Ordinary Differential Equations

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Module No. # 05

Lecture No. # 27

General Systems

Welcome back, again. We have now, completed about 2 by 2 linear system's stability analysis, and is about the phase portraits and all that. Now, when we go to an n-dimensional system, the things are not easy, because there will be n Eigen values and according to n Eigen values, there will be certain Eigen values, which are simple, and there will be Eigen values with algebraic, different algebraic and geometric multiplicity. There will be complex Eigen values, which are distinct. There will be complex Eigen values with multiple multiplicity; higher multiplicity.

Accordingly, the decomposition of matrix is much more complicated, but this is again, one can really, linearly make it equal to another matrix, but there will be various types of blocks what I call, the block way of diagonalising it. What we have seen in 2 by 2 systems, we have three types of block. One is in the diagonal form, lambda 0, 0, mu. The second one is the diagonal form, is lambda, 1, 0, lambda, and third one is of the form, a, minus b, b, a. Now, we are going to have different types of blocks according to the multiplicity and things like that. We will appeal to the Jordan decomposition theorem, and later, you will see that there are typically, two types of blocks, but that two types of blocks can occur in different ways with different order, and that is the difference. Then, each of that blocks computing your exponential a, c, c, and that is what you are going to present here.

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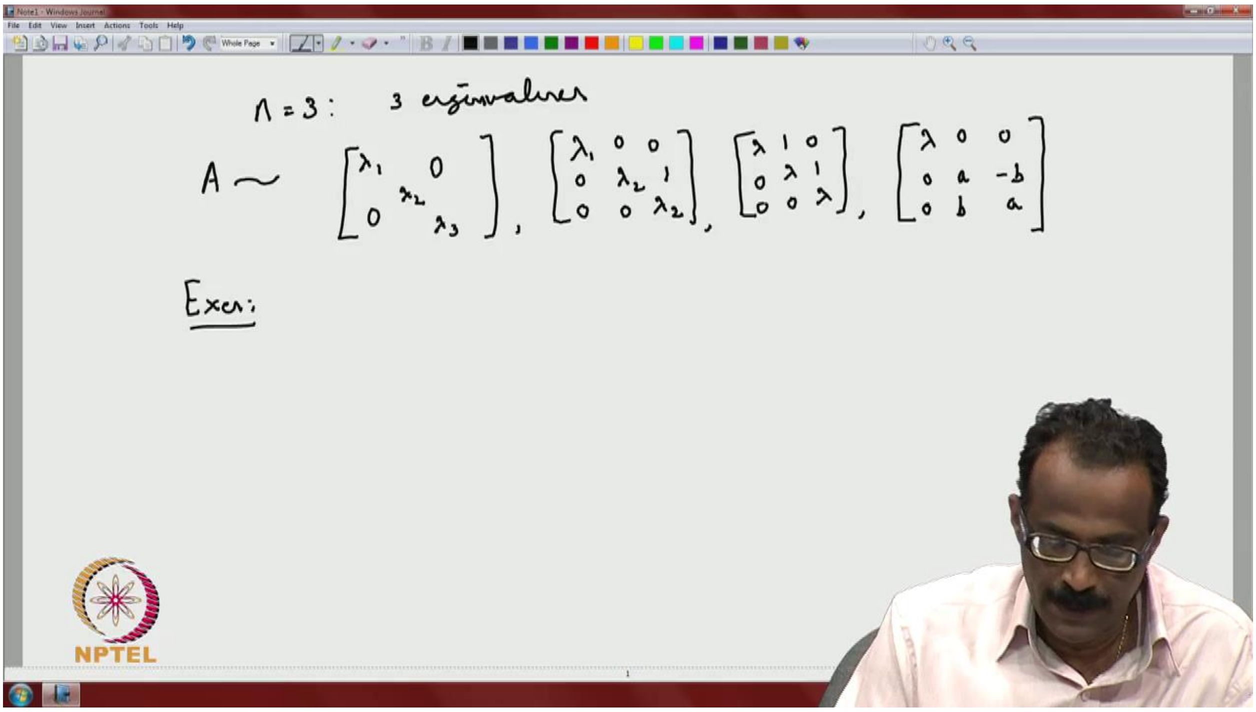

For example, when n equal to 3, you have 3 Eigen values. In this case, the only possibility is that all the three can be real, or one can be real and other two will be complex conjugates. These are the only possibilities will come. So, the typical block matrices will be equivalent to something, like that; you can have the form; it may reach the form, lambda 1, lambda 2, lambda 3. So, you make a block of this form. Typically, this is the case. You have Eigen values; three Eigen values with three Eigen vectors. You will have that. You have the main issue is that the lack of Eigen vectors. It can be of this form, but the another form; it can be lambda here; lambda 1 here, and then, another one; lambda 2; this is a block of this you already studied. You can have a block of this form and here, you have 0, 0, 0. This is the case. You have a block of that form. You will also have a block of that form, another one; lambda, lambda, lambda, 0, 0 here, with all diagonal elements, 1; this is another block of this one.

Typically, these are the blocks we are going to get; high dimension will be complicated. You will have different multiple of blocks and other type of blocks. These are the cases where, you have real Eigen values. Another one, one real and one complex, you will have an Eigen value of this form, and you have a block of this form, a minus b, b a; you have 0 here, 0 here, 0 here, you see. In all these things, you can see that you can compute e power t. So, I will give an exercise here to start with. You consider this type of systems; you take B 1.

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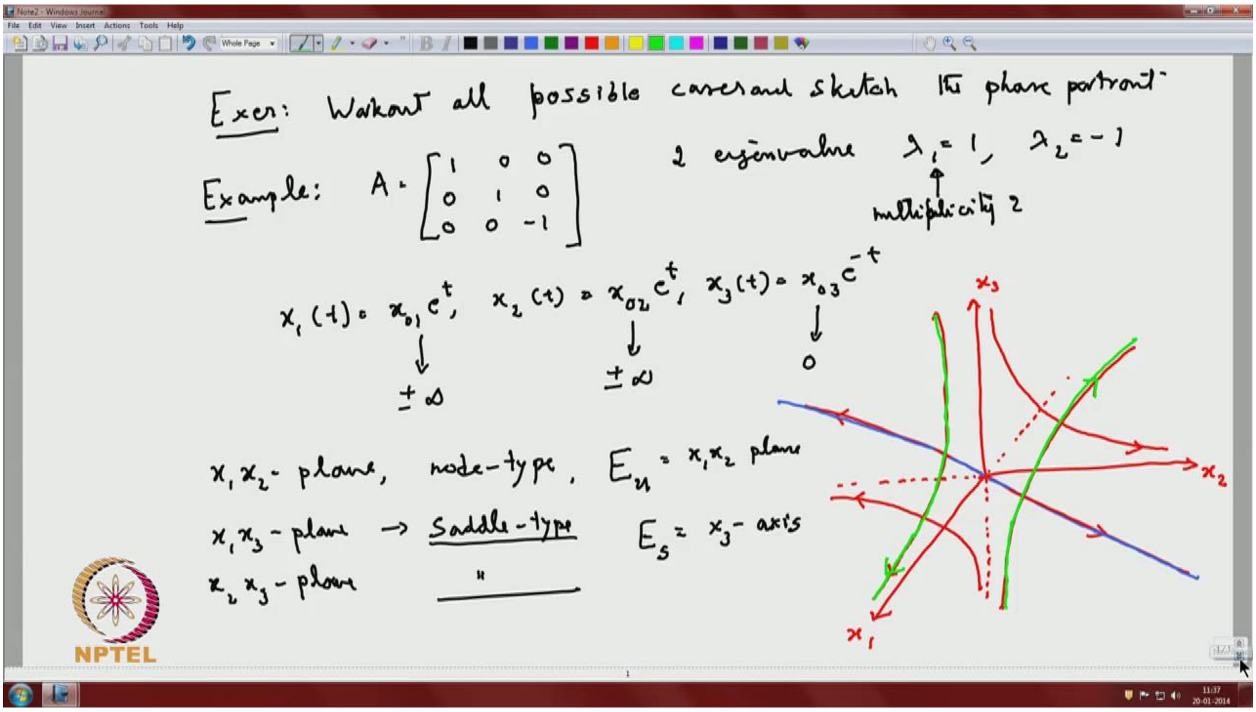

