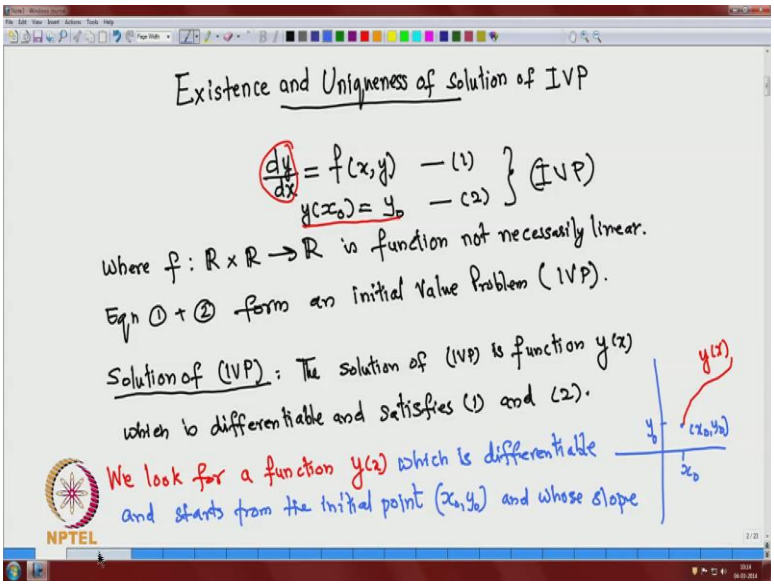
Ordinary Differential Equations

Prof. Raju K George
Department of Mathematics
Indian Institute of Science, Bangalore

Module - 4
Lecture - 15
Well-posedness and Examples of IVP

(Refer Slide Time: 00:38)

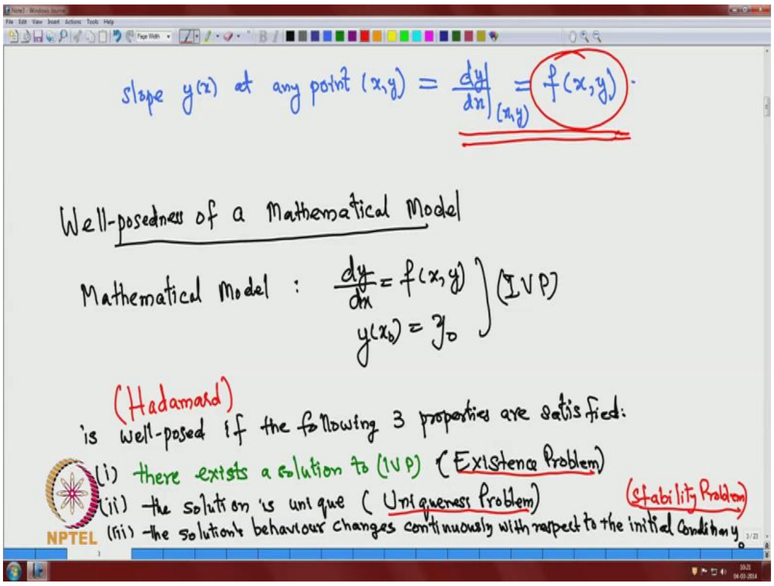


Welcome back to the series of lectures on ODE and its applications. Today, we will discuss about the existence and uniqueness of solution of initial value problems. You have already seen that in the theory of first order and second order differential equations, the need of existence and uniqueness of solution. In this session, we deal with an initial value problem that is a differential equation of the form, d y by d x is equal to f of x y. So, call this as a given differential equation where, f is a function, is defined on r cross r to r, is a function, not necessarily linear.

So, this model can handle both linear and non-linear differential equations. Along with differential equation, we also give an additional initial condition; y at x 0 is equal to y 0. So, call this as equation number 2. This equation 1, along with a initial condition 2, is known as a initial value problem, and equation 1 plus 2; this forms an initial value problem, and commonly, written as IVP; initial value problem. Now, this model, this first order differential equation could be a model, derived from a physical system, and a very important question; once a model of this form is framed, is does there exists a solution to this equation? If the solution exists, will that solution be unique and also, will the solution vary continuously, with respect to the initial condition y 0? So, these are the major three problems, which we will discuss in details in this session. Say, first let us see what do you mean by a solution of this differential equation?

