Ordinary Differential Equations

Prof. Raju K Geroge
Department of Mathematics
Indian Institute of Science, Bangalore

Module - 4
Lecture - 19
Picard's Existence Continued

In the previous lecture, we were trying to prove the Picard's existence and uniqueness theorem.

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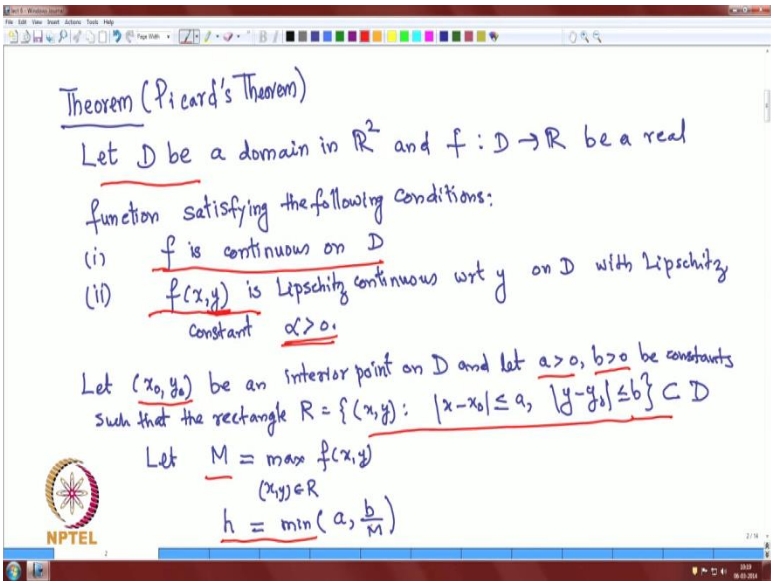

The proof was divided into 4 parts and we proved part a and part b. So, let me just recall the Picard's existence and uniqueness theorem. Let d be a domain in r 2, and f is a function from d to r; a real valued function, satisfying the following conditions; f is continuous on d and f x y is lipschitz continuous with respect to y with a lipschitz constant alpha, greater than 0, and x 0 y 0, the initial point of the initial value problem, that is assumed to be an interior point on d and we take two constants a greater than 0 b greater than 0, such that the rectangle defined by this, is fully inside the domain d and we use a notation m is a maximum value of f in the rectangle, which is attained, because of it is continuous and h is the minimum of a and b y.

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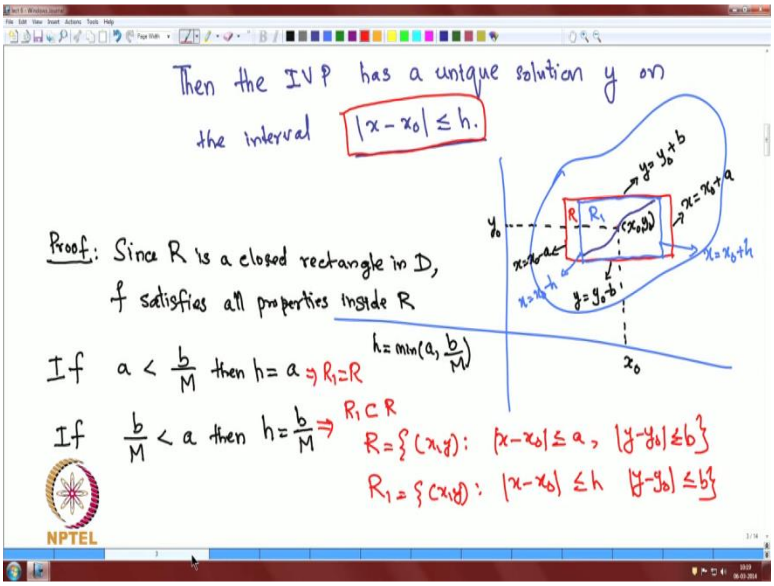

Then, the Picard's theorem is that the initial value problem has a unique solution in the interval x minus x 0 is less than equal to h. The main idea in the Picard's theorem is we defined what is known as Picard's iterative scheme; the iterants.

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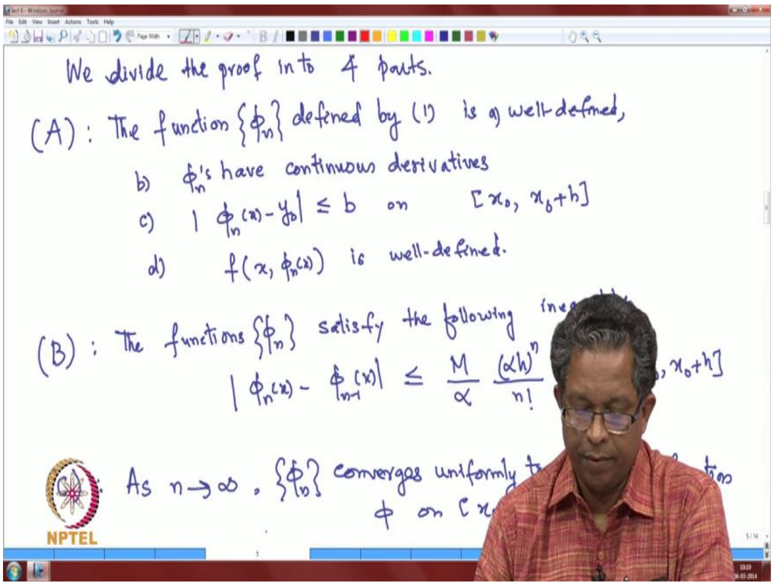

