Ordinary Differential Equations
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Lecture - 20
Cauchy Peano Existence Theorem

Welcome back. We are trying to prove the Cauchy Peano existence theorem where we use only the continuity assumption on the function f.

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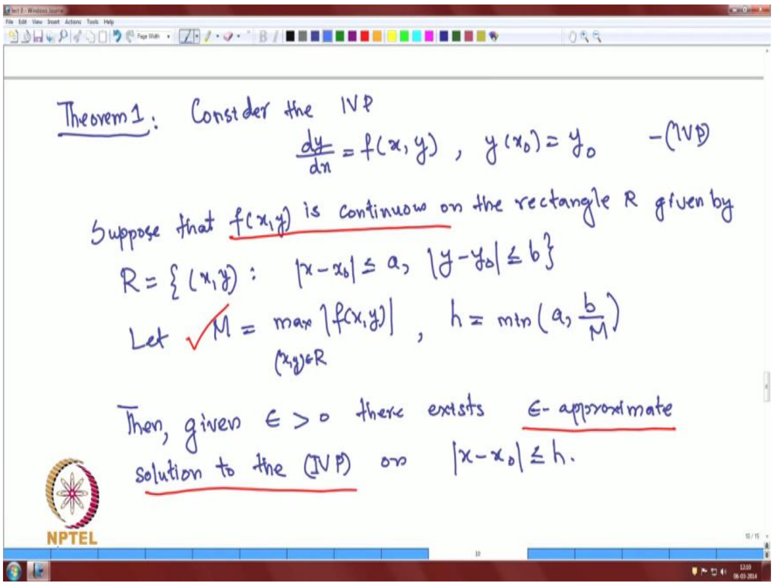

To prove the Peano theorem, first we prove the existence of an approximate solution to the initial value problem. Let us recall the theorem we have stated that is theorem one. Consider an initial value problem d y by d x with initial condition d y by d x is f of x y with initial condition y at x 0 is y 0, where the function f is continuous on a rectangle defined by x minus x 0 is less than equal to a and y minus y 0 is less than equal to b.

And we take some constants m is a maximum value of f in the rectangle; point h is minimum of a and b by m. Then the conclusion of the theorem is that for a given epsilon greater than 0, there exist an epsilon approximate solution to the initial value problem on the interval x minus x 0 less than equal to h. So, we will prove this theorem now.

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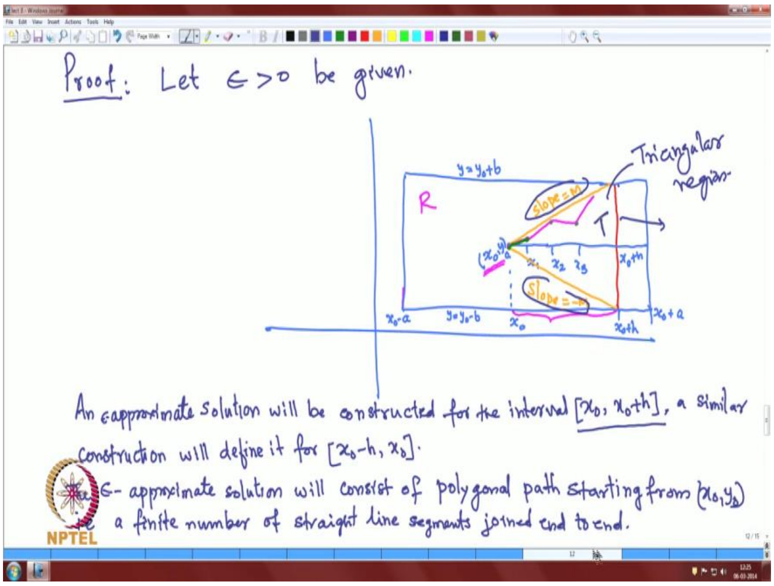

So, the proof is that for a given epsilon. So, let epsilon greater than 0 be given, and let us consider the rectangle r. So, this is a rectangle r we defined where this side is y 0 minus b, and this side is y 0 plus b. This side of the rectangle is x 0 plus x, and this side of the rectangle is x 0 minus a. And we are looking for an approximate solution on the interval x 0 to x 0 plus h; the given initial point is here x 0 y 0 is a given initial point. And now we make polygons and join them to get an approximate shape of the solution curve, and approximate solution will be constructed.

