Ordinary Differential Equations

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Module - 4
Lecture - 16
Gronwall's Lemma

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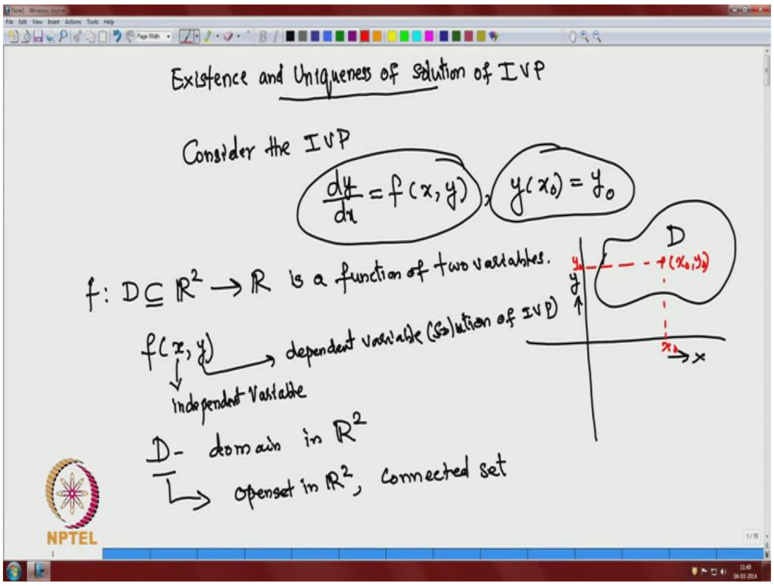

Welcome back. We will continue the discussions on the existence and uniqueness of solutions of initial value problem. The previous lecture, we have seen several examples of initial value problems. Some are having infinitely, many solutions, and some are having no solutions, and others having infinitely, many solutions. Now, we will characterize under what condition, an initial value problem will have unique solution, like the problem posed by Hadamard. Once an initial value problem is given, under what condition, this system has a solution, and under what condition, the solution is unique, and conditions for which, the solution continues continuously, varying with respect to the initial condition. So, consider the initial value problem given by the first order differential equation, d y by d x is equal to f of x comma y, with a initial condition y at x 0 is y 0. So, f is a function. Here f is a function, which is defined on a domain d, a subset of d, which is a subset of r 2 to r is a function of two variables, as it is defied on r 2; f is a function of two variables x and y.

With respect to the initial value problem, the first argument x that is an independent variable, and y is a dependent variable, which is of course, the solution we are looking for. So, f is defined on a domain d. So, d is a domain in r 2. By domain, we mean d is an open set in r 2 and also, it is connected; it is a connected set. Any two given points in d, that can be joined by a continuous curve, which is completely, inside d. So, d is a domain where, r 2 is. So, initial value problem given by d y by d x is equal to f of x y and y at x 0 is y 0. You have, this is your x axis and this is your y axis and a point. So, this point is x 0. This point is y 0. So, initial point is x 0 y 0. We are looking for a function starting from this point, and f is defined on the domain d, and to make sure that first of all, the solution, the initial value problem has a solution; we will require certain conditions on f. So, let me first, state the conditions and we will do the proof later.

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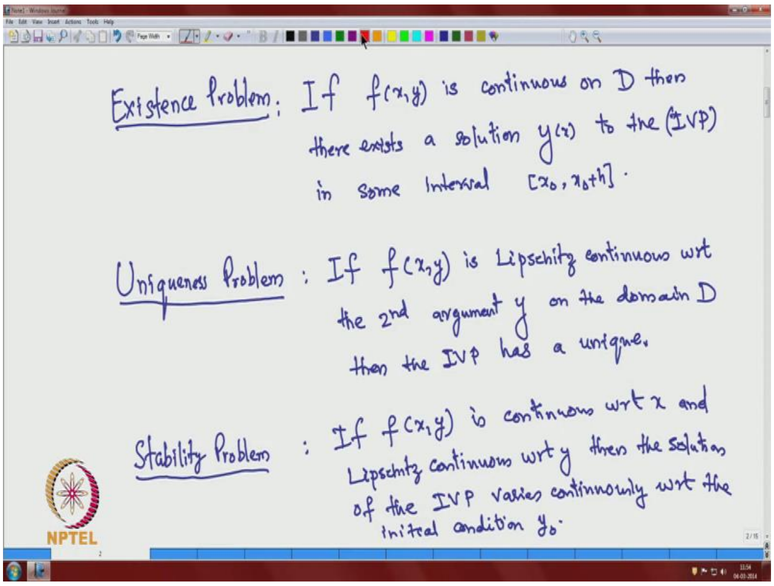