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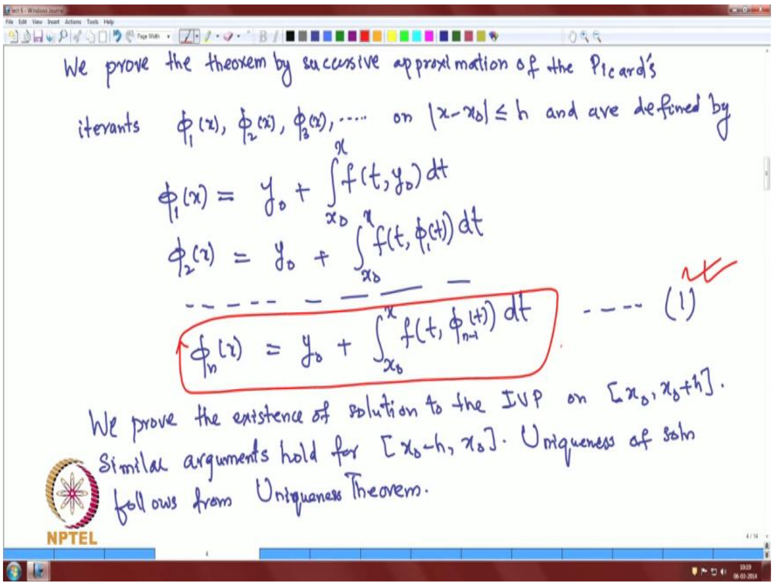

We defined the Picard's iterants by phi n is equal to y 0 plus integral x 0 to x, f of t, phi n minus 1 t, d t, and varies from 1 2; n is 1, 2, 3, etcetera. So, we get a sequence of functions. The main idea of the proof is we prove that this sequence of functions, converges uniformly, to a function phi in the interval x 0 plus h and then, we show that that limit function is a solution to the initial value problem, and by uniqueness theorem, which we have already proved where, we use lipschitz condition; the solution is unique; the limit function is a solution and the solution is unique. So, we divided the proof into four parts. The first part is we have shown that phi n, defined by the iterative scheme is well defined, and phi n is a sequence of functions, having continuous derivatives and phi n x for every n is inside the rectangle r. In part b, we found that the sequence of function phi n that satisfies an estimate; phi n x minus phi n minus 1 x; the absolute value of if it less than equal to m by alpha, into alpha h to the power of n by n factorial, for n going from 1, 2, 3, etcetera.

This happens on the interval x 0 x 0 plus h. Now today, we will prove part c and part d. So, part c; what we want to prove is as n goes to infinity, we will prove that the sequence of function phi n that converges uniformly, to a function phi on the interval x 0 x 0 plus h, and part d, we will show that the limit function phi, which is limit of the sequence of function phi n; that is nothing but the solution of the given initial value problem on the interval x 0 x 0 plus h. So, let us start with the proof of c.

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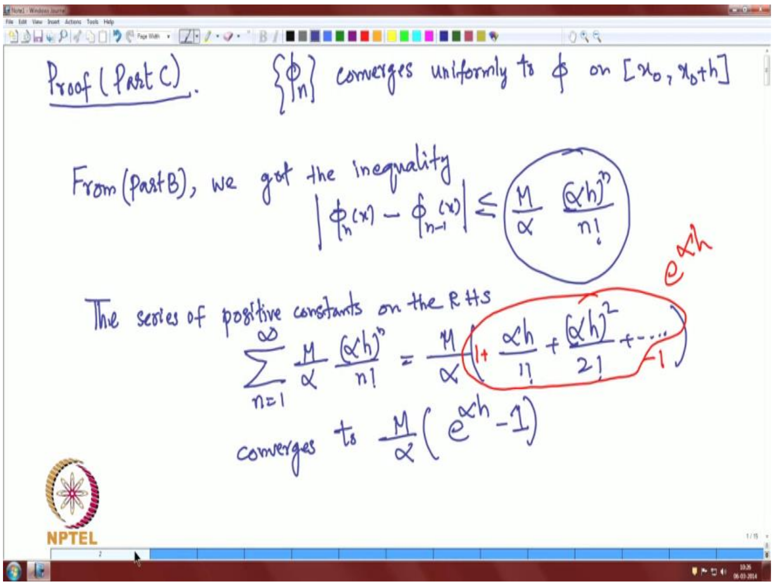

