Ordinary Differential Equations
Prof. Raju K George
Department of Mathematics
Indian Institute of Science, Bangalore

Lecture - 17
Basic Lemma and Uniqueness Theorem

Welcome back to the lectures on existence and uniqueness of solutions of initial values problems. We have discussed about the Lipschitz continuity of a function of two variables, and also we proved in the last lecture the Gronwall's lemma which will be used in proving the uniqueness of solution of an initial value problem.

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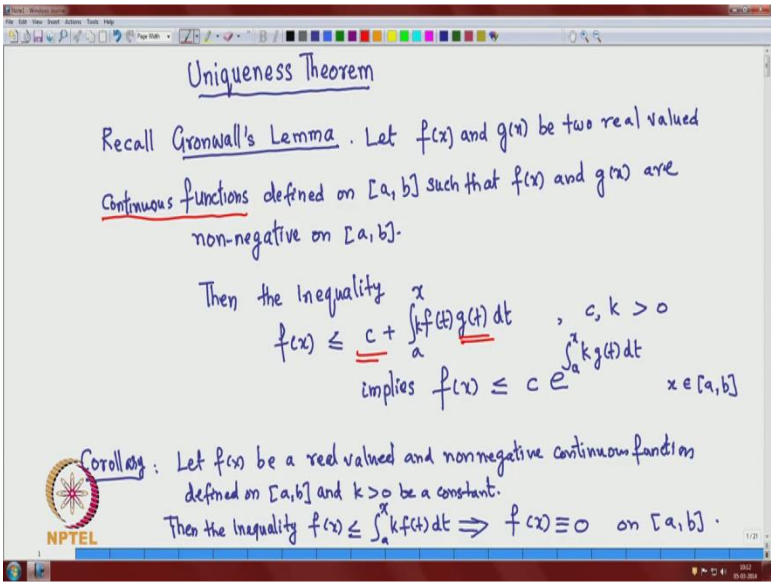

Let us recall the Gronwall's inequality which we did yesterday. The Gronwall's inequality recall Gronwall's inequality or Gronwall's lemma. So, let f x and g x be two real valued function defined on some interval a, b such that both f x and g x are non-negative on the interval a, b. Then the inequality f x is less than or equal to c plus integral a to x f t g t d t with a constant k k times, where c and k are positive constants implies, then this inequality implies f x is less than or equal to c times e to the power integral a to x k g t d t for x in the interval a, b.

So, this inequality says if whenever we have an inequality, f x is less than or equal to c plus integral a to b k times f t g t that implies that f x is less than or equal to c times this constant c times exponential of the integral a to x that into g t d t. The advantage of this inequality is that in the first inequality f x is coming on both sides, and the second inequality gives a bound for the function f, the right hand side is independent of f. And a special case of this Gronwall's equality, we will use improving the uniqueness theorem.

So, that I state as a corollary; let f x be a real valued and non-negative continuous function defined on a, b. In the Gronwall's inequality also we require f x and g x be two real valued continuous function, then also continuity is required for giving a meaning to the integral. So, here in the corollary let f x be a real valued and non-negative continuous function defined on a, b and k be constant. Then the inequality f x is less than or equal to integral a to x k times. So, k times f t d t implies f x is equal to 0 on the interval a to b.

So, this is a special case of Gronwall's inequality where the constant c is 0 and d t or d x is 1, but c we assume to be a strictly positive strictly greater than 0 and that can be tackled.

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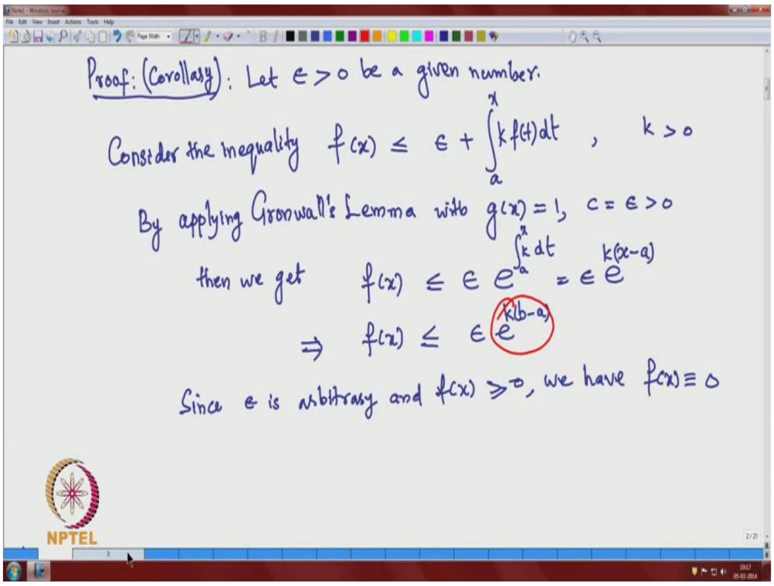