Consider the system with B 1. The exercise is workout all possible cases with above matrix, with above that systems. So, you can write down your system; why it is system with respect to B 1, B 2, B 3, B 4, various things, and see that trajectories how it is look like, and sketch the portrait; sketch the phase portrait if possible. So, we will start with an example here, an easy case; a is of the form with a diagonal form, 0, 0, 0, 1, 0, 0, 0; this is a decoupled system already, in the decoupled system. So, two Eigen values, lambda 1 equal to 1, and lambda 2 is equal to minus 1. So, here, you have your multiplicity, algebraic multiplicity 2, luckily, it is in that format, you do not know. So, it is a decoupled system. You can write down your solution immediately, x 1 t is equal to x 0 1 e power t; x 2 t is equal to x 0 2 e power t and x 3 t is equal to x 0 3 e power minus t, you see. This goes to plus or minus infinity as t tends to infinity. This goes to plus or minus infinity, but this goes to the origin 0. How does it is look like?

If you look at your x 1 x 2 plane, you will have unstable node type thing. Forget about the x 3 component; look at only x 1 x 2 plane. It is something like a node, because x 0 1 e power t is 0 to nothing, but straight lines in the x plane. So, that graph, if you do not have, if you looking only your initial values are in the x 1 x 2 plane, if you are starting an initial, the third component is 0, then by this thing, x 0, 3, 0; x 3 t will be 0 for all t and hence, if you are starting a solution in x 1 x 2 plane, it will remove there, and in that case, it is a node type singularity. So, it is plane, node type singularity and this has straight line, because you have the same Eigen value 1. On the other hand, you will get your, and which is also unstable in this particular case. So, you will get u is equal to x 1 x 2 plane; that is your unstable thing. If you look at it, x 1 x 3 plane, you have one Eigen value. Again, all these are decoupled system. So, x 1 plane, you have the Eigen value 1, and x 3, corresponding to that, you have Eigen value 1.

So, you have your saddle type. This is the same thing, when you look at it, x 2 x 3 plane. So, if you look at these trajectories restricted to this plane, the trajectories will remain there itself, and you will have again, the saddle type and your stable part, which is stable, because you have minus 1 as the Eigen value, which is not stable, sorry, it is a saddle type and you do not have it. But if you look at only the x 3 axis, if you take something, a point, initial point in the x 3 axis that you will get a stable space. So, you will have x 3 is the stable sub space. This is the unstable sub space and this is x 3 stable; x 3 axis is your stable sub space, which is 1. Now, just let us try to plot these things in. So, let me plot this. This is my, if you do that one, this is your x 2; this is your x 1; and this is your x 3. So, you can extend, of course; you can extend like this. So, let us look at only, x 2 x 3. So, you look at x 2 x 3; you have the saddle point. It will meet there, and x 2 goes to infinity. So, you will have x 3, and this is your x 2 x 3.

Correspondingly, this is your x 2 x 3. So, you will have trajectory will be going; In this, trajectory will go here, you see. So, that is the corresponding to x 2 x 3 plane; this is your x 2, you see. Similarly, if you look it at x 1 x 3 plane again, it is a saddle point type. So, you will have this one, which is going to, x 1 going to infinity. So, you will have your x 1 x 3, if you go here. So, you will have your trajectories something like this. So, again x 1 going to infinity, so that, trajectory will move there. So, that is in the x 1 x 3 plane. Now, let us look at the x 1 x 2 plane. In x 1 x 2 plane, the solution if you plot is a straight line, which is a going to infinity. If you start from anywhere, then it is a straight line, you see. You have your straight line; if you look, plot it here, you will have plot. Let me (( )). So, this is the x 1. The axis will be; of course, if you start an arbitrary point accordingly, it will behave. So, whenever you start a point, it will give you a trajectory, and if you project that to the plane, you will see that saddle point be of behaving it.

Let me put it a different color also, for this thing. So, you have that, you see. That is a complete picture about that plane. This is the phase portrait will look like. What the graph and curves are given on the respective projector planes x 1 x 2, x 2 x 3 and x 1 x 3. So, you will have the all that kind of picture. So, if you start arbitrarily, it will move accordingly.

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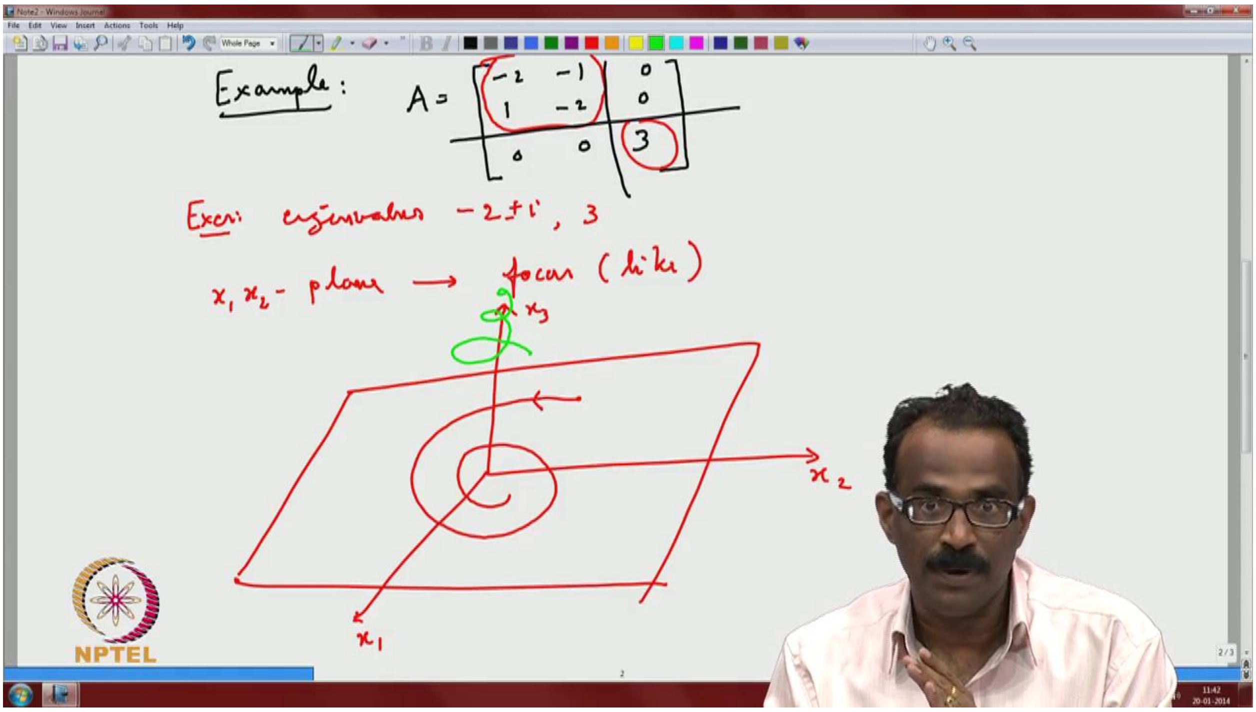

Now, with this one, may be, one more example to give you. So, that you slowly, get used to it, how things will be. 3-dimension; you can do this little more, with little more imagination. I am going to take a matrix, which has some complex Eigen value. So, that still, you know that the complex Eigen value; you have minus 2; this is a b minus b; b minus b minus 2. So, you will take 0, 0, 0, 0, anything, 3. So, if you do this one, you see the corresponding to, you a system, you see. This corresponds to a 2 by 2 system. This corresponds to a single system, and this gives you the first part, gives you a complex Eigen value. So, you can do the exercise again. Each time you can do the exercise; Eigen values are minus 2 plus I, of course, minus 2 minus I and 3; you have these Eigen values.

