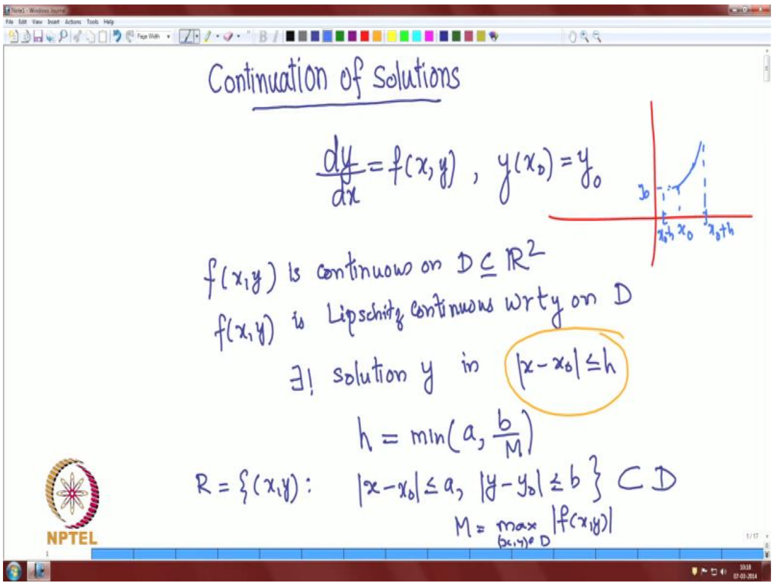
Ordinary Differential Equations
Prof. Raju K George
Department of Mathematics
Indian Institute of Science, Bangalore

Module - 4
Lecture - 22
Continuation of Solutions

Welcome back. We have been discussing about the various aspects of existence, uniqueness and stability of initial value problems. Today, in this lecture, we will see the possibility of continuing the solution.

(Refer Slide Time: 00:43)



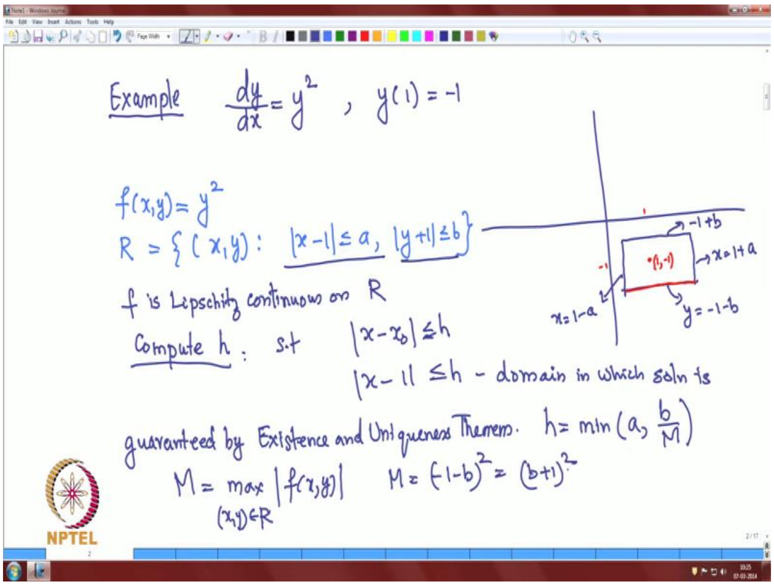
What we have discussed so far is given an initial value problem say d y by d x is equal to f of x y, and initial continuation y at x 0 is y 0, and the existence and uniqueness theorem; that gives sufficient conditions on f, say f is, f x y is continuous on some domain in r 2, and f is lipschitz continuous with respect to y on d.

Then, the existence and uniqueness theorem guarantees that there exists a solution. So, there exists a unique solution y in some interval x minus x 0, is less than or equal to h where, h is a small quantity, is a quantity defined by h; h is equal to minimum of a and b by m. What are these a and b? Here, a is defined; a and b are the parameters which, we use to define a rectangle inside the domain d. So, rectangle inside the domain d is defined by set of all x ys, such that x is less than or equal to say, x minus a; x minus initial point is less than equal to a; y minus y 0 is less than equal to b.

So, these are the parameters, a and b. They are chosen in such a way that this rectangle is inside the domain d, and m, where, m is a constant, which is the maximum of the modulus of the function value, when x y in d or in r. So, the existence and uniqueness theorem guarantees that there exists a solution. That solution is defined on an interval x minus x 0 is less than h. If h is a very small quantity, then the solution which we obtain is very; that is defined only on a small, very small interval; see, for x if you visualize it, say for this is your x 0 point, and this is y 0; you are looking for a solution and the solution is now defined, some small interval x 0 plus h and x 0 minus h. So, the question, natural question; what about the solution outside this interval to the right side of x 0 plus h and to the left hand side of x 0 minus h? Can we continue, can we extend the solution towards the right and towards the left? If it is possible, we say the solution can be continued, and if it can be continued indefinitely, that is for all x, then we say solution exists for, or a solution is defined for all x on the x axis.