Can we see a proof of it, proof of corollary. Let epsilon greater than 0 be a given number; let epsilon be an arbitrary number given. Now consider the inequality f x is less than or equal to epsilon plus integral a to x k times f t d t. Now f x is a continuous non-negative function and k is only given case a positive constant k positive constant, and epsilon is strictly positive. Now by apply Gronwall's inequality on this. So, by applying Gronwall's lemma with some Gronwall's lemma is f x is whenever f x is less than equal to c plus integral a to x k into f t g t d t, we have the inequality f x is less than or equal to c times expansion of the integral a to x k into g t d t.

So, if you apply this Gronwall's inequality here with g x is equal to 1 and c is epsilon which is of course greater than 0. Then we get f of x is less than or equal to c times; c is epsilon times e to the power integral k times integral a to x k times, g is 1. So, this is k into d t which is equal to epsilon e to the power if you integrate x minus k times x minus a. So, therefore, this implies that this non-negative function f x is less than or equal to epsilon times e to the power k and x varies from a to b.

So, the upper boundary is b minus a. So, e to power b minus a is a finite quantity, and this is true for all epsilon any given epsilon. So, since epsilon is arbitrary and f x is greater than or equal to 0, we have f x is identically zero. So, this quantity is a finite quantity and this is true for all epsilon and f is non-negative. So, it implies that f x is equal to literally zero. So, this form of Gronwall's lemma we will apply in the uniqueness result.

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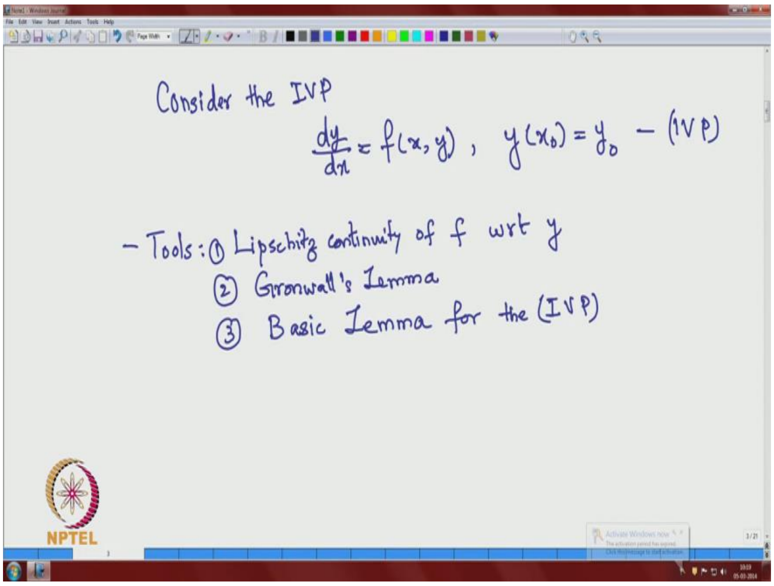

Okay, let us come to the uniqueness result. So, consider the initial value problem. So, d y by d x is equal to f of x comma y with the given initial condition y at x 0 is y 0. So, this is initial value problem, and we will prove that if f is Lipschitz continuous with respect to x, then the solution of this initial value problem is unique. So, by making use of the Lipschitz types Lipschitz continuity condition, we will prove that the uniqueness of the result and also we will invoke they are Gronwall's lemma. So, tools. So, Lipschitz condition Lipschitz continuity of f with respect to y and Gronwall's lemma. So, we also invoke a basic lemma for the initial value problem.