So, what you do you is that you already seen this one. This is already in a decoupled form. So, 2 by 2 system, the x 1 x 2 plane; here, it is like a focus. You have to see that. I do not classify, because like focus, if you take only x 1 x 2 plane, and the other things, you have to see that one. If you look at here, if you are trying to do the phase portrait to this one, you have your phase portrait of x 1 x 2 plane. So, I have my x 1 x 2 plane like this. So, if I take any trajectory, you have here; this is my x 1 x 2 plane; this is x 1; this is x 2, and this is the x 3. If I start any point here again, it is a decoupled; the x 1 x 2 plane and x 3 part is decoupled, because of this particular form of a. So, if I start here anything, any point here, if I start; it will remain again. So, if I start anything in x 1 x 2 plane; that means, the x 3 coordinate is 0, and by looking at the way we constructed your matrix and solution; it is a decoupled. The x 3 part component is decoupled and hence, it will remain in that plane itself, and it will behave like a focus. Since, you have minus 2 as the real part of the Eigen value, minus 2, in the other case, which is a converging thing; it converges.

So, if you start from here, it will remain in the plane, and it will come something like that. It will, and you have an orientation, according to the sign of t, you see, you have that rate in. So, that trajectory, this is the x 1 x 2 trajectory plane, but on the other hand, suppose, I start a point from anywhere, arbitrarily, what will happen? The x 3 component, the Eigen value is 3. So, the x 3, 2 will be x 0, 3 into e power t. So, the x 3 component, as t tends to infinity, the x 3 component will tend to infinity, but then, x 1 and x 2 is something like a focus. So, it will move around that one, and move around the x axis, but moves away along the x 3 direction. So, if you do this thing, if you start from anywhere other than in the plane, it will move like this. It will go around that thing, but the same time, x 1 and x 2; it will move and it will become smaller and smaller, around the x 3 axis, because x 1 and x 2 go to 0. It is the x 1 and x 2, the real part will go to 0. So, it will move around the x 3 axis and move and reach, go up, but the amplitude becomes smaller and smaller.

On the other hand, instead of minus 2, if you replace 2, and it will be diverging. So, again, then, it will be moving around like that; it becomes larger and larger. So, that is how you can focus, you can sketch the graph according to the model. So, we have two examples are given. One; you have the focus behavior on restricted plane or another example in which, you have a saddle point behavior. You can give all kinds of things with node and all that. So, we will skip this. We will do only that one. Now, you will go to the general case and Jordan form.

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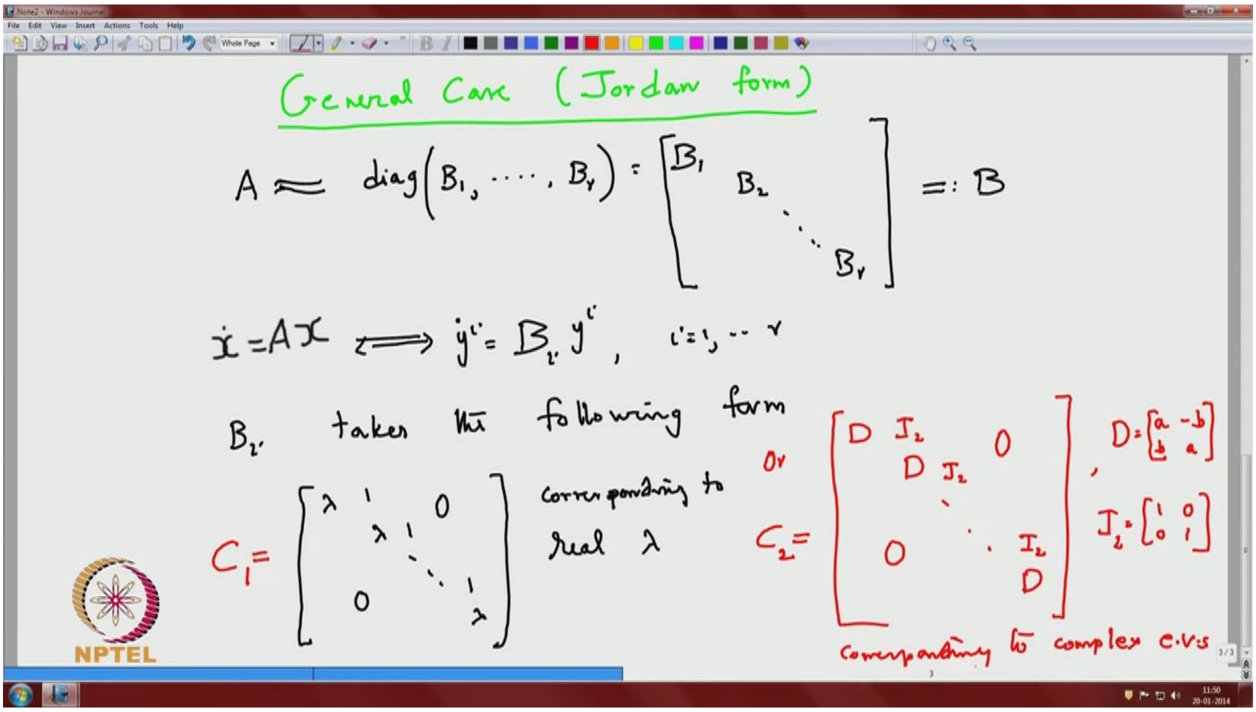

General case, typically, you want Jordan form; I am not going to explain the Jordan form in detail. That you have already, studied in the either, preliminaries or you also, studied in the general linear algebra course. So, Jordan form typically, says that you look for all Eigen values. According to Eigen values, you have to classify the real Eigen values, you have complex Eigen values. Then, the idea is that if you have buoyancy system, you essentially, need n Eigen vectors, but if there is n Eigen value, if all the n Eigen values are all distinct; whether, real or complex; if you can produce n, you have the diagonalization, but then, there will be Eigen value, real Eigen value with algebraic multiplicity, but less geometric multiplicity. Then, you may not have enough Eigen vectors to diagonalize. Then, the idea is to look for what are called generalized Eigen vectors, studied in your preliminaries. According to the deficiency between a geometric multiplicity and algebraic multiplicity, you will be having a lack of sufficient number of Eigen vectors.

This makes, just like in the previous case of 2 by 2 system, when you have an Eigen value repeated twice, you got a block of the form, lambda, 1, 0, lambda, and which is not diagonalizable. Now, as I said, in n equal to 3, you have all kinds of thing, but what Jordan decomposition tells you that every matrix a, if you start with a, you can linearly, make it equivalent to a diagonal form; not with a diagonal matrix; diagonal with entries with blocks B 1, etc. some number B r. You can have the blocks. Each one, each B I will have some node. These are all belong to certain multiplicity and other thing. So, you will be able to write this matrix B 1. You will have some order matrix say k 1, and B 2 will have some order 2, and etc. B r will some order k j. Of course, k 1 plus k 2 plus, etc. k r will be equal to B r. So, the total order will be n by n matrix. How does my B look like? That is more important. The idea is that if you want to solve your system, if you have this called my B, big B, if you want to solve your system a x equal to y; because of this diagonal form, it reduces to, it is enough to write down your solution to solve the system of the form each one. So, if you want do it, you can have a transformation p and p inverse. Here also, it exists, but essentially, reduces to study each B I, a sub system, which may be of smaller order and this also, behaves some special forms, which I am going to describe here. So, it is enough to study B I of some by I, is equal to something. So, y I dot is equal to B I of y. You can have solving for I equal to I to r.

