Ordinary Differential Equations
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Module - 5
Lecture - 25
2 by 2 systems and Phase Plane Analysis

Welcome again. In the first lecture in this module, we have introduced the linear systems, and we were trying to understand the autonomous linear system where a is independent of t. Autonomous, as I mentioned yesterday, the autonomous system has many advantages.

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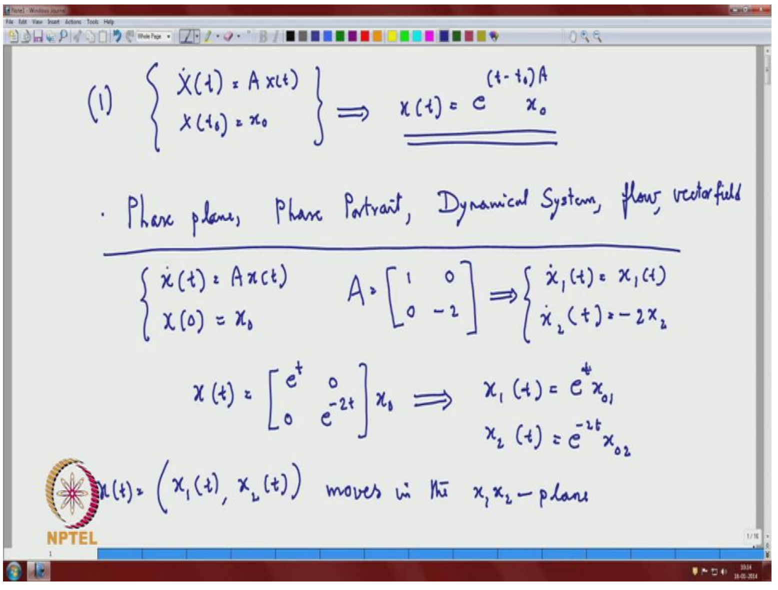

We got the existence of solutions in this. Most of viewers are trying to study this problem, x dot of t is equal to a x t with x at t naught is equal to x naught. This has a solution in the form of exponential representation, and solution; this is a unique solution given by e power t minus t naught a x naught. As I said, it has disadvantages. One is the thing; computation of an exponential of the matrix is generally, difficult and secondly, it does not reveal anything about the solution trajectories. So, today, yes, this is the scenario. In this scenario, we have introduced what is called the linear equivalence; whether, the matrix a can be linearly equivalent to another matrix, and this is nothing but the similarity of the transformations.

When you have a is similar to b, via an invertible matrix b, the system corresponding to a, as I called the system 1, can be converted to a system corresponding to b, something like, y dot equal to b y t. If the system b is corresponding to a system, corresponding to b, is easy to, the exponential is easy to compute, then the solutions will change, and this change is only a coordinate change. The nature of the equilibrium point, which we have introduced yesterday, will not change. If the equilibrium point is stable, it will be stable for even, for the transformal equation. So, what we will be going to see is a detailed analysis of 2 by 2 systems, which is much more simpler. In particular, if you can get the matrix b to be diagonal, immediately, solution can be written anything. This is called the diagonalization of the matrix, but as we know from the linear algebra, not that every matrix can be diagonalized. That is what we have seen, a particular matrix lambda 1, 0, lambda; cannot diagonalized; that is one simple example.

Because the diagonalization is something, like equivalent to the existence of n independent Eigen vectors, which may not be available to you. What best we can do is called the Jordan decomposition, but this Jordan decomposition in 2 by 2 systems is much more easy, and we will have a complete analysis, today. So, in this scenario, we are going to introduce what is called a phase plane. We introduce the phase portrait. We will do all this in the form of examples, and we will introduce after that, what is called a dynamical system, corresponding to this one. These notions, we will see through examples, we will also show the concept of flow and vector field, you see. All these notions, quickly I will recall, but you will also see these notions in more detail, when we study the non-linear system. As I said yesterday, this module is a precursor to the non-linear system, and that is what more interesting problems; you will see more interesting applications, we will be able to see this thing. So, let me start with an example. That is the best way to ease.

So, let us look at a very simple example, x dot of t is equal to a of x t with x at; say, you can put at any point; in particular, you can put at 0, because of the autonomous system. What is a; a, I am taking to be a very diagonal matrix of the form 1, 0, 0, minus 2. So, this is a decoupled system. This system is nothing but you have x 1 dot t is equal to x 1 t, and x 2 dot t is equal to minus 2 x 2. Now, you know how to write the solutions. Either, you can directly use this formula, because this is a diagonal matrix. You know how to compute that, or you can solve here. Whatever it is; the solution will look like this. The solution of x t is nothing but it is at the origins; it is at the 0; we are trying to do it and hence, the solution is written as e power a t; e power a t is nothing but e power t, 0, 0, e power minus 2 t, x naught. If you want it in the component wise, you will have x 1 t. What if you want to learn this course, you have to work this kind of more and more examples here. So, you will have the solutions, e power t, x naught 1. What is the first component of x naught; x naught has two components. This is x naught 1, and x 2 t is equal to e power minus 2 t, into x naught. Now, here is where, I want to tell you that thing. As I said, t is a time, which is treated as a parameter, but this motion of x 1 t, x 2 t; the x 1 t, the curve x 1 t, x 2 t, if you write it, this is equal to your x or t. This moves in the plane x 1 x 2; moves in the x 1 x 2 plane, you see. So, what is happening is that this is, you can think it as a motion of a particle in the plane x 1 x 2.

