# Ordinary Differential Equations

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**Module No. # 04**

**Lecture No. # 18**

**Picard's Existence and Uniqueness Theorem**

Welcome back to the existence and the uniqueness theorem of initial value problems.

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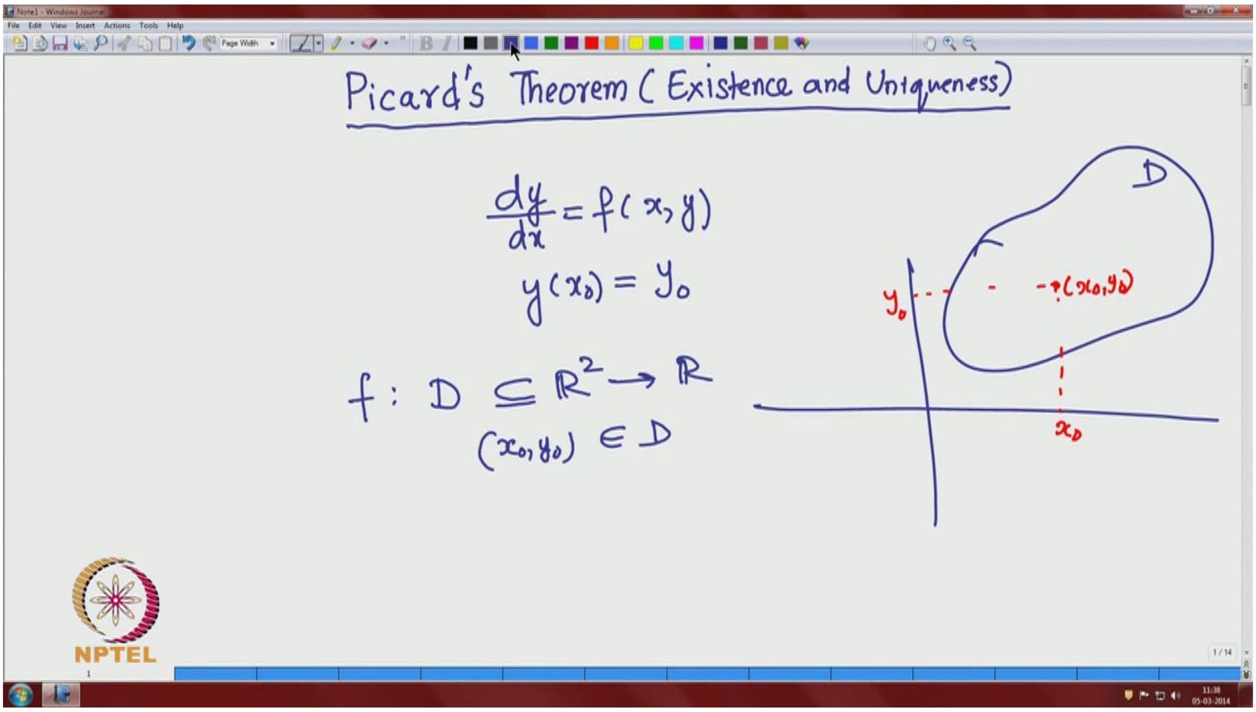


In the last lecture, we have seen that an initial value problem has a unique solution, if the function satisfies a lipschitz continuity condition, with respect to the dependent variable y. So, we deal with initial value problem. Our initial value problem, d y by d x is equal to f of x y and the initial condition is y at x 0 is y 0. So, f is a function, which is defined on a domain, which is a subset of R 2 to R, and x 0 y 0 is a point, an interior point inside d, and this is x 0. This is a point y 0.  So, x 0 y 0 is the initial point, the initial condition. We now, prove the existence of solution and uniqueness of solution, under lipschitz condition with respect to y, and continuity condition with respect to x. So, Picard's theorem uses both existence and uniqueness. Uniqueness part, we have already proved.

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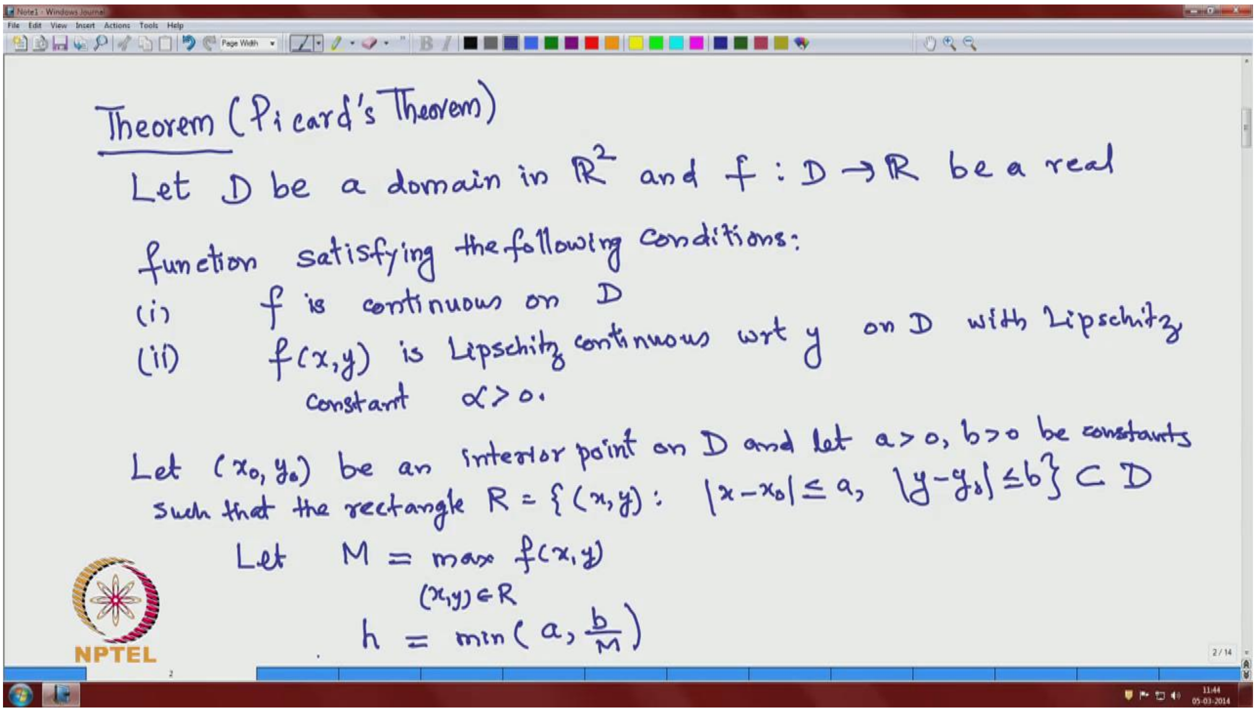


