Ordinary Differential Equations
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Module - 4
Lecture - 23
Series Solution

Welcome back. In this lecture, we will discuss about the solution of variable coefficient differential equations. We have seen that if the differential equation is constant coefficient, whether it is of first order, second order, third order; then, the method of solution we have already seen in the previous lectures, that you find the characteristic equation. So, after being finding the roots, we formulate the solution and we get the general solution and particular solution.

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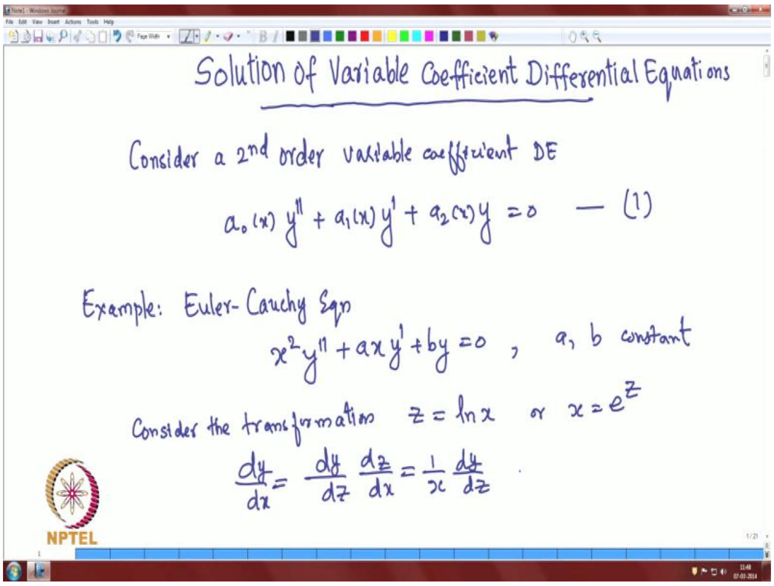

In case if the differential equation is having a variable coefficient then… In case, if the differential equation is having a variable coefficient, then that method fails. For example, if I consider a general differential equation, say a second order. So, consider a second order variable coefficient differential equation, say this is a 0 x, y double prime, plus a 1 x, y prime, plus a 2 x, y is equal to 0; call this equation as 1; then, how to solve it by the method of characteristic equations and finding the roots, will not work. But there are certain situations where, are special forms of this equation, which can be sole, just as how we did in a constant coefficient differential equation. For example, if we can reduce this variable coefficient differential equation into a constant coefficient differential equation, by giving some suitable transformation; for example, if we consider the Euler-Cauchy equation given by x square y double prime plus a x y prime plus b y is equal to 0 where, a and b are constants.

So, this equation, though, it is a variable coefficient; it is a nice form. This can be reduced into a constant coefficient differential equation, just by applying some transformation. So, consider the transformation; say z is equal to natural logarithm of x or x is equal to e to power z. So, you should use this transformation. Then, we are changing the independent variable from x to e z. So, if you do this transformation, then what is d y by d x; d y by d x is given by the chain rule; d y by d z into d z by d x, which is equal to by using the transformation 1 by x, d z by d x is l n x. So, d x by d z is 1 by x; 1 by x d y by d z. Therefore, differentiating a function with respect to x is equivalent to multiplying 1 by x, and taking the derivative of the function with respect to z; that becomes the rule.

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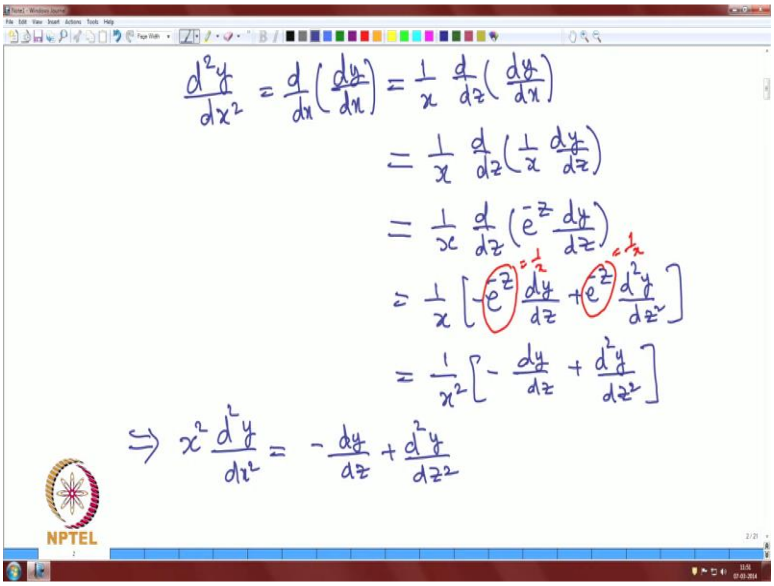

If this is the case, let me find the second derivative. So, d square y by d x square is nothing, but d by d x of d y by d x. So, we apply the rule, stated just above. That is differentiating a function with respect to x is equivalent to 1 by x into differentiating the function with respect to z. Therefore, we get the second derivative, d square y by d x square is equal to d by d x of d y by d x, which is equal to 1 by x times d by d z of d y by d x, which is again, by definition, 1 by x times d by d z of; what is d y by d x; d y by d x is 1 by x again; 1 by x d y by d z, which is equal to 1 by x times d by d z of; 1 by x is, by transformation, e to the power minus e z. So, d y by d z; now, we can differentiate with respect to d z. So, this gives me 1 by x into minus e to the power minus z.