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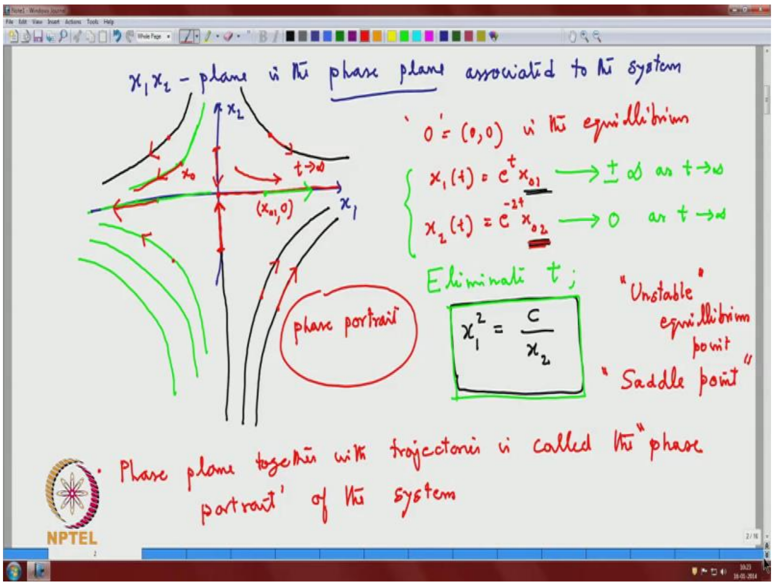

This plane x 1 x 2, is the phase plane; called phase plane associated to the system, you see. You want to understand as this is the motion of the particle, and x 1 t, x 2 t is the parameter representation, with a parameter being time, in the x 1 x 2 plane. So, you want to see the picture, the trajectories in the x 1 x 2 plane, you see. You have x 1 here. You have x 2 here, and then, you want to see how the trajectories will move along that. If you look at that, as you said that a is an invertible matrix in this case, the origin 0; 0 is equal to 0, 0, is the equilibrium point, which is a solution. Any solution, starting with the origin, will remain there, forever. But now look at the trajectory, x 1 t, which I have introduce for this; x 1 t is equal to e power t, x naught 1, and x 2 t is equal to e power minus 2 t, x 0 2. This one, if you look at it, cover be the x naught 1 or x 2. Depending on the sign of x naught 1, this will go to plus or minus infinity, as t tends to infinity. On the other hand, irrespective of which coordinate it belongs to, this will go to 0, as t tends to infinity.

So, here is a situation; wherever be the initial point, the first component will go to; one of the components will go to plus or minus infinity, and the other component will go to 0. If you want to plot this graph in the x 1 x 2, what you do is that to plot this one, you eliminate this one; eliminate t. That is what you do it; you want a clause representation. If you eliminate here, the best way to eliminate; square x 1. So, you will go back to that one. If you square it, x 1 square, will be e power 2 t, but e power minus 2 t, you can put it at the denominator. These are all will be constant; x naught 1 and x naught 2 will be constant. So, you will get basically, c by x 2. This is the closed form of this solution. If you plot here, suppose, you start a trajectory at this point. If you start that trajectory; this is a point nothing but some x naught 1, 0. Since, the second component is 0, these are all, this equation is a decoupled system; x naught 2 will be is 0; the trajectory will remain, x 2 component will remain to be 0, all the time. So, the trajectory will be here. So, this will be a trajectory, basically.

On the other hand, if you start a point here, this is also a trajectory. Only thing, which direction we will move; that direction of the motion is the only important thing you have to see, and the direction will prescribe whether, it will go to plus or minus sign. In this particular case, if you start with a point here, and as x 1 component will go to 0, infinity, the motion will be in this direction. So, if you start from here, the motion from, if you start from here, this will move in this direction. On the other hand, if you look at it here, if you start a trajectory from there, again, the trajectory will remain in the x 2 itself, because x 1 will be 0, all the time, but the x 2 part is going to 0. So, the trajectory will move towards here. If you start from here, the trajectory will move here. So, this is at the time t equal to 0, or any other time, and as t; this arrow represents as t tends to infinity, all the time; you have to understand this arrow as t. So, if you move from here, it will move along this direction to all that equilibrium point, but if you start from here, it moves away from the equilibrium point. Now, let us start from here. Suppose, you take an initial point here; this is your point x naught.

So, if you start from there, your trajectory, the graph tells you something like this, kind of thing, is something like x 1 square equal to x 2. So, it will, the graph, if you plot here; this is the best example to see; it will plot here, your trajectories, if you plot, because this equation is defined not only for t positive; it is defined for t negative as well. So, if you start here, your trajectories will be here. So, all your trajectories will plot here. So, I have your thing. If you have your trajectories like this, if you have your thing, you will have your trajectories here. If you plot here, your trajectory; all the trajectories in. Only thing is that varies the motion, in which direction, this will move. For example, if you start from here, as you know that x 1 coordinate go to infinity and x 2 coordinate will go to 0, as t tends to thing; it moves along this direction.

Because this is the one, x 2 component is going to 0; x 1 component going to infinity. So, if you start, it will match with this. It will move in this direction, if you start from here. So, this is the part where, t tends to greater than 0 part. So, if you move here, it will move in this direction, if you have initial point, it will move, you see. So, everything moves away from that. So, it is t. So, the phase plane together, with all the trajectories, is the phase portrait of the system 1. So, this is the phase portrait; it is a portrait. So, this gives you the complete analysis of your trajectory, and such an equilibrium point; this equilibrium point, because the trajectories is at least, one component posed.