So, proof part c; you may want to prove that phi n converges; the sequence phi n converges uniformly, to some function phi on the interval x 0 x 0 plus h. Note that from part b, we got an estimate for phi n; from part b, we got the inequality. So, part b, we got the inequality; phi n x minus phi n minus 1 x; the absolute value of this is less than or equal to m by alpha times alpha h to the power n by n factorial. Now, we consider the series of positive constants. So, the right hand side, if you look at the right hand side and a make a series of positive constants by using the right hand side, the series of positive constants on the RHS; that is this series m by alpha, alpha h to the power n by n factorial. As n goes from 1 to infinity, which is m by alpha times alpha h by 1 factorial plus alpha h square by 2 factorial plus, etcetera. This converges to m by alpha times e to the power alpha h minus 1. See, in this series, if you add 1 and if you subtract that 1, this summation is e to the power alpha h and subtract 1; you get m by alpha times e to the power alpha h. So, this converges; the right hand side to m as the series that converges to this quantity. Now, we will consider the infinite series.

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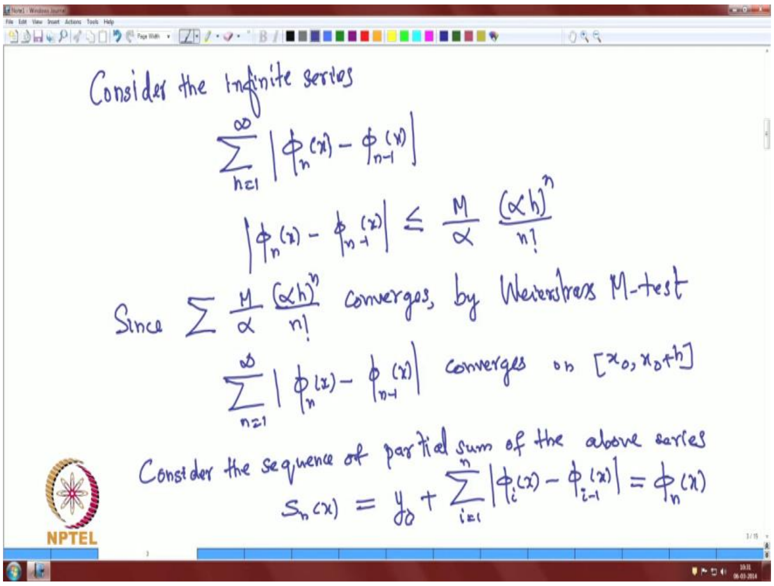

So, consider the infinite series; summation n goes to 1 to infinity phi n x minus phi n minus 1 x. So, this series, we are discussing the convergence of this series. What is the limit of this series? Each time of the series, is bounded by a positive constant, which we got in, we proved in part b. Now, by Weierstrass m tests, each series, each time phi n x minus phi n minus 1 x this is less than or equal to m by alpha, which we proved in part b, to the power n by n factorial. Since, the series found by the right hand side m by alpha, alpha h to the power n by n factorial, converges. We will have invoked the Weierstrass m tests, which we discussed in the preliminaries. By Weierstrass m test, the series n is equal to 1 to infinity phi n x minus phi n minus 1 x; this converges.

It converges uniformly, on the interval. The interval, which we are concerned about is x 0 x 0 plus h, will converge on this. Now, if this series, infinite series converges, what is a limit of it; to what it is converging? For that, let us consider the partial sequence of partial sum. So, consider the sequence of partial sum of the above series. Call it s n; s n x is in the partial sum plus if you add y 0 to it. So, y 0 plus the partial sum; n goes from 1 to; say, i goes from 1 to n; phi i x minus phi i minus 1 x. So, just if you expand the plus terms and minus terms, they cancel each other, and this becomes by definition, this is your phi n x, which is defined by the Picard’s iterative scheme. So, phi n x is the partial sum of the infinite series.

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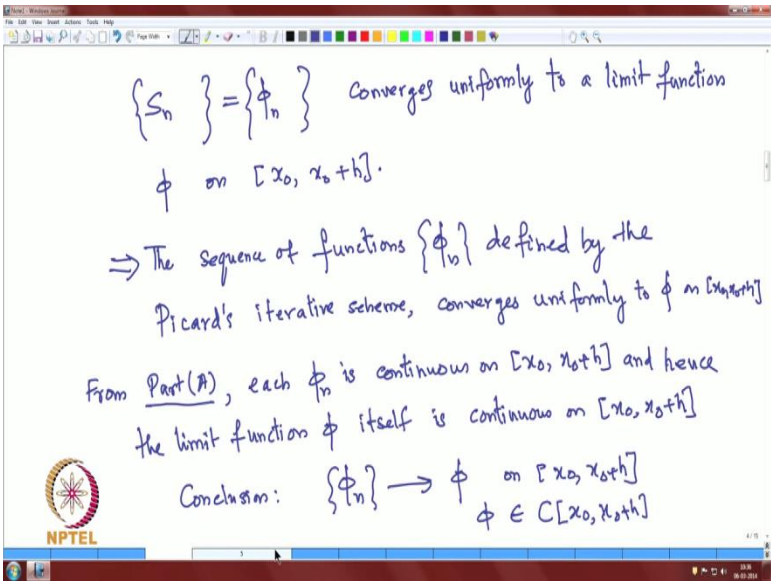