Each y I is a vector, corresponding to k by k vector, k j vector, if k I vector. If B I is of order a I by k I, k by k cross k I matrix, then y I is a vector of order, k I; it is a k I vector. So, solve for all these things. So, at bigger system eventually, reduce to solving a smaller systems. How does B I takes the following two forms? B I takes the following only two forms. That is a whole interesting thing. So, it will be of the form, some lambda; the diagonal entries are lambda; and the half diagonal entries will be 1. The remaining will be 0. This is the case corresponding to real Eigen value, corresponding to real lambda, and it will take the form, the following form. That is another form. It will take the form, d, d, etc. d, and here, I 2 of this itself. This block itself is a block; each one is a 2 by 2 block; everything 0. What is d; d is of the form, a minus b. See, this is a familiar form eventually, reduce to everything, and I 2 is a 2 by 2, is an identity; 2 by 2 identity; 0, 0, 1, you see. So, each block, the only thing is that in the a, b; there are different blocks B 1, etc. B r. Each B I will have different orders, and according to the Eigen values. Whether it is a real, it takes this form or the form d is equal to I 2.

So, this is again, the form is a block, consisting of 2 by 2 blocks. This is corresponding to complex Eigen values. So, it is enough essentially, it is enough to how to know to compute in theoretically, it is still difficult. Theoretically, it is enough to know how to compute the exponential of these two types of matrices, which is what I am going to do it, right now. So, you want to, I call this is equal to the form c 1 for the computation with (( )). This, I call it of the form c 2.

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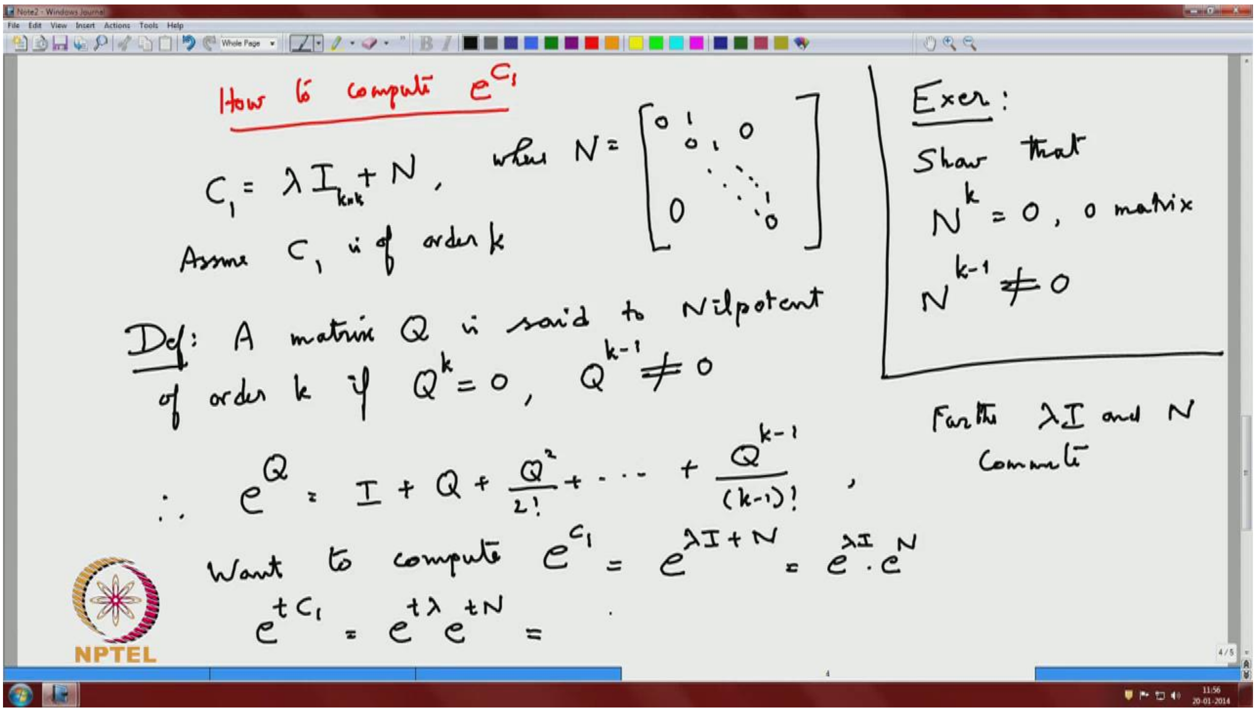

So, I want to know how to compute c 1. How to compute e power c 1? Now, consider this case; we are going to do that one. So, c 1; look at c 1; c 1, I can write it separately, as lambda; I separate this one; lambda into identity, if some order, whatever it is; plus n. I write that one. So, look at here; I separate this one. I keep only the diagonal entry with 0, 0 everywhere, plus, I put diagonal entries also 0, and put 1, 1, 1 as the half diagonal where, some order; I have not told what is order depending on that; where, n takes the form, diagonal entries also 0, 1, 1, etc. 1. The rest of all the elements are 0, you see, it has a very nice matrix. Here is a small exercise again, for you. You have to keep on doing that. Exercise; I give c 1 is of order k, and it will be some order. I assume c 1 is of order k.

So, I is an identity matrix of order k by k, and n is a k by k matrix. Exercise is that show that n power k is the 0 matrix, n power k minus 1 is not a 0 matrix. Just, compute n, n square, etc. You want to compute either, by induction you can compute that one. Such type of matrices are called nilpotent matrix. So, you have a matrix, any matrix. Let me have a matrix a, any matrix q is said to be nilpotent of order k, if k is the first instance or q power k is a 0 matrix, and q power k minus 1 is not a 0 matrix; that is the first instance. Of course, if q power k is 0, that implies q power k plus 1, etc. q power x plus k plus 2, etc. 0. So, that makes the computation of nil matrices easy, because you do not have to compute that after k onwards. So, the exponential term, exponential of nilpotent matrices, reduces to a finite sum, immediately. Therefore, e power k; the computation of a nilpotent matrix is easy; e power q will be identity plus q plus q square by 2 factorial plus; of course, if k is large, you have to do a large thing; still lot of work, but it is finite.

So, you do not have to worry about anything else, you see. So, that e power q, after that, everything is 0, because q power k is equal to 0, implies q power k plus 1 equal to 0, q power k plus 2 equal to 0. It goes on to it. So, you have the nilpotent. Our aim is compute e power c 1. We want to compute e power c 1. Of course, e power c 1, as again, remarked earlier, if you have two matrices now c 1 is of the form two matrices; lambda I plus n. So, if you want to recompute e power a plus b is equal to e power a into e power b, then e power a plus b is equal to e power a into e power a b, if a in b commute.

The interesting fact is that in this case, one of the matrices is identity. Hence, identity and n will always commute. Any matrix with identity matrix will commute. Hence, further, lambda I and n; this is the trivial fact; and compute; that is an important thing. If there is no commutation, you can write it. So, e power c 1 will be e power lambda I plus n. Since, this commute, this will be nothing, but e power lambda I, plus, into e power a. So, you have immediately, e power a, and e power n; you have already the formula. You have already the formula for e power n. So, you have your e power n. In particular, in further, you are interested in computing e power t c 1, because you want to find the solutions, corresponding to c 1.