But if the system is non-linear, then the existence of solution on the all real axis, on the all x axis, may not be possible always, but whereas, if the system is linear, then we may extend or continue the solution to the entire x axis. So, in this lecture, we are going to discuss about the possibility of continuing the solution, outside the domain of interval, which is outside the interval, which was guaranteed by the existence theorem. So, to get a feel of it, let us consider an example.

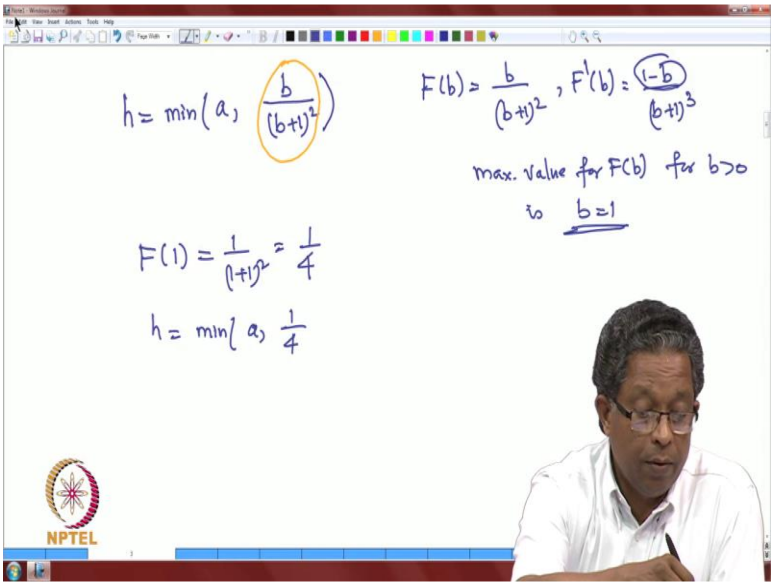
(Refer Slide Time: 06:17)



Take an example; that is a well known example, which we have already solved; see, example. Consider a differential equation, d y by d x is y square; it is a non-linear differential equation and the initial condition given is y at 1 is minus 1. So, this x y plane, the solution, the initial point y at 1; this is point 1; it is minus 1. This is the initial point 1 minus 1. Now, the existence theorem guarantees that there exits an interval on which, this equation has a unique solution. Let us verify these things. So, first of all, our f of x y in this example is y square, which is lipschitz on any bounded rectangles. So, let us define a rectangle r; r is given by set of whole x y point in the x y plane, such that x minus the initial point, x minus 1 is less than equal to a, and y minus minus 1, which is y plus 1, which is less than equal to b. So, obviously, f is lipschitz continuous on this rectangle. If it is lipschitz continuous on this rectangle and obviously, if it is continuous on r; therefore, the existence and the uniqueness theorem applies here, and it says, there exists a unique solution in your neighborhood, x minus x 0, which is x minus 1; it is less than equal to h.

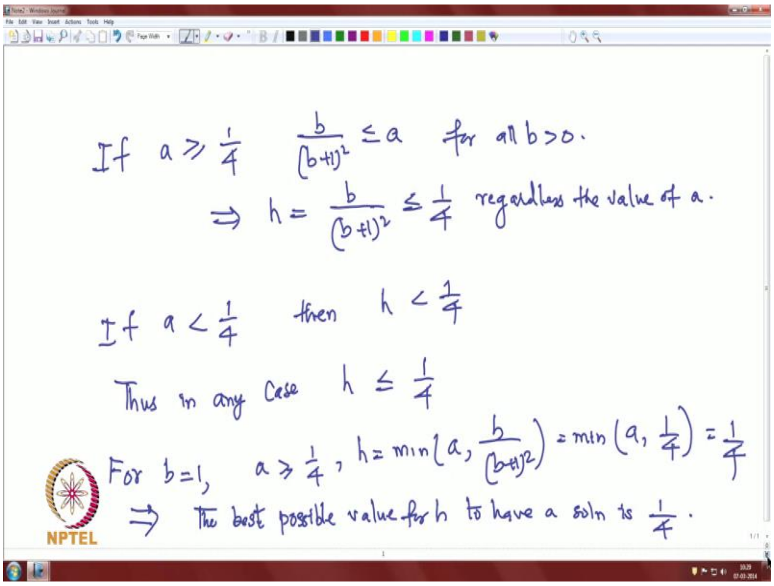
We are going to trying to find what is h; compute h, such that x minus x is less than equal to h. It is the domain on which, the solution is defined. In our case, x 0 is 1; that is x minus 1 is less than equal to h. It is a domain in which, solution is guaranteed by existence and uniqueness theorem; that is right. So, what is h; h is defined as h is minimum of a and b by m; a and b are as given in the definition of r, and m is maximum value of the function f of x y in the rectangle. So, x y belongs to the rectangle. Since, f is just y square; f x y is just y square; we know this f is y square and that you will have maximum value at this point. Let us plot this one. This line is a x is equal to x 0 plus h; that is x 0 plus h is 1 plus a, and this line is y is equal to minus 1 minus p, and this line is minus 1 plus p, and this line is x is equal to 1 minus a. We know that the maximum value of m is attained on these lines. So, maximum is, maximum value of y square is on this side, when y is equal to minus 1 minus b. So, m is minus 1 minus b square, which is same as b plus 1 square.