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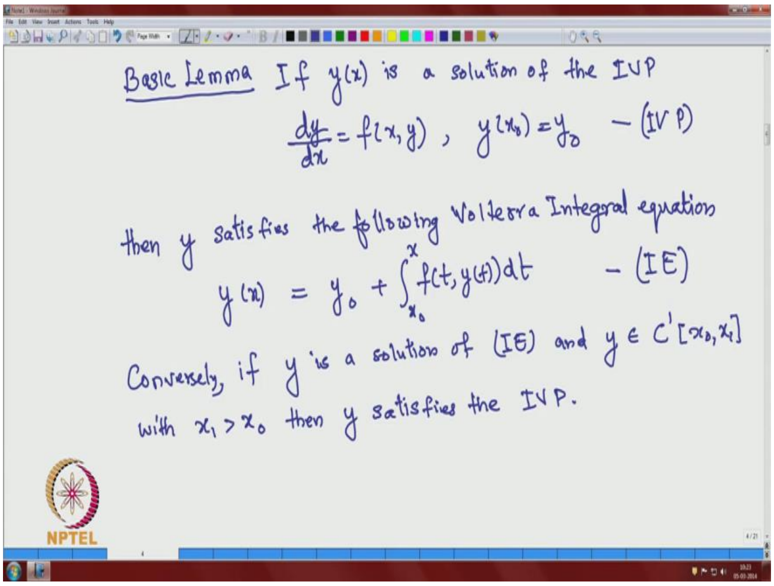

Let me state and prove a basic lemma for the initial value problem. If y x is a solution of the initial value problem d y by d x is equal to f of x y and y at x 0 is y 0, then y satisfies the following Volterra integral equation. Volterra integral equation is given by y of x is equal to y 0 plus integral x 0 to x f of t y of t d t. So, this is the integral equation. We denote this by I e and the initial value problem is I v p. So, the basic lemma is, say, y is a solution of the initial value problem, then satisfies the Volterra integral equation y x is equal to y 0 plus integral x 0 to x f of t y t d t.

And conversely, if y is a solution of the integral equation I e and y belongs to the cos of own continuously differential function if and all some intervals x 0 x 1 with x 1 as some number greater than x 0. So, we use the solution of the integral equation which is continuously differentiable on the interval x 0, x 1, then y satisfies the initial value problem means y is also a solution of the initial value problem.

So, one way we are converting the problem of solution of the differential equation into the problem of the solution of an integral equation. And this integral equation I e, the solution of the integral equation can be treated separately the solution of an integral equation a naught b is differentiable. In case it is differentiable, we can show that it is a solution of the initial value problem.

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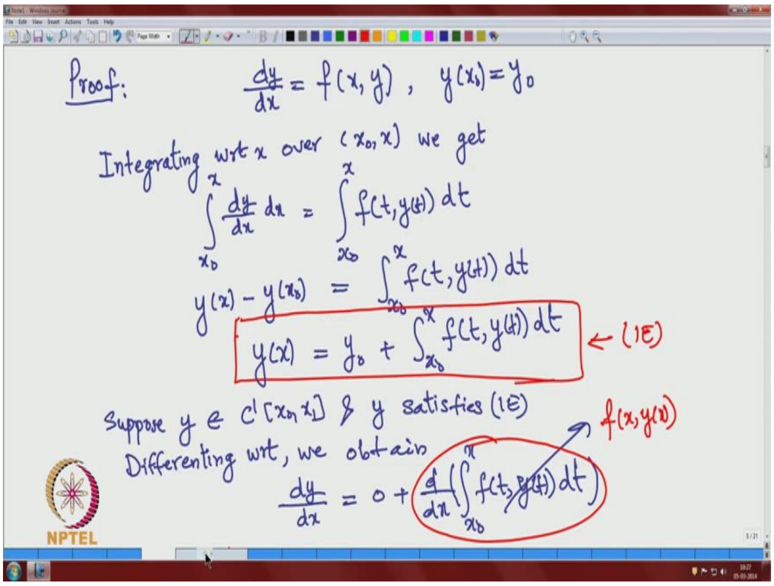

So, the proof of this basic lemma; so what we have is the differential equation d y by d x is equal to f of x y. Now integrating with respect to x over an interval x 0 to x we get. So, integral x 0 to x d y d x x is equal to integral x 0 to x f of we change the variable to t, and y is also a function of x y of t d t. And this is y at x minus y at x 0 is equal to integral x 0 to x f of t y t d t, and we know that the initial condition y at x 0 is y 0. So, therefore, I am pushing this to the right hand side y f x is equal to y 0 plus integral x 0 to x f of t y of t d t.