Applying the product rule, d y by d z plus e to the power minus z times d square y by d z square. So, this is equal to, and this e to the power minus z is 1 by x, and this is also, 1 by x. You can take 1 by x outside. So, this gives me 1 by x square into minus d y by d z plus d square y by d z square. Therefore, the second derivative, d square y by d x square is 1 by x square into minus d y by d z plus d square y by d z square. So, this implies that x square into d square y by d x square is equal to minus d by d z plus d square y by d z square. Now, putting this into the Euler-Cauchy equation; therefore, the Euler-Cauchy equation becomes minus d y d z plus d square y, d z square plus a times d y by d z plus b y is equal to 0.

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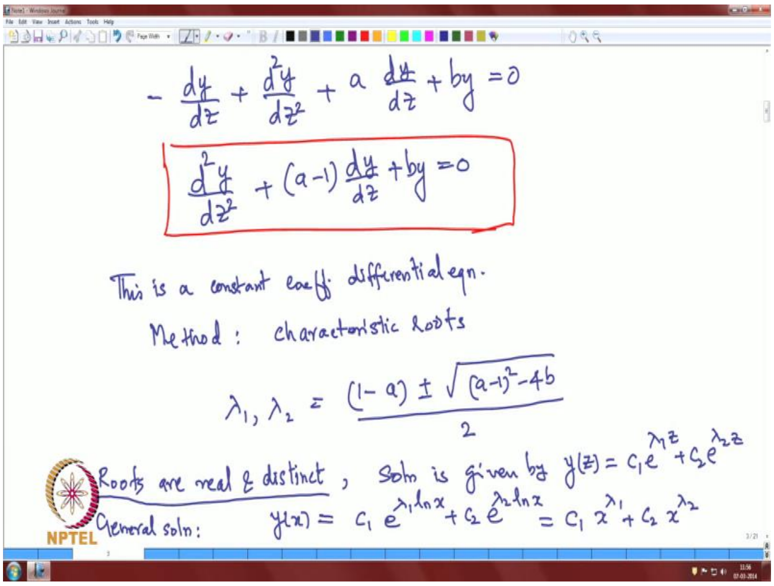

Therefore, if you simplify this, and taking d square y by d z square first, this is d square y by d z square plus a minus 1, d y by d z plus b y is equal to 0. Look at this equation. This equation is a second order constant coefficient differential equation. So, we started with the Euler-Cauchy variable coefficient differential equation. The transformation reduced this into a constant coefficient differential equation, of course, is second order. This equation can be solved. Now, we are taking the characteristic route. So, it is a constant coefficient differential equation. The method is characteristic roots; find the characteristic roots, and three situations where, the roots are real and distinct; the roots are real and equal; and roots are complex, which we have already seen.

Say for example, if we take the characteristic roots, the roots are given by lambda 1. Lambda 2 is equal to 1 minus a plus or minus square root of a minus 1 the whole square minus 4 b, all divided by 2; the discriminant of the equation, b square minus 4 a c by 2. So, if the roots are real and distinct; call it then, lambda 1, lambda 2 are the real roots. Then, the solution is given by y of; remember, the independent variable is now changed to z; y of z is given by c 1 e to the power lambda 1 z plus c 2 e to the power lambda 2 z. Again, if I want to change it back in times of x to change the independent variable back to x, use the transformation. Therefore, y of x is equal to, you change z to x; that is c 1 e to the power lambda 1, and z is l n of x. So, l n of x plus c 2 e to the power lambda 2 l n of x. This is nothing but l n of e to the power lambda 1 l n of x is c 1, if we simplify it; is x to the power lambda 1 plus c 2 x to the power lambda 2. So, this is a general solution. If the roots are real and distinct, and similarly if the roots are real and repeated, and roots are complex, can be treated, similarly. This is a situation where, if the differential equations have a very special form.

And in case, now, we are going to deal with differential equation, which is in a general form; that is a 0 x y double prime plus a 1 x y prime plus a 2 x y is equal to 0. If this cannot be reduced into a constant coefficient equation and no other methods are available to solve it, then one method of solving is by using the power series solution; the series solution of this differential equation. So, let us see how to solve this by using a power series solution.