In this theorem, I will concentrate only on the existence part. So, let me state the Picard's theorem. Now, let D be a domain in R 2, and f from D to R appear real valued function; a real function, satisfying the following conditions. First one is f is continuous on D with respect to both the arguments and that is the meaning of the all domain D, and f of x y is lipschitz continuous with respect to y on D, with a constant, with a lipschitz constant alpha, which is a constant, positive constant, and let x 0 y 0 be an interior point on D. Let a and b be constants, such that the rectangle R, defined by set of whole x y, such that x minus x 0 is less than or equal to a, and y minus y 0 is less than or equal to b. This rectangle is inside the domain D, and let M be a constant. Capital M be a constant, defined by maximum of the function f x y where, x y is in D; this maximum exists, because of it is continuous. We take x y to vary in the rectangle R; rectangle is a closed set inside D. Therefore, this maximum exists, and let us define another constant, h is equal to minimum of a and b by n.

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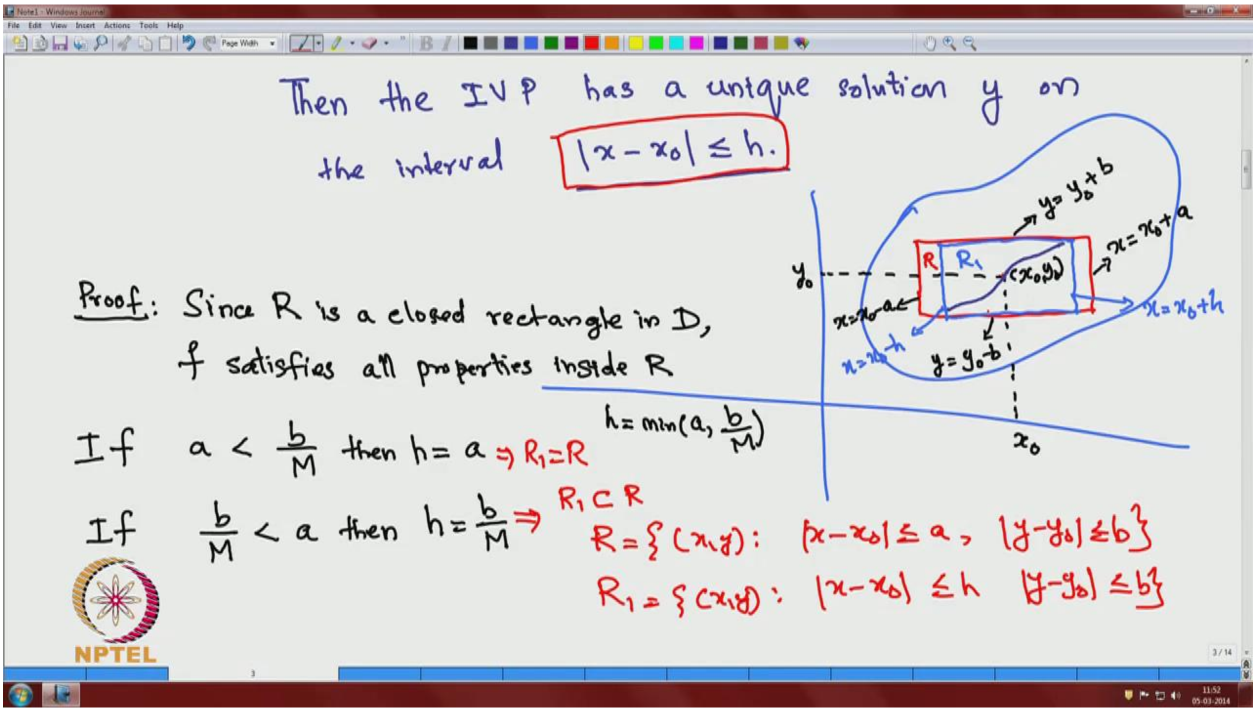


Then, the initial value problem has a unique solution y on the interval, x minus x 0, is less than or equal to h. So, this is a Picard's existence and uniqueness theorem. It gives both existence and uniqueness. The conditions, we assume is f is a continuous on D and f x y is lipschitz with respect to y on D, with lipschitz constant alpha, and the initial point x 0 y 0 is an interior point on D. We take a rectangle, which is inside the domain D; it is a close rectangle inside the domain, such that two constants are defined; M is the maximum of value of the function in the rectangle R, and h is a minimum of a and b by M. So, if I look at, this is domain D and inside the domain, we define a rectangle. So, this point is x 0 y 0, and this is, this side is x is equal to x 0 plus a, and this side x is equal to x 0 minus a, and this side is y is equal to y 0 plus b, and this side y is equal to y 0 minus b.

We now, prove the theorem; proof; the proof has more technical details. Since, R is a closed rectangle, inside the domain D, f satisfies all properties mentioned inside R. Now, there are two situations. One is if a is less than b by M, remember, our definition of the constant h, which is a minimum of a b by M. So, h is defined as minimum of a b by M. In case, a is less than b by M, then h is equal to a, and in that case, we have the full, the same rectangle, which is equal to a. If a is not, if b by M is less than a, then h is equal to b by M. In that case, h is a number, which is smaller than a. So, in that case, we will have another rectangle. So, this line is x is equal to x 0 minus h. This line x is equal to x 0 plus h. So, we have two rectangles. One, this we call it R 1, and the first rectangle is R, or we can write R is set of all x y, such that x minus x 0 is less than or equal to a; y minus y 0 is less than or equal to b; and R 1 is set of all x y; x minus x 0 is less than or equal to h; y minus y 0, the bound for y is the same b.