So, this is the integral equation Volterra integral equation. So, therefore, if y is a solution to the initial value problem, then y satisfies the integral equation. Now we will prove the other way. Suppose that y is the solution of this integral equation or y satisfies this integral equation and y is also continuously differentiable then we can. So, suppose y is in c 1 x 0 x x 1 and y satisfies I e, then differentiating with respect to x we obtained. So, d y by d x is equal to derivative of y 0 is 0 plus d by d x of integral x 0 to x f of t y of t d t using the Leibniz's rule for differentiating an integral. So, we get this turns out to be f of x y of x. So, that derivative of this integral is f of x y of x.

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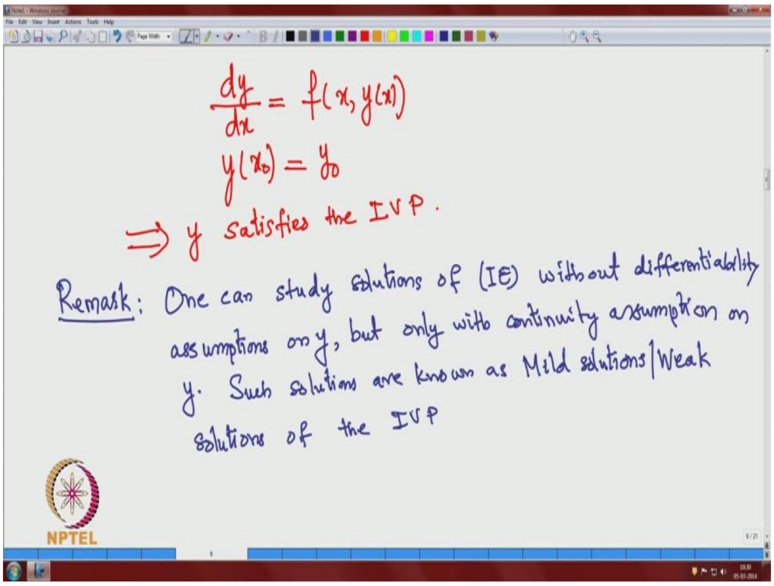

So, therefore, we get d y by d x is equal to f of x y of x. So, therefore, the function satisfies the differential equation and it is very easy to verify that y at x 0 for the integral equation. See in this integral equation if you look at the integral equation and if you substitute if you replace x by x 0, then this is integral x 0 to x 0, okay, that vanishes and y at x 0 is y 0. So, obviously, y at x 0 is y 0. So, this implies that y satisfies the initial value problem.

Now as I pointed out a remark. So, one can study solutions of the integral equation without differentiability assumptions on y but only with continuity assumptions on y. So, such solutions are unknown as mild solutions or weak solutions of the initial value problem. So, if we do not require the differentiability condition on y, then the solutions of the Volterra integral equation just defined are known as mild solutions or weak solutions.

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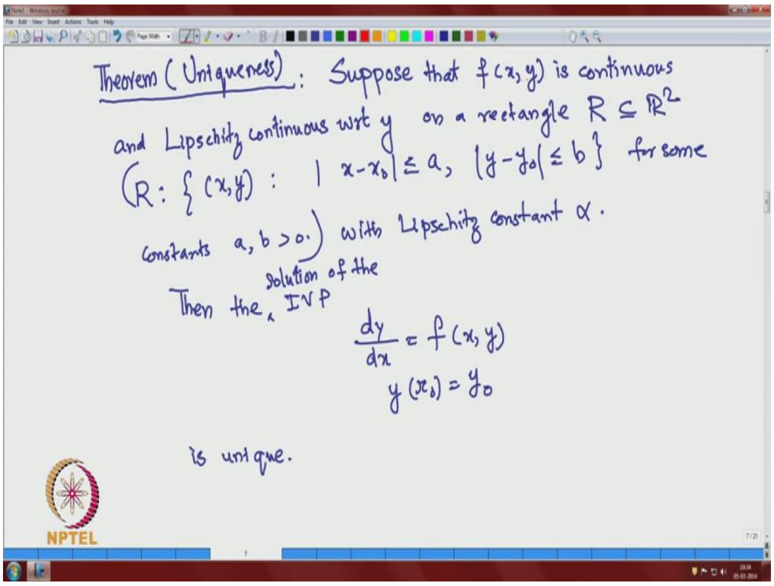

Now we are ready to prove the uniqueness result, so theorem. We now state and prove the uniqueness theorem for the initial value problem. Suppose that f of x y is continuous and Lipschitz continuous with respect to y on a rectangle r which is in r 2, where r is defined as r is a rectangle set of all x, y such that x minus x 0 is less than or equal to a, y minus y 0 is less than or equal to b for some constants a and b positive.