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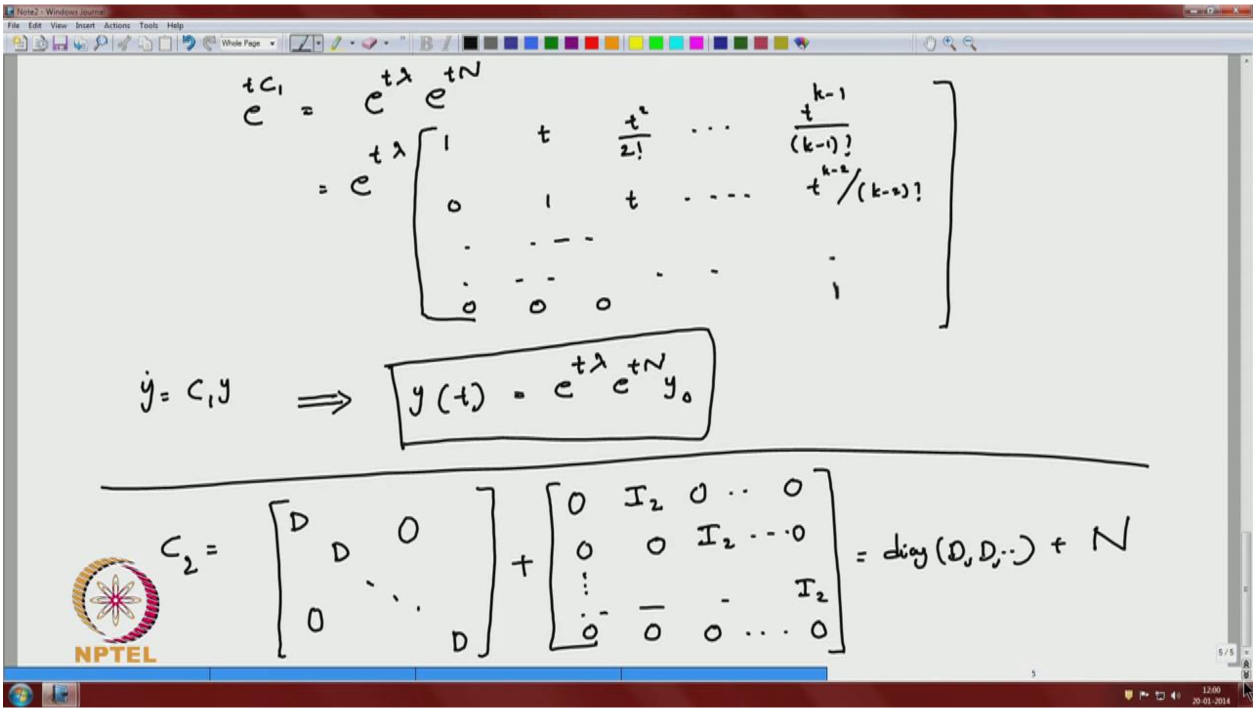

So, that e power t c 1 will be e power lambda t lambda e power t n. So, if you do a simple computation, it is; I will go to the next page. So, e power c 1 will be equal to e power lambda I, into e power; it is e 1; we want to introduce. So, you will get this as if you do this one, you get e power t lambda, into e power t n and that immediately, can be written as e power t lambda. So, we have a complete thing. If you do, because t will change the thing; you get 1, t, t square by 2 factorial, etc. up to t power k minus 1 by k minus 1 factorial. Then, 0, 1, t, the last element will be t power k minus 2 by k minus 2 factorial. So, if you go like that, the last, but one, the last here 1, and the last element row will be 0, etc. 0, you see.

So, we have an immediate solution. For the system with c 1, if you go back to the system with; not this one, next one; c 1, yes, if you go to the system with c 1 here, you see, you want to solve the systems. So, you have that this is the system, a particular form of the matrix. These are the only two things will be coming up. So, if you go here, if you want to solve your system y dot is equal to c 1 y, this will immediately implies, your y t is equal to e power t lambda, into e power t n, this matrix, e power t n of y naught. You see, you have your solution. You have a nice solution here. Now, we go to the next case. Second case; another one is of the forms c 2; c 2 is again, we can write it as in a nice way, but not with an identity. You will write only d on the diagonal; you write your d here, 0 here, 0 here, plus; on the half diagonal, these are all 2 by 2 zeros; 0, etc. 0. This is

2 by 2 zero; 0, 0, I 2. So, you go here, 0. The last, but one, the last element will be I 2 and then, you have 2 by 2 things; 2 by 2, you have 0, you see. So, this is what again, you can write this as two matrices and this can be written as something like you call this diagonal; diagonal of d, d, d, plus some n where, n in this form we saw. So, let me write it r. So, you can write this here. Again, these things are commuting. You have the commutation; there is no problem.

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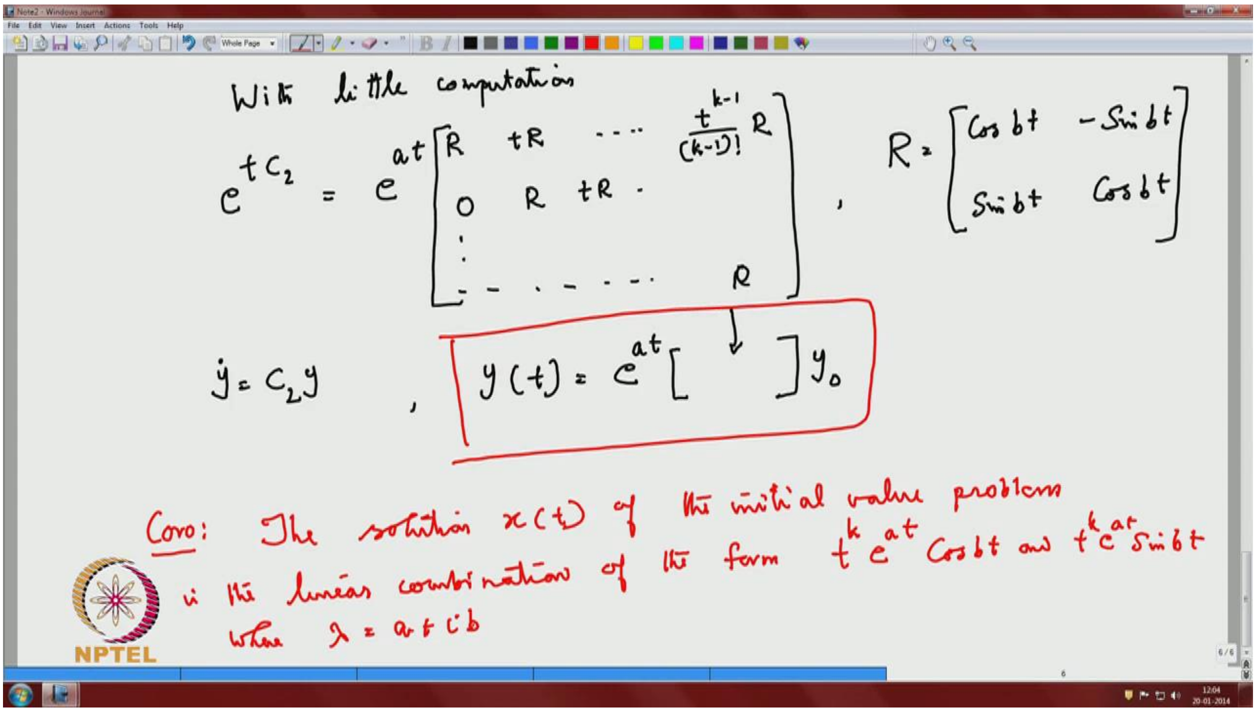

So, if you write your solution with little computation, I will skip here, with little computation, which is an excise, with little computation. The similar thing, you can write your e power t c 2 is equal to; see, that matrix is of the form; what is the matrix; e power a t, into everything will come 2 by 2 blocks are t r, etc. t power k minus 1 by k minus 1 factorial, r 0, r t r, etc. up to that. The last element will be 0. Last, but one will be, last, yes, it is correct, R. So, what is R? R will be of the form, the same formula; your cos b t minus sin t, you see, nothing, you are not able to separate it; cos b t, you see. Again, you have your solution, y t. If you have the solution y equal to c 2 y of this form, your solution will be of the form, y 2 is equal to e power a t, into this matrix, whatever it is. Yes, this matrix. What do you call it; whatever it is; this matrix, into this matrix y naught. So, you have your solution representation, again, you see.

