Ordinary Differential Equations
Prof. A. K. Nandakumaran
Department of Mathematics
Indian Institute of Science, Bangalore

Lecture - 28
General Systems Continued and Non-homogeneous Systems

This is the last lecture of this module. So, let us complete in the... So, today we have two parts, we consolidate all the results we have discussed so far in the last 4 lectures, and in the second part of this particular lecture. We will try to discuss a quickly about the non homogeneous equations so far we were discussing the homogeneous system. But, at the end of it we will also discuss the non homogeneous system today. So, to introduce that we need a concept called, we to consolidate results and write it in the form of theorems. We are going to write now whatever we discussed in the form of theorem, we need to have the concept of generalized Eigen vectors.

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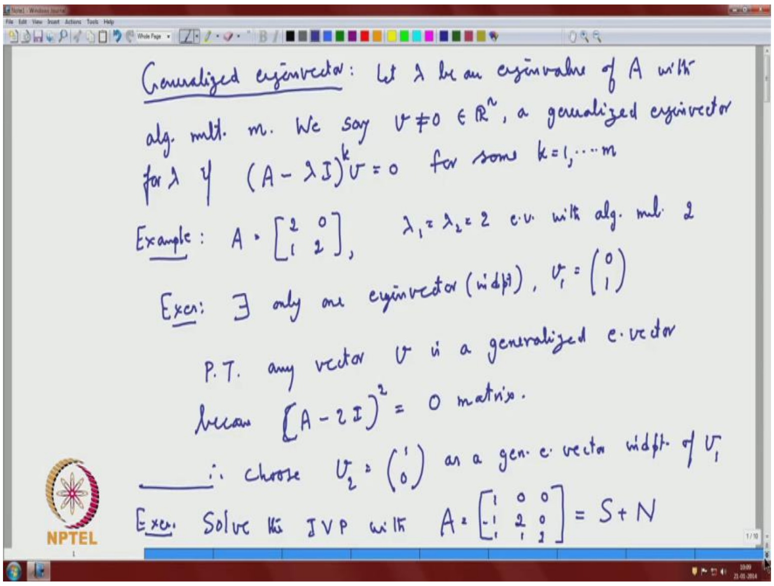

So, generalized Eigen vector as you see, our whole trouble is that when there is an Eigen value, which is repeated. And is an Eigen values repeated with order m and we may or may not get m Eigen vectors. If we get that many number of Eigen vectors corresponding to it is algebraic multiplicity. Then, there is no problem we can always diagonalize the matrix and we can write down the solution.

So, the problem comes for a any particular Eigen value. If it is geometric multiplicity is strictly less than the algebraic multiplicity. Then, there will be a deficiency of a Eigen vectors to find a basis of r n. So, that the matrix can be diagonalized. So, in this scenario we introduce the concept of generalized Eigen vector. So, what is called... So, let lambda be an Eigen value with algebraic Eigen value of A which is an n by n matrix with algebraic multiplicity m.

So, we call we say a vector v not equal to 0 is in R n a generalized Eigen vector of lambda. Rather generalized Eigen vector for lambda, if A minus lambda I power k of v equal to 0 for some k equal to say 2 to m, of course when k equal to 1 it is an Eigen vector. So, we generalized Eigen vector we include even the usual Eigen vectors.

So, for some k this should happen, when it happens for k equal to 1 it is an Eigen vector. When, it happens for higher k it is called a generalized Eigen vector. So, we generalized we use it for both the normal Eigen vectors and more generalized Eigen vectors. So, there is some interesting results which you can... So, for example we will start with an example before going to the...

We will not do example most of the parts are exercises. Look at this matrix A is equal to 2 0 1 2. So, lambda 1 equal to lambda 2, the Eigen value with algebraic multiplicity 2. So, the exercise is that, there exists a only one Eigen vector. And that eigenvector can be chosen to be independent, when I say that one Eigen vector means one independent Eigen vector.

Of course, if you have an Eigen vector every multiple of that is also an Eigen vector. But, you do not have a thing and can be taken to be say v 1 equal to 0 you can prove that. If v is an Eigen vector, every Eigen vector will be multiple of that one. Now, prove that, that is easy any vector is a generalized Eigen vector. Prove that any vector v is a generalized Eigen vector.

This is, because you can see that A minus 2 I whole square is nothing but, 0 matrix. You see, once it is a 0 matrix for every vector it satisfy. So, you can therefore, one can choose v 2 is equal to 1 0 as another as an generalized Eigen vector independent of v 1, you see independent of v 1. So, you got a basis consisting of generalized Eigen vectors and this is the thing we are used it.

So, I will give one more exercise here, here it is a let me write, it is a similar thing solve the problem. Solve the problem in, solve the initial value problem with A is equal to 1 0 0 1 minus 1 having 2 0 1 1 2 this is the same thing. So, in fact we are going to write this of course, this exercise is valid after the description. This you can write it as of the form as S plus N, we are going to say that what is S and what is the N. So, you have to complete the problem I will recall this problem again as an exercise later.