So, we assume that suppose f is a continuous and Lipschitz continuous with respect to y on a rectangle r and the rectangle is defined this way with Lipschitz constant alpha, then the initial value problem d y by d x is equal to f of x y y at x 0 is y 0, then the solution of the initial value problem is unique. If it has solution then the solution is unique, continuity is assumed with respect to both x and y, because of the continuity there exist a solution. And here our major emphasis is on the uniqueness and since f is Lipschitz continuous with respect to y, we are going to prove that the solution is going to be unique. So, if solution exists, then the solution is unique; it cannot have more than one solution if f is Lipschitz continuous with respect to y. We will prove it.

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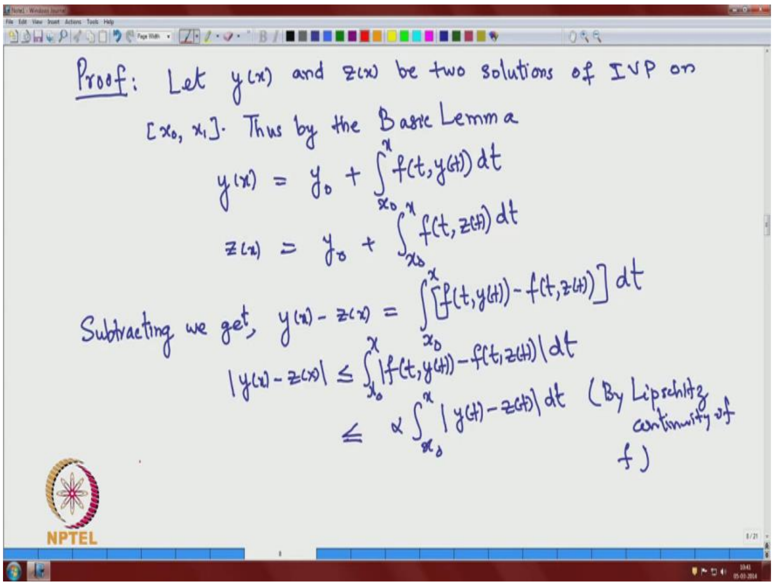

Proof, we invoke the Gronwall's lemma. So, let assume that it has got two solutions; let y of x and z of x be two solutions that y and z be two solutions of the initial value problem and on some interval x 0, x 1. So, thus by the basic lemma, basic lemma says any solution that is satisfying the initial value problem will also satisfy the integral equation.

So, basic lemma says a solution satisfying the initial value problem will also satisfies the integral equation. So, therefore, we are using the basic lemma. So, thus by basic lemma y x is equal to y 0 plus integral x 0 to x f of t y of t d t, and similarly z is a solution. So, z x is equal to y 0 plus integral x 0 to x f of t z t d t is also a solution. Now subtracting one from the other one; so subtracting we get y x minus z x is equal to y 0 on y 0 get cancelled and the integration lemma of integration is common, so integral x 0 to x f of t y t minus f of t z t d t.

Now if we take the absolute value y of x minus z of x is less than or equal to integral x 0 to x, absolute value f of t y t minus f of t z t d t. Now we have assumed that if is Lipschitz continuous with respect to the second argument y. So, f is Lipschitz continuous with respect to the second argument. So, therefore, we can use a Lipschitz condition or Lipschitz continuity. So, this is less than or equal to. So, by Lipschitz continuous alpha times integral x 0 to x y of t minus z of t d t; this is by Lipschitz continuity of f.

So, now you have an inequality that absolute value of y x minus z x is less than or equal to alpha times integral x 0 to x, absolute value of y t minus z t d t. So, now if we use a corollary of Gronwall's lemma, see the corollary of Gronwall's lemma if you have a situation like this if f x is less than or equal to integral a to x. So, if f x is less than or equal to integral a to s k f t d t then f x has to be 0; here the conditions on f is f is non-negative, continuous, and k is a positive constant. So, if the conclusion is f t is equal to zero. So, if we come to our inequality.