So, if you start a trajectory anywhere here, it is moving away, as t tends to infinity; it moves away, you see; that is what; so move away from the equilibrium points. Whenever, that happens, such things are called unstable. So, this is a unstable equilibrium; it is not stable. However close it is; it will eventually, move away from that unstable equilibrium point, and in this particular thing, there are different types of stable and unstable equilibrium. This is something like as saddle; it is sitting something like that, the trajectories moving, and it is called the saddle point, actually. This equilibrium is a saddle point, you see, this the situation about this.

So, you have the phase plane where, your trajectories are moving in r n space, in n dimension; this phase plane is called the phase space for r 3. The phase space is x 1, x 2, x 3 phase, and r 4 is 4-dimension of phase plane. If you are studying a system in a dimension, and it is you want to see the motion of the trajectories; that is the corresponding phase portrait of the system.

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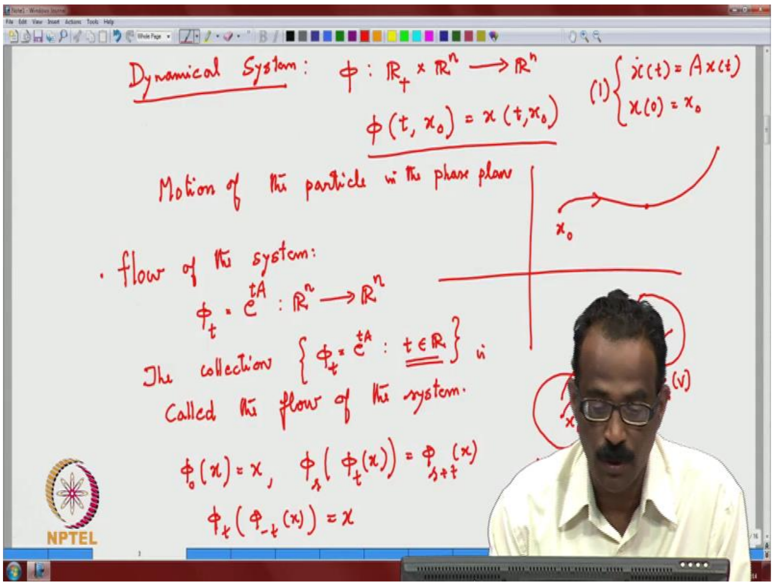

With that, we have another notion called dynamical system. You can also call the given system to be dynamical system, but you can also give more things. Consider the map t from r plus or r, if the solution exists; r plus cross r n to r n, given by phi at t and x naught, is nothing but x t x naught. What is your system; x dot t is equal to a x t, x x; let us start with 0; does not matter as far as the system is concerned; whether, you start with at t naught or start with at t; does not matter. This is nothing but the solution at the time t, starting from the point x. So, if you start in r n, r 2, you will have an x dot here. The solution will move along this as thing, and this is nothing but the; so you can imagine the dynamical system is nothing but the motion of the particle. Once, you fix x naught, is nothing but the motion of the particle, along a trajectory and it gives you the position of the trajectory at time t, starting from x naught at time 0. For this particular system, x dot is equal to a x t with x at 0 is equal to t.

In fact, it is the motion of the particle. Dynamical system is the motion of the particle in the phase plane. That is what you have to understand. There is another related notion. This gives you a better feeling about this that what is called a flow of the system. Now, what you do you is that we will have a physical motivation for this. You look at fix t, you consider phi t; phi t is namely, e power t a for this system; this is a mapping from r n; you can view that this is a mapping from r n to r n from any dimension. So, you collect everything. This collection is called the flow, actually. The collection phi t is equal to e power a t as mapping t a, such that for all t in this system, as you see that not only for t r plus, you can concern, is called the flow of the system. What is the flow doing? What is the advantage of the flow here? For example, the main advantage, when we look at it fluid flow, you can think that fluid at time t equal to 0 in one position, then you view the fluid as particles.

Then, for each particle, after some time, the particle will be at another point, but this does not give you, this view of a motion of the particle, if you view just a particle in the fluid; for every particle, you move to some other position at after some time t, but for a lemon point of view or a natural point of view, when you look at a flow; we never see the flow as particles are moving; rather, we see a collective motion of particles. So, this forces you to understand the motion of just one point, we want to see the motion of that point together, with its neighborhood. So, if you start with at x naught, this is a different phi x naught, and look at this neighborhood, some neighborhood v; this is nothing but at the position phi x 0. So, phi x 0 of v is v; that is it, because t 0, it moves there. Then, for each t, look at all the particle in the neighborhood; you collectively, try to see, this will be moving here, and this will be moving here; this will be moving something. So, you want to see this motion and all this will be at different positions; this may be at a position somewhere here, and you get a neighborhood.

So, this is nothing but your phi t at v. If you look at that, not only the point and x naught and all the neighborhood; you try to see that how the neighborhood together, moves and that gives you the better feeling of the motion of the particle. That is actually, what you are doing; try to do that through that of the systems. You see that a column or a neighborhood of the fluid, moves in that direction. So, that is what the flow, gives you the concept of the flow. Flow has some nice advantages, like kind of semi group properties, and all that which, you may probably, learn in some other thing; phi naught of x is always x. The more important thing is that suppose, flow move from at time, flow from x; this essentially, tells you that the flow from x phi t of x, gives you from the position x, it moves to the position t at time and then, composite it with that phi of thing, will be the same as phi composition, s plus t of x; this is a very important thing. So, your flow moves from one position to another position and then, it moves, is the same of flow moving from to other position.

Another interesting property, this also has a group structure, in fact, for this particular system. You will have phi t of phi minus t of x, is equal to x. In general, many of the flow is the reversible systems, you may not get the last property, but for this system, you can also reverse the flow. That is why it exists for all t in r, but in a semi group, when you go to partial differential equations, etcetera. when you try to understand this kind of motions, you may not get a group structure; you may only get a semi group structure. That is anyway, not the topic of discussion here.