So, for existence, if a f of x y is continuous on d, then there exists a solution y of x to the initial value problem in some neighbourhood, in some interval, say x 0 to x 0 plus h. So, we will prove this existence results later. Broadly speaking, if it is continuous in the given domain, the domain contained initial point x 0 y 0, then one can show that there exists a solution, at least, in some neighbourhood that contain the initial point x 0 y 0. Now, uniqueness problem; if f x y is lipschitz continuous, with respect to the second argument, argument y on the domain d, then the initial value problem has a unique solution, rather the solution of the initial value problem is unique, if solution exist. If x y is lipschitz continuous with respect to the second argument that is y, then the solution is unique. The third problem is stability. If f of x y is continuous with respect to x and lipschitz continuous with respect to y, then the solution of the initial value problem varies continuously, with respect to the initial condition y 0. So, these are the 3 major solutions to the Hadamard problem.

We will address each one of them separately, and we will give sufficient proofs for each of them. See, in the existence problem, what we recover is the continuity of f with respect to both x and y; continuous with respect to x and y, because f is defined on the domain d, which is a subset of r 2. If f is continuous with respect to x and y, then there actually, we can show that there exists a solution in some neighbourhood, containing the initial point x 0 y 0. The second problem, uniqueness, is guaranteed if f is lipschitz continuous with respect to the second argument y, and of course, continuity is required for the existence of a solution. So, continuity is also, required for the existence of solution, and the uniqueness is guaranteed, if f is lipschitz continuous with respect to the second argument.

The stability, the continuous dependence of the solution on the initial condition y 0 is also, guaranteed if f is lipschitz continuous with respect to y, and continuous with respect to x. We we will show it. For a lipschitz continuity, we have already seen the definition and examples of a lipschitz continuous function in the preliminaries. There, we treated function of a single variable, a function of x only; f of x and also, we have seen sufficient conditions under which, a function is lipschitz continuous and examples of lipschitz continuous functions, and examples of functions, which are not lipschitz continuous. Now, since we deal with functions of two variables x and y, we will briefly deal with the lipschitz continuity of f, with respect to the second argument y.

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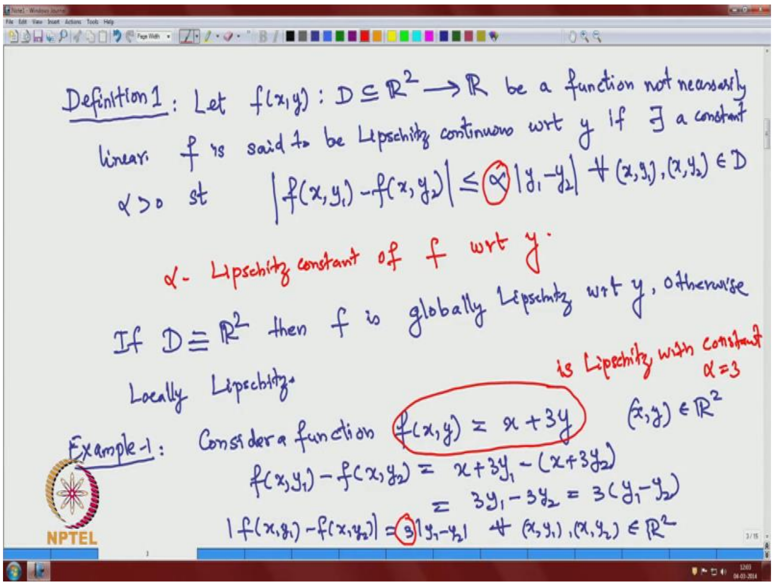

Definition; definition 1; let f x y be a function, which is defined on d with a subset of r 2, x y plane, and it is mapping to r. Let it be a function, not necessarily linear; could be a linear or non-linear function. Now, f is said to be lipschitz continuous with respect to y, if there exists a constant; call it alpha greater than 0, such that f of x y 1 minus f of x y 2; the absolute value of this is less than or equal to alpha times y 1 minus y 2 for all x y 1 and x y 2; x y 1 comma x y 2 are in the domain d. If there exists a constant alpha of satisfying this condition, you take the least of all such alphas, and we call that alpha as a lipschitz constant. So, alpha is a lipschitz constant; constant of f with respect to the argument y. Now, as in the single variable case, if d is the odd space r 2, if d is the entire space r 2, then we say that f is globally, lipschitz continuous with respect to y; otherwise, locally lipschitz. For an existence and uniqueness; for uniqueness result, a local lipschitz continuity is sufficient.

