. And if you look at the equation, it will be in this direction, so it just goes around. So, if you change the sins in the equation for example, if you take x 1 dot equal to x 2 and x 2 dot equal to minus x 1.

Then, we will be reversing the orientation instead of counter clockwise, we will be I will clock wise will be going counter clockwise. So, these are two simple examples giving you the idea of graph of a solution curve and then it is orbit. So, now we are going to discuss some important properties of autonomous systems.

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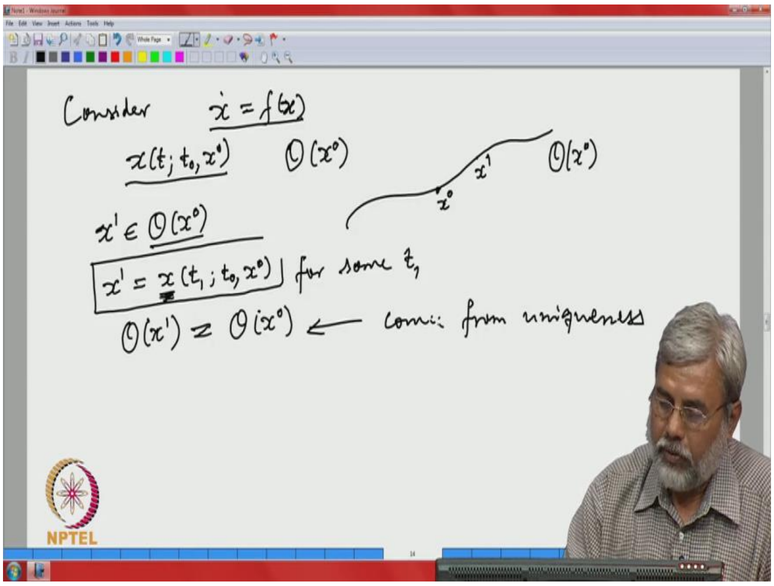

So, let me begin with that thing. So, before that, so again consider. So, now we are going back to system one, so remember system 1 is that x dot equal f of x. And we have a solution, so this is our system 1. So, we have this x of t, t 0 x 0, so solution which takes the value and given initial point t 0. And we define, so the orbit let me just. So, I ask you a simple question. So, now I pick any vector, so this is remember this is an R n in this orbit.

So, for just picture sake, so this suppose this is o of x 0. So, it is passing through x 0, so let me just say this is x 0. So, that is the vector in R n, so I am just drawing it in the plane and the x 1 is here somewhere, this side, that side it does not matter. So, this means what if you look at the definition of the orbit. So, x 1 is equal to x of t 1, t 0, x 0.

So, remember this x of t etcetera is the solution of this system of differential equation. So, that you remember is not just any ordinary functions. So, it is a solution of that differential equation, that you remember. And now I ask the question, what is the orbit of x 1. So, here is the picture, so I am considering an orbit through a given point x 0.

And now I am picking a point in that orbit on that orbit x 1. And by definition x 1 is this for some t 1 are some t 1 with r 1. So, that is now I am asking question, what is this o x 1? So, your guess is very right and it has to be just the same orbit, just by looking at the a rigorous mathematical proof come from the uniqueness, so that you should remember.

So, now you see that x is a solution of the given differential system of the equations and that t 1 it takes the value x 1. So, interpret this one as the solution is taking the value x 1, that x 1 is given to us at time t equal to t 1. And by uniqueness there is no other solution, that is the only solution passing through a x 1 at given time t 1. So, the same x the same solution works for this point also since it is already on the orbit.

And hence the two orbits go inside. So, this is an important thing, so if I start with an orbit and then I take any point on the orbit. So, orbit through that another point is the same as this orbit and that is coming from uniqueness. So, this remember that comes from uniqueness. So, these simple things, but you should know that. So, now will go to describe some simple properties.

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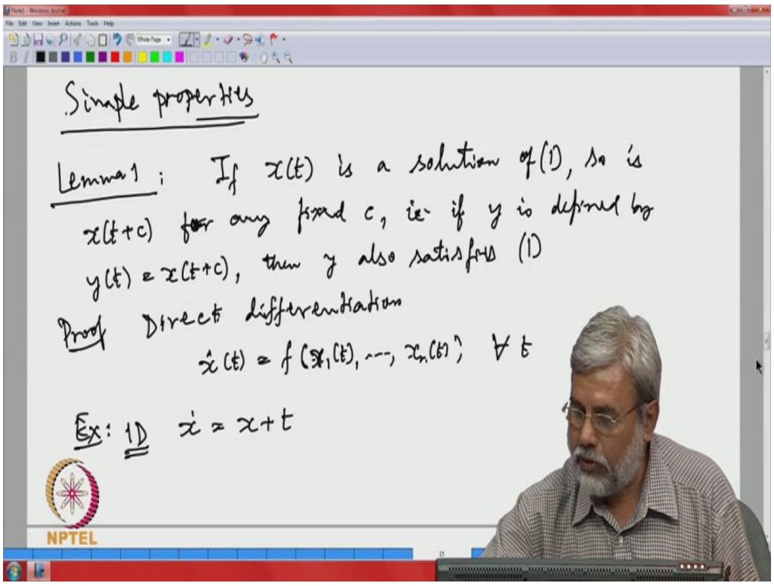

So, simple properties simple, but important… So, I will state these things in the form of lemma. So, lemma I will put 1 lemma 1, so I emphasize the dependence on t, because we are going to change that t if x t is a solution of 1. So, is x t plus c for any fixed and this is very special property for the autonomous systems. So, it is certainly not true for the non autonomous systems.