If a is less than b by M, then these two rectangles coincide. In this case, R 1 is same as R. If b by m, if a is greater than b by M, in this case, let us see that R 1 comes inside R. So, Picard's existence and uniqueness theorem says that it has a solution. The solution starting from x 0 y 0, is a solution, and the solution exists in the interval, x minus x 0 less than or equal to h. So, existence of solution is on R 1. Depending upon the value of a, if a is less than b by M, then the solution exists for the larger rectangle. If a is greater than b by M, then the solution exists on a smaller rectangle R 1. So, we prove the theorem by the method of successive approximation.

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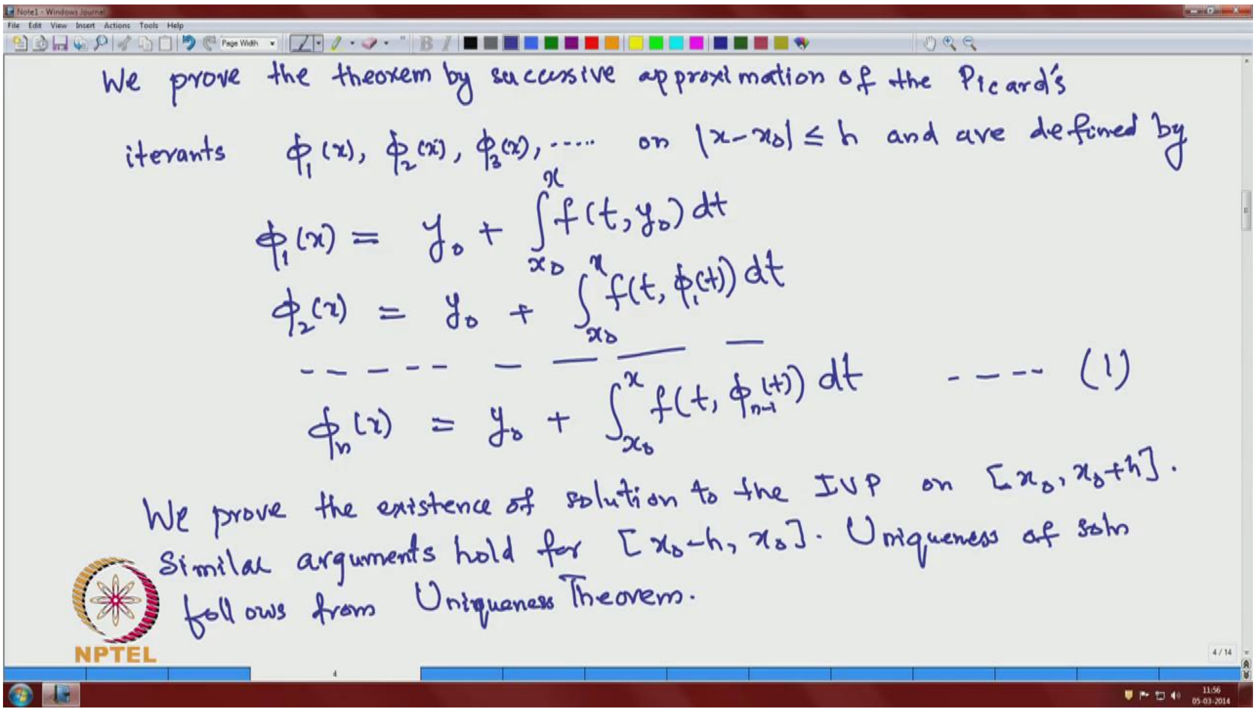


We prove the theorem by successive approximation of the Picard's successive approximation of the Picard's iterants, denoted by phi 1 x, phi 2 x, phi 3 x, etc. These iterants, these Picard's iterants are defined on the interval, x minus x 0 less than or equal to h, is defined on x minus x 0 is less than or equal to h, and are defined by phi 1 x. So, phi 1 is equal to y 0 plus integral x 0  to x, f of t, y 0, which is an initial condition that y 0, dt is a constant function, dt and phi 2 x is y 0 plus integral x 0 to x, f of t, phi 1 t, d t, etc. And phi n t; so, phi n of x is y 0 plus integral x 0 to x, f of t, phi n minus 1 t, d t; call this as equation 1. So, we prove the existence of solution; solution to the initial value problem on one side of the interval, x 0 x 0 plus h. We will prove the existence on one side of the point, x 0 to x 0 plus. Similar arguments hold for x 0 minus h to x 0. Therefore, we proved only on one side, and uniqueness is already being proved; uniqueness of solution follows from uniqueness theorem, which we proved in the previous lecture..