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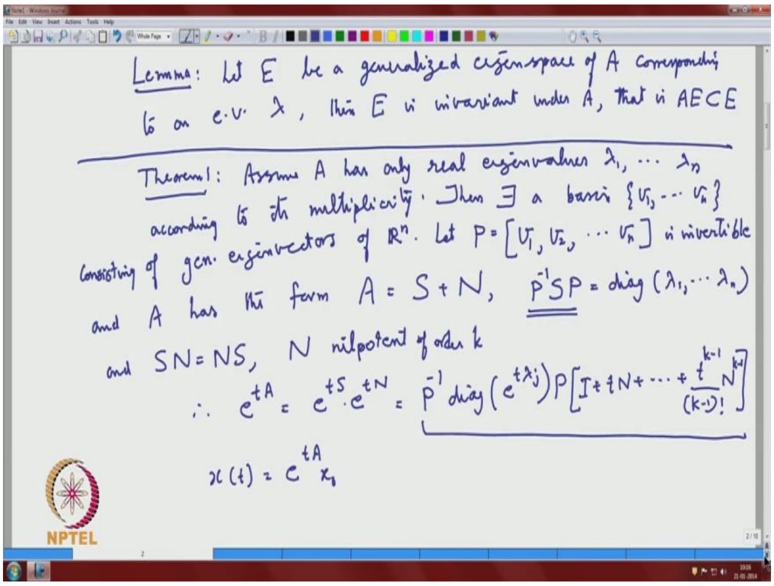

So, there is an interesting lemma can be proved, which is trivial which is not difficult. So, I will skip it I will not prove it here, we do not have that kind of time, that much of time let E be a generalized Eigen space. A generalized Eigen space means, an Eigen space corresponding to generalized Eigen vectors of a particular lemma be a generalized Eigen space.

Just like it is an Eigen space, an Eigen space is the space spanned by the all the Eigen vectors corresponding to a particular Eigen value. Generalized Eigen space of A, then they this is already we are discuss corresponding to an Eigen value lambda. Then, E is invariant under A. That is a quickly you can prove it that is A of E contain the E, see A will act on the elements.

So, whenever you have something, if you take any point in E, it will remain there. So, this kind of things which we know about the Eigen space, will also work for generalized Eigen way. So, I will not prove the results here, because I will not have time to prove it.

So, because we want to complete something, with this we also yesterday wrote that every R n can be decomposed into stable, unstable and all that, which are invariant subspaces under the flow, not under A the flow is E power T a.

So, the spaces E s, E u and E c which are invariant under the flow E power T a. That is what we have discussed yesterday. So, we will leave let me complete all the theorems, this is whatever I am writing is essentially we have discussed in detail 2 by 2 in more detail. And so let me complete for the sake of, let me write down for the sake of completeness.

So, I will do first I will write. So, in the form of theorem. So, I will write three theorems, the first theorems are special cases of actually the final theorem. So, it is a special case theorem 1. Suppose a has only real Eigen value, I am not talking about it is a simplicity, it can have multiple Eigen values.

But, I am say that assume A has only real Eigen values lambda 1, etcetera lambda n according to it is multiplicity. So, lambda 1 can be lambda 2 equal to lambda 3, if there is a multiplicity, taken according to it is multiplicity, then the result what we have discussed so far. Of course, which requires a proof we did not prove it everything, which is nothing but the part of Jordan t composition.

Then, there exists a basis v 1 etcetera v n of generalized Eigen vectors. That is it generalized Eigen vectors of then of R n. That means, a basis consisting of generalized Eigen vectors of R n. And so you take the... So, each v 1 is a vector put that vector as a column vector. So, v 1, so this is the first column, v 2 is the second column, etcetera v n which is a matrix.

And it is a matrix formed by the independent vectors, it will be invertible is invertible. So, you have a basis of generalized Eigen vectors and A has the form. That is what we are so far discussing. The given matrix will not be diagonalizable, if you want to have diagonalizable, you need a basis consisting of Eigen vectors. But, what we are getting is a basis consisting of generalized Eigen vectors.

So, what it says that you can decompose A into two matrices, in which S is diagonalizable and N is nilpotent. So, you see, so you have the problem of computing e power a. But, every matrix can be written as a sum of two matrices, one s which is diagonalizable. And for once it is diagonalizable, you can compute it.

And N is nilpotent and again computing e power n is easy and not S plus N nil will with... So, it is diagonalizable mean with this matrix. So, it is diagonal you see your diagonal lambda 1, etcetera, lambda n. More than and it is not only that S and N commute that is very important N S N nilpotent.

So, you have all the properties, nilpotent. And hence, you can compute e power A therefore, e power A t A is nothing but, e power t s into e power t N. So, you can compute this one and t s can be t s is of the form p inverse of a. So, you can write down this is p inverse of diagonal of e power t lambda j. So, diagonal of p and e power t n is n will be a nilpotent of order sum k order k.

So, in that case you will have e power t n will be of the form, this we computed yesterday t power N, etcetera t power k minus 1 by k minus 1 factorial into N. So, you see n power k minus 1 of course, So, if you want to find a solution. So, solution is x t is equal to e power t a of x naught.

So, you see you have the complete solution, you can directly write down using this one. If this is a matrix it is acts on x naught you get that one, so you have to compute that one. So, this is the case when A has only real Eigen value.

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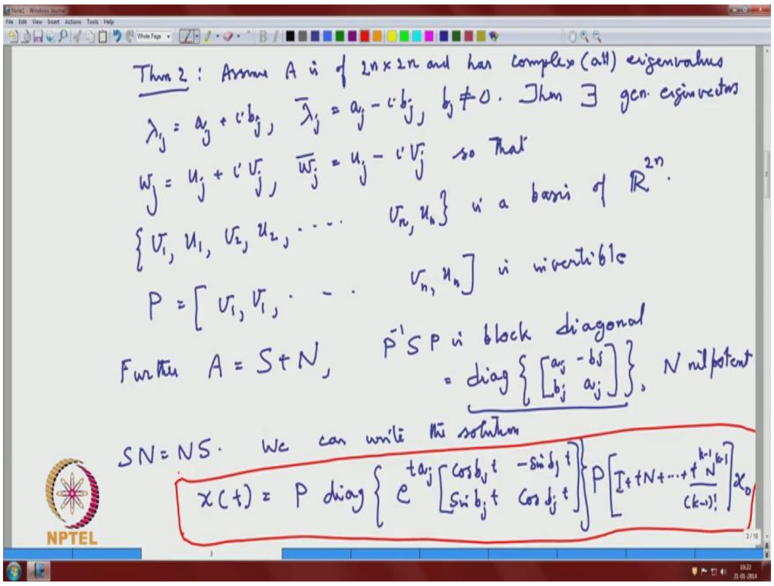

You can also write a theorem 2. Suppose assume A is of order 2 n by 2 n instead of n I am taking even order. That is what? To assume that because complex roots occur in pairs. So, if I want to assume that a has only complex Eigen values it is order has to be even. And has all complex Eigen values.