So, that gives you more or less, a complete description. Of course, the things are not that easy as we think, but then, you can write your solutions, completely here. You are able to write, completely here. So, the important point in this thing that for every solution, you have reduced your system of smaller systems, and the only things are coming is the exponential function in the solution, the polynomial functions, because of t, t square, etc; polynomial functions, and the trigonometric function. So, every solution of your linear system is a linear combination of these factors, something like t power k, e power a t, and cos b t, or t power k, e power a t, sin b t. Only these elementary functions and combinations will come in the representation of the solutions; that is a small remark. May be, we will write a corollary that form immediately, what analysis, you can do that. The solution x t of the initial value problem; this is an important observation; initial value problem is the linear combination of the form, t power k, some number t power k. There will be many different types e power, but it will be of that form, cos b t and t power k, e power a t, sin b t.

So, this also where, lambda is of the form, a plus I b or some form. So, that gives you, you know that only, the expressions of the form, t power k, e power a t, cos b t or sin b t are available and can immediately, see that it can occur stability only, when y is negative for all the Eigen values. So, the stability of this system, if one of the terms, the real part of the Eigen value is non 0, and positive, greater than equal to 0, if any Eigen value, any one of the Eigen values has a real part, which is greater than equal to 0; you do not have the stability. So, the x t goes to 0 only, when the real Eigen values, all the real part of the Eigen value is negative. In that case, it goes with the origin exponentially, because there is an exponential term; that is what one remark. Secondly, what you want to do is that you want to do one or two examples; may be, one or two examples at this stage. May be, I will give you one more thing. One more remark I want to make it here, probably, a thing.

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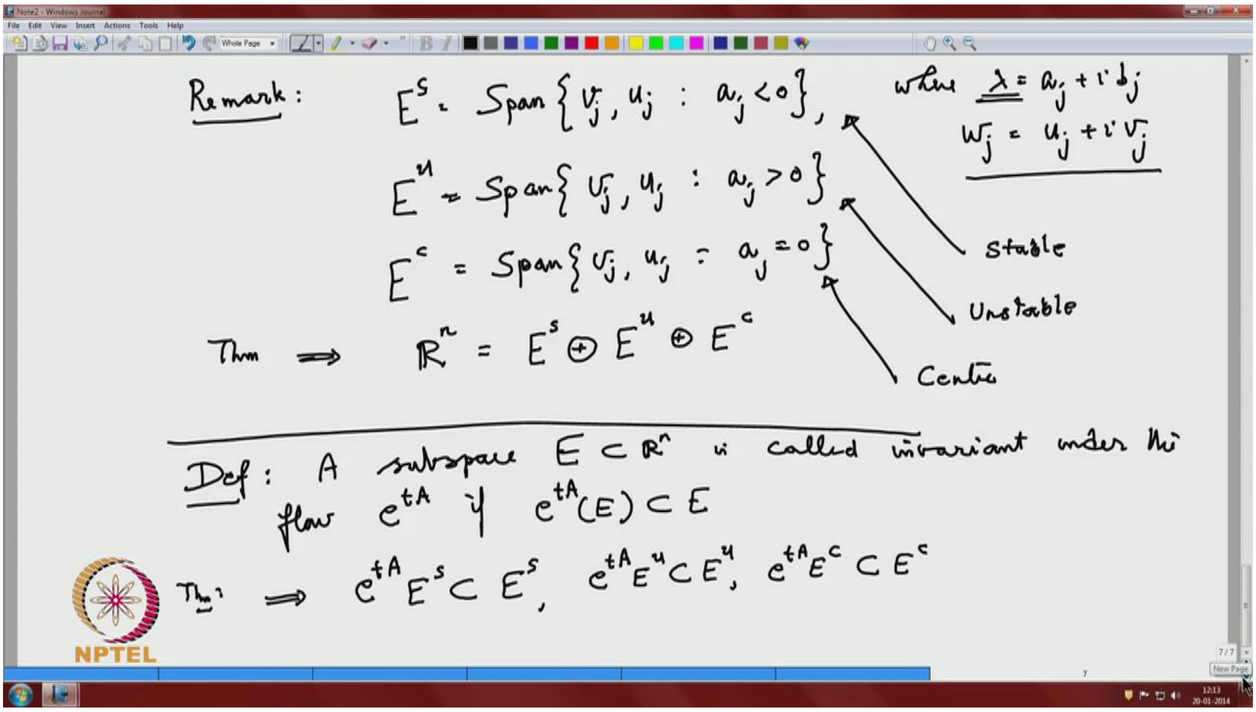

The other remark I want to think, corresponding to that, I do not prove here, but I want to make. In the general case, what will happen is that, you look at this all the subspaces e s, is equal to the span of; I am writing in general complex. If the complex part is 0, it will become real part. So, you look at v j u j where, u j plus I v j is an Eigen vector, corresponding to lambda, such that your e a j is negative. We look at your Eigen value is always, lambda is equal to a j plus I b j, and Eigen vector; this is the Eigen value. Eigen vector is equal to u j plus I v j; you write it that way. In the complex way, you write everything. So, you have, if v j is equal to 0, it becomes a real Eigen value; there is no vector. So, there is no problem. It incorporates even, the real Eigen values with b j equal to 0. Look at all the Eigen values, all the Eigen vectors; real and complex part of the Eigen vector, corresponding to a real Eigen value, corresponding to an Eigen value, whose real part is negative. Collect all that and span it. That are called the stable part and then, similarly, you have your E u; this is span of v j u j with a j positive and E c, which we have not seen in example; may be, we will see an example; span of v j u j with a j is equal to 0.

What the interesting theorem will tell you which, I do not prove; your space R n can be decomposed into E s; these are called direct sum, E u direct sum. Yes, of course, one may need to use the other E c. This is called the stable space. You will learn more ever these things in the non-linear study; stable. This is the unstable substitute; unstable. This is called the centre. What I have not mentioned here is about the generalized Eigen vector. Once you do the Jordan decomposition as I said, you may not have the enough Eigenvectors. So, one has to work with what is called generalized Eigen vector. So, we will be studying in that, the whole analysis in the generalized concept which, we have not introduced, or we do not have time in this course to get into more details about the generalized Eigen vector here, but we have to work this decomposition to happen. In that form, Jordan decomposition to happen, you need to work with generalized Eigen vector. So, we will not give you, may be, probably, if you have time, I will look. The interesting one more thing is that one more; some of the notions I am introducing it, you will understand more about it, when we go into the non-linear analysis.

A definition to start with; a subspace E is called invariant, under the flow; recall the flow; called invariant, under the flow. What is flow? Flow is e power t a. If e power t a of e, if you add to e, it will remain itself. So, it will remain in e. The other results about the proposition of theorem, part of previous theorem which, I have not stated is that these spaces are invariant; e power t a of e s contained in e s; e power t a of e u contained in e u. What it shows that if you start with a point in say, e power s, and if you follow the trajectories, starting from there, under e power a, it will remain any e power s itself. It will not move out of e power s. That is the stable subspace. On the other hand, if you start with initial point from e power u, then e power t a of that element will remain, all the time in e power u. So, it will not leave these spaces like invariant thing. So, the e power s, e power u are invariant subspaces under the flow. So, the flow will not do that one. In particular, if x naught is in e power s, you can see that e power; that means, all it say, real part of the Eigen values are negative and hence, it is stable, exponentially. So, if you start with x naught in e power s, e power t a; a has negative Eigen values. The trajectory will go to 0; that is an interesting thing.