So, by approximate means epsilon approximate, epsilon is given. Solution will be constructed for the interval x 0, x 0 plus h. The right hand side to the given point x 0 and a similar construction will define it for the left hand side left interval; that is for x 0 minus h to x 0. We will prove the epsilon approximate solution through this interval x 0, x 0 plus h. The approximate solution will consist of the epsilon polygonal path starting from x 0, y 0 the initial point.

So, it starts from here this point x 0, y 0. There is a finite number of straight line segments joined end to end to form the polygonal path; that is a graph of the approximate solution. So, what we do is.

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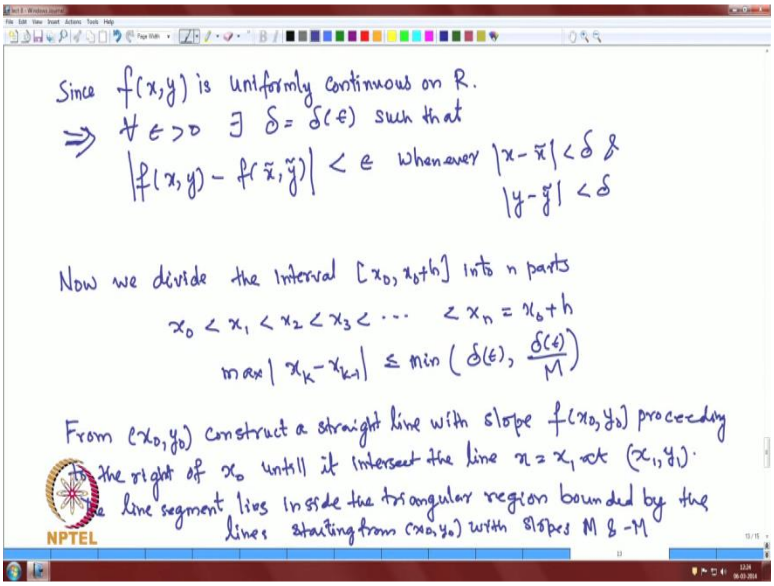

So, since f of x, y is uniformly continuous on the rectangle R; rectangle R is utmost unbounded certain R 2, and if it is continuous that gives a f is a uniformly continuous function on R. So, therefore, this implies that for every epsilon. So, we started with an epsilon for that epsilon greater than 0 there exist a delta. Of course, the delta will be a function of epsilon such that f of x, y minus f of, say, x tilde, y tilde, and this difference is less than epsilon whenever x minus x tilde is less than delta, and y minus y tilde is less than delta.

So, this follows from the continuity in particular uniform continuity f on R. Now what we do is now we divide the interval on which we are looking for a solution x 0, x 0 plus h; this interval we divide into n parts. Say, x 0 is a first point, then right to it x 1 less than x 2 less than x 3, etcetera, etcetera x n that is the last point which is x 0 plus h and make the maximum of x k minus x k minus 1 for any k, which is less than equal to minimum of delta which is a delta we define just above delta epsilon and delta epsilon by m. So, the procedure is as follows.

So, from x 0, y 0 the given initial point, construct a straight line with a slope f of x 0, y 0 which the slope we get from the differential equation. We are starting from the point x 0, y 0, and we know the slope of the solution at the point x 0, y 0 is f of x 0, y 0 from the differential equation as d y by d x is f of x, y. So, you construct a straight line with a slope f of x 0, y 0 proceeding to the right of x 0 until it intersect the line x is equal to x 1.

So, what we do is we start from the point x 0, y 0 and with slope f of z 0 y 0, you draw a straight line; let it touch the line x is equal to x 1 at this point.

So, we got a line segment having slope f of x 0, y 0, okay. So, let this line intersect the line x is equal to x 1 at some point, say, at the point x 1 y 1; x 1 is the point right to x 0, and y 1 is a point at which the straight line hits the line x is equal to x 1. Now this line segment lies inside the triangular region bounded by the lines starting from x 0, y 0 with slopes m is equal to slope m and minus m.