(Refer Slide Time: 12:43)



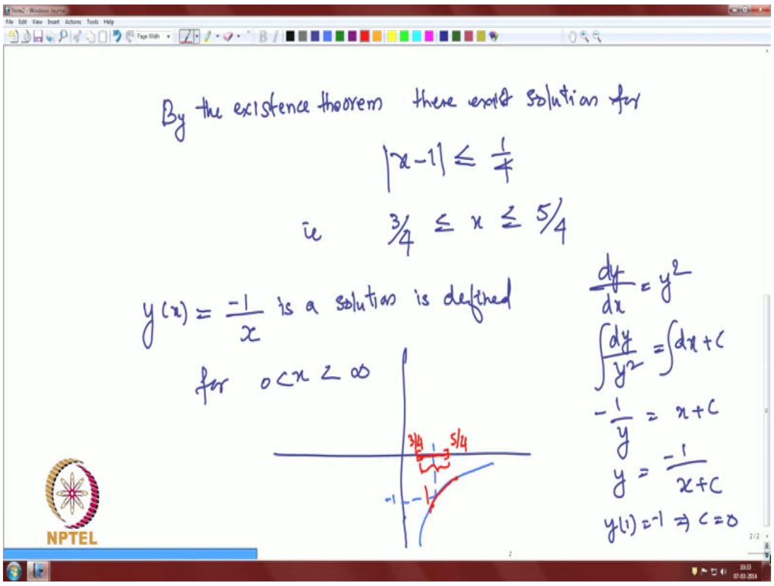
So, we are looking for h, which is a minimum of a and b by m; m, we got as b plus 1 the whole square. So, we want to know which is minimum. So, what is the minimum and maximum? Let us compute what is the value of this. The maximum value of f b, if I define, f of b is equal to b by b plus 1 square. Then, f prime b with respect to b, is 1 minus b by b plus 1 q. So, maximum value for f b for b greater than 0 is b is equal to 1. So, maximum is obtained at when b is equal to 1. So, f of 1, the maximum value of f of 1 is 1 by 1 plus 1 1 by 1 plus 1 square; it is 1 by 4, and h; therefore, h is minimum of a and the maximum value for b is 1 by 4.

(Refer Slide Time: 14:48)



Therefore, if we conclude, if a is greater than equal to 1 by 4, b by b plus 1 the whole square is less than equal to a, for all b positive. This implies that h is equal to b by b plus 1 the whole square, which is less than equal to 1 by 4; so, does not matter what value of a is; regardless the value of a. On the other hand, if a is strictly less than 1 by 4, then by definition, naturally h is strictly less than 4. Thus, in any case where, a is greater than 1 by 4 or a is less than 1 by 4, in any case, the value of h we obtained is 1 by 4. There is now, for b is equal to 1, a greater than equal to 1 by 4; gives h is equal to minimum of a comma b by b plus 1 the whole square, which is minimum of a 1 by 4, which is equal to 1 by 4. So, this the best, this tells us the best possible value of h, value for h to have a solution is 1 by 4.

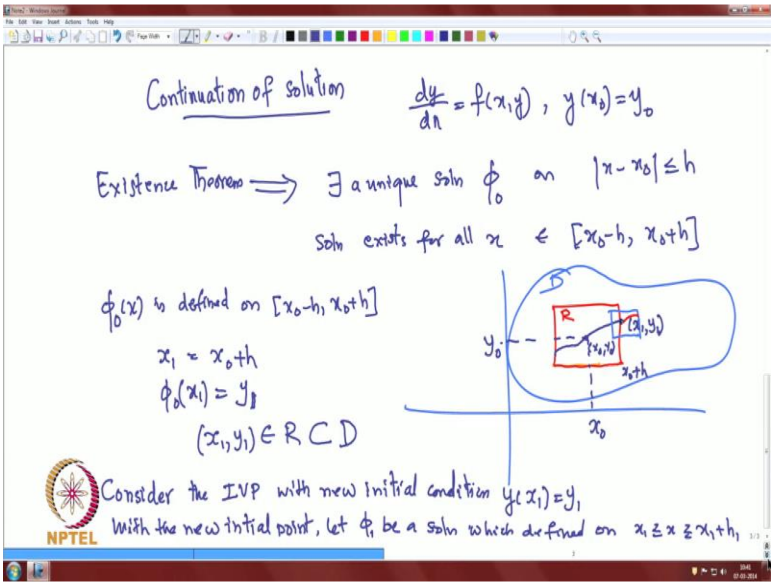
(Refer Slide Time: 17:39)