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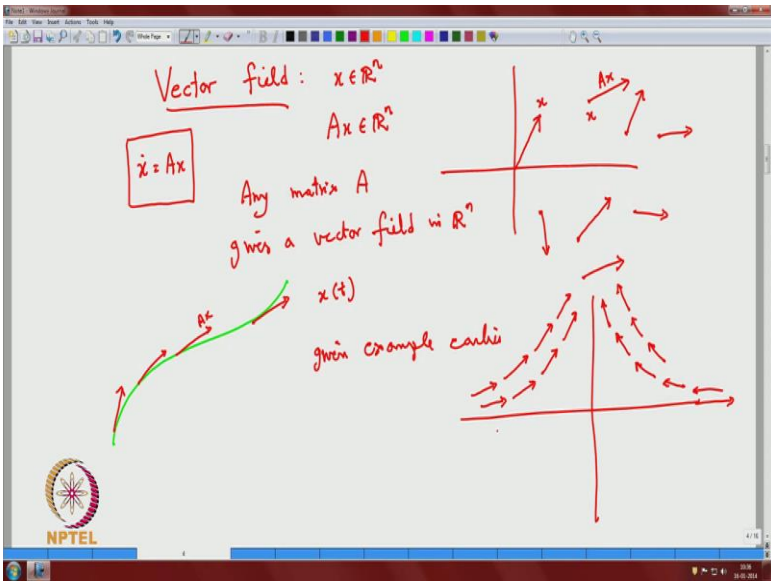

One more notion, which you probably, would like to think; these are all some geometric way of understanding vector field. So, to understand the thing, as you have seen now, is not only understanding the solution; how to interpret the solution; how to give the solution; all part of the study. Vector filed for given x in r n, a x is another point in r n, but in the beginning of our discussion long back in the previous lectures, we said in r n, we can view either, the points of r n s points or you can also view points as a vector; just like we have seen. Either, you can view this point x or you can view that point as a vector v here. The advantage of this vector, if you view this as a vector, I can put this vector with a same length parallel to that. So, you can get that view. For example, when you have a fluid flow of a particle, the position of the particle you would like to view it as a particle; but on the other hand, the velocity at that point is also, a point in r n, but then, is better to view the velocity is a vector, placed at that point. So, when you are given a dynamical system x dot is equal to a x; x t, you view it as a position and a x, you view it as a vector.

So, that way, any matrix a gives a vector field structure in r n. What you can say is that for all the points here, x; you have a point here x; you can associate a vector. This vector is nothing but x. If you take another point, you will have another vector. All the points can be associated with vectors. If you want to know for a given dynamical system, if you look at it here clearly, what are we trying to do? We are trying to associate x dot equal to a x. When you are looking for solution x t to this dynamical system x dot equal to a x, you are looking for this trajectory a x. So, that x dot; x dot is equal to nothing but your tangential vector field. This tangential vector field should be a x; that is what we are looking at it. So, the vector associated to this point, each point should be; we are looking at a trajectory. Solution associated to that one; you want it; these are all should be your a x, you see. So, that way, a solution, a trajectory x t, is a curve x t is a trajectory to the solution at each point, the tangent is given from the vector field.

The moment a is given to you, you have a vector field given; a vectors are associated with all r n. So, you are trying to find trajectories in such a way, so that, each point on that curve, the tangential vector coincides with a vector field given to you. So, the moment, you are given a linear systems like x dot equal to a x, you want to plot your vector field in that way. So, for the given example earlier, if try to plot your vector field, you can see that the vector field; you already know the trajectory; your vector field will be like this. So, you plot your vector field; it will be something like that. So, you get all your vector fields like this. This is your vector field associate. So, if you plot it here, you see, we plot here.

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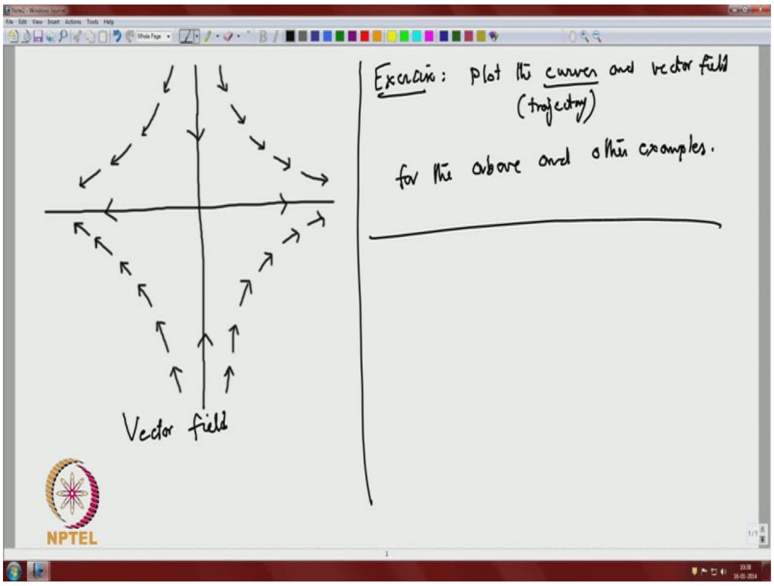

So, that is a vector field. Let me plot again, the vector field. If you have your vector field to that example, are looking something like that. This is for the given example. It may vary like that, is your plot form here; that is, have curves. Similarly, if you plot here like that; these are, this point, this is the vector field, and now these directions have changed as now, the direction is in this direction. So, if you plot here, the direction will be like this. If you plot here, this will be the direction. You get each point, you will have the corresponding; this will be here, because it is a stable one, you see. So, this is the vector field. So, every system, you will have a vector filed associated with it.