See this line segment which we just drawn from x 0 to its right of x 0 that is going up to x to 1, and this line segment is inside the triangular portion; this is a triangular region. This is bounded by the straight line having slope minus m and another straight line having slope minus m and also by the line x is equal to x 0 plus this line; this line is x 0 plus h. So, it is a triangle formed by three lines, and this post of the approximate solutions starting from x 0, y 0 lies inside this triangular portion.

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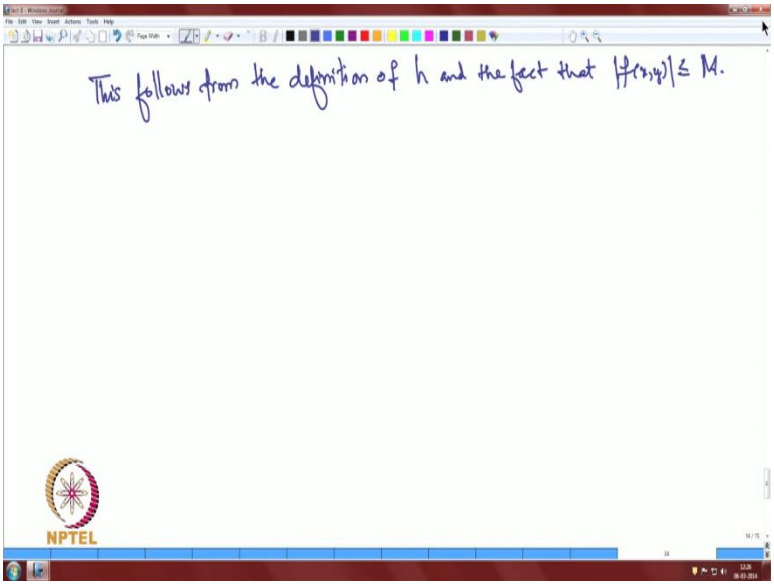

And what is the guarantee that this line segment will lie inside the triangular portion? So, this follows. So, from the definition of h which is minimum of a and b by m and the fact that the bound for f of x is h less than equal to M if it is bounded by M. Now we define solutions on each of these subintervals.

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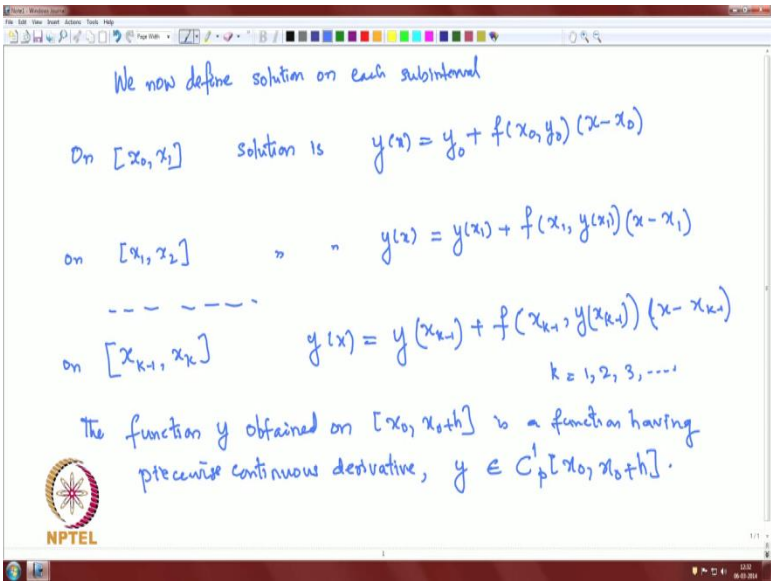

Now we define solutions. We now define approximate solution on each subinterval in the same way. So, in the first interval that is in the interval x 0, x 1 is a first interval. There the solution is y of x is equal to starting from y 0 plus slope is f of x 0, y 0, and x varies up to x 1. This is from x minus x 0; this is a straight line, equation of a straight line having slope f of x 0, y 0 and starting from the point x 0, y 0 this solution on the interval x 0, x 1 and similarly. So, on the interval x 1, x 2 the solution is y of x is equal to.