This is node general, probably, you may learn in for most stable manifold theorem which, we will not do it here. Probably, we may do it in a module of non-linear analysis, is possible; otherwise, we may not cover, but basically, it is a content in the linear system of that. The stable thing, anything you start from a stable subspace, then even, get the subspace, when you go to non-linear analysis, you start this thing. But in the linear system, if you start from this stable e s, it will go to the thing, here. To give, probably, I want to give some more examples; one more lemma or something, I may skip it here, but then, I want to start two more examples in the remaining time of this lecture. We will give one example or two examples. Let us see, we will give one more example.

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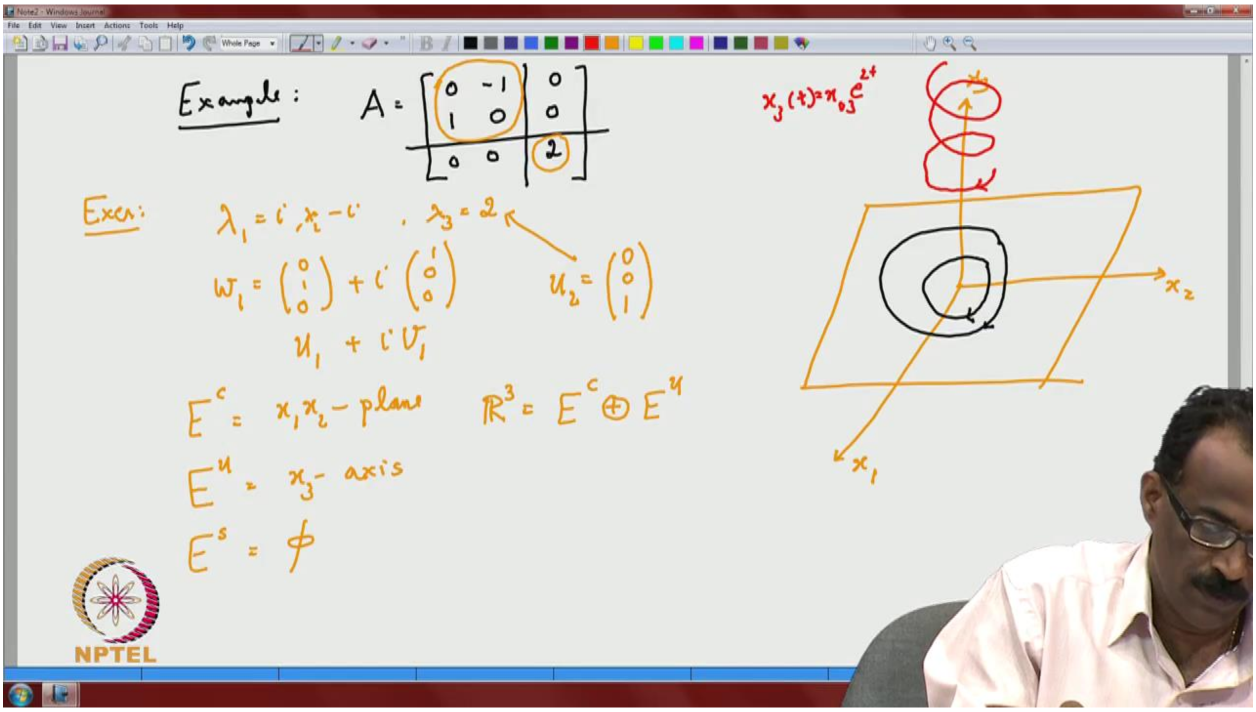

Let us start with an example here, and may be, one more example in the next class. We will do it thing. So, with this example, we can do this thing. So, let us consider a system, A; 0, minus 1, 1, 0, 0, 0, 0, 0, 2. If you look at the system A, this is again, actually, decoupled into a 2 by 2 system here. You look at here; this is a 2 by 2 system, decoupled with A equal to 0. So, it is something like a centre, you see, and then, x t component is separate. So, what are your Eigen values? You have your Eigen value I, of course, minus I. The other Eigen value, lambda 2, and you have lambda 3 is equal to 2, and you can also see that your Eigen vector. You can see that. These are all exercise. Whenever, I leave, I do not work out; is called the exercise part. You have your 0, 1, 0, plus I into 1, 0, 0, and that we will call it this is your u 1 plus I v 1, and you will have, corresponding to this, you have another Eigen vector u 2, because that is a real Eigen value. So, you have only one Eigen vector 0, 0, 1, you see, you have.

So, if you look at it here, this part is 0; it is corresponding to a centre. Your e c is nothing, but your x 1 x 2 plane. It is a center; it is a form of a center with the real part of this Eigen value is 0. So, for this lambda 1, lambda 2, the real part is 0. So, it will behave like a center. Your other Eigen value 2, that is corresponding to the x 3 part and since, it is 2 and positive, it will go to infinity, and it is unstable. You have your unstable part, is the x 3 axis. You see, these are always, decomposing and there is nothing like stable here; stable is empty. Again, complete x 1 x 2 plane, and decompose your R 2. Of course, your R 3 is E c plus direct sum, into E u; E s is empty. So, you plot this graph here, if you want to see a plot of this graph, if you, this is your x 1, 2; your x 1, and there is your x 3, and x 3 is a decoupled part. So, whenever you start a solution here, because x 3 in the x 1 x 2 plane, if you have an initial value in the x 1 x 2 plane, and by the solution, is a decoupled; x 3 is decoupled. It will remain in the x 1 x 2 plane itself.

In the x 1 x 2 plane, the real part of the Eigen value is 0. So, it will remain like a center. So, it will be a circle; the solution will be a circle. So, if you start from here, it will be like this, you see. It will be in the x 1 x 2 plane. So, what will happen if this is a point, if you starting from above or below; what does it show? This shows that it will rotate the x 1 x 2 plane; it should rotate, but then, the x 3 component, x 3 t is equal to x 0, 3. If you look at here, x 3 t is nothing, but x 0, 3 into e power 2 t. It goes to infinity without reducing this thing. So, if you start from any point here, if you start from here, it will move like a center. But then, x t will go on up, you see. So, if anything, it will curl around with the same like a center, and will move up. If you have below, it will also move below with the same radius. So, can work; the exercise for you to take all kinds of things in 3-dimension and see all possibilities. May be, one more example, and then, in the next class, I will give one more example, if I have time.

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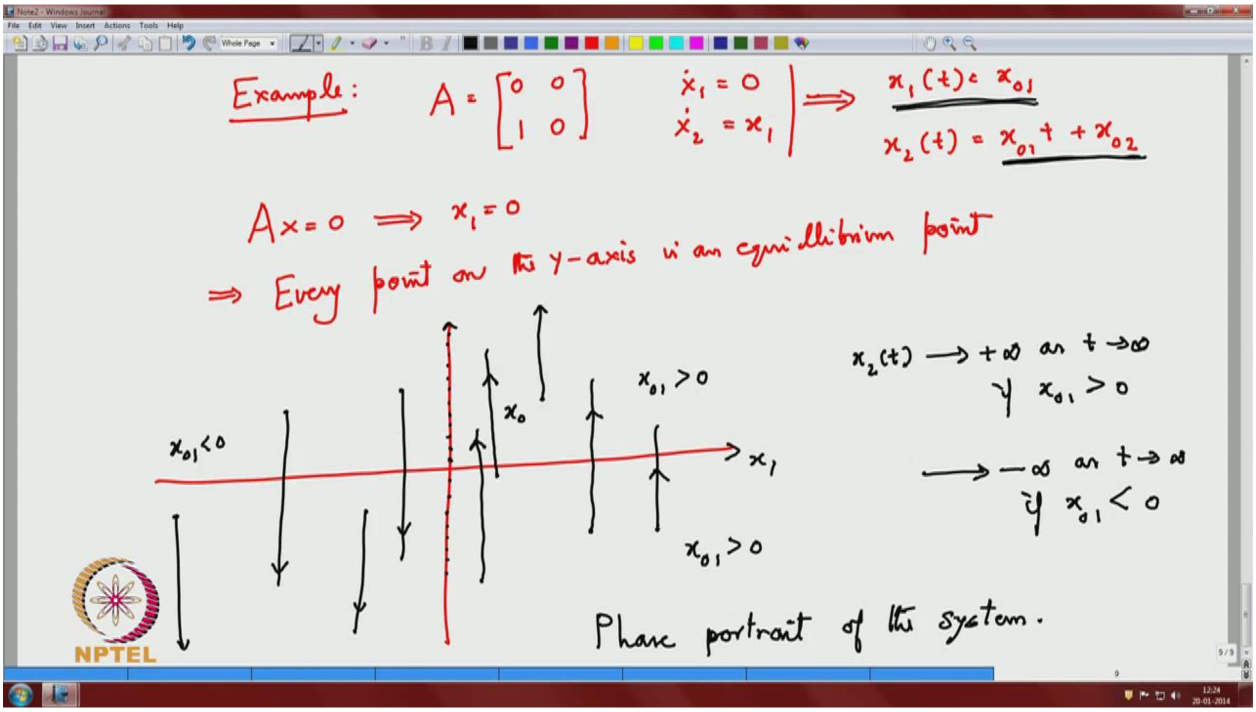