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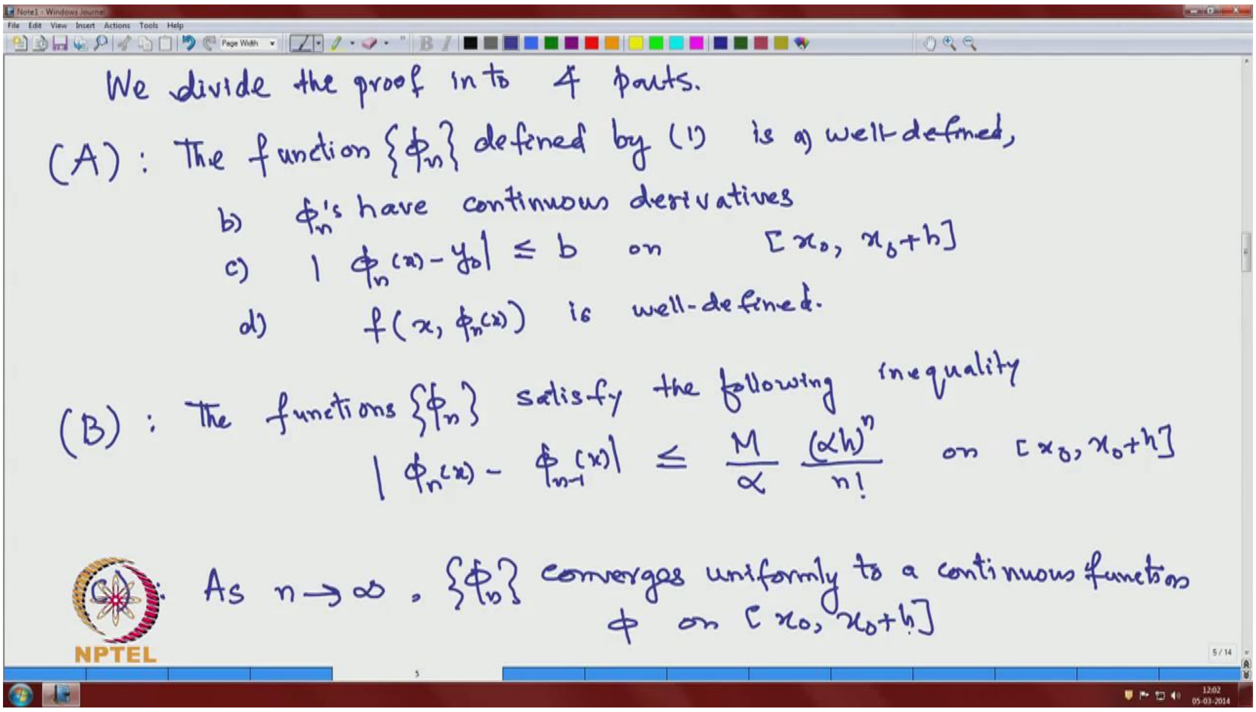


Now, we divide the proof into four parts; so, part a, part b, part c and part d. So, part a is, in part a, we will prove that the functions phi n is the sequence of function; phi n defined by equation 1, is well defined; that is there and it is also, continuous and is called a part, is well defined and b is phi n; phi n's have continuous derivatives. C; it obeys an estimate phi n x minus y 0, the difference between the initial point and all and phi n x. This is less than or equal to b; means all phi n's are within the rectangle; b on x 0 x 0 plus h, the interval on which, we are proving the existence.

And d, when you evaluate the function f at this phi n's, phi n x. So, this is well defined. So, this is part a. In part a, we will prove that the Picard's iterants, defined by equation 1, this phi n x, this equation, this is well defined and phi n's of continuous derivatives on the intervals x 0 x 0 plus h and phi n obeys an estimate, phi n x minus y 0 is less than or equal to b on this interval, and f of x phi n x is also defined. That is part a we will prove it and part b; part b is the functions, phi n x, the sequence of functions we defined by 1, satisfy the following inequality, following estimate; that is absolute value of phi n x minus phi n minus 1 x is bounded by M by alpha; M is the maximum value of the function on the rectangle and alpha is a lipschitz constant times alpha h to the power n, divided by n factorial. This happens for all x in the interval x 0 x 0 plus h. So, the Picard's iterants defines a sequence of function. That sequence of function satisfies this estimate. Now, part 3 that is a part c, is we will prove that as n goes to infinity, the sequence of functions phi n; this converges uniformly. So, this sequence of functions converges uniformly to a continuous function; call it phi, a continuous function phi on the interval x 0 x 0 plus h.

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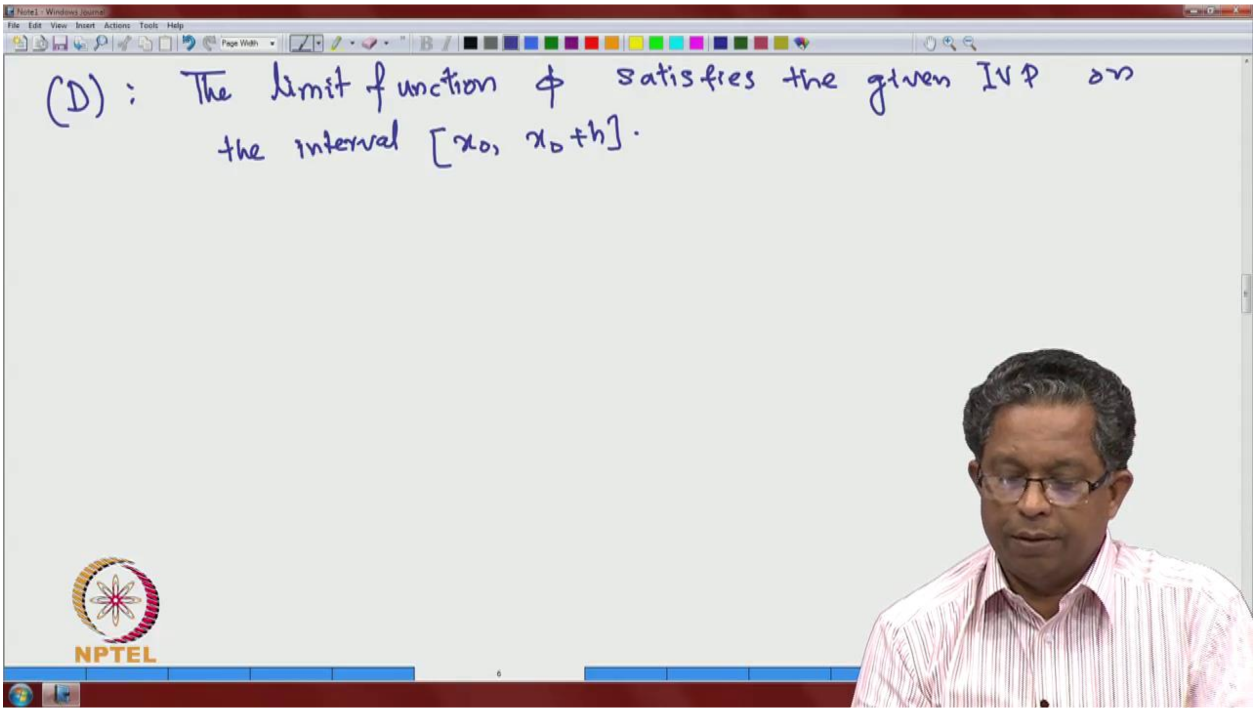


Now, the fourth part, part d of the proof is that we will prove that the limit function, which is obtained in part c, phi satisfies; limit function phi satisfies the given initial value problem, IVP on the interval x 0 x 0 plus h. So, in short, if we prove all these four parts;

part a, b, c and d, then we obtain a limit function phi, which happens to be the solution of the initial value problem that proves an existence of solution to initial value problem, and uniqueness is already been proved in the uniqueness theorem. So, let us prove part by part.