So, that tells us by the existence theorem, there exists a solution, in fact, unique solution for x minus 1 is less than equal to 1 by 4. There, the interval is 3 by 4 less than equal to x, less than equal to 5 by 4; it is the interval on which, we have a solution, but this non-linear initial value problem, we have solved by using analytical method, and we found that see; if you solve the equation d y by d x equal to y square; separating out the variable, y square is equal to d x, and integrating. This is 1 by minus 1 by y is equal to x plus c or y is equal to minus 1 by x plus c. Putting the initial condition, y at 1 is minus 1, gives us c is equal to 0. Therefore, the solution, a general solution which, we get by analytical method, is y x is equal to minus 1 by x, is a solution, but this solution, we know that this solution is defined for all x between 0 and infinity. If we look at the graph of it, the solution, the given point is 1. This is minus 1 and interval in which, we got solution is only this. So, 3 by 4 and this is 5 by 4. So, existence theorem guarantees that there is a solution only, on this region; that the solution is given by this portion.

But look at the solution, minus 1 by x, say is the solution is defined between 0 and infinity. Therefore, the question, natural question is can we continue the solution? The existence theorem guarantees that there is a solution in the interval, 3 by 4, 5 by 4; can extend the solution to the right and also, to the left to get a solution in a more large interval? The solution of the initial value problem, we want to continue to the right and to the left. This continuation, this process is known as a continuation of the solution. So, we deal with this one.

(Refer Slide Time: 21:26)

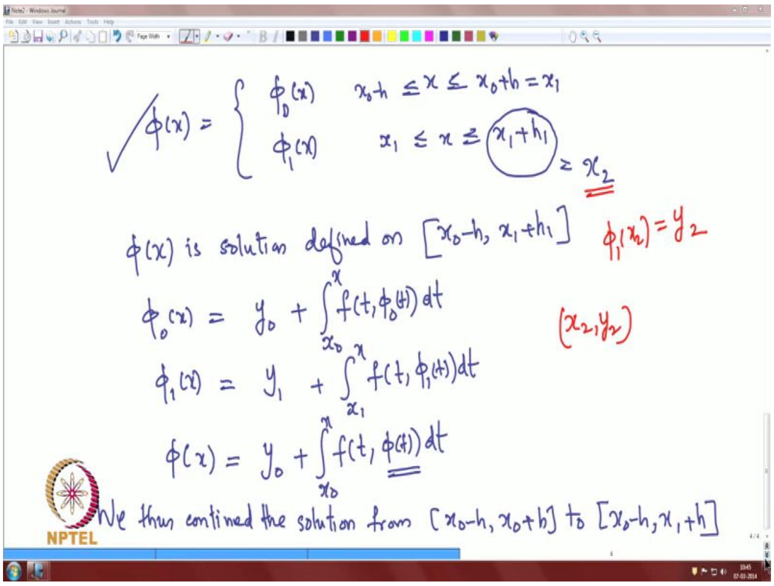


So, continuation of solution; consider the differential equation d y by d x is f of x y, and y at x 0, is the initial continuation, y 0, and existence theorem; this theorem implies that there exist a unique solution. Existence and uniqueness theorem implies that there exists a unique solution, y on x minus x 0 less than equal to h. Therefore, the solution exists for all x in the interval x 0 minus h to x 0 plus h. Now, the idea is to continue the solution to right and to left. So, we will deal with the continuation of solution to the right. The similar argument holds for continuation to the left. Graphically, we have the domain d, and inside the domain, we have a rectangle r, and the initial point is inside the rectangle. Existence theorem says that the solution exists to right and to left. This is x 0 plus h. So, solution exists; the point is x 0, and we have a solution exists. So, this is x 0 y 0; it is a point x 0 y 0.

So, existence theorem says there exists a unique solution y. Let me just call it, instead of y, it is a solution say, phi 0; phi 0 on this interval. So, phi 0 is defined, phi 0 x is defined on x 0 minus h and x 0 plus h, even at the end point also. At the end point, x 0 plus h and if I call, let x 1 is equal to x 0 plus h; that is this point; and the value of phi 0 at x 1; the end point, if you call as y 1. So, this point is x 1 y 1. So, x 1 is x 0 plus h and y 1 is the value of the solution at x 1. So, the point x 1 y 1 is a point, inside the rectangle and the rectangle is inside the given domain d, and the function satisfies all nice properties, like continuity and the lipschitz continuity, on the all domain d. Therefore, the question is now; can we start the solution? Can we take the differential equation with a new initial point? Consider the initial value problem with a new initial condition, y at x 1 is y 1. Therefore, we are trying to start a solution; we are going to start a solution from the new point, x x, you know x 1 y 1.