And there we start from the last point of the previous interval which is y at x 1, which we obtained from the solution of the first interval y at x 1 plus the next slope is f of x 1 and the point there is y of x 1. So, this is a slope and x minus x 1. So, this is a solution defined on it is a straight line starting from the last point x 1, y 1 that is x on y at x 1, and similarly if we keep on doing on the interval x k minus 1, x k on this interval. So, y x is given by the solution is given by y of x k minus 1 plus the slope at y k minus 1 f of y x k minus 1 into x minus x k x k minus 1.

So, this interval is a solution on any k'th interval of two points x k minus 1 x k, where k is equal to 1, 2, 3, etcetera. So, if you join all these solutions, these are the solutions straight lines we obtained on each of the interval, and if we join them and we get a function which is a having piecewise continuous derivative. So, the function y obtained on the interval x 0, x 0 plus h is a function having piecewise continuous derivative. So, that is y is an element of c 1 p x 0, x 0 plus h. So, in the property of approximate solution, we want the solution to be in c 1 p on those points at which the derivatives are defined.

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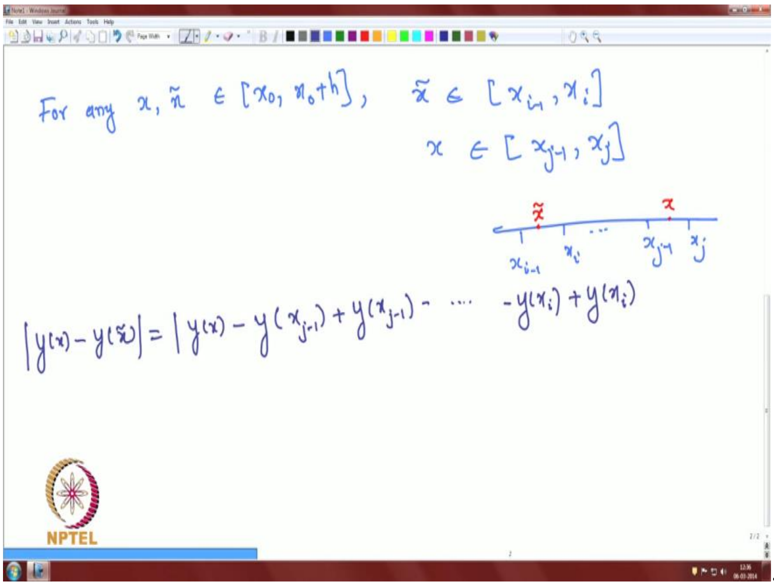

Now let us consider. So, for any x let us take two points x and x tilde in x 0, x 0 plus h to study further properties of y lets take two points x and x tilde in the interval x 0 and x 0 plus h; without laws of generality let us assume that x tilde is an element in one of the subintervals. Say, it is in x i minus 1 x i and x is in x j minus 1 x j and assume that this is x i minus 1; this is x i. So, this is x j minus 1; this is x j, and we take x from here, okay. So, x tilde is somewhere in the interval between x i minus 1 and x i, and x is from the interval x j minus 1 x j.

So, consider the difference y of x minus y of x tilde; okay, we can write this as y of x minus y of x j minus 1, I will rub out this. I will write it fresh. Y of x minus y of x tilde is equal to y of x minus y of x j minus 1 plus y of x j minus 1 adding and subtracting j minus 1 minus y of i plus y of x i minus y of.

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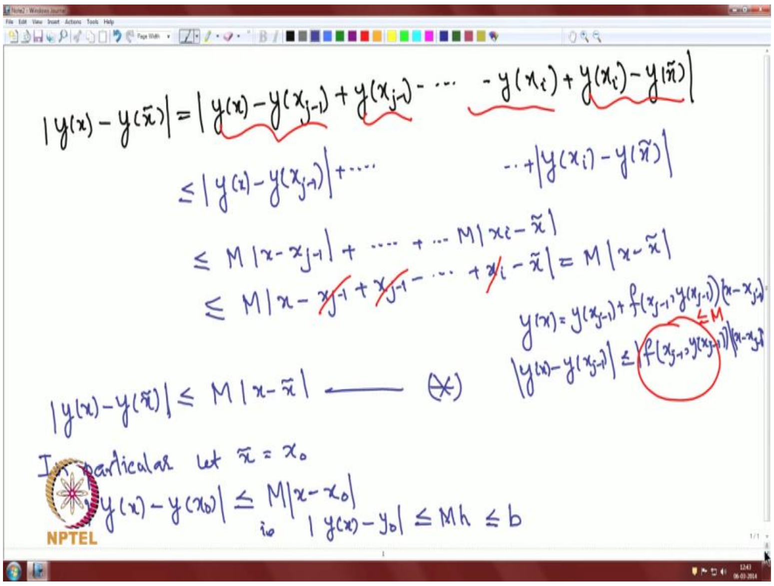