Since, infinite series converges uniformly, on to this interval; say, the partial sum s n x, which is phi n x; that converges the sequence of partial sums. If you suppress x, this converges uniformly, to a limit function, say phi on the interval x 0 x 0 plus h. Therefore, this implies that the sequence of functions phi n, defined by the Picard's iterative scheme converges uniformly, to phi on this side interval, x 0 x 0 plus h and also, from part a, which we have proved; each phi n is continuous on x 0 x 0 plus h. Therefore, the sequence of functions converges uniformly, to phi and each phi n is continuous. Therefore, we can invoke the theorem, which we discussed in the preliminaries to conclude that the limit function phi itself, is continuous. So, from part a, each phi n is continuous and hence, the limit function phi itself, is continuous on x 0 x 0 plus h. So, in conclusion, therefore, we have, conclusion is the sequence phi n converges to phi on x 0 x 0 plus h, and phi is an element of the set of whole continuous functions, defined on x 0 x 0 plus h. Now, we will prove the next section, that is part d.

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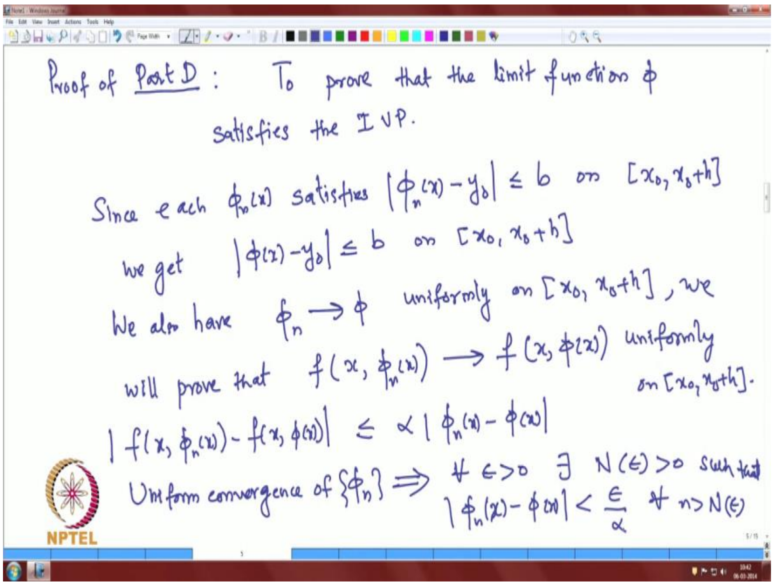

Proof of part d; there, we will show that this limit function phi, which we just got as a limit of the sequence of functions, defined by the Picard's iterative scheme, is a solution to the initial value problem. So, to prove to that the limit function phi satisfies the initial value problem; since, each phi n x satisfies the estimate phi n x minus y 0 is less than equal to b on the interval x 0 x 0 plus h, which we have proved in part a. Since, each phi n x is inside the rectangle r or r 1, we get phi x minus y 0 is less than equal to b on x 0 x 0 plus h. So, this we can; phi n converges uniformly, to phi on the interval. Therefore, the limiting function phi x satisfies phi x minus y 0 is less than equal to b on this interval. We also have the convergence phi n to phi; that is a uniform convergence.

So, this converges uniformly, on the interval x 0 x 0 plus h. We will prove that the function f x phi n x; this converges uniformly, to f x phi x, uniformly on x 0 x 0 plus h. How this is done? We have only proved that phi n converges to phi uniformly, on x 0 x 0 plus h, and by using that, and f is given to be continuous, and f is a having nice properties; lipschitz continuity and continuity with respective to x and lipschitz continuity with respective to phi. Therefore, if we find f of x phi n x minus f of x phi x, which is a limit function, this is less than or equal to by using the lipschitz continuity of f with respective to the second argument, is alpha times phi n x minus phi x.

So, uniform convergence of phi n implies that for every epsilon greater than 0, there exist a delta; there exist a positive number n and that n dependence upon epsilon only, such that n positive; such that phi n x minus phi x; this difference is less than epsilon for all n greater than n epsilon. So, we can take this epsilon by alpha; also, another epsilon, this to get epsilon at the end.