Now, let us see an example; say 1. Consider a function f x y, which is given by x plus 3 y where, x and y, they are on, then they are x y, is on the entire r 2. So, let us check the lipschitz continuity of f. So, f of x y 1 with respect to y, f x comma y 2, which is x plus 3 y 1 minus x plus 3 y 2, which is equal to 3 y 1 minus 3 y 2; 3 times y 1 minus y 2. Therefore, modulus of x y 1 minus f x y 2 is equal to 3 times y 1 minus y 2 for all x y 1 and x y 2 in r 2. So, this shows that f is lipschitz with a lipschitz constant 3. Therefore, this function f given by f is equal to x plus 3 y, is lipschitz with lipschitz constant alpha is equal to 3, and this lipschitz continuity is global, because it is true for all points, x y 1 and x y 2. Therefore, it is, f is globally lipschitz with respect to y. Now, consider another example.

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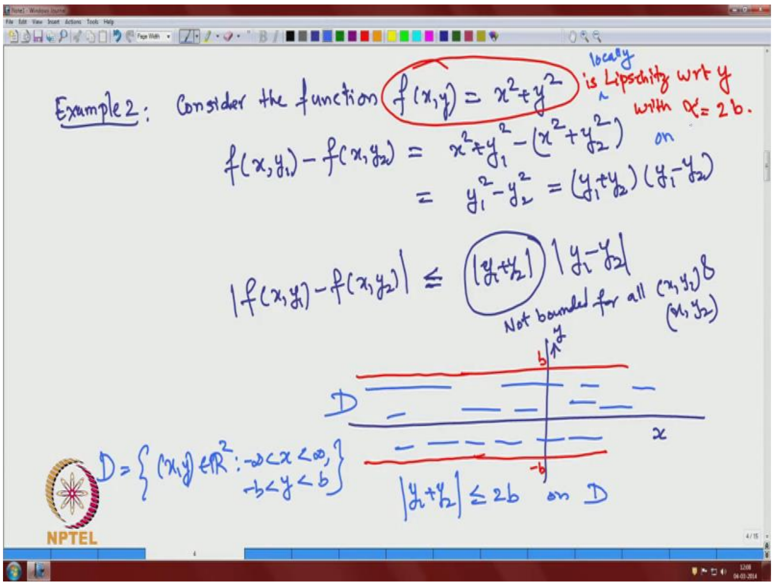

Consider the function of two variables x and y given by x square plus y square. So, these two variable functions; let us check the lipschitz continuity. So, f of x y 1, the first point and f of x y 2, the second point; these two points are arbitrary points of the x y plane; that is in r 2, which is equal to x square plus y 1 square minus x square plus y 2 square, which is equal to y 1 square minus y 2 square, which is expanded as y 1 plus y 2 into y 1 minus y 2. Therefore, f of x y 1 minus f of x y 2 is equal to, if it is less than or equal to y 1 plus y 2, if you can bound times y 1 minus y 2. We know that this quantity y 1 plus y 2, as long as these points are arbitrary, varying in all r 2, is not bounded for all x y 1 and x y 2. Of course, if y is bounded, the y coordinate is bounded, then these quantities can be bounded by a constant, say for example, if you take an infinite strip, this is x axis and this is y axis. If I vary y between say, minus b and b, and if I take this infinite strip, if I call this as d; d is set of all x y in r 2, such that x is varying from minus infinity to plus infinity, and y is varying from minus b to plus b.

On this d, the given function say, this y 1 plus y 2, is bounded by 2 b. So, now, y 1 plus y 2 is less than equal to 2 b on d. Therefore, we conclude that this function, given by f x is equal to x square plus y square, is lipschitz continuous with respect to y with constant, the lipschitz constant, alpha is equal to 2 b. So, a function of two variables x and y and we have verified the lipschitz continuity, and this lipschitz continuity is local. So, this is lipschitz continuous and this function is lipschitz, and the lipschitz continuity is local. So, this is locally, lipschitz with respect to y, and that is on the domain, we just defined d. So, on the domain defined by this, set of whole x y, such that x is varying from minus infinity to plus infinity, and b is varying from minus b to plus b; it is an infinite strip parallel to x axis. We now, see a function, which is not lipschitz.