So, it is a this need not be the case I told you, as this has a special case complex Eigen values lambda j is of the form a j plus i b j lambda j bar is equal to a j minus i b j complex Eigen value b j not equal to 0. Then, there exists generalized Eigen vectors, complex generalized Eigen vectors w 1 j is equal to u j plus i v j, w j bar is equal to u j minus i v j here this is that one.

So, that if you write down you can write v 1, u 1, v 2, u 2. You write that way real and complex parts, complex and real part v n, u n is a basis of R 2 n. So, we are working with 2 n is a basis of R 2 n. So, write your matrix P is this matrix P, P is a column now, you v 1, v 1 these are all column vectors in r n, v n, u n. So, it is a 2 n by 2 nothing is invertible.

Again, further you have the same representation. The only thing is that, it is a complex further you have A is equal to S plus N, S is diagonalizable. That means, P inverse of S P is equal to you get diagonal of lambda 1 it is a diagonal. So, let me write is diagonal of course, diagonal entries with lambda 1, lambda 1 bar, lambda 2, lambda 2 bar, etcetera is diagonal in nilpotent. There is something is wrong, not exactly what I said, it is diagonal is block diagonal it is block diagonal.

What is the meaning of block diagonal I am telling. This is of the form, you will have diagonal of each block will have 2 by 2 entry a j minus b j, b j a j. So, that is a block diagram. So, that is a correct way of writing P inverse of S P is block diagonal. So, the first d which we have seen yesterday. So, the first entry is first block 2 by 2 and then 2 by 2, then 2 by 2 like that along the diagonal.

So, it is 2 by n and a nilpotent of course, of some order nilpotent. That is it, S N equal to N S this always you need it to commute, then only you can compute. So, you can again write and we can write the solution as. Again you see, we can write the solution as complete solution as x t is equal to P diagonal, because you can compute the e power of this one easily. That is nothing but, e power this we have already done for each one cos b j t minus sin b j t sin b j t cos b j t. This is the diagonal entry, this is the e power of that one into P. And then you will have your e power t N, e power t N is nothing but, I plus t

N plus etcetera t power k minus 1 N power k minus 1 by k minus 1 factorial of x naught. So, you have the complete solution.

So, you see of course, this involves of the interesting point to be remarked here is that to write down the solution. Even though this is diagonalization this block diagonalization is done, via the finding the generalized Eigen vectors. At the end you do not need the generalized Eigen vectors to be computed.

To write down the solutions it is enough to find the Eigen vectors, all the Eigen vectors. And it is order algebraic, you do not need anything further. So, you if this is the situation, you do not need to find the generalized Eigen vector in these special cases. So, see that is a interesting point.

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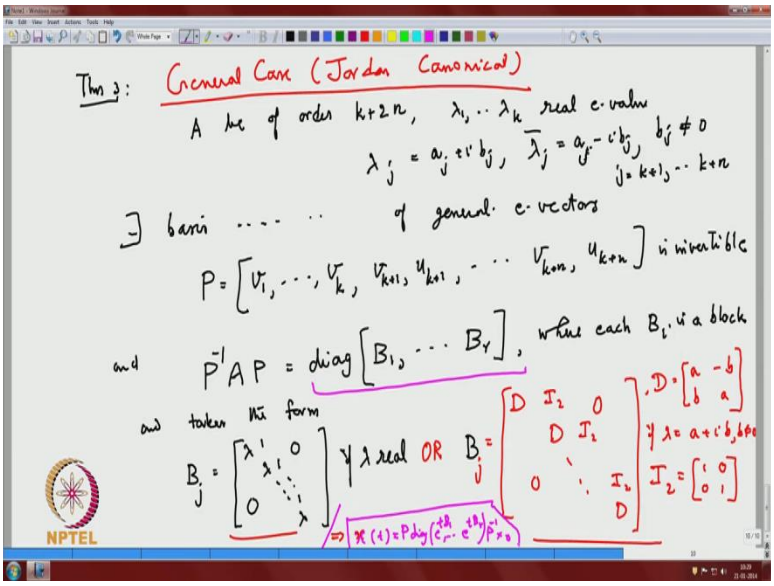

So, now we will write the last part of it before going to the ((Refer Time: 20:55)). So, the general case, which you are already seen. This is nothing but, your Jordan canonical form, which we have already written yesterday. Today we have written the special cases separately, that is what it says that.

So, you start with I do not write it in the form. So, this is a theorem again, you can write it if you want it in the form of the theorem. Let A be of order to separate of order k plus 2 n. Now, this does not reduce any generality, this is just to separate your real Eigen values and complex Eigen values. Here, I am going to assume that it has lambda 1, etcetera lambda k real Eigen values.

So, you can have any number any n you can decompose in the form k plus 2 n real Eigen values lambda k plus j is something like or lambda j itself you can write it lambda j equal to a j plus i b j lambda j bar is equal to a j minus i b j b j not equal to 0 it is a complex Eigen value and j runs from k plus 1 to k plus n.