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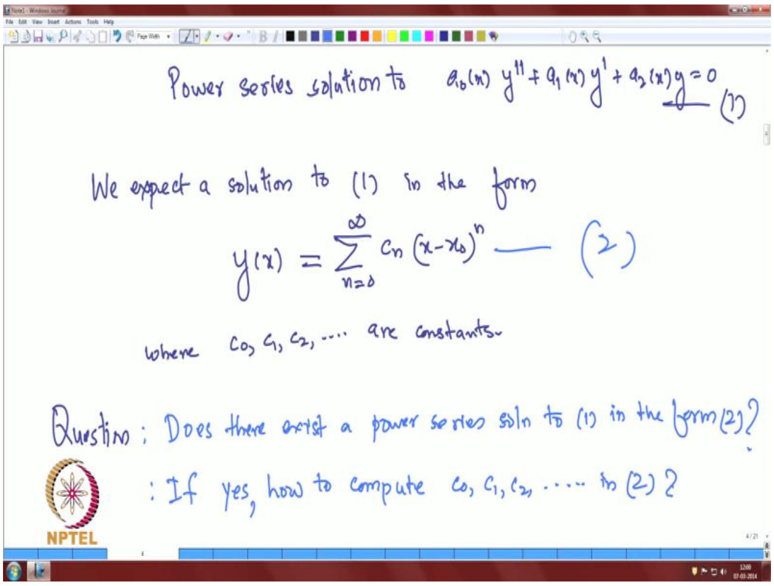

Power series solution to the equation a 0 x y double prime plus a 1 x y prime plus a 2 x y is equal to 0. What we expect is we except a solution to this equation; call it 1; in the form y of x is equal to summation n, goes from 0 to infinity; c n x minus x 0 to power n where, c 0, c 1, c 2, etcetera. are constants. So, we expect a solution to the variable coefficient differential equation 1, in a power series, from infinite power series, about a point x 0. Now, the question is does there exists a power series solution to 1 in the form 2? Call this form as 2. Does there exists a power series solution to the above differential equation in the form 2? If it exists how to compute the constants, c 0, c 1, c 2? If yes, how to compute c 0, c 1, c 2, etcetera. in 2? So, these are the two questions.

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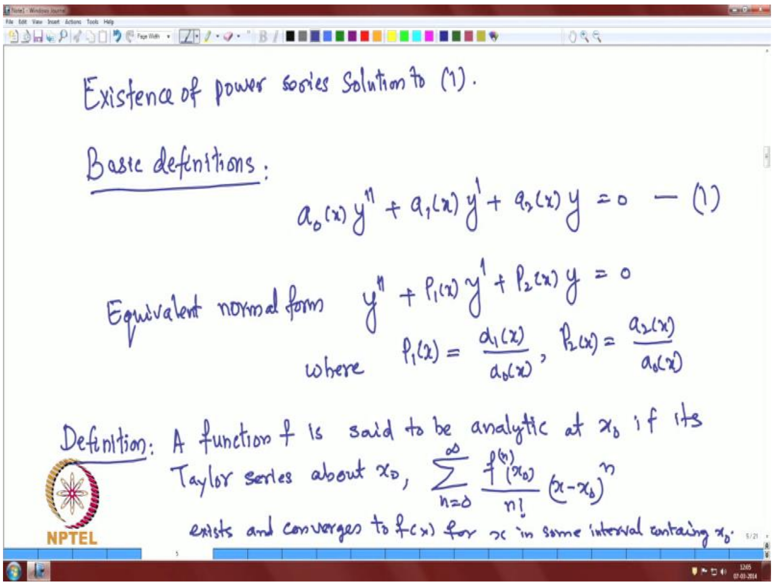

First, we address existence, existence of power series solution to 1. Under what condition, the power series solution for 1 exists, and that is series is convergent. To answer this question, we need some of the basic ideas, some basic definitions. So, to address existence problem; that is under what condition, 1 has a power series solution of the form 2, we require some conditions and we define. Let us again, state the form of this equation, a 0 x y double prime, plus a 1 x y prime, plus a 2 x y is equal to 0, and equivalent normal form is, we call this equation 1, and if we take the equivalent normal form, y double prime plus; p 1 x prime plus p 2 x y is equal to 0 where, divide throughout by a 0, where p 1 x is equal to a 1 x by a 0 x and p 2 x is a 2 x by a 0 x. Now, we define what you mean by an analytic function. Definition; a function f is said to be analytic, a real valued function, f is said to be analytic at a point x 0, if its Taylor series expansion about x 0, given by summation, n goes from 0 to infinity; the n th derivative of f evaluated at x 0, divided by n factorial into x minus x 0 to power n. The Taylor series about x 0 exists and converges to f x for all x in some neighborhood of x 0, containing x 0. So, in some neighborhood, in this case is interval; some interval; some interval containing x 0. In this case, we say that a function is analytic.

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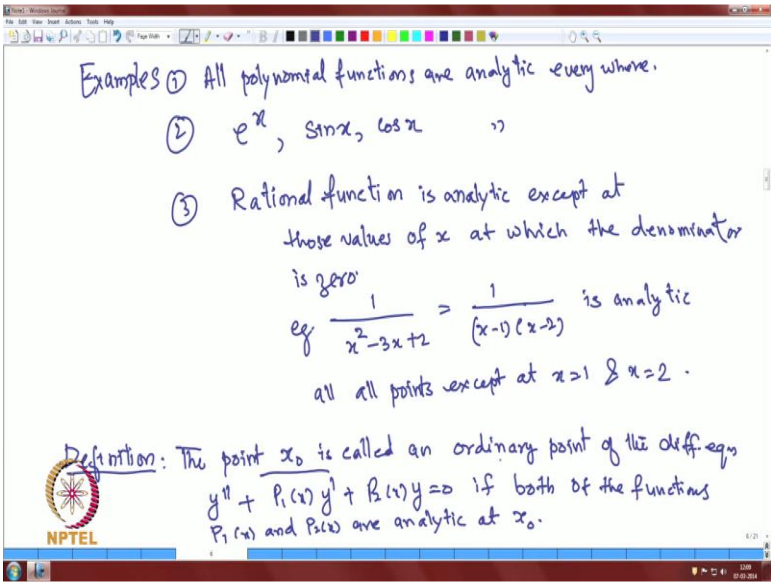