So, what I would like; those, who are learning here as an exercise, plot these things. Exercise; plot the curves and vector fields. Curves means solution trajectory; I mean a trajectory and vector field for the above and other examples. So, what we have seen is that when you have the solution trajectories, and if you plot all the tangent vectors, you will get your vector field. On the other hand, if the vector field is given, you take any trajectory, so that, the tangent vector to that curve should be from the vector field. It will be a vector field associated to that, and that curve will be the solution to the systems. With these basic notions, we will see more and more examples, as we go along. What we have seen is only, a saddle point example. Now, we are going to see the entire analysis for the 2 by 2 systems which, we will do that. So, basically, we want to do a phase plane analysis of 2-dimensional system; that is what we are going to do now.

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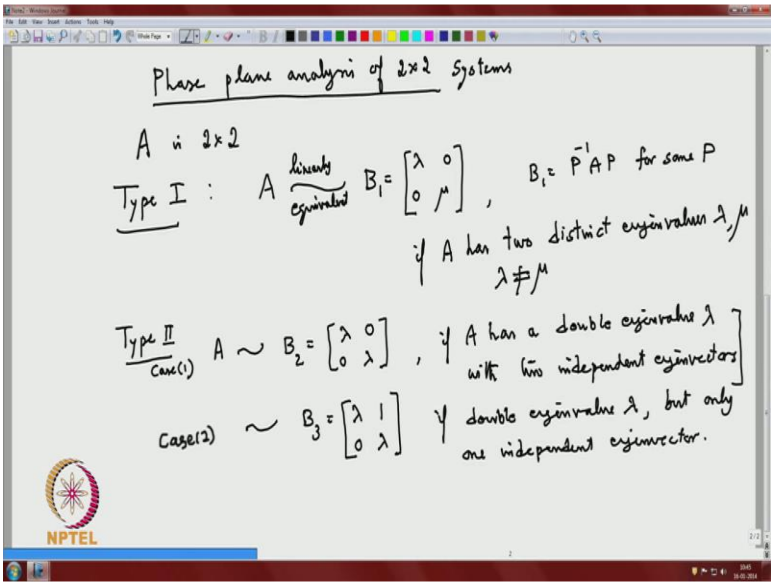

So, this phase plane analysis of 2 by 2 systems; you have some terminologies of phase plane and you are going to see, not only here, even, there other module of non-linear systems. So, what are the things? Now, we want to recall from the linear algebra. We want to know if a is 2 by 2; we want to know what are the possible 2 by 2 linear equivalent matrices. Typically, this can be classified into 3 different categories. What I am saying that linear algebra tells you that every 2 by 2 matrix is linearly equivalent to one of the 3 categories, according to the existence, according to the Eigen values; whether, is a real Eigen value and distinct; is a real Eigen value, but multiple real coinciding Eigen value or the Eigen values are complex.

So, I am going to write the 3 things which, I will not do it. Probably, some of you will have learnt in the basics. So, that is what is called the type 1; we want to call type 1. In this case, a will be; let me put a notation; this is for linear equivalence to b 1; b 1 is of the form, lambda, 0, 0, mu. When is there that this notation means that a, linearly equivalent; that is the meaning of this linearly equivalent; means B 1 can be written as p inverse of p. What that means; here B 1 is of the form, p inverse of a p, for some invertible matrix p or some p; that is the meaning of this. When does this happen? This happens when a has two distinct real Eigen values. If a has two distinct Eigen values, lambda, mu; that means, lambda not equal to mu; this is the type 1 matrix. What are the matrix p, in this case, is easy; you look for since, there are two distinct Eigen values; it has two Eigen vectors, which are independent. Put that Eigen vectors as column vectors of p, you get this thing.

So, p also, can be constructed by obtaining the Eigen vectors corresponding to lambda, mu, because it is independent; it is invertible. So, that is a case, type 1 situation. What is type 2? Type 2; a will be linearly; there are two cases in type 2 also. It can be of this form, lambda, 0, 0, lambda; there are two cases. So, in type 2, there is a case 1, and this happens, if a has a double Eigen value; means, Eigen value with multiplicity, algebraic multiplicity 2; double Eigen value 2; that is the meaning of double Eigen value; means, Eigen value with algebraic multiplicity to with; that is the thing. Now, when has an Eigen value, a coinciding Eigen value, there are two cases. There can be still, it can have two independent Eigen vectors or it can have only, one independent Eigen vector. That is the problem. When you have two distinct Eigen values, you have two distinct independent Eigen vector. When it is a coinciding Eigen value, it is depending on the matrix. You can have two independent Eigen vectors and that refers to as the geometric multiplicity.

The classification within this comes, because of this geometric multiplicity. So, this is the case where, it gives you two independent Eigen values. If it has two independent Eigen values and still, it forms a basis and you have the diagonalizability with two independent Eigen vectors. This will be equivalent to B 3. In that case, B 3 takes the form and non diagonalizable matrix. Diagonalizability, in general, is not possible; 0, lambda, lambda; again, double Eigen value with algebraic multi Eigen value, lambda, but only one independent Eigen vector, you see. The whole troubling diagonalizability is the lack of independent Eigen vectors; one independent; that means, this is the first case, you have algebraic multiplicity 2, geometric multiplicity 2.