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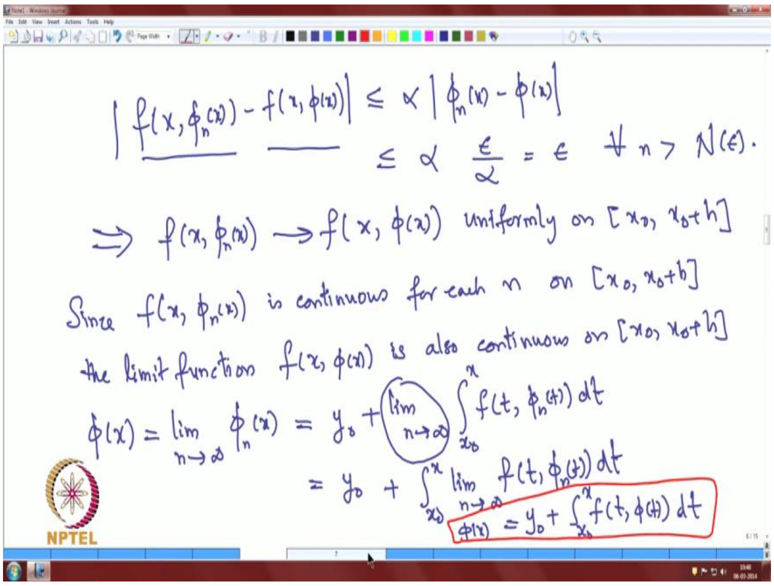

Therefore, we find the estimate f of x phi n x minus f of x phi x, which is less than or equal to alpha times phi n x minus phi x, and uniform convergence of phi n implies that whenever, x is for all n small n larger than the capital n; this can be made less than epsilon by alpha. So, this is less than or equal to alpha times epsilon by alpha, which is epsilon, for all n greater than capital n, which is a function of epsilon. So, for given epsilon greater than 0, we could prove that their exist an n, such that f of x phi n minus f of x phi x; this difference, the absolute value of the difference is made less than an epsilon for all n greater than n of epsilon. So, this shows that f x phi n x converges to f x phi x uniformly, on x 0 x 0 plus h.

Now, since f of x phi n x is continuous for each n; this is continuous for each n on the interval x 0 x 0 plus h, the limit function f t phi t function, f x phi x is also, continuous on x 0 x 0 plus h. The convergence is uniform and for each n in the sequence, each time of sequence is continuous, and the sequence converges uniformly. Therefore, the limit function is also continuous follows from the theorem; we discussed in the preliminaries. Therefore, phi of x is equal to limit of n goes to infinity phi n x, which is equal to, by definition, phi n is y 0 plus limit n goes to infinity integral x 0 to x, f of t, phi n t d t.

Now, we invoke theorem 3 that we did in the preliminaries, on the interchange of limit and integration of sequence of functions. Since, a convergence is uniform and each time, f of t phi n x phi n t is continuous; we can interchange this limit and the integration. So, this is equal to y 0 plus integral x 0 to take the implement inside; limit n goes to infinity; f of t phi n t d t. Now, limit n goes to infinity f t phi n t is your phi f of t phi. Therefore, this is equal to y 0 plus integral x 0 to x f of t phi t d t. So, your left hand side is phi of x . Therefore, the limit function takes a form, phi x is equal to by 0 plus integral x 0 to x f of t phi t d t.

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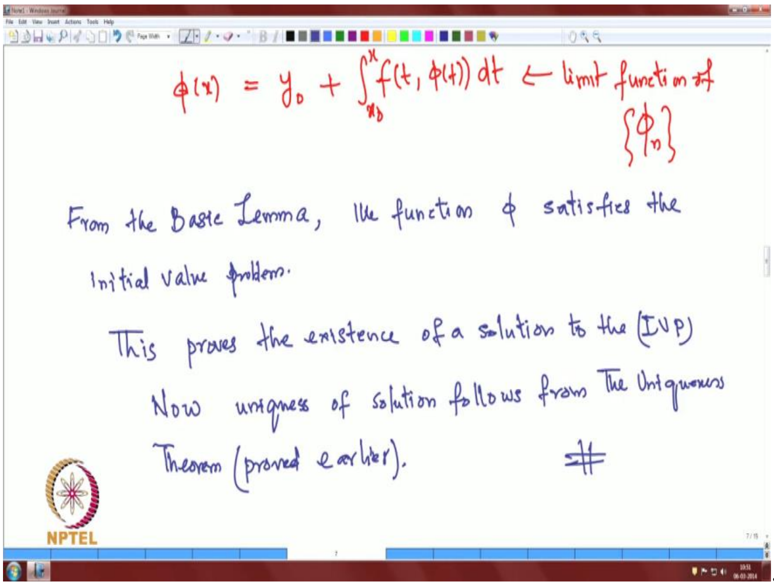