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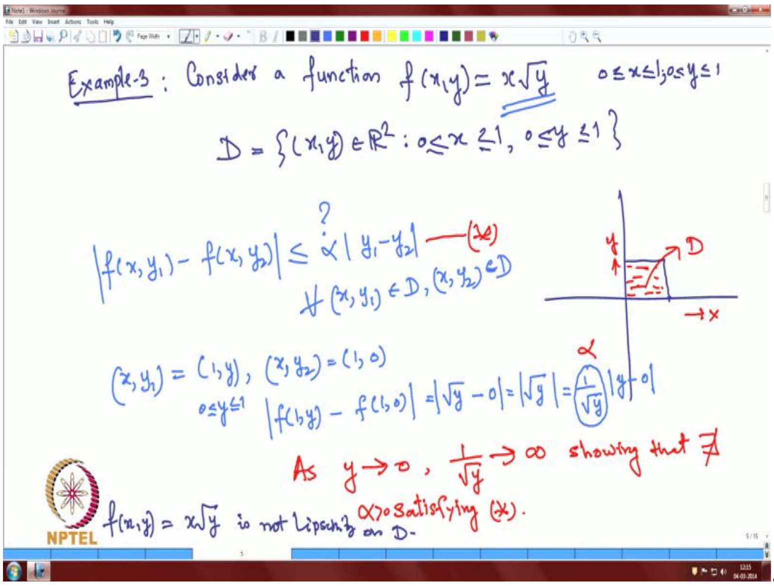
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Example of a non lipschitz function, example 3; consider a function f of x y, a function of two variables, which is given by x into square root of y. Suppose that this is defined on domain where, x varies from 0 to 1 and y varies from 0 to 1. So, the domain you are looking for, domain is set of whole x y in r 2, such that x is varying from 0 to 1 and y is also, varying from 0 to 1. So, this is a domain. Now, let us check the lipschitz continuity of this function on this domain. So, let us compute f of x y 1 minus f of x y 2. Now, we are looking for a condition like this; f of x y 1 minus f of x y 2; value of this 1 is less than equal to alpha times y 1 minus y 2, for all x y 1 in d and x y 2 also in d, for all points x y 1 and x y 2 in d. We show that it is not possible for this function to have a condition like this. So, this is a question mark; whether, this condition is satisfied or not for any arbitrary point x y 1 and x y 2.

Let us take two points x y 1; I am taking as 1 0, or 1 y, arbitrary y, and x y 2; let us take that as 1 0; 1 y where, y is a value where, y varies between 0 and 1; y could be any value between 0 and 1, and x 2 or y 2 is a 0, and x is 1. In both points, the x coordinates are 1; 1 y and 1 0. Let us compute. Therefore, f of 1 y minus f of 1 0; this is equal to f of 1 y, by definition of your function. So, x is 1 and this is square root of y minus f of 1 0; y coordinate is 0; this is 0. So, this is equal to square root of y and y is positive. This, I write as y is 1 by square root of y times. This is, times y minus 0. So, if we can bound this quantity by a constant finite number alpha, then it satisfies lipschitz condition for this point. We know that as y is approaching to 0 from right side of course, the quality 1 y square root of y; it goes to infinity, showing that there does not exist an alpha, satisfying a finite number, alpha positive, satisfying the condition; if I call this condition as star, satisfying star.

So, as y approaches to 0, as this is a x axis and this is y axis as y approaches to 0, then this quantity 1, 1 upon square root of y; that blows up. Therefore, lipschitz condition is not possible. So, the conclusion is the function f x y, given by x square root of y, is not lipschitz on the given domain d. Therefore, this function is not lipschitz continuous and not locally lipschitz continuous on d. Therefore, always, to check lipschitz continuity by this method, may not be that very straight forward. Now, as in the case of one variable, there are sufficient conditions, which ensure the lipschitz continuity of a function of two variables. Now, I will state a theorem that ensures or guarantees the lipschitz continuity of the function of two variables with respect to y; so theorem.

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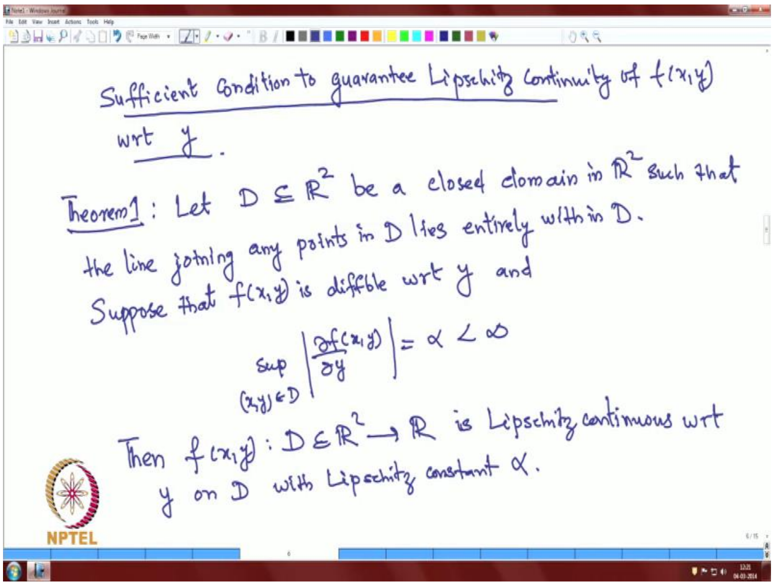