So, you see. So, you have all the Eigen values and then there exists basis that is all is there exists basis, we do not write it of generalized Eigen vectors. You see, we can write your complete theorem. And you can write down your P. So, you will have P you write your P as the matrix. So, you have the first k Eigen vectors generalized Eigen vectors corresponding to k, this is not a corresponding to the real Eigen values.

So, this v 1 to v k is the Eigen vectors generalized Eigen vectors, corresponding to lambda 1 to lambda k. And then v 1, u 1 v k plus 1, u k plus 1 like that to v k plus n, u k plus n is the are the Eigen vectors, generalized Eigen vectors corresponding to the complex Eigen values and this matrix is invertible. So, you considering what the most general case. And a and with this P, you cannot diagonalize you can write down the diagonal with blocks.

Now, you will have the problem earlier, you had a very particular format in the specific cases. But, you will have diagonal of b 1 etcetera, for some r we do not know that are all depend on it is multiplicity and algebraic multiplicity, geometric various factors, where each B i is a block, block may be can be a 1 by 1 block also. So, if you have a only one simple Eigen value with one vector, it will be a single 1 lambda it will come.

And it takes one specific form and takes the form, that is a thing and takes the form. And this we have computed already, yesterday we have elaborate way computed. And any B i j will be of the form lambda, lambda, etcetera, lambda. And the upper diagonal elements 1 etcetera 1. The rest of the elements are 0. And the order again depends on the multiplicity there are some estimate product we will not discuss here.

But, any typical b j will look like this particular form. And the order of b j may vary depending on the lambda just. This is for the case, if lambda real or it can have the block or... So, it can have the block B j will be of the form, again we have discussed how to compute the given, this will be D I 2, D on the diagonal I 2 here D I 2.

These are all 2 by 2 matrices inside, where D is equal to of the form, D will be of the form a minus b b a for if lambda is equal to a plus I b b not equal to 0. And I 2 will be of the form, it is the identity matrix 2 by 2 matrix. So, this is the typical Eigen value. So, using this, this will be immediately write down your solution as x t.

Because, so you have decoupled a system a x equal to x dot equal to a x, it is enough to solve the system for y dot is equal to b j of y. For each b j you solve it separately and each b j will have one either this form or this form. So, with that you can immediately write your solution x of t is equal to... So, let me finish in this page itself, you will have P since this is of the diagonal form, even for the block the diagonal form works.

So, it will be of the form diagonal e power t b j, t B 1, etcetera e power t B r at P inverse of x naught. So, you see, so you have your complete solution. So and you have to compute each t power B j each t power B j is a matrix set matrix, which is a same order as the B j. And how to compute t power B j is what we had discussed yesterday. So, your this gives you the complete analysis of the Jordan composition and representation of the solution for the linear system. So, with this we now move on to the last section of this talk, namely for the non homogeneous system.

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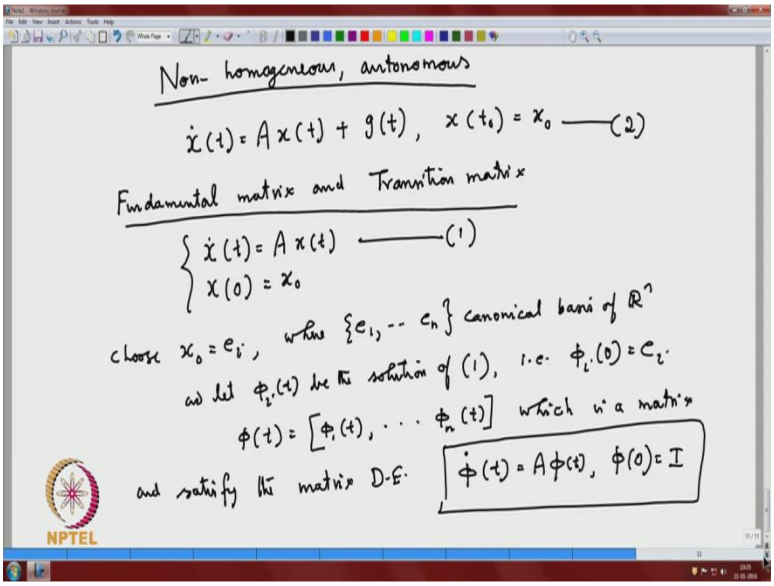

So, we will move on to the next section non homogeneous equation. Again autonomous, we are in autonomous only because all these facilities are available known. So, what would be your equation, you have your equation x dot is equal to A which is independent of t. So, x t plus g t where a the elements g t. So, you can your g t and your initial condition x at t naught is equal to x naught.

So, we are going to introduce a two concepts called fundamental matrix and transition matrix. This is crucial even in understanding and we you can write down the solution in a without introducing this notions, which I am going to describe soon. But, that is the thing used to understand non autonomous systems, because this represents. So, we are going to represent the solution of this system, in a using the fundamental matrices, which can be generalized to the non autonomous system.

So, let me introduce this notation, which is crucial fundamental matrix and transition matrix. So, let me solve is the system 2 let me call it. And it we call it the same notation I am using, I am not using in a different notation. I am using this notation A for even for the homogeneous system. So, this let me call it 1, this is my 1.

So, I want to solve this equation with x at 0 is equal to say x naught. I can solve for various initial condition. So, choose x naught equal to the basis vector e i, where e 1 to e n canonical basis, that is 1 0 etcetera of basis of R n. So, for each of this initial condition I will have a solution and let phi i t be the solution of 2, solution of 1 I am talking about the homogeneous equation.