Examples; see, all polynomial functions are analytic, everywhere, and functions containing e to the power x, sine x, cosine x; they are also, analytic. Another important class of functions, analytic is all rational functions. Rational functions of the form p x by q x where, p x is a polynomial; q x is a polynomial. So, rational functions are also analytic at all point except, at the point at which, denominators are 0; except, at the point denominator is equal to 0. So, rational function is analytic except, at those values of x at which, the denominator is 0. For example, if we take a rational function 1 by x square minus 3 x plus 2, which is written in the form, x minus 1 into x minus 2. So, this is analytic at all points except, at x is equal to 1 and x is equal to 2. In these 2 points, x is equal to 1, and x is equal to 2; the rational function is not analytic; elsewhere, it is analytic. Now, a point is, another definition is ordinary point for a differential equation. So, definition; the point x 0 is called an ordinary point of the differential equation 1.

If differential equation 1 is in the normalized form; call it y double prime plus p 1 x y prime plus p 2 x y is equal to 0, if both of the functions p 1 x and p 2 x are analytic at x 0. A point x 0 is said to be an ordinary point of the differential equation in the normal form, y double prime plus p 1 x y prime plus p 2 x y, if both of the functions, p 1 x and p 2 x are analytic at x 0. If that fails, if it is not true, then we say the point is a singular point.

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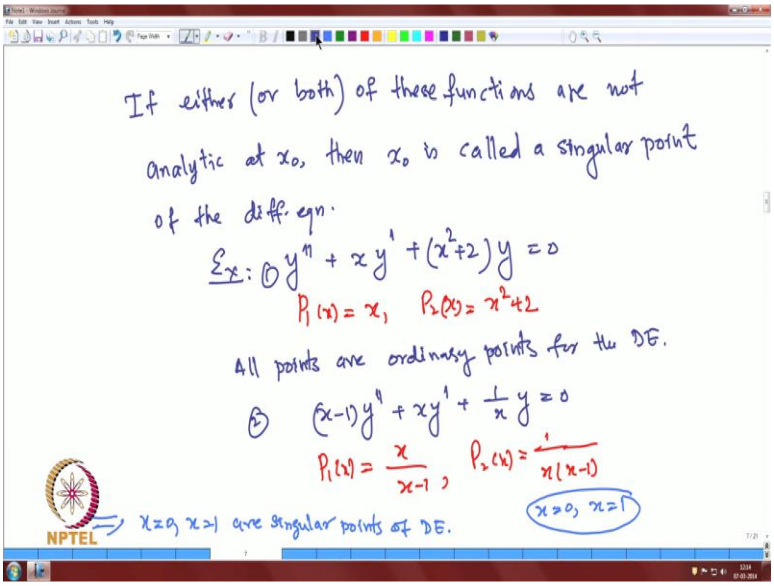

A singular point; if either or both of these functions are not analytic at x 0, then x 0 is called a singular point of the differential equation. See, quickly, let us look into an example; y double prime plus x y prime plus x square plus 2 y is equal to 0. Here, p 1 x is this x, and p 2 x is x square plus 2. So, obviously, both are analytic for all points. All points are ordinary points for this differential equation; all points are ordinary points. If we consider another differential equation x minus 1 y double prime plus x y prime plus 1 upon x y is equal to 0. Here, p 1 x is x upon x minus 1, and p 2 x is 1 by x into x minus 1. So, p 1 is analytic for all points except, at 1. So, for p 1, x is equal to 1, is a point at which, p 1 is not analytic. Here, x is equal to 0 and x is equal to 1, for p 2 is not analytic at these two points. Therefore, the differential equation is analytic for all points except, 1 and 0. So, the conclusion is 0 and 1 are the only singular points. So, this tells us x is equal to 0 and x is equal to 1 are singular points of three differential equations. Now, if x 0 is an ordinary point of a differential equation, then we have a sufficient condition to guarantee (( )) series solution. Now, I state the form of the theorem.

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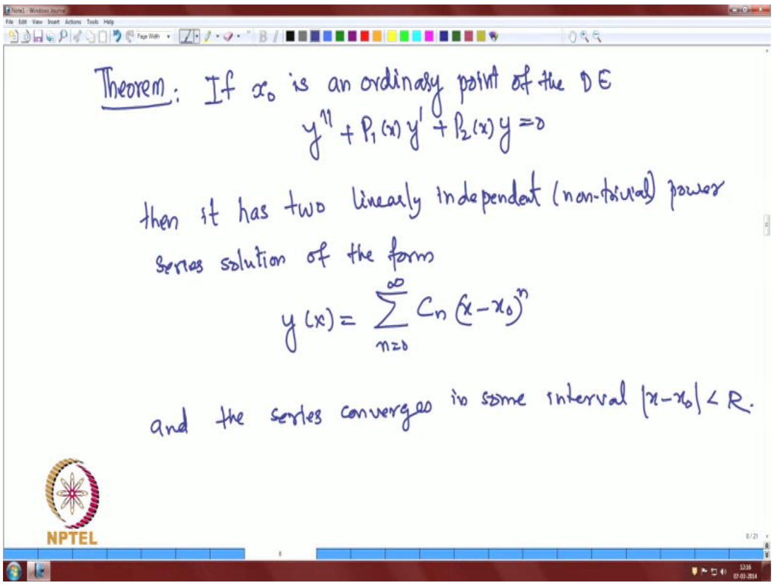