Say, sufficient condition to guarantee lipschitz continuity of f, which is f a function of two variables with respect to the second variable y. So, we deal with sufficient conditions. So, we state this. Therefore, the theorem1; let d be a closed domain in r 2. So, that is of domain; it is a connected set and open connected set, which is also closed. So, we look for a closed domain in r 2, such that the line joining any points in d, lies entirely within d. So, we take a domain, a closed domain, such that any two points, the line joining them, should lie completely inside d. Suppose that the supremum, suppose that f x y is differentiable, with respect to y and the supremum of the partial derivative, del f by del y, the supremum of the absolute value of the partial derivative of f with respect to y where, x and y lies in the domain d; the supremum is attained and which is equal to alpha, is a finite number, if the partial derivatives are now, bounded in this sense.

Then, the conclusion of the theorem is then, f of x y, which is from of course, from d to r is lipschitz continuous, with respect to the second argument, with respect to y on the domain d with a lipschitz constant alpha. So, the sufficient condition that guarantees the lipschitz continuity of a function f is that function f x y, if f is differentiable with respect to the second argument y, and if the partial derivatives with respect to y, is bounded of supremum of absolute value of the partial derivative is a finite number in the domain, then the function f itself, is lipschitz with respect to y. The proof is very simple; I will just give the proof.

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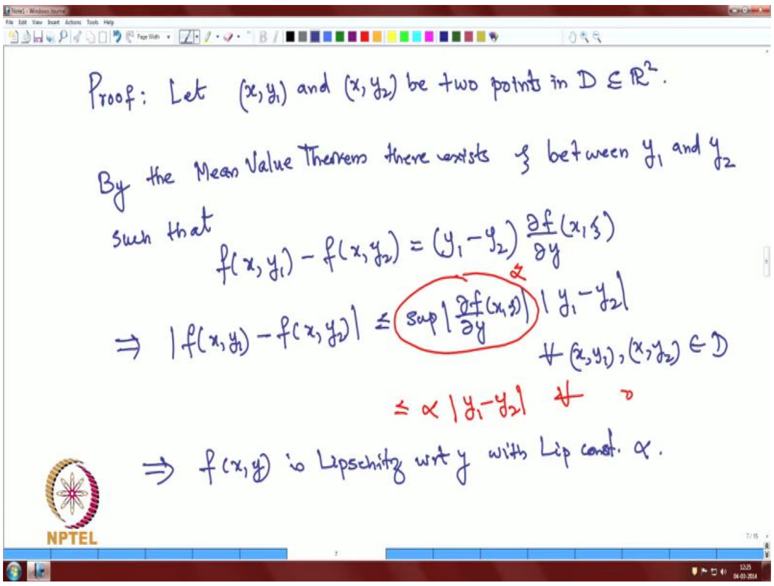

The proof is just by using the mean value theorem. So, let x y 1 and x y 2 be two points in d. Now, by lagrange, now, mean value theorem, we have there exists a number psi between y 1 and y 2, such that f of x y 1 minus f of x y 2 is equal to y 1 minus y 2 times del f by del y, at the point x psi. So, this psi is a point between y 1 and y 2; that is guaranteed, that is assured, because we assume that in the domain d, the line joining any two points that lies entirely, inside. Therefore, psi is a point between y 1 and y 2, which is totally, inside the domain d.

Now, taking the absolute value; this implies the absolute value of f of x y 1 minus f of x y 2, which is obviously, less than or equal to supremum of del f by del y, at x psi times y 1 minus y 2 for all x y 1 and x y 2 inside d. By hypothesis, this is bounded by, and this is alpha. So, this is less than equal to alpha times y 1 minus y 2 for all x y 1 and x y 2 in d. So, proving that f x y is lipschitz with respect to y, with lipschitz constant alpha. Therefore, if a function is differentiable, it is easy to check the lipschitz continuity, if we can find the bound for a partial derivative with respect to y.