Now, if you recall, therefore, phi of x is equal to y 0 plus integral x 0 to x f of t phi of t d t; it is a limit function of the sequence phi n, which is the sequence obtained from the Picard's iterative scheme. Now, if you invoke from the basic lemma, any function satisfying this integral equation, has to satisfy the initial value problem. The function phi satisfies initial value problem. Therefore, the Picard's iterants converges uniformly, to the solution of the initial value problem. This solution is unique. That uniqueness follows from the uniqueness theorem, which we proved. So, this proves the existence of a solution to the IVP. Now, for uniqueness what you require is the lipschitz continuity, which is already assumed in the theorem. Now, the uniqueness of solution follows from the uniqueness theorem, proved earlier. So, this completes the proof of Picard's existence and uniqueness here.

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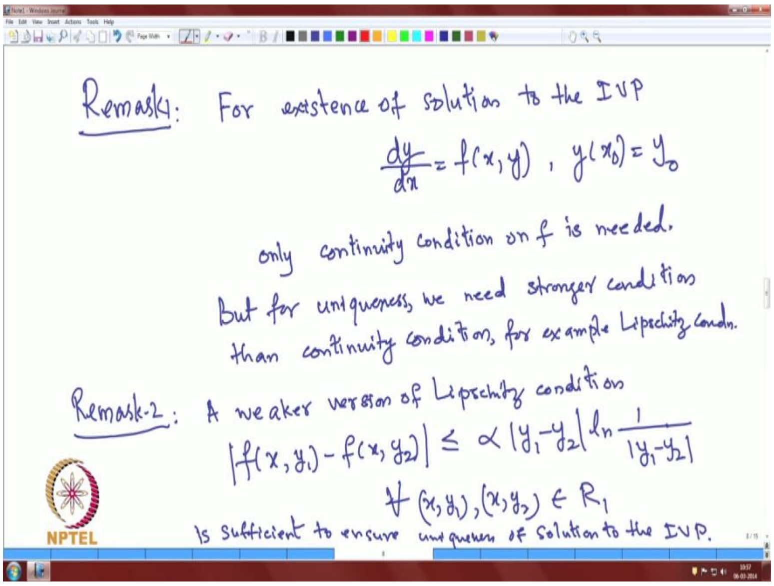

Few things to remark; although, lipschitz continuity was used in the above theorem to establish the existence result, it is possible to establish existence theorem, just by assuming only continuity assumptions on f. However, to establish uniqueness of solutions, one need to use conditions like lipschitz continuity of f with respect to y, or some other conditions, weaker or stronger conditions. So, for existence of solution to the initial value problem; that is we call d y by d x is equal to f of x y, y at x 0 is y 0 for the existence of solution to the initial value problem; only, continuity condition on f is sufficient. Only continuity condition is necessary; only continuity condition is needed, but for uniqueness, we need stronger condition than continuity. Say for example, lipschitz type; for example, lipschitz conditions and one can also, use a weak version of lipschitz type condition to ensure the uniqueness; that is the remark 1.

Remark 2; a weaker version of lipschitz type condition, say, it is something like f of x, y 1 minus f of x y 2 is less than or equal to some constant, alpha times y 1 minus y 2; this is a lipschitz condition, but this is replaced by times l n of 1 by y 1 minus y 2; this is a weaker condition than the lipschitz condition; for all x y 1 and x y 2 on the domain of the rectangle. A weaker version of lipschitz condition is sufficient to ensure uniqueness of solution to the initial value problem. But still, continuity condition is enough to prove the existence. Now, we look into the Peano, Cauchy-Peano theorem, on existence of solution that requires only, continuity condition on the function f.

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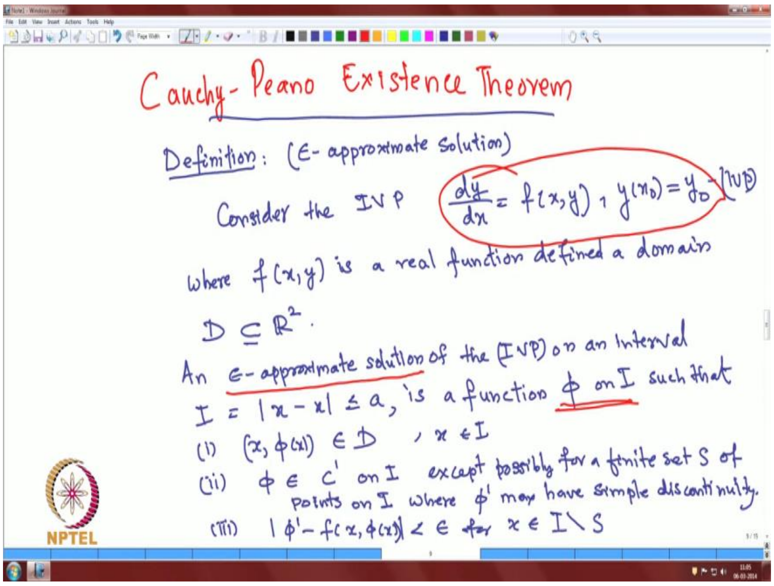