Theorem says if x 0 is an ordinary point of the differential equation; call it y double prime plus p 1 x y prime plus p 2 x y is equal to 0; then, it has two linearly independent non trivial power series solutions of the form, y x is equal to summation c n x minus x 0 to the power n; n goes from 0 to infinity; and the series converges in some interval; the interval of convergence; x minus x 0 less than some r. So, this is an important theorem, guarantees the existence of a power series solution to a differential equation. If x 0 is an ordinary point of a differential equation, then above that point, we can find if we have a second order equation, then we can find two linearly independent power series solutions. So, existence of two linearly independent power series solutions at an ordinary point is guaranteed by this theorem. Now, the method of solution; how to compute the solution?

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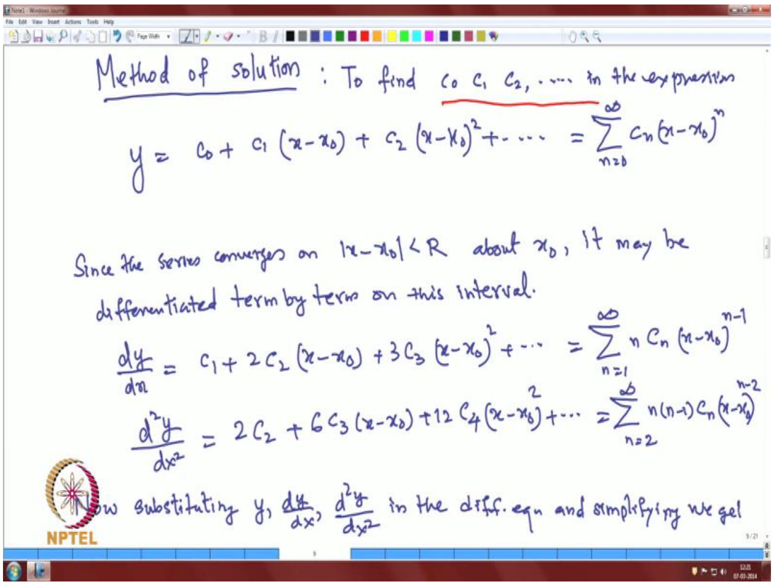

Method of solution is what we want to find? We want to find c 0, c 1, c 2, etcetera. all these coefficients in the expression y is equal to (( )) series solution is of the form c is c 0 plus c 1 x minus x 0 plus c 2 x minus x minus x 0 square plus, etc, which is in compact form; you write as c n x minus x 0 to the power n; n goes to 0 to infinity. So, our aim is to find these coefficients. Since, the series converges on x minus x 0 less than r, some number r, by the existence theorem, about the ordinary point x 0; the series can be differentiated; it may be differentiated term by term, on this interval; see, d y by d x is derivative of y, which is c 1 plus 2 c 2 x minus x 0 plus 3 c 3 x minus x 0 square plus, etcetera. which is summation; n goes from 1 to infinity; n c n x minus x 0 to the power n minus 1.

Similarly, second derivative, d square y by d x square, the derivative of the about series, which is 2 c 2 plus 6 c 3 into x minus x 0 plus 12 c 4 into x minus x 0 square plus, etcetera. which is written as summation; n is equal to 2 to infinity; n into n minus 1 into c n, x minus x 0 to the power n minus 2. Now, substituting these values of y d y by d d x d square y by d x square in the original equation; now, substituting y, d y by d x, d square by d x square in the differential equation and simplifying.

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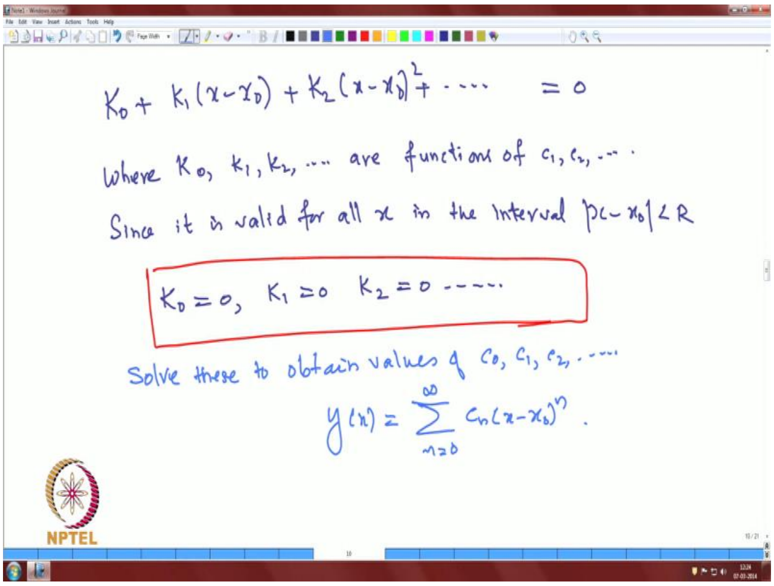