Let us consider y of x minus y of x tilde is equal to y of x minus y of x j minus 1 plus y of x j minus 1 minus etcetera minus y of x i plus y of x i minus y of x 2, if you group them. So, this is less than or equal to y of x minus y of j minus 1. So, remember y x j minus 1 is less than x plus etcetera plus y of x i minus y of x tilde. And if you look at the properties of the function y because y is a solution, and y is a solution satisfied by the formula given that is the equation of a straight line.

So, this is equal to less than or equal to m times x minus x j minus 1 plus etcetera. So, remember the form for y x; y x is equal to y of x k minus 1 or in this case j minus 1 plus f of x j minus 1 y x j minus 1 y of x is y of x j minus 1 plus f of slope is f of x j minus 1 y x j minus 1 into x minus x j minus 1, and we know that by definition these quantities. So, there is if you take the difference y x minus y x j minus 1 is less than equal to the slope x j minus 1 y x j minus 1 into x minus x j minus 1, and this quantity is bounded by this is bounded by m.

So, therefore, we get the first m is less than equal to m times x minus x j minus 1. And similarly, the other terms are coming as a solution from the each interval; I can compute it to get this is also m times x i minus x tilde. Now use a fact that this x 1, x 2, x 3, they are all in the increasing order. So, therefore, this absolute value we can get rid of that; this is less than equal to m times x minus x j minus 1 plus x j minus 1 minus etcetera plus x i minus x tilde, and we will be able to cancel out these terms to obtain m times absolute value of x minus x tilde.

So, what we have shown is the absolute value of y of x minus y of x tilde, where x and x tilde are from anywhere in the interval x 0, x 0 plus h is less than or equal to a constant m times x minus x tilde in particular, okay. So, now you call this inequality as star; we will be using it again. So, in particular let x tilde is equal to x 0 the first initial point x 0, then what we have is y of x minus y of x 0 which is less than equal to m times x minus x 0.

So, there is absolute value of y of x minus y of x 0 is y 0 is less than equal to m times x minus x 0 is bounded by h and which is again bounded by b by definition of m; by definition of h this is bounded by m. So, therefore, what is the conclusion?

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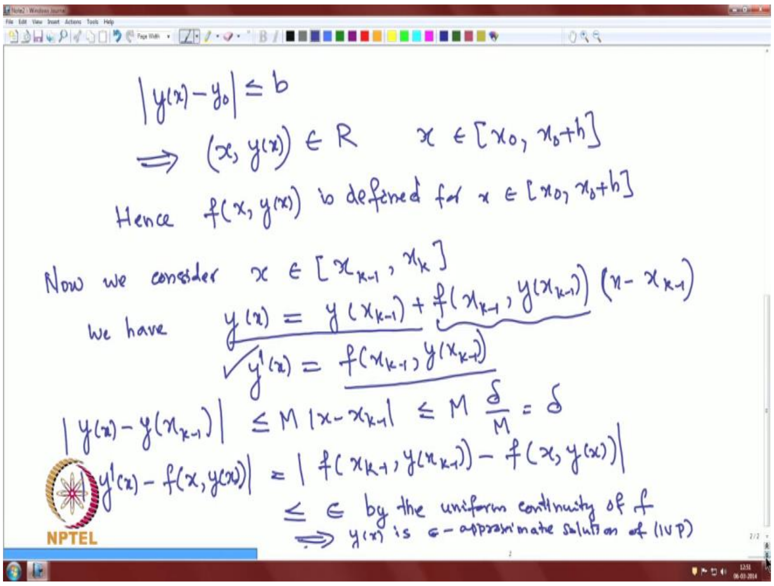

The conclusion is so we obtained y x for any x minus y 0 is less than equal to b. So, therefore, this implies that x, y of x, this is a point inside r for x in the interval x 0, x 0 plus h. So, hence f of x, y of x is defined; y of x is an approximate solution we defined we constructed and f of x, y x is defined for all x for x in the interval x 0, x 0 plus h. Now we consider an x in the interval, say, x k minus 1, x k a particular interval.