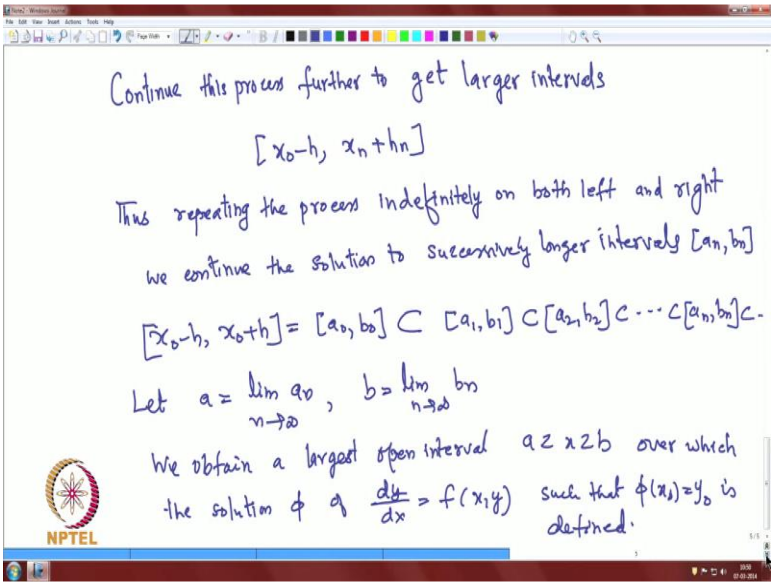
Now again, apply the existence theorem. That theorem says that we can find another rectangle, such that x 1 y 1 is a point, inside the rectangle and continue the solution or get a solution; so, a new point. So, this point is now, solution on the first interval is, we denote by phi 0, with a new point, with the new initial point. Let phi 1 be a solution; that is defined on x 1 less than equal to x less than equal to x 1, plus some h 1; h 1 is a similarly defined constant for the new rectangle. Now, because of the existence theorem, we could find another solution, phi 1, which starts from x 1 y 1, which was the end point of the previous solution, and it is extended up to x 1 plus h 1; that is the maximum interval, guaranteed by the existence theorem. So, we got 2 solutions, phi 0 and phi 1.

(Refer Slide Time: 29:12)



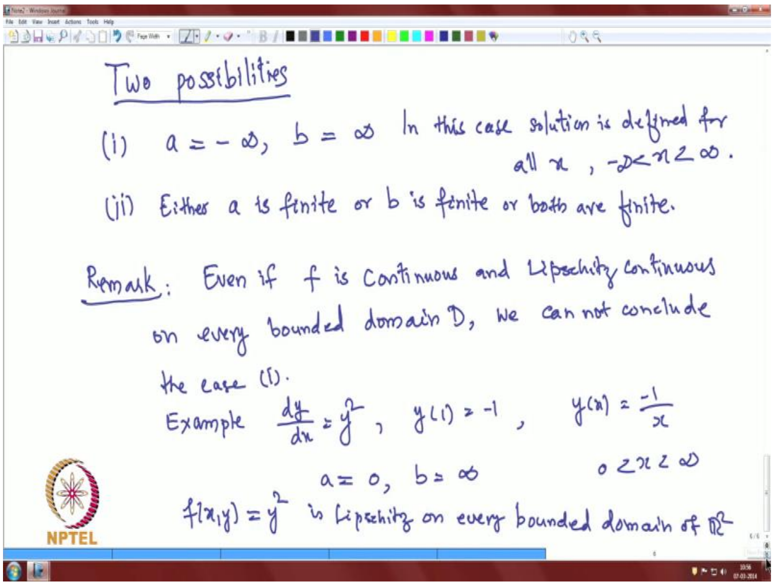
Therefore, we have two solutions, and if I combine these two solutions into one solution that phi x, I defined; phi x is a solution, which is phi 0 x for x varies from x 0 minus h and its upper limit is x 0 plus h, which we denoted by x 1. The second solution is phi 1 x. So, phi 1 x is defined from x 1 is less than equal to x, which is going to x 1 plus some h 1. Now, this combined solution, this is now, phi is a solution, defined on the interval x 0 minus h to x 1 plus h. If we write on the formula for phi 0, phi 0 x is equal to, by the basic lemma, y 0 plus; that was a initial condition; plus integral x 0 to x, f of t, phi 0 t, d t and phi 1 x is equal to, starts from y 1, plus integral x 1 to x, f of t, phi 1 t, d t; right, and phi t, if you combine these two, this is phi x is equal to the solution, starting from y 0 and y 0 plus integral x 0 to x, f of t, phi t, d t. So, this phi t is defined by phi x here. Therefore, we have extended the solution too. We thus, continued the solution from x 0 minus h to x 0 plus h to a larger interval, x 0 minus h to the x 1 plus h. Now, what we can do is we can repeat this process. Now, take x 1 plus h 1 as a new point. If I call this point as x 2 x 1 plus h 1 as a new point x 2, and the value of the function phi 1 at x 2, if I call as y 2; I got a new initial condition, x 2 y 2. Again, if x 2 y 2 is inside the domain and f of x 2 y 2 is defined, it has got all properties, since, f is defined on d. The conditions of the existence theorem are satisfied. Therefore, you can find an h. So, continue further to get the interval of definition, enlarged. So, we continue this process.