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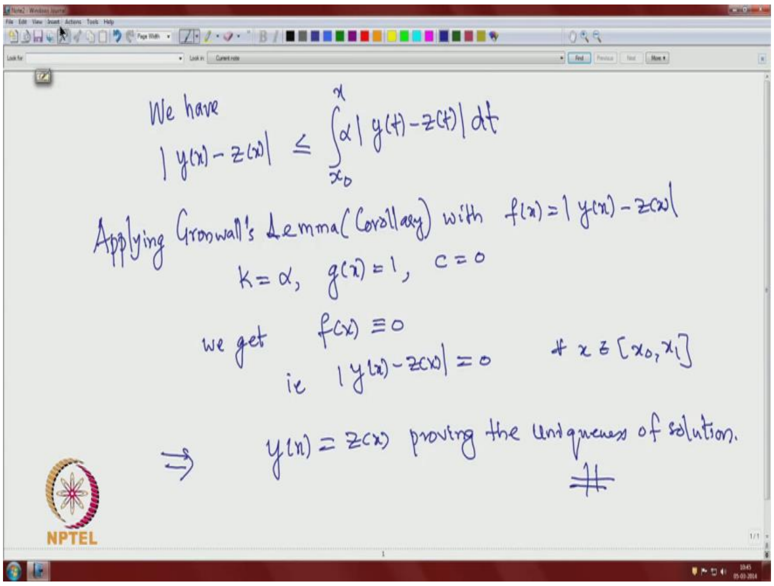

So, now by using Gronwall's, we have we obtained that absolute value of y of x minus z of x is less than or equal to integral x 0 to x, alpha times y of t minus z of t d t. So, applying Gronwall's inequality Gronwall's lemma with, so Gronwall's lemma we use a corollary with f of x is absolute value of y of x minus z of x and k is equal to alpha and g x the corollary is 1 and c is 0. So, we get f of x is literally 0. So, that is y x minus z x is equal to 0 for x in the interval x 0 to x 1.

So, the conclusion is. So, this implies that. So, we started with two solutions y and z and we have come to a conclusion that the difference between these two solutions for all x is 0. So, there is y of x is equal to z of x proving the uniqueness of the solution. So, the solution is unique. Now we will consider a few examples.

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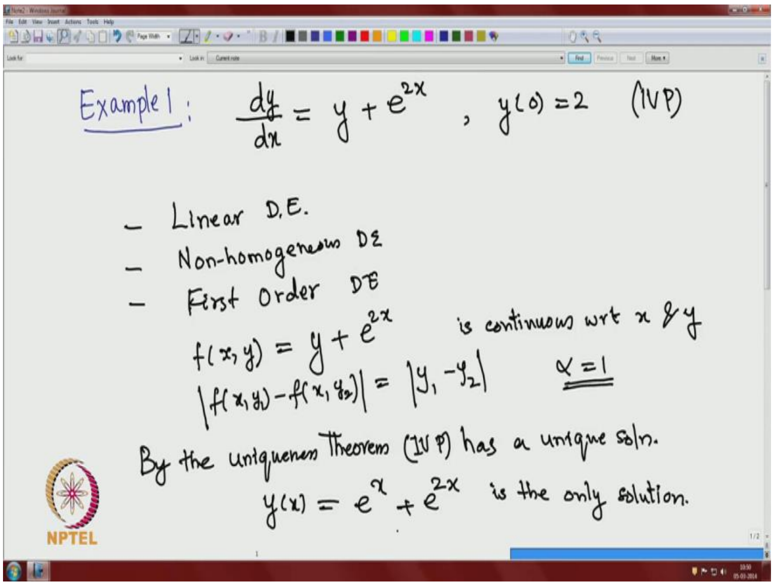

So, example call it example one, d y by d x is equal to y plus exponential 2 x with initial condition y at 0 is 2. So, look at this simple initial value problem. See, note that it is a linear differential equation and non-homogeneous. So, linear and non-homogeneous differential equation and this equation is first order. So, sufficient theory has already been developed and you have seen how to solve this equation by using the method using the integrating factor techniques and all. So, let us analyze the existence of solution uniqueness of solution. Okay, here the function f of x, y in the right hand side of this equation is y plus e to the power 2 x, and this is continuous.

So, with respect to x and y; it is linear in y and it is an exponential function; in x it is a continuous function in the all r 2. So, f is continuous, and what about the Lipschitz continuity? So, f of x y 1 minus f of x y 2 which is equal to y 1 minus y 2; so it is linear. So, therefore, it is Lipschitz continuous, here the Lipschitz constant alpha is 1. So, it satisfies all the conditions of the uniqueness theorem. So, the function is continuous in x and y and the function is Lipschitz continuous with respect to y with the Lipschitz constant y. So, therefore, y is the uniqueness theorem.