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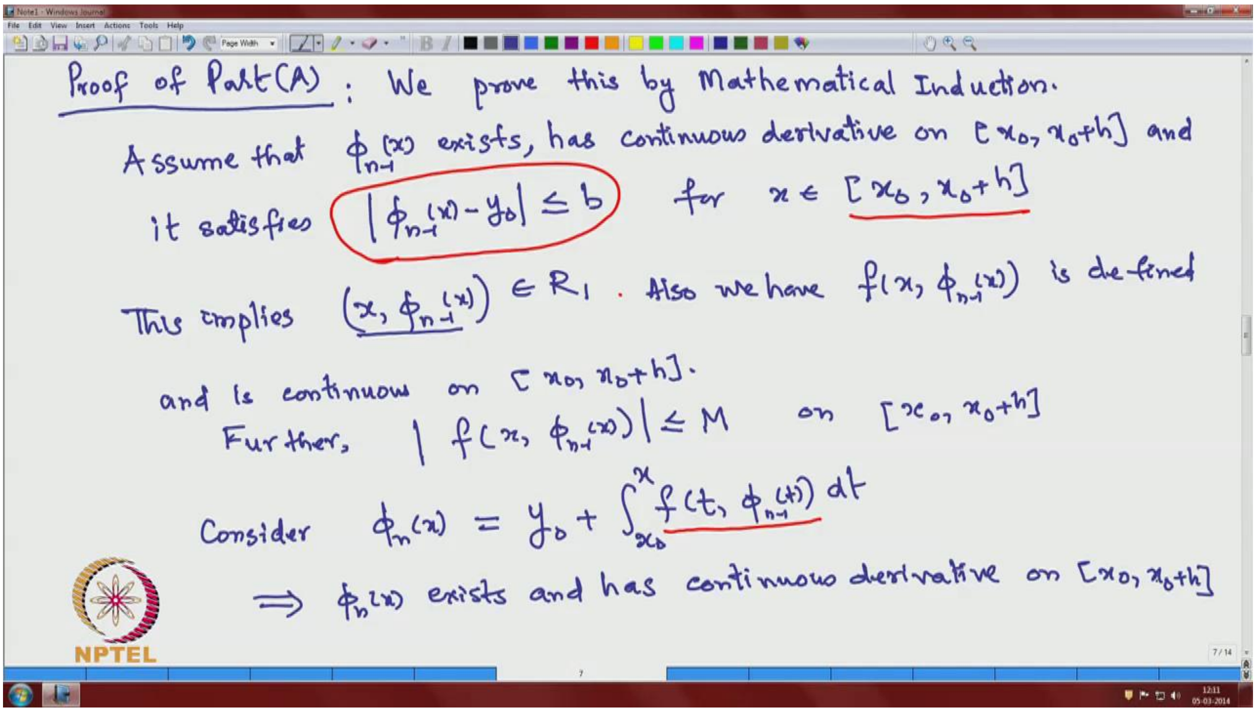


So, proof of part a. This, we will we prove by mathematical induction. We will assume that the result is true for n minus 1 and then, we will prove that it is also, true for n. Then, we will check this is correct for n is equal to 1 and then, by the method of mathematical induction, we conclude that this is true for all n. So, assume that phi n minus 1 x exists. So, phi n minus 1 x exists and it has continuous derivatives on the interval x 0 x 0 plus h, and it satisfies the estimate; it satisfies phi n minus 1 x minus y 0, is less than or equal to b for all x in the interval x 0 x 0 plus h. Now, after assuming this, we are going to show that these properties are also true for phi n. If we assume that these properties are true for phi n minus 1, we are going to show that this property is true for n, phi n.

So, this implies, this above condition implies that x phi n minus 1 x, the point x phi n minus 1 x is in the rectangle R 1, because you have this bound, phi phi n minus x minus y 0 is less than or equal to b. So, that shows that it and that x is in the interval x 0 x 0 plus h. Therefore, the point x phi n minus 1 x is in R 1. Now, also, if this point is in R 1, we can evaluate this function at this point. So, f of x phi n minus 1 x is defined and phi n minus 1 is continuous and f itself is continuous, is defined and is continuous with respect to x on x 0 x 0 plus h. So, further, if we evaluate the function f x phi n minus 1, since that is in the rectangle R 1; this can be evaluated for the function f. because f is defined on R 1 and this is going to and this will have a bound less than or equal to M by hypothesis.

So, maximum value of f on R is M; that is the maximum of it. So, this is on x 0 x 0 plus h and all this discussion, helps us to look into the integral. So, consider the function phi n x, which is defined as y 0 plus integral x 0 to x, f of t, phi n minus t d t. Now, we have seen that the function f is continuous with respect to x, and it is well defined. Therefore, with the properties mentioned above, helps us to conclude that phi n x exists. So, y 0 plus integral x 0 to x and therefore, the continuous function f. This integral exists and has continuous derivative. Since, f is continuous with respect to the second argument, therefore, this is integral exists and that can be differentiated to get a continuous derivative on the interval x 0 x 0 plus h.