(Refer Slide Time: 33:27)



So, continue this process further, to get larger intervals; intervals like the end point is x 0 minus h; the right end point is x n plus h n where, x n and h n is obtained at each step. So, repeating this indefinitely; thus, repeating the process indefinitely, on both sides. So, what I have shown is only on the right hand side; the same process can be done on the left hand side. On both left and right, we continue to get the solution to successively, longer intervals. So, if I call, those intervals are a n and b n where, x 0 minus h, x 0 plus h is the one, which we started, which we denote by a 0 b 0, and the near one, a 1 b 1, continued on both sides, a 1 b 1; then, which is a subset of the next one, a 2 b 2 and so on to a n b n, and if I take the limit of these an s; let a is equal to limit an; n goes to infinity, and b is equal to limit; n goes to infinity, b n. Then, we obtain the largest open interval, a less than x, less than b over which, the solution phi of d y by d x is equal to f of x y, such that the initial condition, phi at x 0 is y 0, is defined. So, you can find such a larger interval, a less than x less than b on which the solution of this initial value problem is defined.

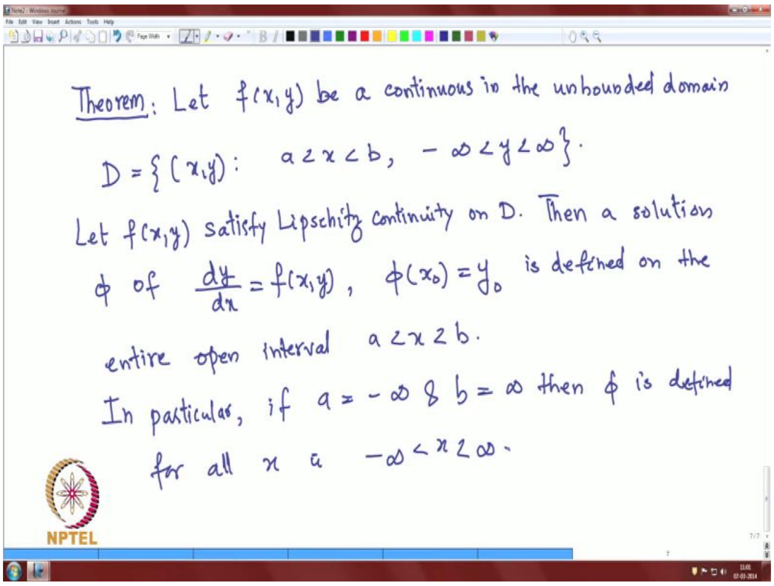
(Refer Slide Time: 38:10)



Now, in this process, there are two possibilities. One is this a is minus infinity, and b is plus infinity. In this case, we say solution is defined for all x; that is x is between minus infinity plus infinity. So, this is one of the possibilities that solution can be extended both to right and to left. So, that the entire the solution is defined on the entire x axis. Now, the second possibility is either, a is finite or b is finite, or both are finite. So, a is a finite number or b is a finite number or both are finite, or one is infinite, and the other one is finite. So, you remark that even, if f is continuous and it is lipschitz continuous on every bounded domain d, we cannot conclude the case 1, above. So, f could be continuous and lipschitz continuous on every bounded domain d. Still, we may not be able to conclude that the solution is defined for the entire x axis.

Taking an example, the same example of d y by d x is y square; the same example, if we if we choose with an initial condition say, y at 1 is minus 1. So, we have seen that the solution exists, y x, which is minus 1 by x. The existence theorem says that the solution exists only, in the interval 3 by 5 and 3 by 4 and 5 by 4, but we can now; further extend to left and right; that this solution is defined between 0 and infinity. So, what we have here, is there were two possibilities; a is equal to 0 and b is infinity. Remember, f of x y, which is equal to y square, is lipschitz on every bounded domain of r 2, but note that it is not lipschitz on an unbounded domain. If f is globally lipschitz, then there is a possibility of extending it to the entire; the solution can be extended to the entire x axis, but here, the non-linear function is not globally, lipschitz; it is only lipschitz on bounded domains. So, if it is a lipschitz on an unbounded domain, then we can expect larger open interval over which, the solution of the initial value problem is defined. So, we state this in the form of a theorem and which, we will now prove it. The proof follows, if we use a successive approximation method or theorem.

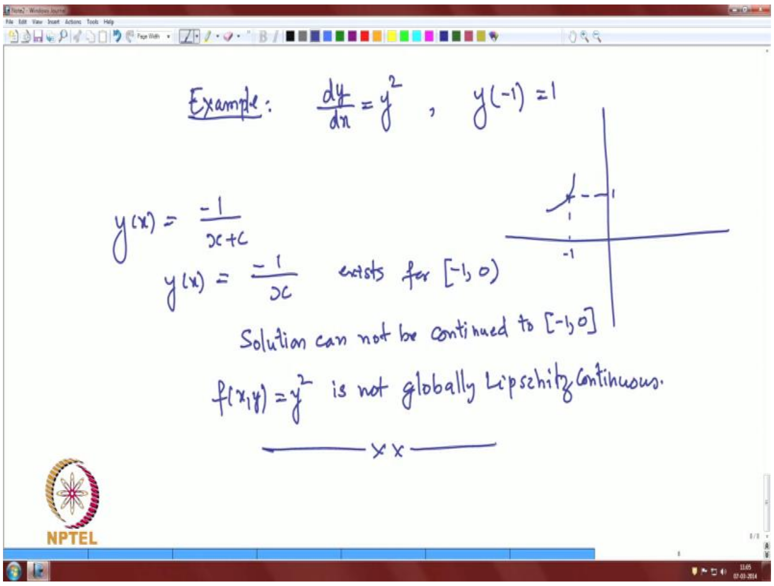
(Refer Slide Time: 44:13)