That means, phi i t satisfy the equation 1 with, that is phi i t at 0 will satisfies the equation e i that is all. So, phi i t phi i dot is equal to a phi i and phi i 0 is equal to e i. So, with this I can introduce phi i phi t a matrix. Because, phi i t is a vector it is a vector solution. So, each vector I will put it in the first component phi 1 t, etcetera, phi n t. You know that these are all independent solutions and inevitability.

Now, you can think you can immediately understand this equation. And what is this phi t will satisfy. So, phi t will satisfy each phi i t will satisfy an equation with 1 0 0 as the initial condition, phi 2 will satisfy with the initial condition 0 0 etcetera. So, phi t which is a matrix and satisfy the matrix differential equation phi t phi dot of t phi is a matrix now.

So, you can still that one with... So, you are arranging phi your each row properly. And what is your phi at 0 phi at 0 is nothing but identity. Because, phi i at 0 is 1 0 0, etcetera phi t at 0 is equal to 0 1 0, etcetera; it will satisfy the matrix differential equation. So, you can do it instead of the initial condition.

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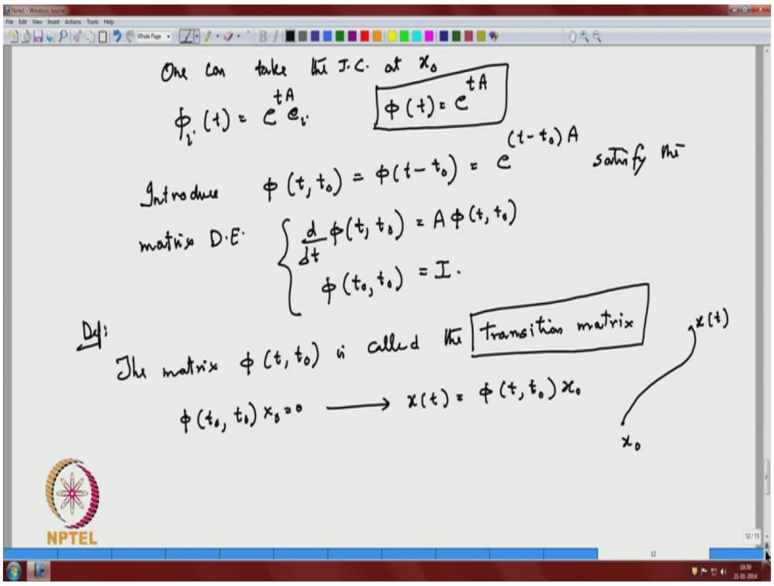

I can do the initial condition, one can take the initial condition at t naught. So, but by the way what is a solution of phi i t, phi i t is nothing but, e power t A e i, e i is the basis the other e is the here that is it, that is not a problem. And what is your phi t, your phi t is nothing but, arranged in columns it is nothing but, your flow you see.

So, phi t is in this particular case. So, I can take your phi t, t naught. So, I introduce phi at t, t naught is nothing but, phi at these are all very special k works for the autonomous system. This is nothing but, e power t minus t naught of A and this will again a t.... So, this phi t, t naught again will satisfy your matrix differential equation. But, with the initial condition at t naught.

That is all the difference, we just translated satisfy the matrix differential equation d by d t to avoid that t naught are so there which we are going to vary phi t, t naught. So, this nothing but A of phi t, t naught. T naught is fixed t is varying, with the initial condition at t naught phi t naught is identity, you see that is a thing. So, you have your solution essentially I am translating t, you know you can write down.

This is called the transition matrix, the matrix definition phi the matrix phi t, t naught is called the transition matrix. Because, it is the name is suits translation, this is where what is it is happening is that, it takes the point at t naught. If the position is at t naught is at x naught, it tends to phi t, t naught of x naught at time t. So, that is what happening.

So, phi t, t naught of x naught is equal to you look at that phi t, t naught of x naught is equal to x naught. And then at time t the solution will be at... So, x at t is equal to phi t, t naught of x naught. So, you take the trajectory. So, if you have the trajectory at the initial point it is takes to the trajectory translate. So, the phi t, t naught takes the trajectory from x naught to x t at time t.

So, the name is correct the transition matrix is correct. This also have the other properties, which we may probably tell a little later we will try to say something later at the end of it. So, there is also a notion of fundamental matrix.

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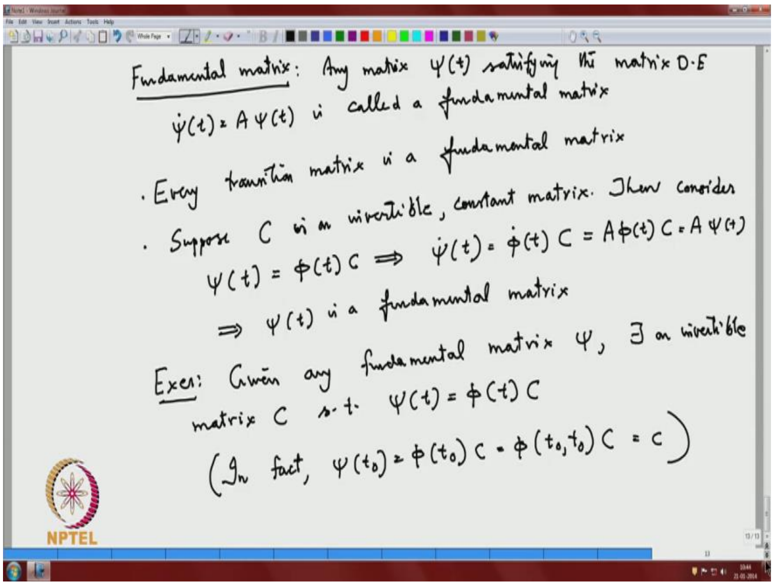

It is more or less like a transition matrix, but we do not put a condition on the initial value. So, any matrix psi of t; that means, consisting of functions. So, the entries of psi t are a functions matrix, satisfying the matrix differential equation psi dot of t is equal to a psi t is called a fundamental matrix fundamental matrix. So, the trivial fundamental matrices are transition matrices.