So, one more example; examples are the best way of imagine. So, I want to have A with both; we have already seen; with one Eigen value 0. You have one with both Eigen values 0, you see. So, what is this system corresponding to that; x 1 dot is equal to 0; x 2 dot is equal to x 1. That implies immediately, x 1 t is equal to constant. That is nothing, but x 0, 1, and your x 2 t is equal to x 0, 1 t, plus x 0, 2. Let us find out equilibrium point. If you want to find out equilibrium point, you look at A x is equal to 0. When A x is equal to 0, the first one give any information, because x 0 equal to 0; you will get it. The second one will gives you an information; your first component is 0. So, the first component is 0; second component is arbitrary. That implies, every point; earlier, we got every point on the x axis; here, every point on the y axis is an equilibrium point, you see. So, if you try to see that here, plot the graph here, all the points here, are equilibrium poinst.

If you recall the earlier example which, I have given in a similar situation with an Eigen value with one Eigen value 0, and the other Eigen value, non 0 Eigen value; you have seen that all the points on the x axis thing, and anything you start above, it goes towards that thing, but here, you will see something different. When there is a degenerate case, the situation will differ. Now, let us look at it; any point x naught here. Again, what does it says that this tells you that again, x 1 t is x 0, 1. So, it should remain in the perpendicular line still, and the second one, because of the x 0, 2; this looks as it. It depends on this t tends to infinity. You say, x 0, 1. If you take x 0, 1 in this; this is a quadrangle. On this quadrangle, x 0, 1 is always positive, you see. So, if you look at the first, if you take the upper half plane, the first and second quadrangle; your x 0; x 0, 1 positive or x 0, 2; yes, if you take this, sorry, if you look at this portion, this is the x 1, x axis, and that portion; on this portion, here also, x 0, 1 is positive. If you look at here, in this case, x 2 t here, tends to plus infinity as t tends to infinity, if x 0, 1 positive, and it tends to minus infinity as t tends to infinity, If x 0, 1 is negative.

So, if you start any point here, the first component will (( )) here. So, it has to move in a line, perpendicular to x 1 axis, but it should move to plus infinity. Even from here, your x 0 component is positive. So, it will not go towards the equilibrium point. Anywhere, you start it; it will move like this. It will move again like that only. On the other hand, this side x 0, 1 is negative. In that case, it will go to minus infinity thing and again, it should remain in this perpendicular to x 1 axis. So, it will move here. If you start from here, it will move, you see. Near the centre, the behavior will be like this. Anyway, where it starts; does not matter. Even, if any point you start it; it will move like that; you can see. These are all equilibrium points. Also, on the y axis if you start it, it will remain there, because every point is an equilibrium point. In the right side if you start it, this is going to the infinity, along that one. So, this is a phase portrait of this thing, of the systems. With this, we will have finished more or less everything.

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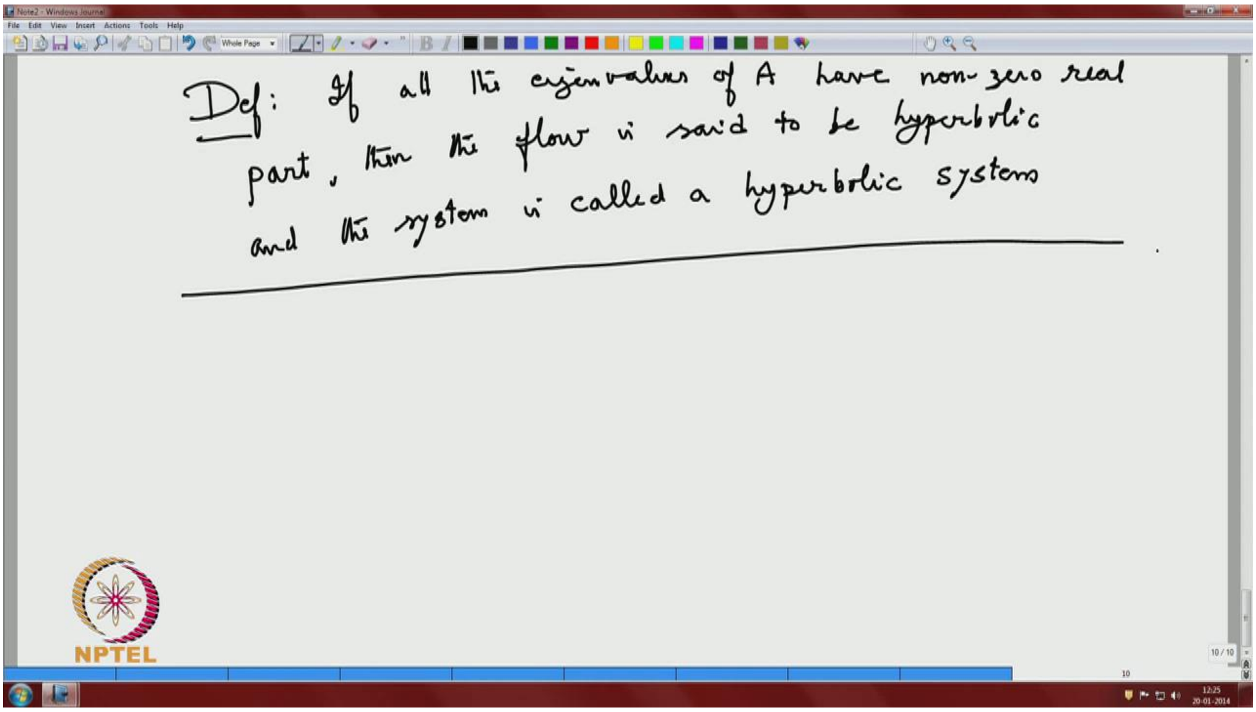

With one definition, I will end this class. Definition; if all the Eigen values of a; this is because this terminology will be used in the non-linear systems. If all the Eigen values of A has non 0 real part; 0 real part leads to center; that is why you have non 0 real part. Then, the flow is said to be hyperbolic, and the system is called as a hyperbolic system. So, you will study more about this hyperbolic system, etc. in the non-linear thing. In the last lecture, if we have time, we will try to present one more example, but then, main aim of the my next lecture, the last lecture of this particular module is to see how to represent the solutions in the non homogeneous system. So far, we were studying x dot equal to a x t. So, we want to see how to use this, to represent a solution of the form, x dot equal to a x t plus g x, and we make few remarks when it is a non autonomous system by a depends on a t. Then, we do not have a representation like that, but we can represent a solution in the form of something in this; what are called the fundamental and transition matrix. So, with that, we stop this lecture.

Thank you.