Let f x y be continuous in the unbounded domain; denote this by d, which is equal to set of whole x y in r 2, such that a is less than x, less than b; x is bounded between a and b and minus infinity less than y, less than infinity. It is an infinite strip where, x is varying between a and b, and y is varying in the entire y axis. Let f of x y satisfy lipschitz continuity on d. Then, a solution, call it phi; solution phi of d y by d x is equal to f of x y with phi at x 0 is y 0. The initial condition is defined on the entire open interval defined on the entire open interval, a less than x less than b. So, if this is the case when the function f is continuous and lipschitz continuous on a given bounded domain; domain is not bounded; bounded with respect to x and its lipschitz continuous on an unbounded domain of y axis, then the initial value problem has a solution on the entire open interval x between a and b, and in particular, if a is minus infinity and b is infinity, then the solution phi is defined for all x. That is x is varying between minus infinity plus infinity. Here, the solution exists globally. Therefore, if f is lipschitz on an unbounded domain, then one can expect a solution on the entire real x axis.

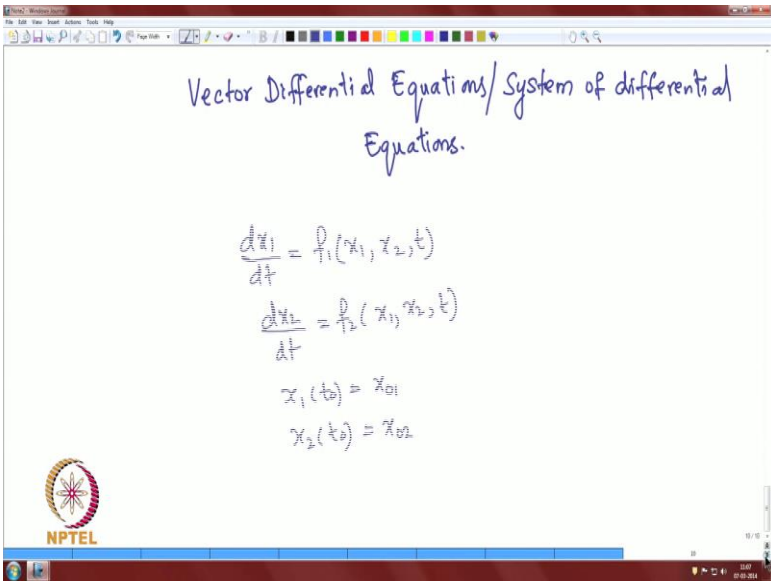
What we have discussed in this lecture so far, is the existence and uniqueness result, guarantees a small interval on which, a solution exists and the solution is unique, and we have seen that the solution can be extended towards the right and also, towards the left, provided the function f is good, smooth. If the function is lipschitz continuous, globally lipschitz continuous, then the solution can be extended further, on an open interval on which, the function is assumed to be continuous and lipschitz; otherwise, still, we have to have a a small interval x minus x 0 is less than h.

(Refer Slide Time: 50:10)



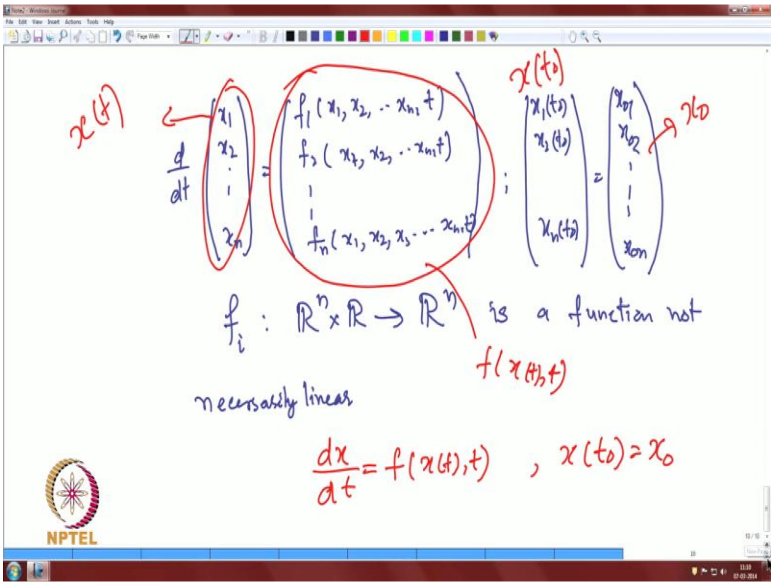
Let us look at one example. Consider an example; d y by d x is equal to y square; the same example. This time, the initial condition is y at minus 1 is 1. So, the initial condition is given y at minus 1 is 1. We know that the solution is y of x, which is equal to minus 1 by x plus c, and putting the initial condition; we get c is equal to 0 and the solution is y is equal to minus 1 by x. If you look at the interval, see, that solution exists; this solution exists for; if you try to continue the solution further, see this solution exists for minus 1 to 0. So, at 0, that blows up. Therefore, the solution cannot be continued beyond this point. So, up to 0, it cannot be extended; solution does not exist or cannot be continued to minus 1 0; it cannot include 0. That is natural, because it is not violating the previous theorem. Previous theorem says if the function is lipschitz continuous on the unbounded domains; globally lipschitz continuous, but here, if it is not f of x y, which is y square, is not globally lipschitz; it is not globally lipschitz continuous. Therefore, we cannot expect that the solution is, solution can be continued further. So, we stop the discussion of continuation of solution at this point. Now, we came back to a system of equation, come back to initial value problem, which are not scalar, but a vector differential equation. So, we spend some time in dealing with the existence of solution, existence and uniqueness or solution of vector differential equation.