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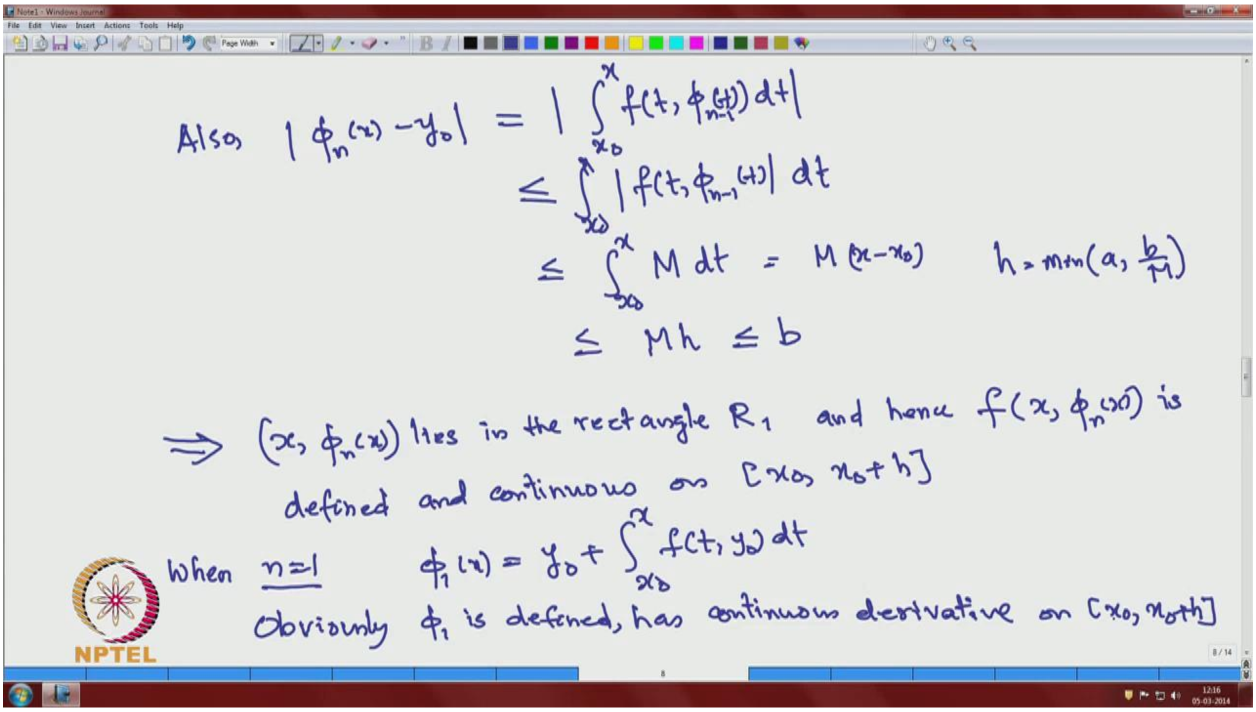


Now, also consider so, also if you find the estimate phi n x minus y 0, the absolute value phi n x minus y 0, by definition; this is the absolute value of integral x 0 to x, f of t phi n t d t and this is less than or equal to integral x 0 to x; absolute value of phi t phi n minus

1; phi n minus 1 t, d t and we have seen that if f of t phi n minus 1 t is bounded by a constant M. Therefore, this is less than or equal to integral x 0 to x m d t, which is integral x 0 to x. We can straight away integrate this one and get M into x minus x 0 and x minus x 0 is less than or equal to h. So, this is M of less than or equal to h and M h is basely less than or equal to b, by definition of h.

So, h is minimum of a b by M. So, these all arguments help us to conclude that x phi n x; this point, this lies in the rectangle. So, x phi n x lies in the rectangle R 1 and hence, f evaluated at x phi n x is defined, since f is continuous. So, f is continuous; phi n is continuous and this f of x phi n x is continuous, is defined and continuous on the interval x 0 x 0 plus h. So, what does it say? That it says that all properties says that this phi n is also satisfying all the properties we assumed in part a. Now, with assumptions, that if phi n minus 1 satisfies the properties, then phi n also satisfies the properties. Now, let us check for the case when n is equal to 1. When n is equal to 1 case, we have phi n x, which s equal to, by definition, y 0 plus integral x 0 to x, f of t y 0, d t. So, obviously, phi 1 is well defined. Obviously, phi 1 is defined and f is continuous, and y 0 is a constant function, which is continuous. Therefore, f is continuous and therefore, this phi 1 is defined and has continuous derivative on x 0 x 0 plus h.

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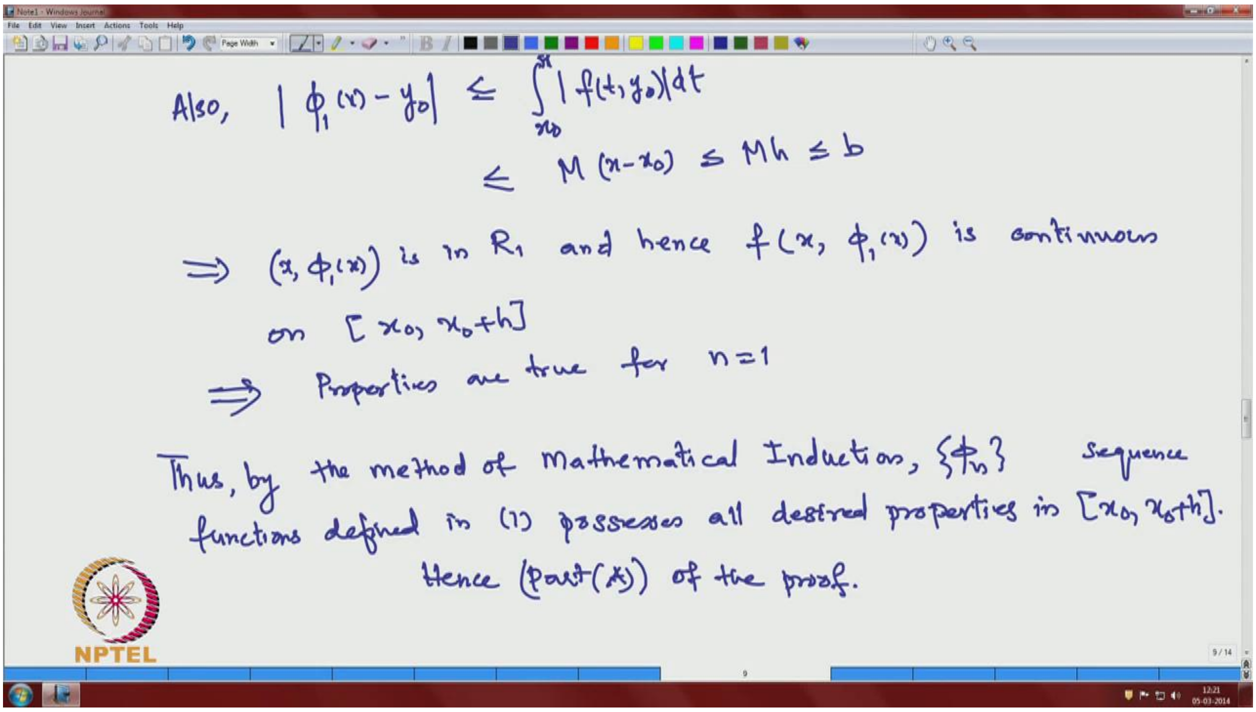