Now, we look into the Cauchy-Peano existence theorem. We state and prove Cauchy-Peano existence theorem for the initial value problem where, the function f is continuous on a domain d in r 2, but not lipschitz continuous with respect to the second argument; that is why. To prove the existence theorem, we first define what is called an epsilon approximation to solution for the initial value problem, just by using continuity on f. Subsequently, we define a sequence of approximate solutions. We define a sequence of approximate solutions for the initial value problem. We show that this sequence of epsilon approximate solution; that is uniformly, bounded and equi continuous. So, once we have a uniformly bounded and equi continuous sequence of functions, we make use of Arzelascoli theorem to extract a subsequence of the sequence that converges uniformly, to a function. Later, we should prove that the limit function is a solution to the initial value problem; that is the old idea of Cauchy-peano existence theorem.

Cauchy-peano existence theorem; we first define what is known as an epsilon approximate solution to the initial value problem. So, definition; it is called epsilon approximate solution. Consider the initial value problem, which is d y by d x equal to f of x y with initial condition, y at x 0 is y 0; this is a initial value problem where, the function f x y is a real function, real valued function, defined on a domain; call it d in r 2. An epsilon approximate solution of the initial value problem, IVP on an interval; called it I, which is z of 4 x, absolute value of x minus x 0 is less than equal to a, is a function, the epsilon approximate solution is a function; call it phi, defined on I, such that the following properties are satisfied; first, x; for every x in the interval I, is in the given domain d; x on I. Second property; phi is c 1 class; this on I; this is one time continuously differentiable; phi as continuous first derivative except, possibly for a finite set; call it s of points on I where, the derivative phi prime may have finite discontinuity or simple discontinuity; have simple discontinuity. It is a kind of jump discontinuity. Third condition for approximate solution is the difference between phi prime minus f of x phi x, less than epsilon for all x in the interval I except, for the points on this finite set s.

So, for a given initial value problem, we define an epsilon approximate solution. So, phi is said to be a function phi, is set to be a function phi on an interval I, given by x minus x is less than equal to a, said to be an approximate solution, epsilon approximate solution, if this property x phi x is in the domain, and phi is continuously differentiable except, on a set of finite number of points. The difference between phi prime and f of x phi x; this is difference they are; this is made smaller than epsilon on the interval I minus s. So, what we do now, is we will prove that under continuity assumption on f, just by continuity assumption on f with respect to both x and y, there exist epsilon approximate solution to this initial value problem.

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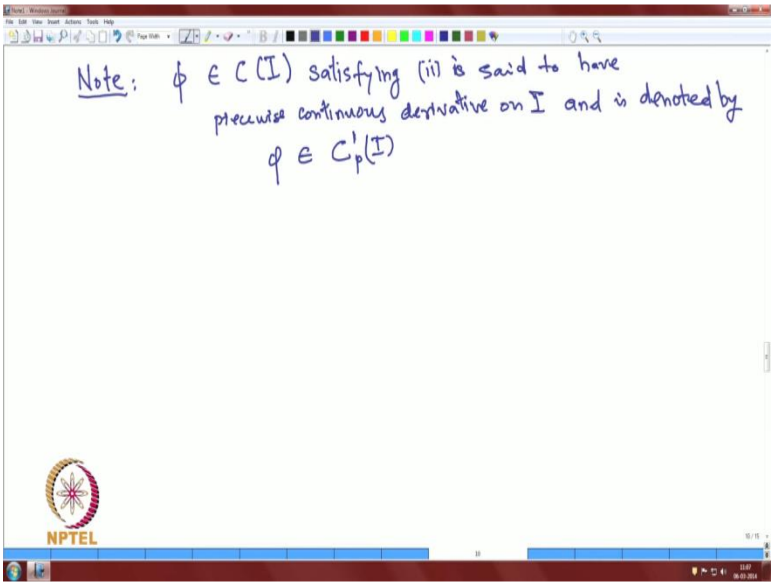

So, we prove the following theorem and one note is the solution phi or any function phi, which is an element of c I, continuous functions defined on the interval I, satisfying the second property, which we just have seen; the property 2; this property; that is phi is in c 1 of I except, possibly for a finite set s of points on I where, phi prime may have simple discontinuity. So, if any function satisfying the second property, phi is said to have piecewise continuous derivative on the interval I, and is denoted by phi is in the set, the class c 1 p I; a class of functions having piecewise continuous derivatives.