We get some constant k 0 plus k 1 into x minus x 0 plus k 2 into x minus x 0 square plus, etcetera. is equal to 0 where, these constants k 0, k 1, k 2; these are functions of our other constants c 1, c 2, etcetera. Since the series, it is valid for all x in the interval x minus x 0 less than r; that is interval of convergence. Now, equating right hand side and left hand side, the coefficients of x, x square, x cube, etcetera. And also, the constant terms; we get k 0 is equal to 0; k 1 is equal to 0; k 2 is equal to 0, etcetera. So, we get equations k 0 is equal to 0; k 1 is 0; k 2 is 0; if we solve these equations, then we get the values of c 0. So, solve these to obtain values of c 0, c 1, c 2, etcetera. Once we have c 0, c 1, c 2, etcetera. plug into the series form; we get the series solution. The series solution y x is equal to summation c n x minus x 0 to the power n. So, this is the method. Let us illustrate this by an example.

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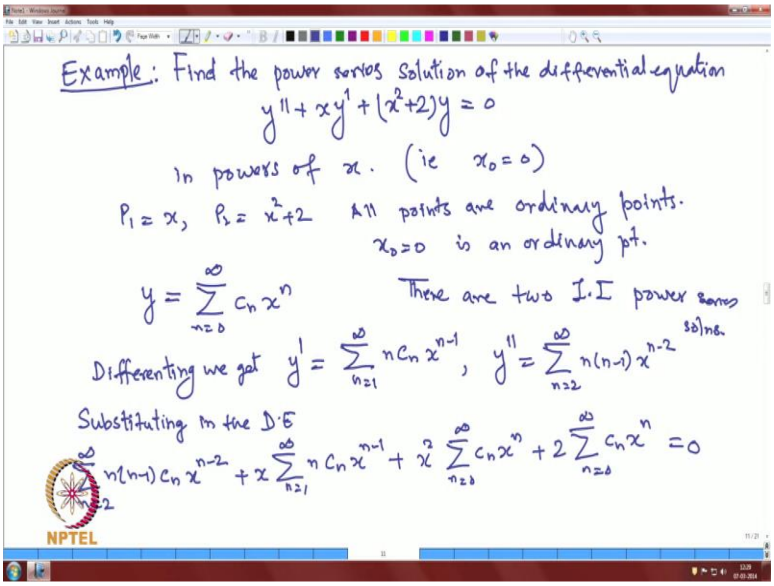

Let us consider an example. What is a problem? The problem is, the question is to find the power series solution of the differential equation given by y double prime plus x y prime plus x square plus 2 y is equal to 0. So, find the power series solution of the differential equation in powers of; the question is to expand or get the power series solution in powers of x. That is ask to expand the solution, get the solution in powers of x minus x 0 where, x 0 is 0. Thus, check whether, 0 is an ordinary point. Obviously, your p 1 is x, p 2 is x square plus 2. So, all points are ordinary points. So, x 0 is equal to 0, is an ordinary point. Therefore, by the existence theorem, there exists a power series solution, two linearly independent power series solution of the form summation c n x to the power n.

So, we look for a solution y x; y is equal to summation c n x to the power n; n goes from 0 to infinity. This is guaranteed by the existence theorem. So, there are two linearly independent power series solution. We want to do now; our aim is to find this assumes c 0, c 1, c 2, c n. So, what we do is by differentiating, we get y prime; y prime is summation; n goes from 1 to infinity, n into c n x to the power n minus 1. Similarly, y double prime is summation, is going from n is equal to 2 to infinity, n into n minus 1 x to the power n minus 2. Now, plug in these values to the given differential equation; substituting in the differential equation, we get summation n into n minus 1, c n x to the power n minus 2; the first term, that n goes from 2 to infinity; the first term, and the second term is x into y prime, which is x into summation; n goes from 1 to infinity; n into c n x to the power n minus 1. Now, the third term is the sum of two terms, x square plus 2. I take x square into summation into y, summation c n x to the power n; n goes from 0 to infinity, plus constant term 2; 2 into summation; n goes from 0 to infinity; c n x to the power n, which is equal to 0. Since x is independent of the index, we may rewrite this.

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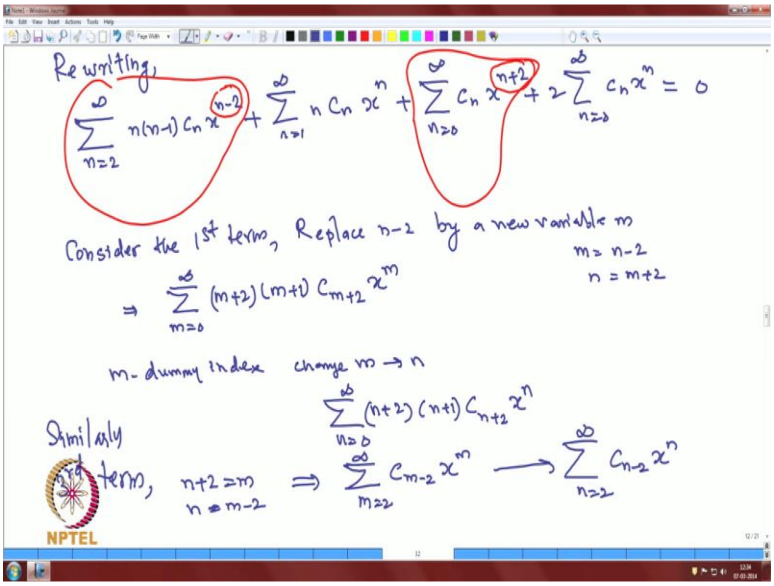