So, also we know that in this interval the solution y x is given by, okay. So, we have this interval; the solution is y x is equal to y of x k minus 1 plus f of x k minus 1 the slope f of x k minus 1 times x minus x k minus 1. This is an equation of the straight line or the solution defined in the interval x k minus 1 x k; obviously, f of x is a slope is f of x k minus 1 y of x k minus 1 which we will use later. So, let us see what is y of x minus y of x k minus 1 here.

Y of x the solution minus y of x k minus 1 the solution at the starting point. So, this is less than or equal to if you make use of this equation y of x minus y of x k minus 1 is bounded by the bound of f which is bounded by m. So, this is m times x minus x k minus 1, which is of course less than by definition of our partition m into delta by m which is delta. So, that delta is a function of epsilon. Now if you take the difference of the derivative y prime x minus f of x y x.

So, in this interval if you find the difference of this solution, the derivative of the solution with a right hand side of the differential equation y prime x minus f of x y x. In fact, this is going to be the error, the difference between the solution, which is equal to we have seen that y prime x is f of x k y x k minus 1. So, this is f of x k minus 1 y x k minus 1 minus f of x y of x and from the uniform continuity of f we have seen that. So, this is less than or equal to epsilon by the uniform continuity of f.

So, whenever x minus x k k minus 1 and x the difference between x k minus 1 x is less than delta and y x k minus 1 and y x delta is less than delta; this difference is less than epsilon by the uniform continuity of f. So, what does it say? So, this implies that if y now satisfies all the properties of an epsilon approximate solution. So, this implies that y x. So, y x is an epsilon approximate solution. So, y is in c 1 p and y prime minus y it is less than epsilon for all points at which the derivatives are defined.

So, that is not satisfied only at a finite number of points at which the derivatives having gen type discontinuity. Now with this background we are going to prove the celebrated theorem of Cauchy and Peano; so, Cauchy Peano theorem on the existence of solution.

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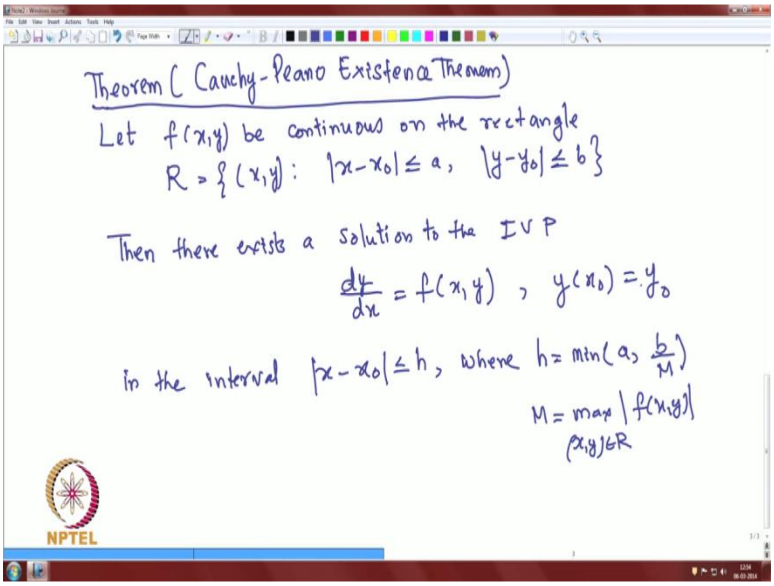

So, theorem Cauchy Peano existence theorem; so, in the previous theorem under the same conditions of the previous theorem, we are going to show that the initial value problem is going to have a solution, okay. So, the statement of the theorem is let f x, y be continuous on the rectangle r which is defined as set of whole x, y such that x minus x 0 is less than equal to a, y minus y 0 is less than equal to b.