So, solution of IVP, say equation 1 and 2, did this formal IVP; the solution of I v p is a function, say y x, which is differentiable and satisfies; it satisfies the equation 1 and the initial condition 2. So, a solution; since, the equation is d y by d x is equal to f of x y, we recover that the solution y, has to be a differentiable function. It has to satisfy the differential equation. Therefore, it is a differential function and also, satisfies a given initial condition; y at x 0 is equal to y 0. In other words, we are looking for a function. So, we look for a function y x, which is differentiable, and starts from the initial point x 0 y 0; that is, if this point is x 0 and this point is y 0, say this point x 0 y 0; we are looking for a function; we start from x 0 y 0 and whose slope of y x at any point x y in the x y plane, is d y by d x at x y, which is given by the differential equation f of x y.

(Refer Slide Time: 07:32)

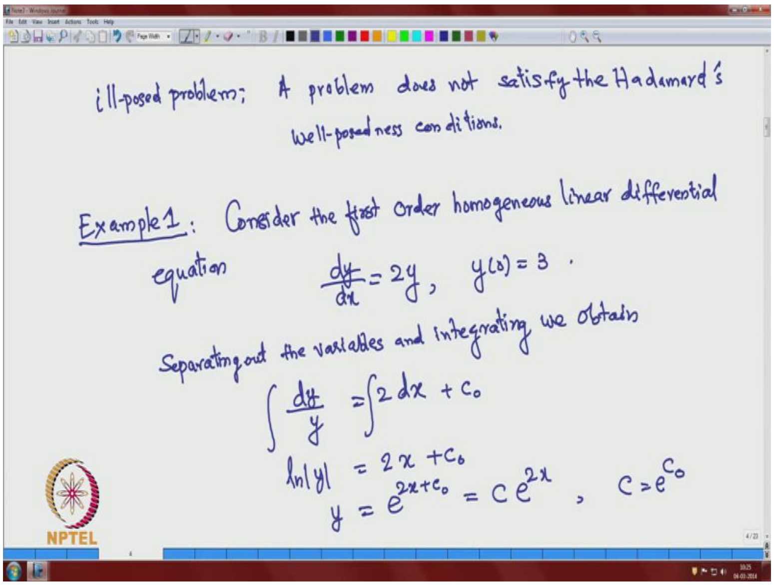


Therefore, the solution of a differential equation 1 with the initial condition 2, is a function that starts from the initial point x 0 y 0. Of course, that is differentiable, and the slope at any point is given by d y by d x is equal to f of x y, which is coming from the differential equation; f of x y is given to us. So, slope at any point x y is known to us, and the initial point is known to us. We are looking for a function, which starts from x 0 y 0, and whose slope is at any point is f of x y. Now, we deal with a well-posedness of this problem; well-posedness of a mathematical model. The mathematical model is given by the initial value problem. So, let the mathematical model is given by the differential equation d y by d x is equal to f of x y, with initial condition y at x 0 is y 0.

So, this model is well posed, if the following 3 properties are satisfied. So, this well-posedness was raised by the famous French mathematician J Hadamard. So, the well-posedness problem was introduced by Hadamard, and Hadamard raised the issue of well-posedness, which says that given a mathematical model, if the model satisfies the following 3 properties, then the model is well posed. The first property is there exists a solution to the initial value problem. There exists a solution, so that is an existence problem, which is known as an existence problem. So, given a mathematical model, the first question is to decide whether, there exists a solution or not; that is the existence problem. Second problem is a solution unique; the solution is unique. So, that is uniqueness problem. If the solution exists, will there be more than 1 solution or the solution is unique? The third problem or third property of a well posed problem is the solution; it is a behavior.