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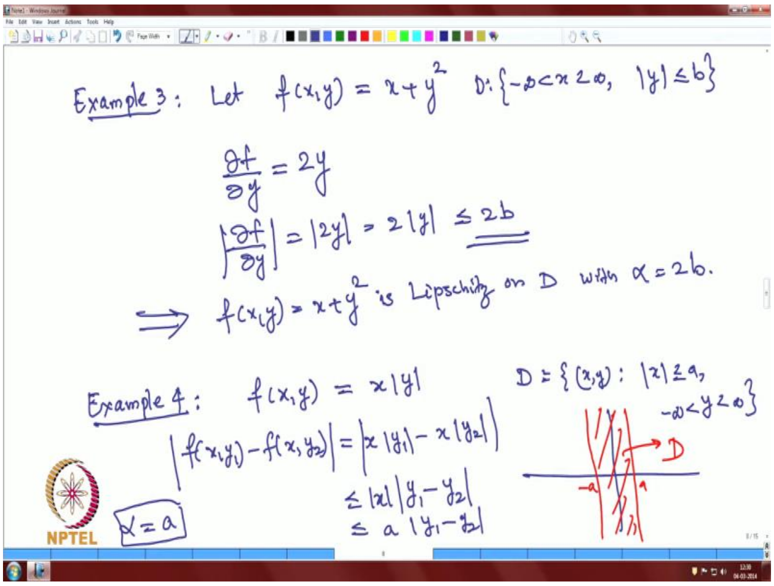

So, let us take the example, a similar example, which we considered; example 3. So, let f of x y is equal to x plus y square where, x varies from minus infinity to plus infinity and y is bounded by b. So, absolute value of y is less than equal to b. So, obviously, del f by del y, the partial derivative of f with respect to y, is 2 y and del f by del y is equal to 2 y, is equal to 2 times y, and y is bounded by b. Therefore, this is less than equal to 2 times

b. So, this shows, this implies that f of x y, given by x plus y square, is lipschitz on if we say this is a domain, if domain d is given by this on the domain d with alpha is equal to 2 b. At this junction, we know that the condition in the theorem is not a necessary condition for a lipschitz continuity; it is just a sufficient condition. If the partial derivates are bounded, then the function is lipschitz continuous. There is a strong sufficient condition.

There are functions, which are lipschitz continuous, but at the same time, not satisfying the hypothesis of the above theorem. So, we give a counter example. Consider a function f of x y is given by x into absolute value of y, but the domain is, which is defined on a domain where, x y, such that mod of x is less than equal to a, and y is between minus infinity plus infinity. It is also an infinite strip where, y varies from minus infinity to plus infinity. This is bounded between minus a to a. So, this is a domain. So, if we look, check for a lipschitz continuity, f of x y 1 minus f of x y 2 where, x y 1 and x y 2 are two points in the domain, which is equal to x absolute value of y 1, minus x absolute value of y 2. If we take the absolute value of this, this is less than or equal to mod of x into absolute value of y 1 minus y 2, and as x is bounded by a. So, this is a times absolute value of y 1 minus y 2. So, this shows that the function f given by x into mod y is lipschitz with a lipschitz constant, alpha is equal to a. However, this function does not have a partial derivative with respect to y, as it is not differentiable with respect to y at 0. Therefore, the conditions in the theorem is just necessary, just sufficient.

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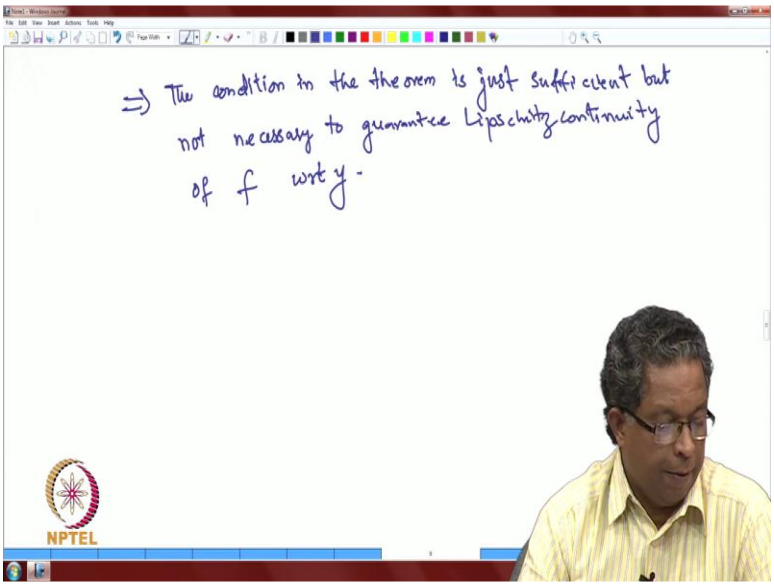