Then there exists a solution to the initial value problem d y by d x is equal to f of x, y y at x 0 is y 0 in the interval x minus x 0 is less than equal to h, where h is some minimum of a b by m with m is equal to the maximum of the function f x, y, where x, y is in the rectangle. So, for all x, y the rectangle if m is a bound it is a maximum of it, then there exists a solution in the interval starting from the point x 0, y 0. So, let us quickly see the proof of it by using the ideas from the previous proof previous theorem.

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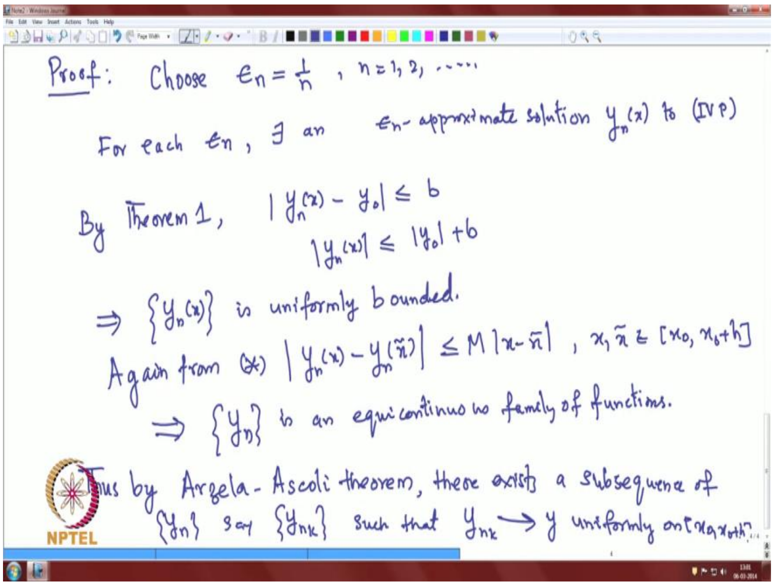

Proof, choose epsilon n to be 1 by n where n is equal to 1, 2, 3, etcetera; P is small sequence of positive numbers. Now for each epsilon n, there exists an epsilon n approximate solution, call it y n x to the initial value problem, okay. So, this is true because of our previous theorem. For every epsilon n, there exist an epsilon and approximate solution which we denote by y n to a initial value problem. Now by theorem the previous theorem one, in the theorem we have shown that y n x the approximate solution y n x minus y 0 is less that equal to b or y n x is bounded by y 0 plus b.

So, right hand side is independent of n. So, therefore, the family of solution; so, therefore, this implies that y n x the solution, this sequence of solution is uniformly bounded. So, this is a uniformly bounded sequence of functions. Again from the previous theorem, in fact, from the inequality from star if you look into previous theorem, we have shown that this is star this inequality. So, from star that y minus y x minus y x tilde is less than equal to m x minus x tilde for all x and x tilde in the interval x 0, x 0 plus h.

So, therefore, again from star we have that y n x minus y n x tilde is less than equal to m times x minus x tilde, where x and x tilde are members from x 0, x 0 plus h. So, what does it say? So, this implies that the sequence y n is an equicontinuous, so equicontinuous family of functions. So, therefore, epsilon n approximate solutions which are denoted by y n's; so, y n's are uniformly bounded and equicontinuous.

Now we invoke Arzela-Ascoli theorem; we here make use of Arzela-Ascoli theorem. So, thus by Arzela-Ascoli theorem, there exists a subsequence of y n, say, y n k. So, there exists a subsequence of y n k such that y n k converges to some function y uniformly on x 0, x 0 plus h.

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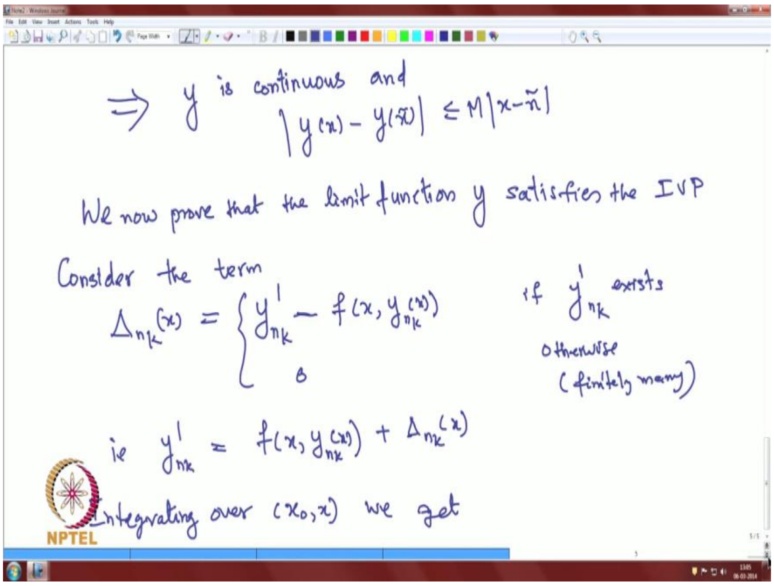