Solutions behavior changes continuously, with respect to the initial condition. So, these 3 problems; the first one is there exists a solution to the mathematical model; the second problem is whether, the solution is unique; and third problem is whether, the solution changes continuously, with respect to the initial condition. Initial condition here, it is y 0. So, the third problem is known as the stability problem; stability. So, the existence problem first; then, the uniqueness problem, and the third one is the stability problem.

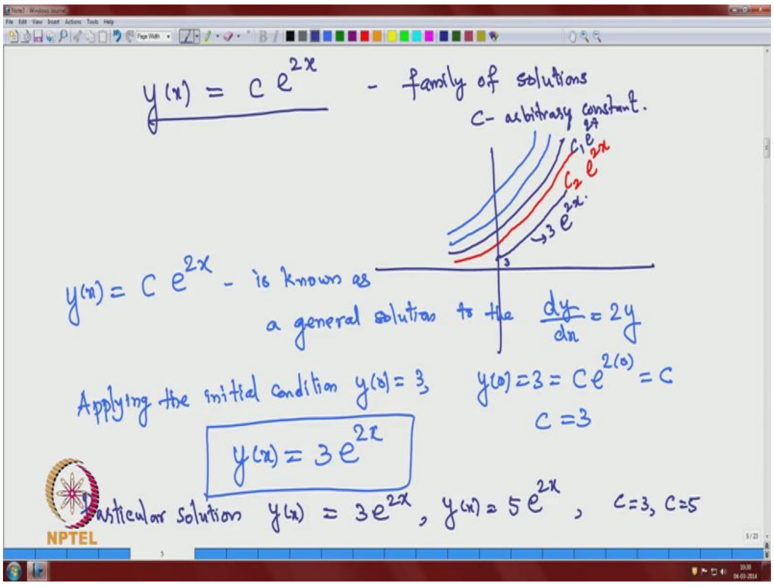
(Refer Slide Time: 14:40)



So, if a model is more satisfying these 3 properties; any model, which is not satisfying these 3 properties, is known as an ill-posed problem; a problem does not satisfy the Hadamard's well-posedness conditions. Now, we will discuss the problem of existence, uniqueness and stability of some simple differential equation, simple initial value problems. So, let us take an example. Example 1; consider the first order homogeneous linear differential equation given by d y by d x is equal to 2 y, along with an initial condition, say y at 0 is 3. You have already seen that how to solve this linear differential equation in the previous lectures.

If you separate out the variables and integrating, we obtain d y by y is equal to 2 d x, if we separate out the variables. If we integrate adding a constant c, we get l n of y is equal to 2 x plus c, and taking exponential on both sides. You get y of x is equal to e to the power 2 x plus c, which is equal to say, the first constant is c 0. I get c into e to the power 2 x where, c is equal to e to the power c 0. Therefore, the solution of this differential equation; I have not applied the initial condition; the solution of this differential equation can be obtained easily, by separating out the variables, or by the method of separation of variables.

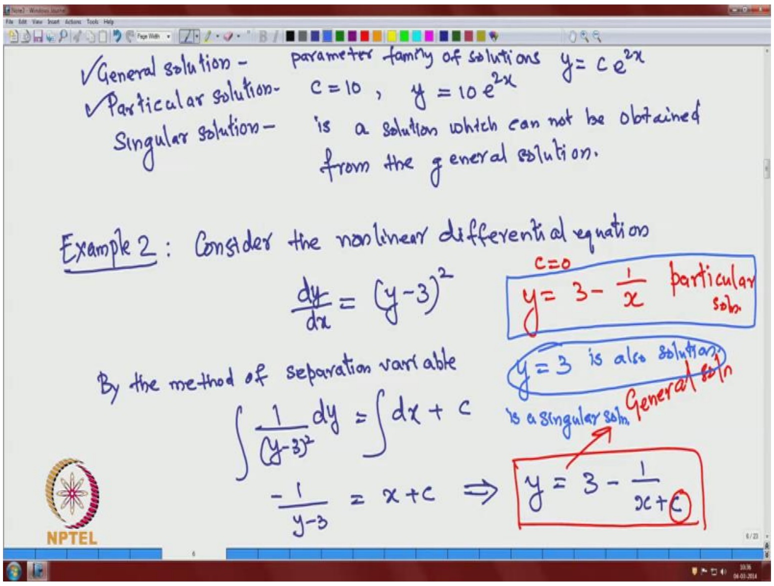
(Refer Slide Time: 18:43)