Also, if we find the estimate, also the bound phi 1 n x minus y 0,  by definition; this is less than or equal to integral x 0  to x, f of t y 0, d t and f is bounded by M. So, this is less than or equal to M into x minus x 0, which is less than or equal to M h and is bounded by b. So, obviously, phi 1 also satisfies these properties. Therefore, this implies that x phi 1 x is in R 1 and hence, f of x phi n x is continuous on x 0 x 0 plus h. Therefore, this implies that the properties are true for n is equal to 1. So, what we have proved in part a is the properties of phi n's, like they are all defined, and phi n's are having continuous derivatives and when phi n is put into f; that is also continuous; and phi n x phi n's are in the rectangle; these all properties are true for n is equal to n minus 1; that is what we assumed. From that, we have proved that it is also true for the case n, and it is also true for n is equal to 1.

Thus, by the method of mathematical induction, phi n of, that is a sequence of induction phi n sequence of functions, defined in one by the Picard's iterants that possesses all desired properties in the interval x 0 x 0 plus h. Hence, part a of the proof is established by mathematical induction.

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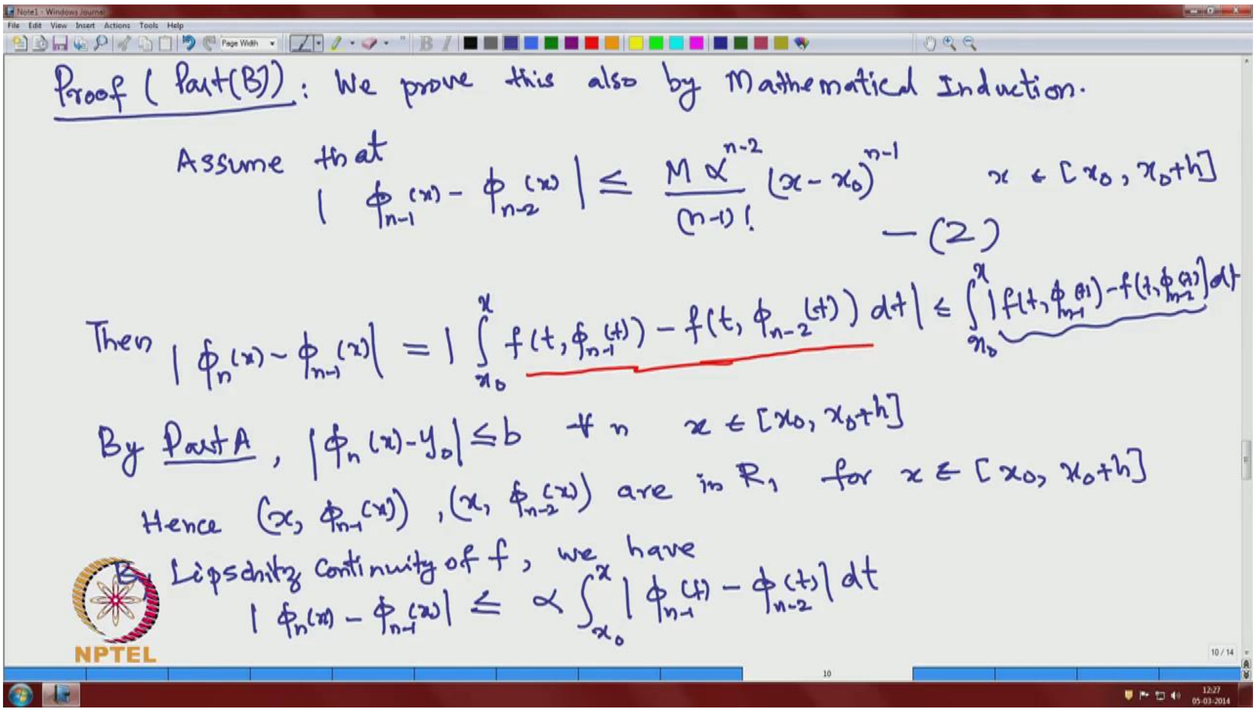


Now, we look into part b; proof of part b. What we are looking for? We are looking for anex for the Picard's iterants. So, we again, prove this again, by mathematical induction.

We prove this also by mathematical induction. So, assume that is true for n minus 1. Assume that the estimate is true for n minus 1, that absolute value of phi n minus 1 x minus phi n minus 2 x is less than or equal to M alpha to the power n minus 2, divided by n minus 1 factorial into x minus x 0 to the power n minus 1 for x in the interval x 0 x 0 x 0 plus h. We assume that the estimate is true. This inequality is true for n minus 2. So, call this inequality as 2. Now, by using this inequality, we find the estimate for phi n. Then, phi n x minus phi n minus 1 x; this, by definition of the iterants is integral x 0 to x, f of t, phi n minus 1 t, minus f of t, phi n minus 2 t, d t; so, by definition of the functions phi n. Now, by part a, what do we have in part a? We have that phi n x minus. In part a, we have shown that phi n x minus y 0, this is less than or equal to b for all n and x in the interval x 0 x 0 plus h, which we have established in part a.