So, by the uniqueness theorem I v p this initial value problem has a unique solution. This linear problem has a unique solution; remember that not all linear problems linear initial value problems are having unique solutions. We have seen example, and further, we will see examples. And by using the integrating factor and the method which we have already seen, we have seen that a solution of this equation is given by y x is equal to e to the power x by applying the initial conditions e to the power x plus e to the power 2 x is the only solution. This solution is found by using the method we studied earlier and by the uniqueness theorem this is the only solution.

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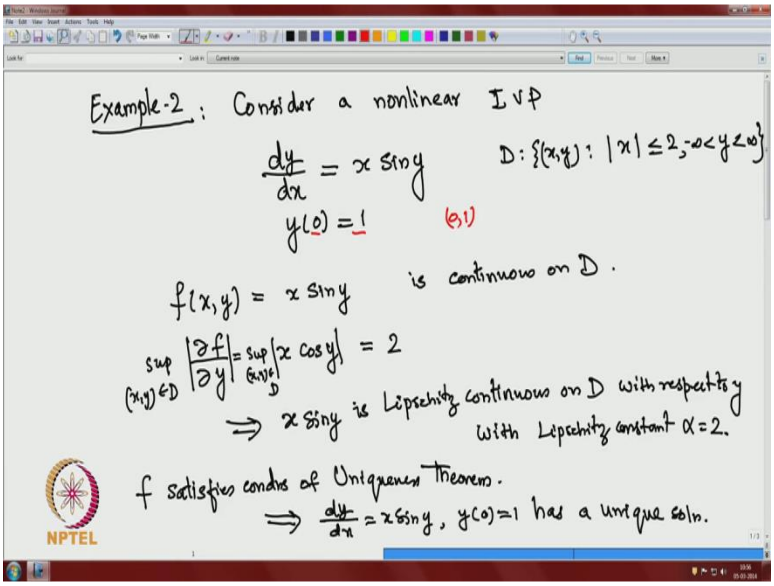

And now let us consider another example, so example 2. So, let us consider a non-linear initial value problem. So, consider a non-linear initial value problem given by d y by d x is equal to x into sin y, where this function is defined on for x on a domain; domain is defined by set of whole x, y where x is bounded by 2 and y is free. So, y varies from minus infinity to plus infinity and x is bounded by 2. So, let the initial condition be y at 0 is 1; we will analyze the existence and uniqueness.

So, first let us write down the function, the right hand side function f of x y is x into sin y is continuous only, and you check whether this function is Lipschitz continuous with respect to the second argument y. To check the Lipschitz continuity of f with respect to y, we have stated and proved a sufficient condition. If the partial derivative of f with respect to y is bounded in the given domain, then the function is Lipschitz continuous with respect to y in that domain.

So, the partial derivative, so del f by d y is equal to x cos y and if you take the bound, say you take the supremum of this one for all x y in d. So, this is hence maximum is enough. So, maximum is attained. So, x y in d; so this is equal to cos y is bounded by 1 and x is bounded by 2. So, the supremum is 2. So, therefore, this implies that x sin y is Lipschitz continuous on d with respect to y with Lipschitz constant alpha is equal to 2. So, therefore, this function satisfies all the conditions of the uniqueness result. So, f satisfies conditions of uniqueness theorem.

So, this therefore, the conclusion is. So, d y by d x is equal to x sin y with y at 0 is 1 has a unique solution starting from the given point of 0, 1. But the existence part as I have already mentioned if the function is continuous with respect to x and y, then existence is guaranteed in some interval starting from the given initial point 0, 1. And now what we have proved or observed is the uniqueness is guaranteed because of the Lipschitz continuity property of f with respect to y. Now we will see an example where we have more than one solution.