So, the conclusion the condition if the theorem is just sufficient, but not necessary to guarantee lipschitz condition, lipschitz continuity of f with respect to y. Therefore, a given differential equation, we can check the continuity of f with respect to x and y, and also check the lipschitz continuity of f with respect to y, to address the question of existence, uniqueness and continuous dependence of the solution, with respect to the initial condition. Also, note that lipschitz continuity is a stronger notion than continuity. Any lipschitz continuous function is continuous, but lit less than that differentiability. For proving the uniqueness result, we employ a very important lemma, which is known as Gronwall's lemma.

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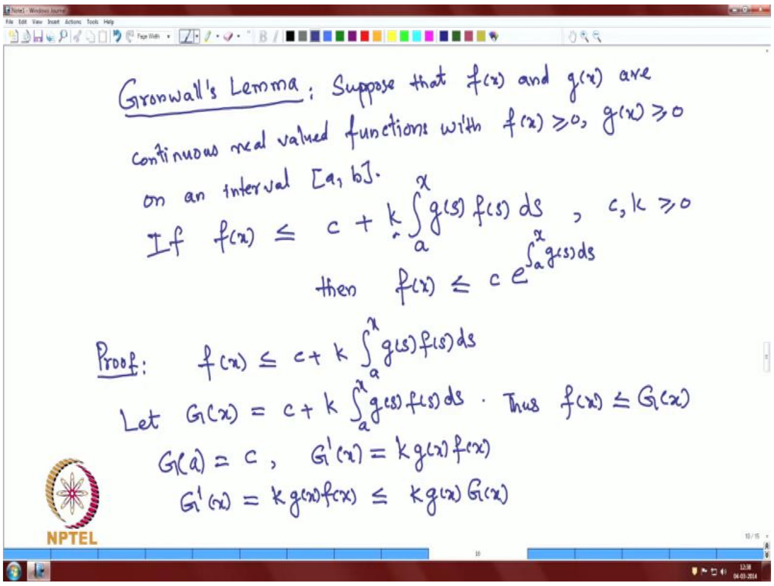

So, this will be invoked to prove the uniqueness of solution of an initial value problem. So, let me state the Gronwall's lemma. Suppose that f x and g x are continuous real valued functions with a condition that f of x is greater than equal to 0 and g of x is also greater than equal to 0 on an interval; call it a b. So, if f x and g x are two continuous real values functions, both nonnegative, and defined on interval a b, then if we have an inequality; that is f x is less than or equal to some constant c, plus k times integral a to x g s, f s, d s where, c and k are nonnegative constants; then, the conclusion is f of x is less than or equal to c times exponential integral a to x g s, d s. So, this inequality is known as a Gronwall's inequality. If f x is less that equal to c, plus k times integral a to x g s, f s, d s, then f x has to be bounded by f x is less than equal to c times, e to the power integral a to x g s, d s. In the first inequality, f x is there on both sides, left hand side and the right hand side, and the conclusion is f x; if this is the case ,then f x can be bounded by a function, which is independent of f.

The proof of this inequality is what we have is, we have f x is less than equal to c plus k times integral a to x g s, f s, d s; this we have. If I take the RHS and denote this by g x, let g x is equal to c plus k integral a to x g s, f s, d s. So, this implies that, thus, what we have is f of x is less than or equal to g x by definition; it is a hypothesis, plus g x is your RHS. So, f of x is less than equal to g x and also, g of a, if you evaluate g at a, integral a to a, which is 0, which is just c. If you differentiate g, g prime x is equal to if use leibniz formula and differentiate this integral; first part is 0, then k times, this is k times g x, f x. Therefore, g prime x is, therefore, we get e prime as this is equal to k g x, f x, and this f x is less than equal to g x. So, this implies that this is less than or equal to k times g x, capital g x.