So, y x is equal to c, e to the power 2 x, which is a family of solutions of curves. So, one parameter see, this is a solution for every value of c. So, c is an arbitrary constant. So, the given linear differential equation has scored infinitely, many solutions and for every c, the given expression y; this y is equal to c into e to the power 2 x is a solution. So, if you plot the solution, this is one solution or one value of c. So, this is, c 1 e to the power 2 x and this could be another solution for c 2 e to the power 2 x, and you have a family of one parameter family of solutions. So, these are all solutions, a family of solutions to the given linear differential equation, and this solution is known as a general solution. So, y x is equal to c, e to the power 2 x is known as a general solution to the differential equation d y by d x is equal to 2 y. So, I have not applied the given initial condition.

Once I give an initial condition, applying the initial condition y at 0 is 3, then this becomes y at 0 is equal to 3, which is equal to; we can plug in the value to x is 0. So, c into e to the power 2 into 0, which is c, or in other words, c is equal to 3. Therefore, we get the solution y x is equal to 3, e to the power 2 x. So, this is a solution of the given differential equation, satisfying the given initial condition; say, if this point is 3, say, then 3 e to the power 2 x. So, this is a solution if this point is 3. For a particular value of c, the general solution, becomes a particular solution, and give a specific value to arbitrary constant c, then the solution we obtain is whereas, a particular solution. A particular solution, y of x is equal to 3, e to the power 2 x is a particular solution and similarly, y of x is equal to 5, e to the power 2 x; they are all particular solutions by putting various values. So, initially, it is c is equal to 3, and second case, c is equal to 5. So, a particular solution is obtained by giving a particular value to the general solution. Now, for a given differential equation, there can be another type of solution, which is known as a singular solution.

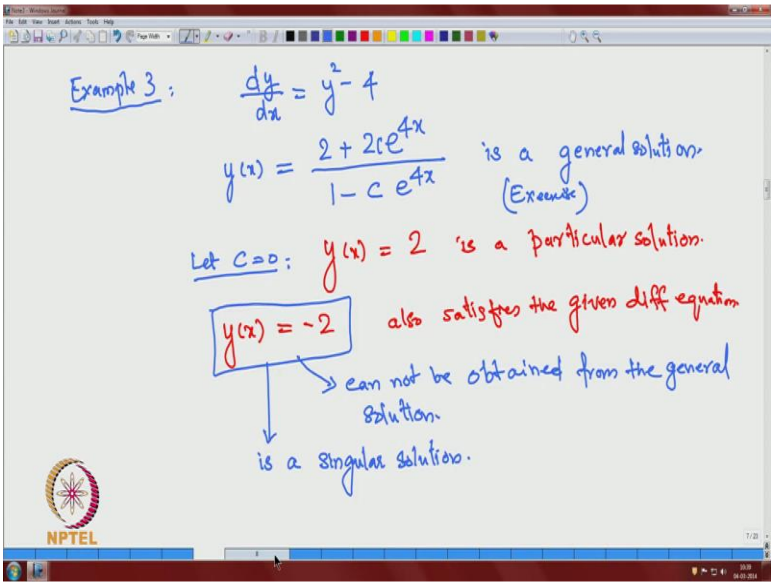
(Refer Slide Time: 23:28)



So, three types of solutions, we have discussed; one is a general solution, and a particular solution, and a third type of solution is singular solution. So, general solution; we obtained and in the general solution, there is a parameter. There is a parameter family of solution. In our case, it was y is equal to c, e to the power 2 x, and particular solution; say, c is equal to you particularize the value of say, c is equal to 10. Then, y is equal to 10, e to the power 2 x is a particular solution, and a singular solution is a solution, which cannot be obtained from the general solution, but in our previous examples, we have only these two types of solutions; general solution and particular solution, and singular solution does not exist for the previous example. Now, we consider an example; example say, we call it 2. So, consider the non-linear differential equation, say d y by d x is equal to y minus 3 the whole square.