So, simple examples we can try, so what does it means let me explain this thing. So, just; that means, that is if y is defined by y of t is equal to x of t plus c, then y also satisfies. So, this is direct differentiation, so just differentiate y with respective to t and then that will translate into the derivative of x. So, this proof is just one line.

So, direct differences, so I am not doing anything here. And you use the fact that this x dot t is equal to f of x 1 t etcetera x and t for all t. So, in particular if I replace t by t plus c, this will be prove and that is what we want for y. So, this you remember this, this is for all t, so it is this equation is satisfied for all t. And simple example if you want to try this just again 1D let me mention that 1D.

So, when I change t to t plus c you see there is a dependence on t. So, that also will be changed to t plus c. So, I will not have this if I define again y t equal to x plus t c, it will not satisfy y dot equal to y plus t, it will satisfy y dot equal to y plus t plus c. So, that is the explicit dependence on of the right hand side on t. So, this lemma 1 is enjoyed by only autonomous systems and from there though it is, so simple a very useful result follows that I will take.

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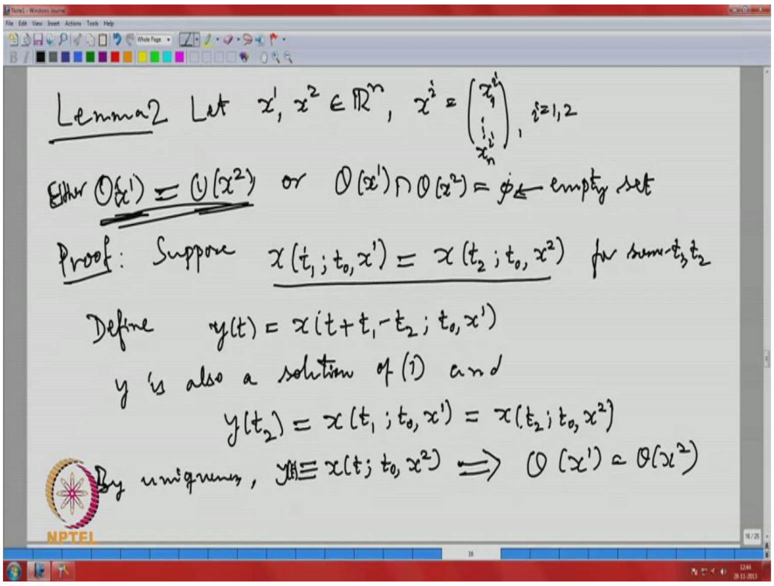

So, you took two points in R n. So, let x 1 and x 2 be in R, so I am writing super scripts. So, that your you know that they are vector there not just a real numbers. So, if you want me to stress that. So, x 1 or x i x i 1, x i n, so they are x i equal to 1. And consider this orbits through x 1 and x 2 and this lemma 2. So, let me put that 2 and the lemma 2 says either this is equal to that or, so this is empty set.

So, this is a very remarkable result this follows immediately from lemma 1. So, that in the remaining 5 minutes I will just indicate the proof of this. So, this is remarkable thing it is slightly more than uniqueness, uniqueness is certainly there, but it is slightly more than that. So, we are not using the graph here, we are projecting the graph on to R n. And there also we are claiming the a kind of uniqueness.

So, either any two orbits are the same or they are disjoint. There is no common point between two orbits, if they are distinct then they will remove distinct forever. So, proof is very simple and this one will be using throughout this discussion on the quality detail. So, suppose they have a common point, this orbit o x 1 and o x 2. Suppose, they have if they are already disjoint I do not have to proof anything, suppose they have a common point.

So, there is point here and there is point there and they are equal. That means, what suppose x 1 t 1 is equal to, let me not bother about that orbit passing let me just. So, this is orbit passing through x 1, so let me just suppose x of t 1. So, let me just write t 0 x 1. So, this is a point in o x 1 orbit true x 1. So, this is x of t semicolon t 0 x 1 is the solution curve passing through x 1 at time t equal to 0.

And suppose that is equal to x of t 2, t 0 x 2. So, in fact we have already seen that is what the that remark was there. So, I if I take any point on the orbit, then orbit through that point is also the same as the original thing. So, that is a, so now define x of t. So, I am just remember this are all x, x, x, so let me this is a new things. So, let me may be I will define y of t is x of t, this we can figure it out plus or minus we will see that, this is for some t 1, t 2.

Then, I would like to conclude that at the orbit of x through x 1 and orbit through x 2 are the same. So, this either of this thing… So, let me take this solution maybe I should have called it x 1 that is fine. So, now by lemma 1, so I am just translating this t 1 minus t 2. So, this y t is also is a solution y is also is a solution of 1.

So, now you just compute y at t 2 just let me figure it out t, if I compute a t 2 then this is will be a x of t 1, t 0 x 0 that is fine an important t naught y of t 2 is just you substitute t equal to t 2 and this will just become x of t 1, t 0 x 1 and that is same as according to our hypothesis, you just see there this hypothesis x of t 2, t 0 x 2. So, we have obtained a solution of again system 1 which satisfy this y of t 2 equal to x of t 2 is 0 x 1.

And now you use uniqueness. So, that is by uniqueness y is identically equal to y of t x of t, t 0 x 2, so this solution. So, this implies the orbit of this solution y the two orbits are same let me. And by if you look at this one, this a the orbit of y is same as the orbit of x. And now with this thing this orbit of y is also same as this one, so this let me just. So, this complete, this implies y of x 1 equal to and completes the proof and will elaborate this in the next class.

Thank you.