In this second case, algebraic multiplicity 2, but the geometric multiplicity is 1, and hence, it will give only one column vector for p. The remaining vector 1, has to construct to find the equivalents. You want to write a is equal to 2, you have to find p all the time. In the other cases, getting p through eigenvectors; that is a concept of generalized Eigen vector, and that is the kind of the whole linear algebra and the Jordan decomposition theorem. In higher dimensions, there may be many complications. Some of the Eigen values are real; some are complex; some are multiplicity; there are certain things, are independent; certain things are simple Eigen value. So, all complications happens; you have to distinguish all that. That is why you do not get the full diagonalizability; you get block diagonalizability. Type 3 is the last case which, you know already, type 3.

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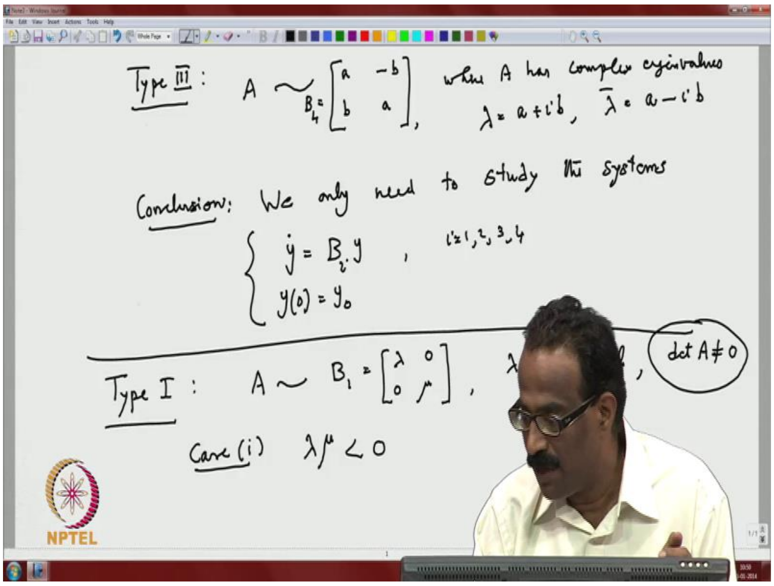

The type 3; this is the case. In this case, a will be linearly equivalent to a minus b, b a where, a has complex Eigen values, lambda is equal to a plus I b, and the other Eigen value, lambda bar, is equal to a minus I b. So, you have two Eigen values; a will be linearly equivalent to a minus b, b a. Let me call it B 4. Our aim now; if you want, as I said again, and again, the if you want to understand the linear system x dot equal to a x, especially, the nature and the behavior of solutions; it is enough to understand the systems, corresponding to b 1, b 2, b 3, b 4. Then, we can record everything. Therefore, in conclusion, we only need to study the systems y dot equal to b, corresponding to each of this for b I c; b I of y where, for all I equal to 1, 2, 3, 4, with some initial values say, y at 0 is equal to y naught.

Once we know that, we understand the nature and the corresponding a will have the same nature of that one, and then, there will be a coordinate change, which we will see; how these things are happening. So, we are going to study these one by one now. So, we going to do type 1 case now; we start with type 1. In type 1, a will be; let me recall again; will be equivalent to, b 1 is equal to lambda, 0, 0, mu. This is the case where, a has; again, recalling; a has two distinct Eigen values lambda, mu; not equal to mu, and real. So, we want to understand these things.

Now, again, we will split into various cases. Again, we are putting the various cases and we are assuming here also, determinant of whole analysis, right now, determinant of a equal to 0. If the determinant of a not equal to 0, lambda and mu cannot be so the only equilibrium point is the origin. No other, because of the invertibility of a. If one of them is 0, or two of them is 0, then there will be many equilibrium points, so that are all special degenerate cases; we will give one or two examples of degenerate cases, later. So, we want to understand the situation where, determinant of a naught is equal to 0. In the type 1, we have two classifications. Case 1; this is within type 1 where, lambda mu, the product lambda mu, less than 0. We will start with the examples. Then, you will see what happens. So, let us take, we have already seen one example.

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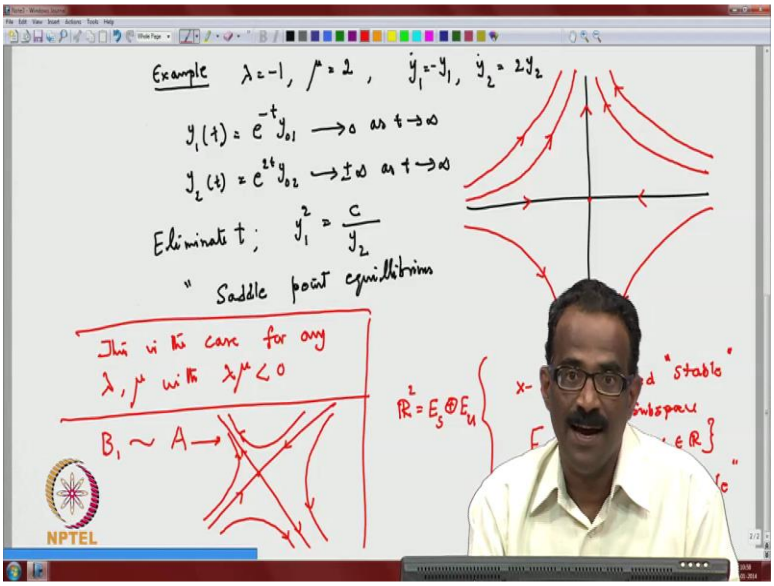

A similar example, we will try to see. Then, we will see that all will work the same way. So, we want to understand the phase portrait. We will take lambda is equal to say, minus 1. We already considered such an example, but let we do it in a slightly, sign change; how the arrows will change. Earlier, we considered a saddle point situation. This is a saddle point situation with 1 and minus 2. It will behave same thing. So, what is a corresponding system? You will have y 1 dot is equal to y 1; this, we already studied. So, there is nothing much to do it. Minus y 1 y 2 dot is equal to 2 y 2, you see, and you have your solution. I can immediately write the solution, y 1 t is equal to e power minus t, y naught 1 and y 2 t equal to e power 2 t, y naught 2. This goes to 0 as t tends to infinity. This will go to plus or minus infinity, depends of the sign of y naught 2; y naught is positive go to plus infinity; y naught to is negative, goes to minus infinity, as t tends to infinity, eliminate x t.