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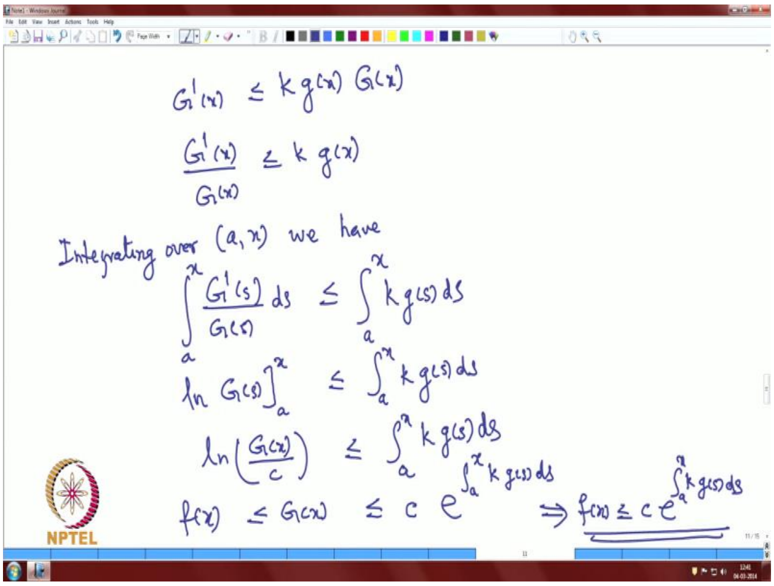

In other words, what we have is g prime x is less than or equal to k small g x into capital g x, and if you divide this by g x; g prime x divided by g x is less than or equal to k small g x. Now, integrating this over the interval a x, we have that integral of g prime x by, which is g prime is equal to g s by g of s, d s, integral a to x, is less than equal to integral a to x, k times g of s, d s, by using the integral of derivative by the function, which is nothing, but the logarithm l n of g s, and evaluated at these two points a and x, is less than equal to integral of a to x k g s d. Since, g of s at a is c, this is l n of g x divided by c less than or equal to integral 0 a to x k g s, d s. Taking exponential on both sides, we get g x is less than or equal to c times, e to the power integral a to x, k g s, d s, and from our hypothesis, we know that f x is less than or equal to g x. Therefore, f x is, this implies f x is less than or equal to c times exponential integral a to x k g s, d s. So, this proves the Gronwall's inequality.

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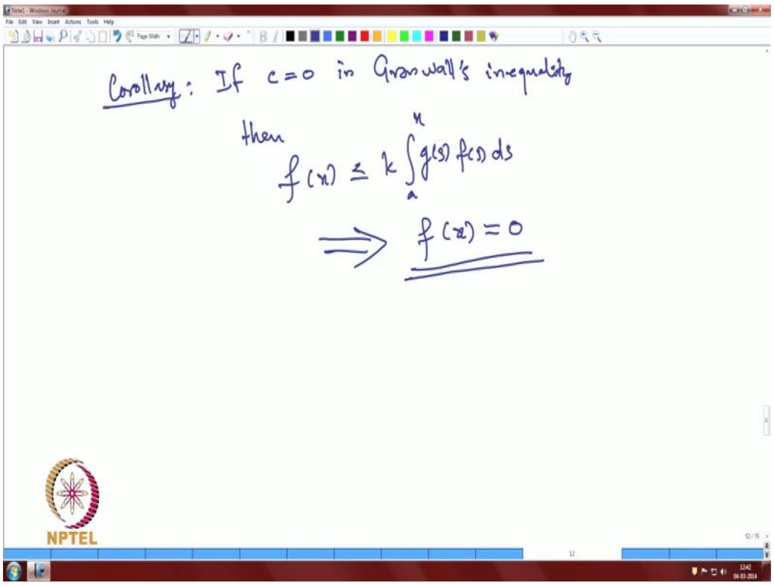

As a remark in the Gronwall's inequality, we can say as a coronary in the Gronwall's inequality is if c is equal to 0 in Gronwall's inequality, then f of x is less than or equal to k times integral a to x g s, f s is d s; this implies that f of x is equal to 0. So, that proof follows very easily, because if we take for every epsilon, if we take an epsilon greater for every epsilon greater than 0, the Gronwall's inequality holds, and if that holds, then you get epsilon times k, integral a to x d s, f s. So, that makes that f x to be 0, since f x is non negative. So, this result, we will be using in proving the uniqueness result; the Gronwall's inequality.

Now, we have build up necessary tools to prove the uniqueness of an initial value problem, if the function f is lipschitz continuous with respect to y. So, the uniqueness will be proved in the next theorem. Therefore, two things we have seen that one is if the function is continuous with respect to x and y, then there exists a solution to the initial value problem. If the function is lipschitz with respect to y, then the solution is unique, and continuous dependence will be shown later. We will see the proof of this in the subsequent lectures.

Bye.