(Refer Slide Time: 53:57)



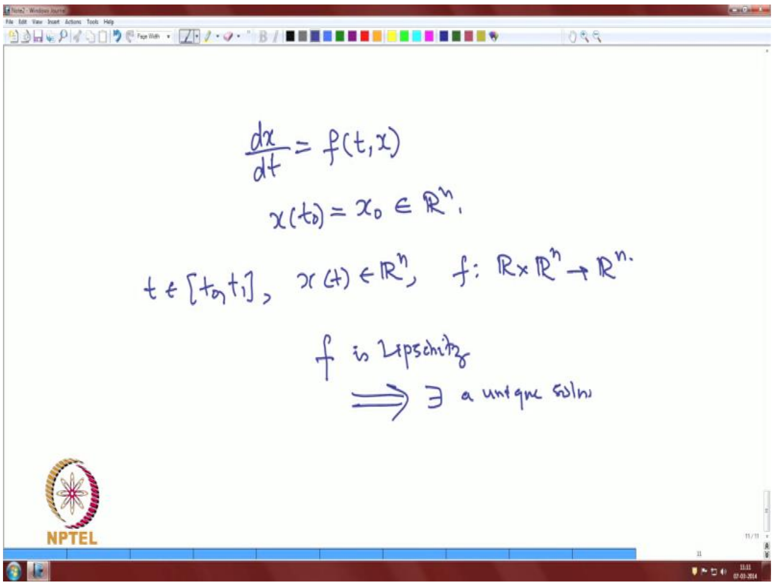
Vector differential equations or we say, a system of differential equations; say for example, if you have say, d x 1 by d t is equal to f 1 of x 1 x 2 and t; d x 2 by d t is equal to f 2 of x 1 x 2 t, with initial conditions, x 1 at t 0 is x 0 1; say, and x 2 at t 0 is x 0 2. See, you have two equations; coupled equations. Then, what can we say about the existence and uniqueness of solution of this equation? In general, you can have n equations.

(Refer Slide Time: 55:32)



If you have n equations, so, d x 1; you will write in a vector form. So, d by d t of x 1 x 2 up to x n, which is equal to f 1, x 1, x 2, up to x n t; f 2, x1, x 2, up to x n t, and f n x 1, x 2, x 3, up to x n t; we have n equations with the initial conditions, x 1 at t 0, x 2 at t 0 and so on, x n at t 0, is x 0 1, x 0 2, etc, x 0 n; where, each f I is a function from r n cross r to r n, is a function, not necessarily linear. If we denote this vector, this x y, just as x t and all this components, this into say, f of x t t, then this equation can be written in the form, d x by d t is equal to f of x t t, with initial condition, x at t 0 is x 0; x 0 is this quantity; this is x 0 and this is x at t 0. So, the question; under what condition, this initial value problem has a solution and solution is unique, and we have a stability of solution, etc; all these things can be similarly, studied along the same line.

(Refer Slide Time: 58:32)



See, here, the conditions are our system is d x by d t is equal to f of; let me write this t comma x; this is also a notation, conventionally; and x at t 0 is x 0. So, for each time point say, t 0 t 1 is a time interval. Then, x of t is in r n, and f is now, r cross r n to r n, and x 0 is in r n. We can show that if this function f is lipschitz continuous with respect to the second argument, f is lipschitz, then this implies that there exists a unique solution. So, if the vector f valued function, the vector function f is lipschitz with respect to the second argument x, then this system has a unique solution, and the solution can be continued, and the solution is stable with respect to the initial condition x 0. So, all these things can be studied and analyzed in a same manner, as we have done for scalar case. So, with this, we finish the existence, uniqueness, continuation and continuous dependence of solutions of initial value problem.

Thank you.