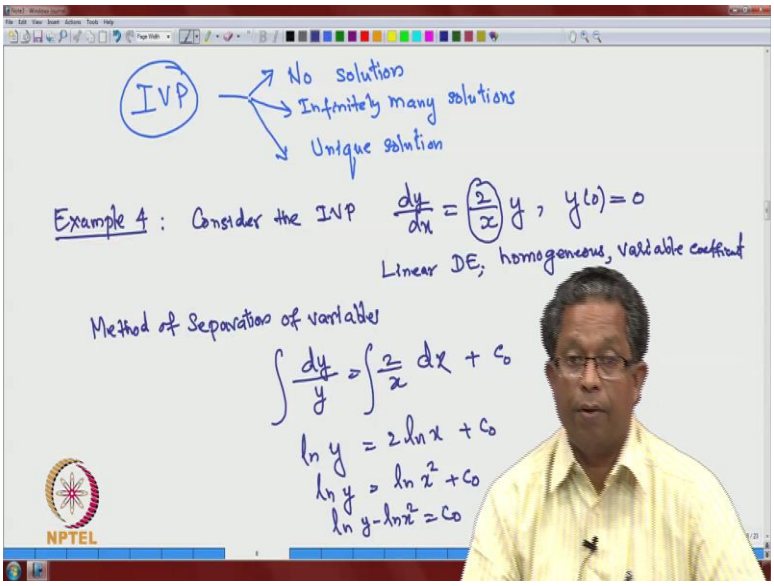
So, we have seen that this equation is obviously, a non-linear differential equation. So, how to solve it? By the method of separation of variables, separating of the variables and integrating, you get this is d y, y minus 3 square d y is equal to d x and integrating, plus adding a constant c. So, integral of 1 by y minus 3 the whole square is minus 1 by y minus 3, which is equal to x plus c. So, this simplifying, we get y is equal to; you take y minus 3 over there, and minus 1 by x plus 3 there. So, y is equal to 3 minus 1 by x plus c. So, this is a general solution. For every value of c, this is a solution. Therefore, this is a general solution. So, this is a general solution and if you give particular values to c; say for example, y is equal to 3 minus 1 by x, is a particular solution where, c is equal to 0. When c is equal to 0, you get a particular solution. So, this is a particular solution. Now, you can verify easily, that y is equal to 3. Once you substitute y is equal to 3, it is a constant function, into this equation, and this equation satisfied, is also a solution, but that is not obtained from the general solution. Therefore, this is a singular solution. So, y is equal to 3, is a singular solution. So, this example, we see three types of solutions; general solution, and the particular solution, and also, a singular solution, and we can see further examples.

(Refer Slide Time: 29:46)



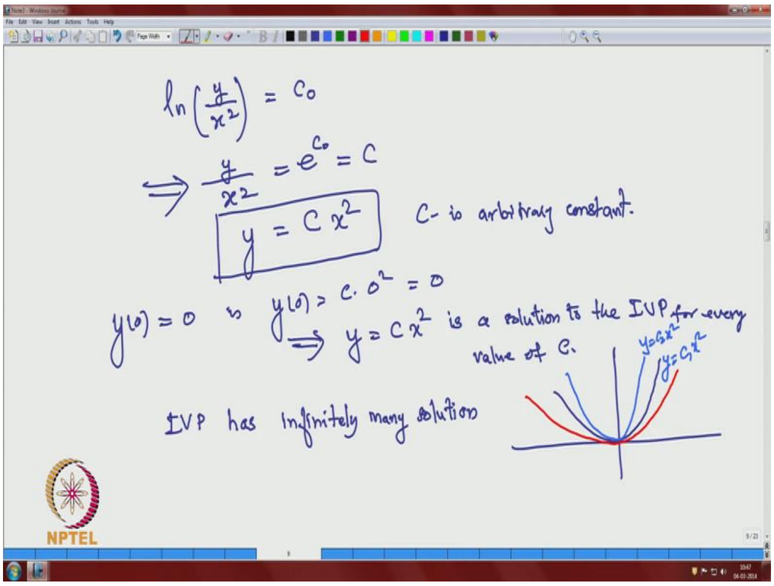
Another example say, example call it 3. So, consider a differential equation d y by d x is equal to y square minus 4. Again, by the method of separation of variables, and you can integrate and you can easily, see that y x is equal to 2 plus 2, some constant c times e to the power 4 x, divided by 1 minus c, e to the power 4 x, is a general solution. I leave this as an exercise to show that y is a general solution. Once you specialize c, when c is equal to 0, then we obtain y is equal to 2, is a particular solution. Now, this can also be shown that observe that y x is equal to minus 2. If you look at the function, constant function, y x is equal to minus 2, also satisfies a given differential equation. So, y is equal to minus 2 is also a solution, which cannot be obtained from the general solution. Therefore, this y is equal to minus 2; this solution, that cannot be obtained from the general solution. Hence, therefore, this is a singular solution. Now, we look into initial value problem where, an initial value problem need not have a solution or another initial value problem have more than one solution or an initial value problem has a unique solution; these three situations.