So, you see so every transition matrix is a fundamental matrix. So, hence fundamental matrix in this particular situation is of the form e power t minus t naught into a. That is, your thing every transition. But, there is also a another interesting thing, which you can easily prove. Suppose, c is invertible there is no, suppose c is a constant matrix, which is c is an invertible, of course invertible constant matrix.

Then, psi t you define psi t as phi t, you post multiply c then consider this one, consider psi t. So, this implies psi dot of t with respect to t, phi t means phi t, t naught if you want to know that one. So, suppressed t naught here psi dot t is equal to phi dot of t and c is independent of t. But, phi dot of t its nothing but, A phi t and c and this is a associative law is true.

So, substitute phi t that you will get a psi t. That means, if you take any invertible matrix c. And then define psi t is equal to phi t c, then it is also that imply psi t is at fundamental matrix, but the interesting result which is trivial to prove. So, I will leave it as an exercise every fundamental matrix is of this form, given any fundamental matrix c psi there exists an invertible matrix C, such that psi t is of the form phi t into C. In fact, if you observe you can compute C you can given psi, you can compute C. In fact, psi t naught is equal to phi t naught psi. But, phi t naught is nothing but, phi t, t naught. I told you already the other one is suppressed phi t, t naught psi it is initial value phi t I represented for phi t, t naught supper.

But, phi t t naught is identity. So, this is nothing but c. So, your c has to be of the form psi t naught. So, you have that this is a trivial exercise with this you can compute and do that one psi is of the form that one you just you need the uniqueness of the solution that is all.

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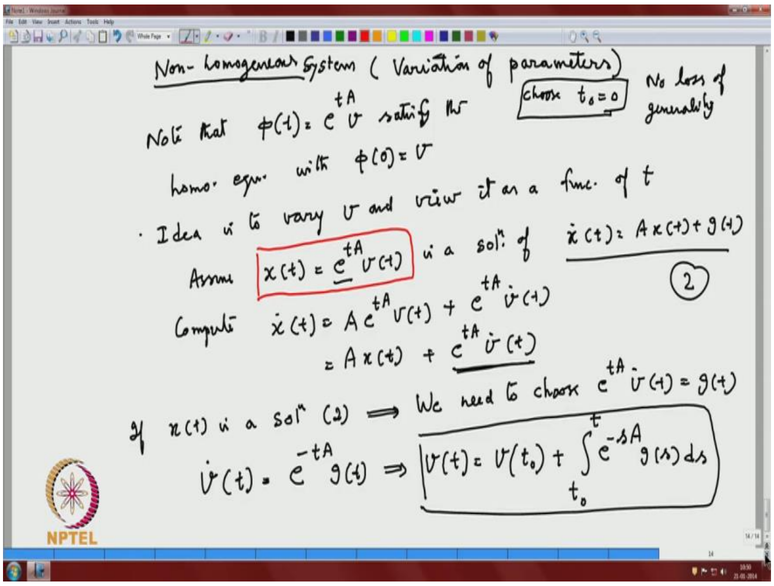

Now, with this we will go to the non homogeneous equation. I use this notation, as I said you do not need all this. But, it is introduced to understand the non autonomous system, which we may not do that one. So, now non homogeneous system, we apply the variation of parameters, which we have done in first and second order equation, we you variation of parameters.

So, note that if you see... So, I use this notation phi, so let me work with that initial condition t equal to 0. That is, enough you can work with any t naught also. So, choose t naught equal to 0, so to avoid, but you can it does not matter. There is no loss of generality, only you have to work thing.

So, note that phi t is equal to e power t A of x naught for v satisfy the homogeneous equation. These are same idea we have used homogeneous equation, with phi at 0 is equal to v. So, the idea which we have used in variation of parameters. If you take your flow e power t A and act on any constant vector v, it will only satisfy the homogeneous equation. So, you can never expect to have a solution e power t A v for a non homogeneous equation.

So, the idea which we have used is to vary v and view it as a function of t, that may give you a hope. Because, if you then do that one you are never going to get a non homogeneous solution, this might give. So, assume x t is e power t A into v t is a solution of non homogeneous equation, solution of x dot of t is equal to A x t plus g t.

Now, let us compute here compute, if you compute x dot of t. So, it is a product formula, once you apply the differentiation to the first term, you get A here A e power t A v t. And if you apply that there, you get e power t A into v dot of t. This is nothing but, your e power t A into v t is nothing but, your x t plus e power t A into v dot of t.

So, if you want x dot t is a solution to this non homogeneous equation 2. So, if x t is a solution to 2 will immediately imply, we need to choose this term is g t, we need to choose e power t A v dot of t is equal to g t. Or otherwise, you have the first order equation for v t, this is an integral calculus problem. You do not have nothing to do it, it is a e power t A into g t.

So, you just integrate that will give you your v t at v at 0. V at 0 is equal to what do I get it? We get x naught you want this to be at 0, you want that to be 0. So, you have v at t naught v at t, if you integrate t naught. If you do that 1 plus integral t naught to t of course, you can work with t equal to 0 or not you integrate that one e power minus s A into g of s d s, this is what you want to do. So, you get that. So, now you substitute this formula, you substitute v t in. But, you substitute, you substitute here.