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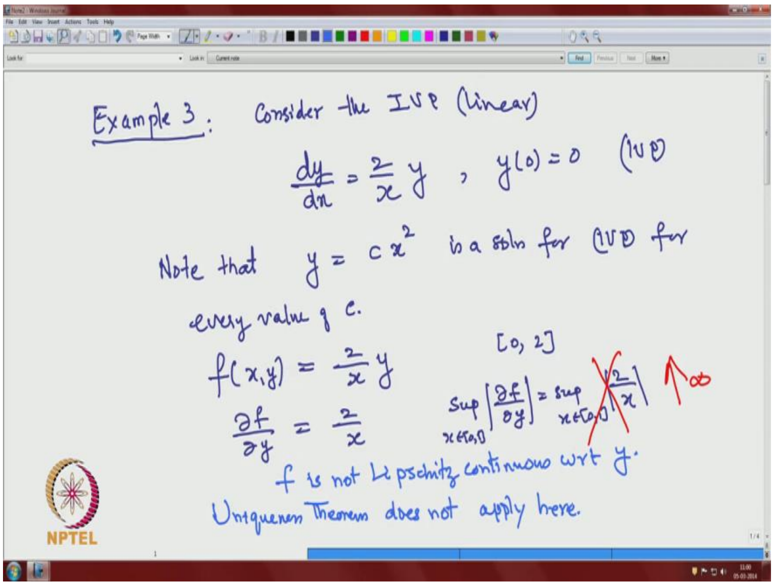

That example we have already seen a linear equation. So, I will take an example, so example three. So, consider the initial value problem; of course, this is a linear one d y by d x is equal to 2 by x into y, and the initial condition y at 0 is 0. So, in our earlier sessions we have obtained its general solution and also its solution satisfying the initial condition. So, note that y is equal to c into x square is a solution for the initial value problem for every value of c.

So, therefore, we have seen that it has got infinite linear solution. Let us just check why or how we compare this with the uniqueness result. So, in this case f of x y is 2 by x into y on an interval, say, 0 to some number, say, 0 to 2; 0 is a point given and del f by del y, obviously, it exists is 2 by x, but if we take the supremum of del f by del y when x and y are here; it depends only on x when x is in the interval, say 0, 1 which is sup.

So, this does not exist; we know that this blows up. So, therefore, the function f is not Lipschitz continuous with respect to y; see at x is equal to 0, you see the singularity is, therefore, this differential equation as x goes to 0 2 up on x blows up. So, if is not Lipschitz continuous with respect to y; so therefore, there is no surprise why this equation does not have a unique solution. So, the uniqueness theorem does not apply here. And we have several other examples which we dealt with.

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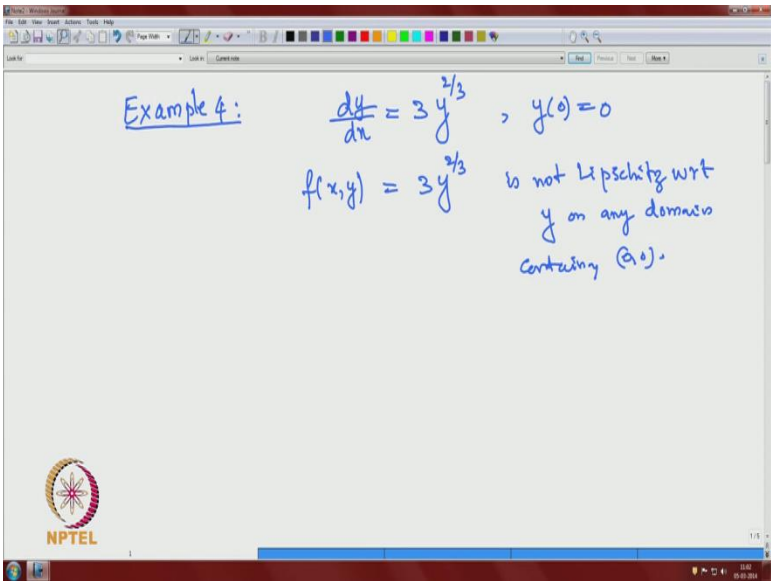

One example which can be done as an exercise, so example four; so consider the initial value problem d y by d x is equal to 3 y's upon 2 by 3; the initial condition y at 0 is 0. We have seen the earlier lecture that this differential equation has got infinitely many solutions including y x is equal to x cube. So, it has got many solutions, and in this case also if we look at the function f of x, y is equal to 3 y 2 by 3. So, it is a continuous with respect to both the arguments, but it is not Lipschitz with respect to y on any domain containing the point 0 series. So, it is not Lipschitz on any domain containing 0 series.

So, therefore, the uniqueness is not assured by the uniqueness theorem, okay. So, given a differential equation, we can make out whether the equation has a unique solution provided the function f is Lipschitz continuous with respect to y; that is one of the sufficient conditions. Again remember is not necessary; there can be a function which does not satisfy Lipschitz continuity, but still thus it can have a unique solution, and we will prove the existence theorem in the coming lectures. So, by various existence theorems also we will see, bye.