So, rewriting, we get summation n goes from 2 to infinity, n into n minus 1, c n x to the power n minus 2; the first term plus, summation n is equal to 1 to infinity, n into c n x to the power n plus, summation; if we have just multiplied each term by x, you get x to the power n, and n goes from 0 to infinity; the third term. There also, we did a multiplication by x square. So, the results c n into x to the power n plus 2, and multiplied by x square plus 2 times summation, n goes from 0 to infinity, c n x n is equal to 0. Now, look at the first and third term; this first term and third term. The first term, the index is n minus 2; the third term, the index is n plus 2. We want to make all the same uniform index that x n. So, we can do the following to rearrange that. Consider the first term, first summation and replace n minus 2 by a new variable, new dummy index, m. Therefore, what we have is m is equal to n minus 2 or n is equal to m plus 2. So, if we use this, then the summation becomes; then, this gives the first term in terms of m; m goes from 0 to infinity. Now, n is going from 2 to infinity; that becomes when n is 2, m is 0.

So, the summation goes from 0 to infinity, m plus 2 into m plus 1. So, m becomes m plus 2, and n minus 1 is n plus 1, into c m plus 2, into x to the power m. Now, remember, that m is just a dummy variable, dummy index. We can change m to n; change m to n notation. Therefore, it becomes summation n is equal to 0 to infinity, n plus 2 into n plus 1, into c n plus 2, x to the power n. Similarly, if you do the same thing for the third term, similarly, there we will make n plus 2 is equal to m or n is m minus 2. So, this gives in terms of m, this will be m is going from 2 to infinity; instead of n is going from 0 to infinity, and m is going from 2 to infinity, c m minus 2 and x to the power m. Again, that m is a dummy index. Therefore, we can plug back the n or we get this is n goes from 2 to infinity, c n minus 2, x to the power n. Now, the index of these terms are n with respect to m.

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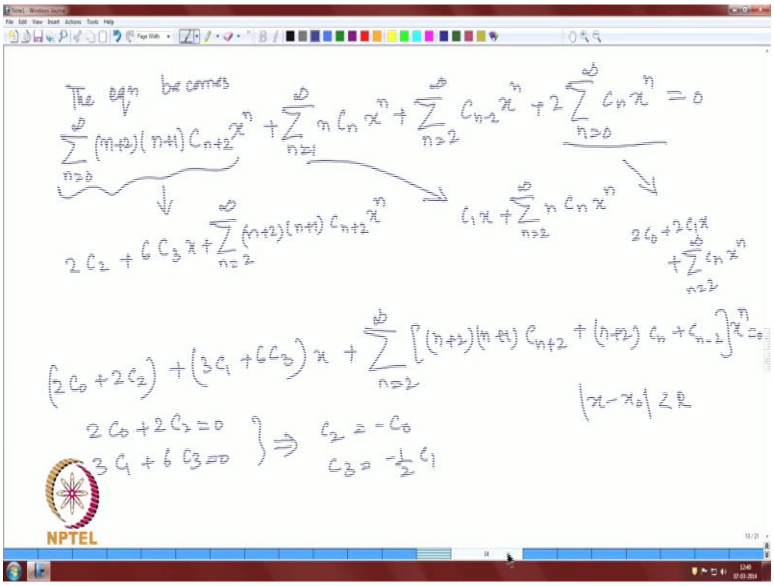

So, what we do is the equation now, becomes summation, n goes from 0 to infinity, n plus 2 into n plus 1, c n plus 2, x to the power n plus, summation n goes from 1 to infinity, n c n; there, we did not do any change; x n plus summation; the third term is also changed; n goes from 2 to infinity, c n minus 2, x n plus the last term, 2 times summation n goes from 0 to infinity, c n x to the power is equal to 0.

Now, we want to make some common range, since, this summation is not uniform; someone is starting from a 2; 2 infinity, another one is 1 to infinity; other one is 0 to infinity. The common summation range is 2 to infinity. So, the other terms, we can separate out. For example, the first term, first summation, the case n is equal to 0, and n is equal to 1, can be separated out. Therefore, what we get is this term, which is 2, c 1; 2 times, when n is equal to 0, get 2 c 2 plus 6 c 3 x plus, the common range, n is equal to 2 to infinity, n plus 2 into n plus 1 into c n plus 2 into x to the power n. Similarly, for the second term, that can be split as c 1 x plus, the common range, n is equal to 2 to infinity, n c n, x to the power n. Third term, you do not have to change. The last term, it can be changed; this is 2 c 0 plus 2 c 1 x plus, summation c n, x to the power n, n goes from 2 to infinity.