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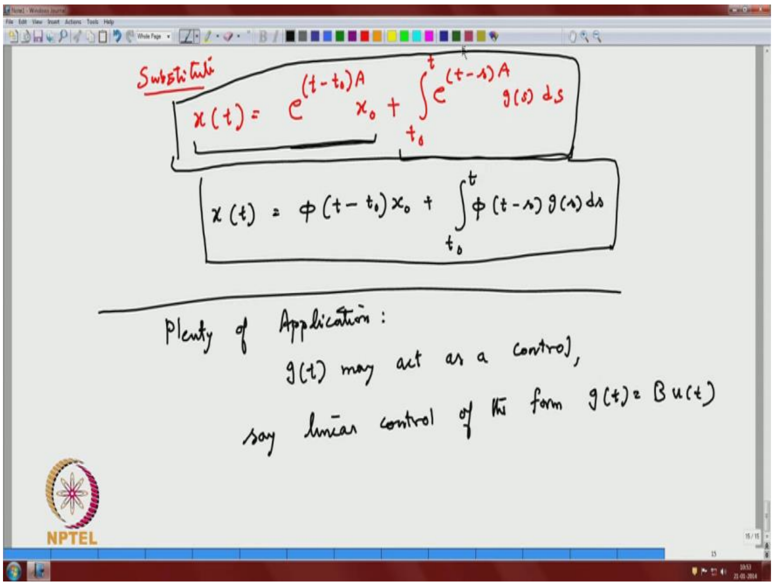

So, if you substitute v t there what you will get is a substitute, what you get here x of t will be... So, e power so you can do small computation which I will skip here t minus t naught of A at x naught. Of course, when you choose t equal to t naught, it will be just x naught do the computation substitute and do the computation. You get plus integral e power t naught to t, e power t minus s this is a just a small computation absolutely no difficulty g of s d s. So, you have that formula.

So, this is your formula for your solution the using the variation of parameters you see. So, when there is no g this term would not be there and you know that this is the solution to your homogeneous equation. So, the first one is exactly getting what you are seen in homogeneous equation. If you recall your second order equation, you have a complementary version which is a solution to your homogeneous equation, we have seen that.

And then you have a kind of particular integral and that is what we have done here. So, let me write it in this form. So, this is of the form phi t this is nothing but phi of t minus t naught x naught an simple form plus integral t naught to t phi of t minus s g of s d s, this kind of representations are very, very important in applications.

For example, the non homogeneous term will appear in the controlled form. So, plenty of applications, which you know plenty of applications. For example, g t may be a control g t may act as a control. So and it may say for example, linear control, we will not get into that one of the form, this is a g t is equal to some B of u t you see. So, you have that control that format.

So, we will not do that one, but it gives you an important thing. And another thing interesting thing is that. Here, you can also define this kind of solutions, in the weak form and other kinds of thing in o d e, it is called the mild form. And which is also important in control theory, because you do not look for control feature continuous.

And when you do not have the continuity, this kind of weaker formulations of the representation of the solutions, you can write as thing. So, we will now get into that, but what we have seen is that, you can write down your solution consisting of two parts. One part is the solution, corresponding to your homogeneous system. And in the last 4 and a lectures, we were actually studying the homogeneous system completely.

You are trying to understand this phi t minus t naught of x naught, completely. According to the various form how to compute that one. Once, you compute you also have a second term, that corresponds to the non homogeneous part with this we more or less end out thing. But, we make a slide or a just one to commence about the non autonomous system and we will stop here.

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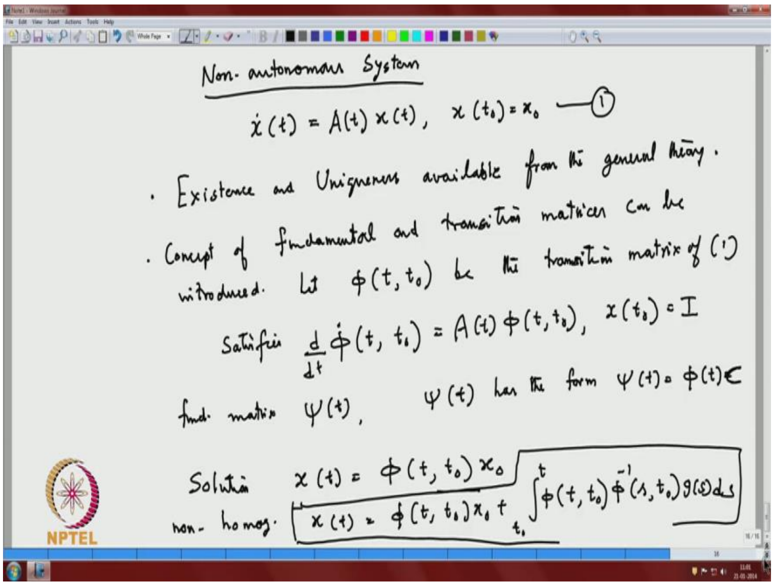

Non autonomous system. You know that non autonomous system works completely different. Even in the second order, you have seen that one, when you have a equation with a constant coefficients, in the second order equation. Linear constant coefficient you have a complete analysis, how to represent the solution. But, when you have an equation x double dot plus p t x t plus q t x dot and q t x, x t equal to 0 or y t or g t then you have no there are some methods.

But, in general finding two independent solutions is difficult. You have the solutions structure and here also we are going to give you representation. What you have seen in even in the n by n system. You have a complete knowledge of the homogeneous system, in terms of the exponential. That exponential representation is not possible. So, you have solution here of this form x dot t.

So, the variation of parameters with non homogeneous is fine, but even with A t x t. So, we are not going to do much on this x t naught is equal to x 0. But, what I want to or the homogeneous counterpart, there will be a g t there. But, the interesting point is an existence and uniqueness available by the same general theory, which you have studied in our earlier module.