We have done exactly, the same thing; eliminate t. You will get it, you square it; you will get y 1 square is equal to some constant, with y naught by y naught, 1 by y naught 2, constant into y 2. So, this is called the saddle point equilibrium. Saddle point, which is an unstable equilibrium; saddle point equilibrium is always, an unstable. So, if you plot your curve here, is an exact thing; if you plot here, this is a equilibrium point and y 1 is always going to 0. So, these things are here; y 1 part. This will be here and y 2 go into infinity. So, it will be here; it will be here. So, if you plot your curves here now, this will go to, y 2 is going to infinity. So, it will be here. So, this is your phase portrait. If you plot here, this will be like this; this will be like this. So, you have only, certain cases, I will draw the entire thing. After that, you should plot it, accordingly. So, if you look at it, the x axis is called stable subspace. So, you will define E s, the stable subspace E s, set of all elements of the form x 0, with x in r, and y axis is unstable subspace, is called unstable subspace; that is called E u, is equal to set of all elements by the 0 y; y is equal to r.

This will give you, this together, will give you; your R 2 can be decomposed into E s plus; this is a general feature which, we will see later. Every R n for a given dynamical system; we can classify into stable, unstable plus one more thing will come, node, when you go to this thing. For this saddle point, this is the case. You can classify this thing. Now, you also know how to plot your thing; how to plot the vector field here. So, this is the phase portrait in this situation by taking lambda equal to minus 1, and mu equal to 2. What I am going to tell you is that this, from here, is not anything specific about lambda and mu; what you need is difference sign. So, this is the case for any lambda, mu, with lambda mu negative; that is what I am saying.

If lambda and mu have opposite signs; this is case 1 within type 1. So, whenever, you have two Eigen values, real Eigen values, this thing, but it will have different sign; you can do this one, instead of y 1 t power minus t y naught, you will get E power lambda t y naught 1 and y 2 t will be E power mu t, y naught 2. Then, if you eliminate t, you will have an expression of the form, some y 1 power alpha is equal to c by y to power. So, the trajectories will be the same; it will remain the same way. Only thing, depending on the sign, which one is negative, which one is positive; the arrows will change. You will have one directional arrows with lambda negative, and mu positive, and the arrows will change, if lambda negative and mu positive. The stable axis; either, x axis will be the stable in this case, and in that case, y access will be unstable, or you will have x axis unstable, and y axis stable. So, this gave the case for all type 1 with distinct Eigen values with different opposite sign, will have this this case. All these equilibrium points, in this situation of lambda mu negative, is called the saddle point equilibrium. So, the behavior will be the same as saddle point equilibrium.

One more point I want to remark here, before going to the next case; if you make this linear equivalence; if you have this particular matrix b 1, which has come from a linear equivalents a 1; I told you, this is just a change of coordinate system. So, instead of having this thing, corresponding to a, you will have new coordinate system, something like that. You may have another coordinate system; need not even be perpendicular, but again, this corresponding point origin, will still remain, because under linear equivalent, this only one equilibrium point, which is the origin and origin will still remain, because the equilibrium point in that case, and there will be a coordinate change, and your trajectories will still, be like that. You see the nature of the trajectories will be like saddle point equilibrium with appropriate t arrows. You have to put appropriate arrows, that is all. That will give you, depending on the sign and thing. So, here, and this will be in this direction, you see. That is what I said; there is no difference in the nature of this saddle point, nature of the equilibrium point. For all the matrices, all coming from b 1 or b 1 corresponding to the matrix a, with lambda mu less than 0, will have the same nature of this solution.

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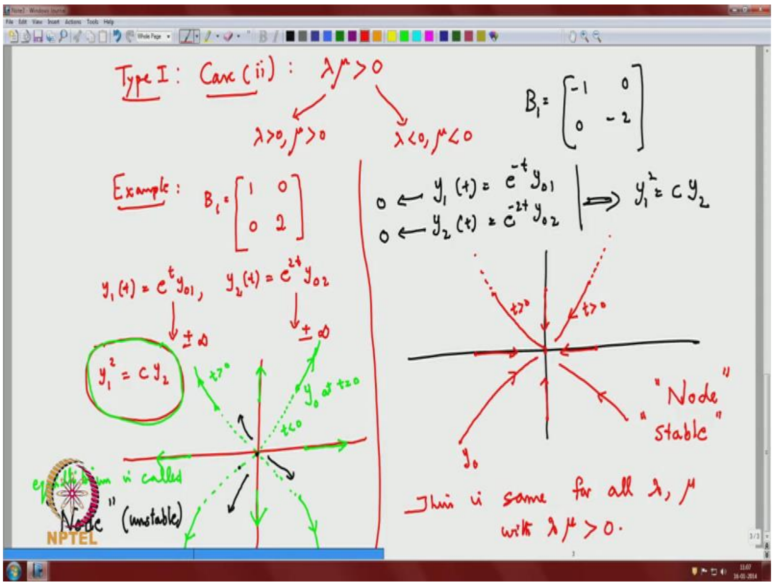

Now, go to the type 1, and type 1 again, with lambda mu; case 2, with lambda mu positive, you see, that is the thing. What we have seen is that with lambda mu, negative mu. Within this, there are two cases, you see. There are two cases with lambda positive, mu positive, and there is another case with lambda negative, mu negative, but to understand both these cases. The best way again is to take up the example; it will behave in the same way. So, let us take, start with an example. This is the case again, B 1. We start with the example. Then, you will see that there is no difference; example, you take B 1 to be of the form; it is all in the case of B 1, type 1. So, you take to be 1, 0, 0, 2, you take it; you want the case.