Now, if we rearrange all this, we get, this implies that 2 c 2 plus 6 c 3 plus summation, the first term; if we combine all this together, if we add them together, we get 2 c 0 plus 2 c 2 plus 3 c 1 plus 6 c 3 x plus, summation, the common summation, we will take n is equal to 2 to infinity, n plus 2 into n plus 1 c n plus 2 plus n plus 2 c n plus c n minus 2 times, x to the power n is equal to 0. This series converges for in the interval x minus x 0 is less than r. So, equating the coefficients of the powers, left hand side to 0, because right hand is already 0; we get 2 c 0 plus 2 c 2 is 0 and 3 c 1 plus 6 c 3 is 0, and this gives c 2 is minus c 0 and c 3 is minus half c 1.

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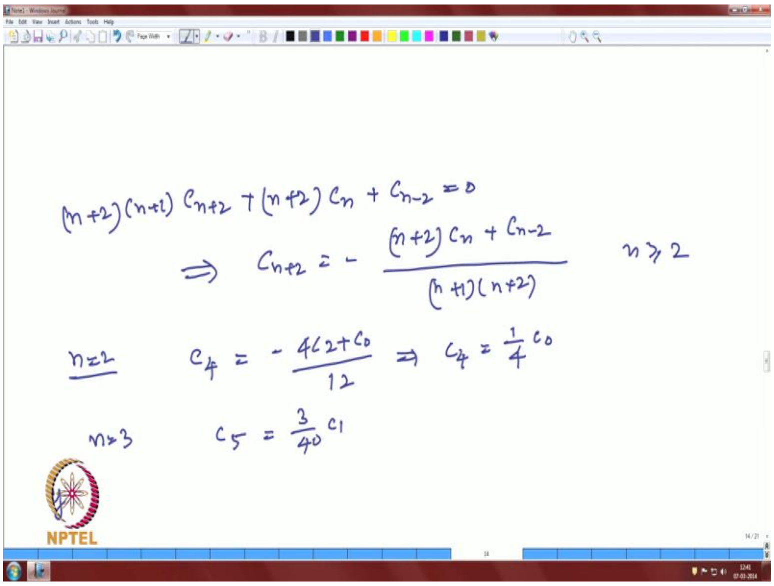

Also we get from the summation term that n plus 2 into n plus 1, c n plus 2 is equal to plus n plus 2 into c n plus c n minus 2 is equal to 0. If we solve it, c n plus 2 is obtained as minus, n plus 2, c n plus c n minus 2, divided by n plus 1 into n plus 2 for n greater than 2. Therefore, for each case, n is equal to 2 case, c 4 can be solved; c 4 is minus 4 c, 2 plus c 0 by 12. So, this implies that c 4 is 1 by 4 c 0, and similarly n is equal to 3 case; c 5 is told to be 3 by 40 c 1.

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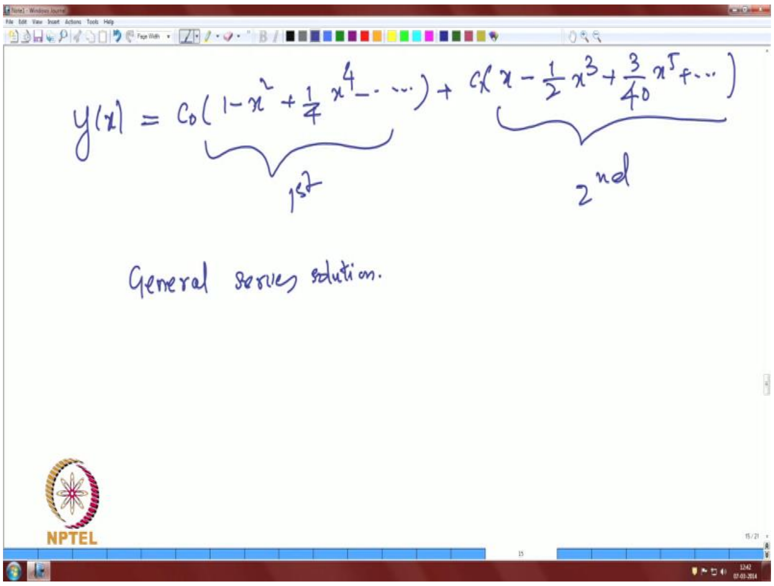

Therefore, the solution can be written as by using these coefficients. So, y of x is equal to c 0 into 1 minus x square plus, 1 by 4 x to the power 4 minus, etcetera. plus, c 1 into x minus half x cube plus, 3 by 40 x to the power 5 pus, etcetera. So, we see that there are two series. So, this is a two series solution. The first one and the second one; they are linearly independent two series solutions and they have the linear combinations, c 0 of the first one plus, c 1 of the second one; it is a general solution, say, general series solution.

Therefore, by doing this method, you can find if a point is an ordinary point, we can get the series solution and two linearly independent series solution of a differential equation, if the point is an ordinary point. If the point is a singular point, there are methods, Frobenius methods, and all, that will come in another series. So, with this, I would like to finish. So, we have seen even, if the differential equation is variable coefficient, we can have solution in power series form, series solution.

Bye.