So, the if of course, the elements in a t are continuous that minimum assumption of the continuity of the matrix a t is required, but existence and uniqueness available from the general theory. Now, the concept of fundamental matrix and transition matrix can be introduced. That is the important thing, concept of we you do not have now e power t A and finding a solution at t naught you cannot ((Refer Time: 54:38)).

So, that is also an important thing, you it is not possible you find the solution at a x equal to 0 and then translate, because of the non autonomous system, which you will see in more elaborate way in non-linear analysis. You cannot just translate find the solution at t naught equal to 0 and then translate like phi t minus t naught. And that is not available in this case.

So, you have to study, if you want to understand that t naught or to study directly at t naught. The facility of this kind of translations are not available, which you will see in non-linear analysis. But, the concept of fundamental and transition matrix can be introduced. That is the whole thing, transition matrix can be matrices can be introduced, that is a important thing introduced.

So, now, you have to introduce. So, let phi now phi this is important now, you have to write separately. You cannot write this is of the form phi t minus t naught, that is not the thing. So, t naught and it is works differently be the transition matrix of one. Means, it satisfies this matrix differential equation the satisfies phi dot that is d by d t this is a matrix phi dot at t, t naught is equal to A t phi t, t naught and x at t naught is equal to identity.

This can be introduced the thing and you can also introduce the fundamental matrix psi. What is this is a matrix psi t, which satisfies the differential equation. And you have the same formula, you can also write your psi t will be of the form psi t has the form. All that results are 2 here psi t has the form psi t is equal to phi t c you can do all that.

And your solution will be all that can be introduced exactly these were. But, never write phi t, t naught is equal to phi of t minus t naught equal to e power t minus t naught you will get. And that portion is not available to you, the rest of the system you can introduce you using the uniqueness phi i t and then put it is a column wise. And you can introduce everything phi t, t naught form and you can write down here.

And your solution also can be written as x t you may think that it is solved. But, it is nothing is solved actually, we are only saying that phi t, t naught is a solution of this matrix differential equation of phi i t, each row column is a solution of your differential equation with a thing, but how to solve that differential equation. And how to find that is not given to you.

In the earlier case, you have a representation of e power t A. So, you have your phi of t, t naught of x naught you see. So, you have your solution that is all you can do it for the homogeneous equation. And now this is called the transition matrix, and for the non homogeneous system. You have your solution x t is of the form phi of t, you have the component ((Refer Time: 59:01)).

All this can be thing x naught plus integral t naught to t phi of t, t naught. You have to write this way not phi t, t naught of phi inverse of I think this probably may be. If I think this may be s there may be some s phi inverse of no, this is t naught only. Here, s t naught g naught g s d s. So, have a representation of that form. So, you have the thing.

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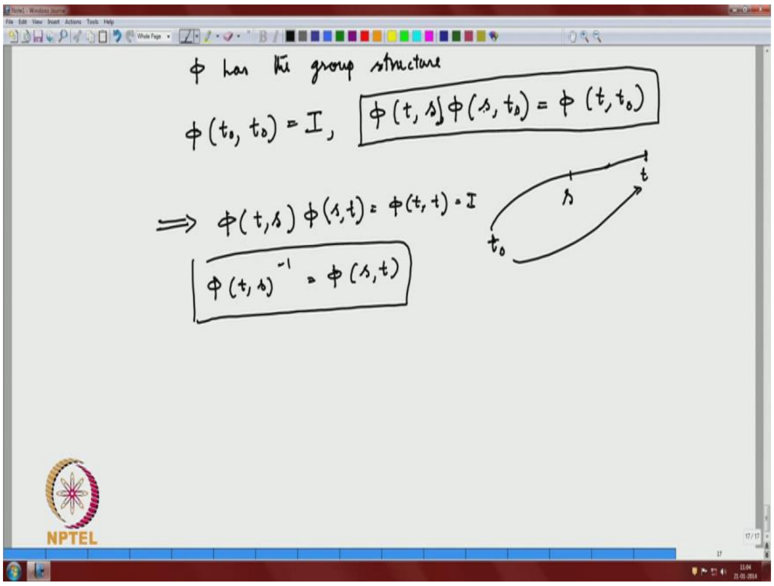

So, the last thing phi has the group structure, phi has the last slide as the group structure, because this is due to the inevitability of the reversibility of your system. When, you node the solution at t naught you can also solve these for the past. But, then there are when you go to p d e's and other things, you may not be able to do it. And you will only get a semi group structure.

So, what are the thing you have phi t naught phi t naught, this property is identity that you have it. And the other interesting property, this is the crucial property phi t s and phi s t naught is nothing but phi of t, t naught. This is the most crucial property of the semi group structure. It means that your solution is going from s to t and s to t naught. And so you have you want to go from t naught to t.

First you go from solution t naught s and then you take an initial condition at s and then go to t. So, that means you can go from s t naught at time t naught here, you reach the time t s here. And then you go there instead of you can go directly here, that is what is this semi group property tells. So, this immediately tells you that this is invertible, the implication is that phi t s. If I put phi t naught in particular if I put a c equal to t I get no s it is a...

And if I put s is equal to t I will get phi t, t or maybe I have to do it some t naught properly, you will get a identity. So, the you will get it as inverse of phi t s is equal to phi s t, you can prove all that it is a same thing. If you want to see that s is equal to t naught identity t, t naught I think that is correct. So, we will skip this, so you have all that property.

So, as far as that non autonomous system is concerned, you have all the representation of the solution using transition matrix. But, there is no way in general to determine transition matrix, as in the autonomous system. But, then there are some conditions under which this transition matrix can be computed in some form of the exponential using certain commutativity of that one. So, with this we will complete the module on linear systems. And you see the use of this thing, in the when you we will study the non-linear systems and it is stability analysis.

Thank you.