If you do these things, what will happen? If B 1 will have, you can write down your solution; your y 1 t will be e power t, y naught 1, and y to t will be e power 2 t into y naught 2. In either case, this will go to plus or minus infinity, because e power t minus, depending on the psi, this will also go to plus infinity. Now, if you eliminate here, your y 1 square will be constant into y 2; this is something like a parabola, you see. So, you have a parabola thing. So, if will plot this curve here, here again, look at it here. Let us look at our first equilibrium point. This is an equilibrium point. This is a decoupled system. So, if you start from here, it will remain in the same axis; it cannot move, because if it is in x axis, your y naught 2 will be 0 and y 2 t will be 0, and hence, it will remain in the x axis, but then, y 1 t goes to infinity, because y 1 t is equal to e power t into y naught 1. So, the arrows will move along this direction. If you move from here, it will move away from here. If you start from here, is a same situation, but both Eigen values are positive; it will move in this direction. In the saddle point equilibrium, if one axis moves away, the other axis moves towards it, but in this case, both moves here.

So, if you move here, it will be something like that. So, it will be like that. What about if you start a point here? If you start a point there, both y 1 and y 2 goes to infinity, but it moves according to this rule, and that something like a curve, which is something like a parabola. So, as t tends to infinity, it will move along this direction, and this portion will be the a t, because these solutions are defined even, for the minus infinity. So, it moves along these curves. So, if you start from here, it will again move here. This is for the t positive side, and this is for the t negative side. We are at this point, is your y naught, at t equal to 0; y naught at t equal to 0, you see. This is for the view part, t negative.

From here, if you move again, it will move along the parabola; this is the region. So, if you start from here, it will move. The parabola may change it. What I am trying to see that this is the same situation for any lambda mu positive; same case for any lambda positive. Only thing that e power t, you will have e power lambda t y naught 1, y 2 t will be e power mu t y naught 1. So, you may not get a relation; y 1 square is equal to c y 2, but you get a relation something, like y 1 power alpha is equal to c y power beta thing, but the curves may change. The behavior will be same. Such a point is called the equilibrium point, is called a node. Let me write down this one; node, and this is unstable. This is called an unstable node, because whatever be the solution, any point use that solution here; solution will move away. However, close it is; does not matter; it will move. Of course, the trajectories do not intersect. So, whatever be the close point, y naught closer to the origin; the solution will, trajectory will (( )), and this equilibrium is referred to as the node. What I am saying that this is a same situation with any lambda mu; for all lambda positive, mu positive, this curve will be, the behavior will be the same thing.

Now, let us look at this case. In this case, if you look at it, you start with the B 1 in this case. So, both with minus 1, 0, 0, minus 2; you want a system. So, the corresponding system you can write. You will have your y 1 t is equal to e power minus t y naught 1, y 2 t is equal to e power minus 2 t y naught 1. So, if you eliminate, you will get again, y 1 square is equal to some constant into y 2. The only thing is that this will go to 0; both will go to 0 in this case. So, if you plot this curve, the curve will be the same. Say, you have your equilibrium point. In this case, if you start from here, only the arrows will change, because it is going to 0. If you start here, you will have here, yes. So, the curve from, if you start here, now this will come towards the origin, along that, you see. This is the negative part. So, this is the positive part now. So, if you start from here, it will be here; this is the t positive part; this is. So, if you start anything from here, from any point; does not matter. If you start from here, it will go to the; it will only go as t tends to infinity. It is the same thing; it will go. So, this is your phase portrait as in, and this is again, a situation of a node. In this case, we call it as stable node. You will have this stable node and you have; this is same for all lambda mu, with lambda mu.

Student: Sir, 1 second.

Prof: Yes.

Student: Yes sir, you can continue.

Prof: Yes.

So, what we have done is that in the type 1, it has two distinct Eigen values, lambda mu negative. You have a saddle point equilibrium, which is unstable; one trajectory goes to 0, and one trajectory goes to infinity or minus infinity. When it is lambda mu positive, either, both trajectories will move away from the equilibrium point, going to infinity, giving an unstable equilibrium. On the other hand, when lambda mu positive with both lambda and mu are negative; both converge as negative, then the solution trajectory will go to, in fact, it goes to the origin in an asymptotic way, giving a stable equilibrium.

We refer to this case with lambda mu positive, is an equilibrium point, is called a node, and your subspace in the unstable case, the whole r 2 is an unstable space. In the stable thing, whole r 2 is a stable case. Now, in the next class, we will continue with this for the other two cases of type 2 and type 3. What you are going to see is that type 2, more or less, behaves like in this fashion with slight change in the shape of the trajectory, but behavior will be like a node behavior, the type 3 case where, the Eigen values are complex; you will have a different behavior. You will also see some periodic behavior, which we are going to see anyway, in the non-linear systems, more on the periodic trajectories. With this, we will end this particular topic.

Thank you.