Hence, what we have is the point x phi n minus 1 x and x phi n minus 2 x; these two, both the points are in R; are in the rectangle R 1 for x, of course, in the interval x 0 x 0 plus h. Since, they are in R 1, f satisfies all the properties in R 1, including the lipschitz condition continuity properties. Therefore, by lipschitz continuity of f, if we apply the lipschitz continuity of f, we have; on this difference if we apply lipschitz continuity over here, we have absolute value of phi n x minus phi n minus 1 x, which is less than or equal to alpha times; alpha is a lipschitz constant; integral x 0 to x phi n minus t minus phi n minus 2 t d t. Here, obviously, this is less than or equal to integral x 0 to x, f of t, phi n minus 1 t, minus f t, phi n minus 2 t, d t. Applying the lipschitz continuity, and the lipschitz constant is alpha; therefore, alpha will come out, is alpha times integral x 0 to x 1, phi n minus 1 t, minus phi n minus 2 t.

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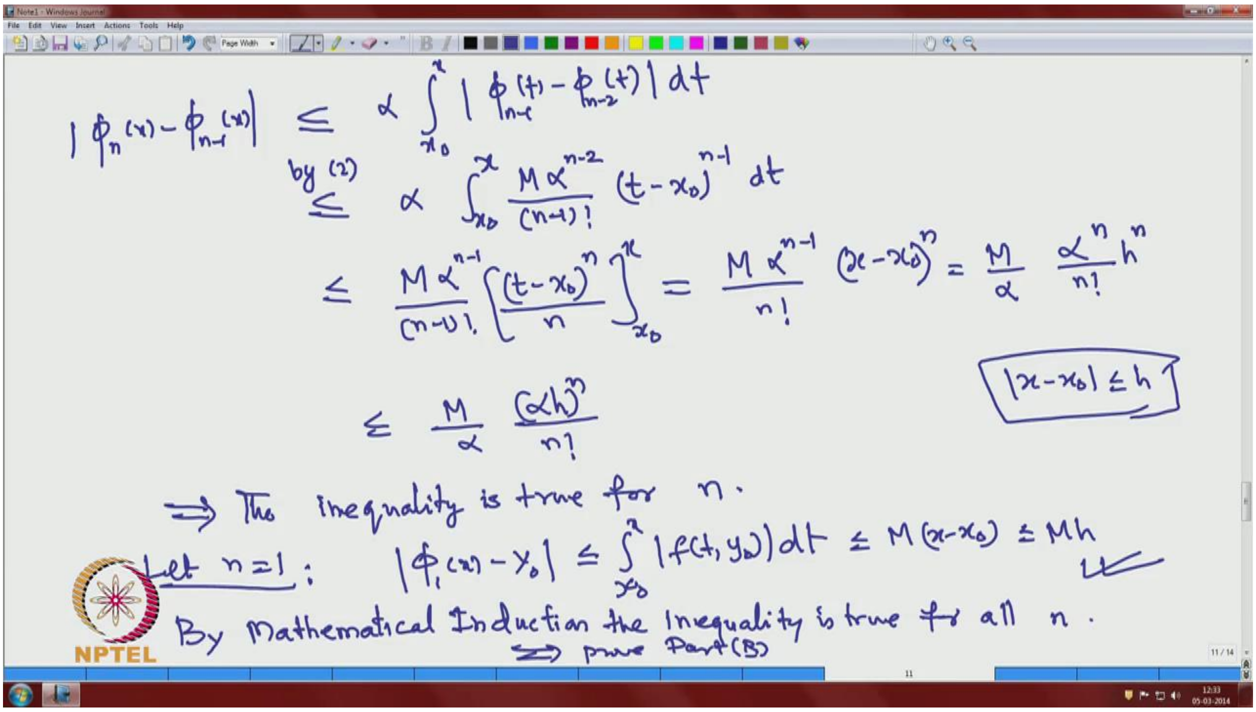


Now, by the assumption, therefore, what is our left hand side? Our left hand side is phi n x minus phi n minus x is less than equal to what we have is alpha times x 0 to x; absolute value of phi n minus 1 t, minus phi n minus 2 t, d t. Now, by using our assumption, by 2; by 2 is our assumption; we assume this is of 2; we assume that for n minus 1 case, this we have. If we assume this, then we get this. This is alpha times integral x 0 to x. So, putting the values, m of alpha n minus 2, n minus 2 by n minus 1 factorial, t minus x 0 to the power n minus 1, d t. If we integrate it with respect to t, this is less than or equal to; we get m into; alpha, I can take outside; alpha to the power n minus 1 by n minus 1 factorial. Then, integrating t minus x 0, we get t minus x 0 to the power n by n.

And you have to evaluate it at the point x 0 and x. So, this gives you that when x is equal to x 0, this vanishes, and t is equal to x; get x minus x 0. So, this is M into alpha to the power n minus 1 by n factorial, n minus 1 into n, n factorial into x minus x 0 to the power n. So, which with a just adjustment of constants, if I divide by alpha, M by alpha into alpha to the power n by n factorial into; x minus x 0 is always less than or equal to h. So, this is h to the power n. Therefore, this is less than or equal to M by alpha into alpha h to the power n by n factorial. So, x minus x 0 is less than or equal to h. Therefore, this implies that the inequality is true for n. We assume that inequality is true for n minus 1, then we could prove that the inequality is true for n. Now, for the case, let n is equal to 1 check for the case n is equal to 1. When n is equal to 1, we have that phi 1 x minus y 0 is equal to integral less than or equal to integral x 0 to x, f of t y 0, d t, and f of t y 0 is less than equal to n. Therefore, this is less than or equal to M into x minus x 0, which is again, less than or equal to M of h. So, it is true for the case M is equal to 1. Therefore, it is true. So, for alpha is equal to 1 case, it is true. Therefore, by mathematical induction, the inequality is true for all n. Therefore, this proves part b.

As we have seen, the total proof of the Picard's existence and uniqueness theorem is divided into four parts. Part a gives some nice properties of sequence phi n, and part b is a very useful estimate for the sequence of functions, phi n and both these parts are proved. Now part c and d; in part c, we will prove that the sequence phi n, that converges uniformly to a continuous function, and part d, we will prove that limit function is a solution to the initial value problem. So, these two parts; we will prove in the next theorem, next lecture.

So, let us just recall; part a gives nice properties on the sequence of functions, phi n, which are the Picard's iterates. Now part b uses a good estimate for the Picard's iterates phi n. Now, remaining part, we will prove in the next theorem.

Bye.