So, invoking Arzela-Ascoli theorem, we get a subsequence of the constructed sequence y n that converges uniformly to some function y in the interval x 0, x 0 plus h. So, this implies that y is continuous. So, again by using the theorem we discussed in the preliminaries a sequence of function converges uniformly and each y n k is continuous, then the limit function is also continuous and y of x minus y of x tilde is less than equal to m of m times x minus x tilde, okay.

So, now our task is to prove that this limit function y is a solution to the initial value problem. So, we now prove that the limit function y satisfies the initial value problem. So, for that consider there are term. So, consider the term denoted by delta n k of x which is equal to y n k which is a subsequence we have extracted from the sequence prime minus f of x y n k x. So, this difference this is error if y prime n k exists; otherwise, we take this 0; otherwise case is only for finitely many points. So, in other words this is y n k prime is equal to f of x y n k x plus delta n k x, by integrating over x 0 x we get.

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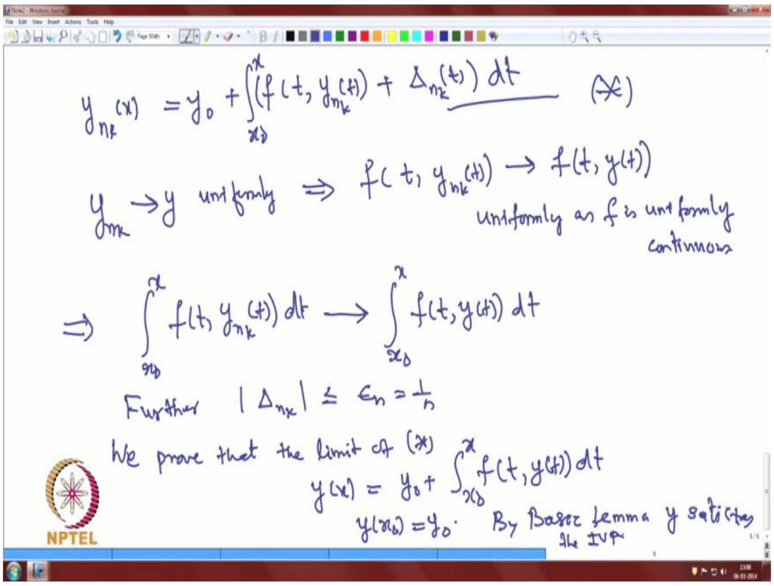

So, integrating we get y n k x is equal to y 0 plus integral x 0 to x f of t y n k t plus delta n k t d t but we know that y n k converges to a uniformity . So, this implies that as f is continuous f of t y n k t converges f of t y t uniformly. So, uniformly as f is continuous f is uniformly continuous. So, this implies that the integral x 0 to x f of t y n k t d t converges to integral x 0 to x f of t y t d t.

Further, we have delta m k which is by definition is less than epsilon n which is 1 by n and passing to the limit, this tends to zero. So, we prove that the limit of star is y of x is equal to y 0 plus integral x 0 to x f of t y of t d t, and obviously, y at x 0 is y 0 and by basic lemma y has to satisfy. So, by basic lemma y satisfies the initial value problem. So, therefore, the limit of this convergence subsequence y n k is a solution to the initial value problem. So, this proves that solution exists. Thus the limit function y n is a solution to the initial value problem. So, in this lecture we have proved that if we have an initial value problem d y by d x f of x, y and f is continuous with respect to x and y, then there exists a solution to the initial value problem, the uniqueness is not guaranteed, bye.