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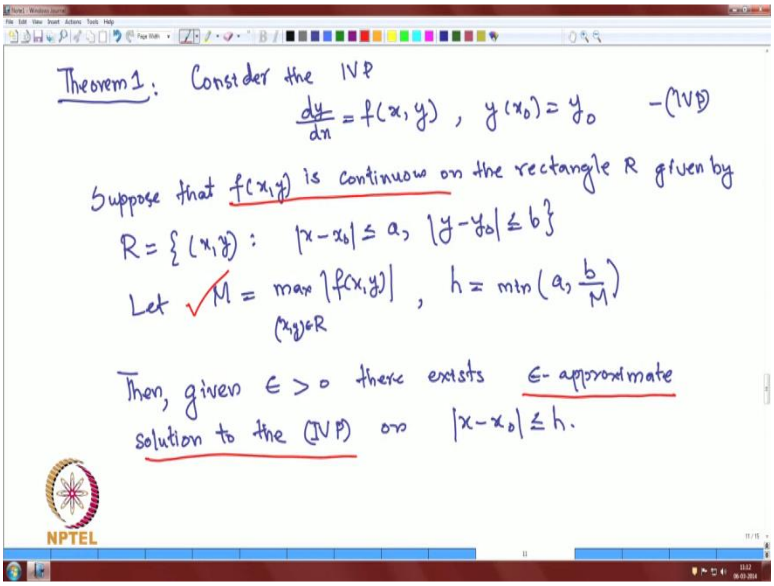

So, we state the theorem; theorem 1. In this theorem, we prove that under continuity assumptions on f with respect to x and y, there exist epsilon approximate solutions to the initial value problem. Consider the initial value problem d y by d x is equal to f of x y with initial condition y at x 0 is y 0; initial value problem. Suppose, that f of x y is continuous on the rectangle r, given by r, is set of four points x y's, such that x minus x 0 is less than equal to a, and y minus y 0 is less than equal to b.

Let m be a constant; m is equal to maximum of the function f of x y, maximum of the function m and x y lies on r, and h is a constant, defined by minimum of a b by m. Then, the conclusion of theorem is then, given epsilon greater than 0, there exists epsilon approximate solution to the initial value problem on the interval x minus x 0 less than equal to h. Theorem does not say anything about the uniqueness; theorem says that there exist an epsilon approximate solution to the initial value problem, under the continuity assumptions on f. The constant m; since, f is continuous on the rectangle; rectangle is a closed set inside r 2. Therefore, bounded and closed come back set, and the maximum is attained. This m is defined and h is the minimum of a and b y m, depends on the value of m. So, we will prove this theorem.

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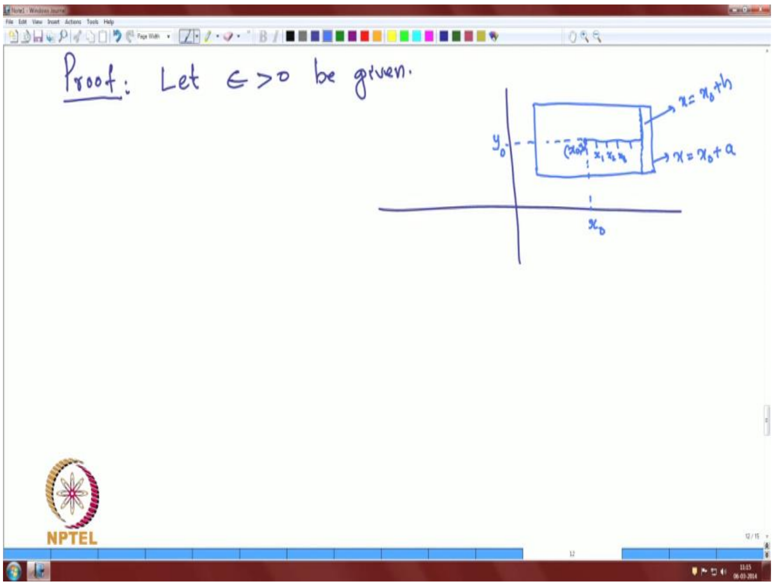

The all idea of this theorem; the proof of the theorem; we take let epsilon greater than 0 be given, then this is x y prime, and this is a rectangle and say, this is point x 0 y 0, and say, this line is x is equal to x 0 plus a, and if h is smaller than a, this is x is equal to x 0 plus h. We divide the interval x 0 to x 0 plus h into n parts. So, the first point is x 0, x 0 y 0, then x 1, x 2, x 3, etcetera. Then, we define approximate solution, starting from x 0 y 0. We will approximate the solution by straight lines. The idea is starting from x 0. At x 0, I know what is the slope of the solution? Slope of the solution is f of x 0 y 0; I make a straight line. Then, from that that straight line, meets a line, x is equal to x 1 and 2; some points from there, I again, find the slope, and I make line segments and join the line segments to get a polygon. That polygon is an approximate solution. The mesh, the difference of x I and x I plus 1 is very small. Then, that difference of the actual solution and the approximate solution can be made small. We will do that detail of the proof in the next lecture.

Thank you.