(Refer Slide Time: 33:16)



So, initial value problem cases; no solution; infinitely, many solutions and unique solution. These three situations; we will look into some examples of initial value problems that has unique solution, and infinitely, many solution and no solution; all these situations, we will see. Let us first consider a linear case. Later on, we will give sufficient condition to ensure under what condition, an initial value problem has a unique solution and under what condition, a solution has a solution and solution is unique, and all such things; we will deal with later. So, example 4, just to get a field of existence and nonexistence of solutions, we take a few more examples. So, consider the initial value problem, say d y by d x is equal to say, 2 by x y, with an initial condition, y at 0 is 0. Obviously, this is a linear equation; linear differential equation, and it is a homogeneous; linear homogeneous, and variable coefficient. So, this coefficient is variable; 2 by x. So, is there, and there is no terms, which does not depend upon y. Therefore, it is a homogeneous and obviously, it is linear with respect to the unknown function y. Therefore, it is a linear differential equation. Now, let us look into the solution. Again, by the method of separation of variables, we have d y by y is equal to 2 by x into d x, separating of the variables and integrating and adding a constant, call it c 0. So, you can write this is l n of y; this is 2 l n of x, plus a constant c 0, which is l n of y, which is l n of x square; 2 l n of x is l n of x square plus c 0, and bring this term over here; l n of y minus l n of x square is equal to c 0.

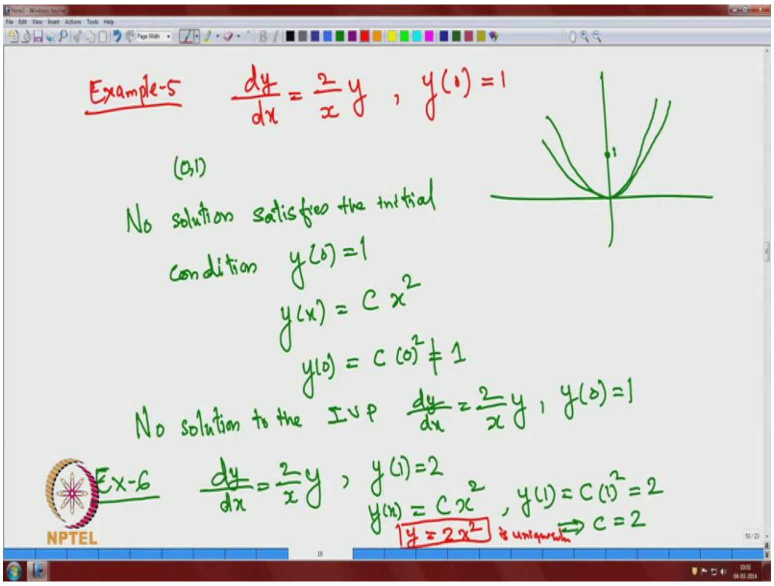
(Refer Slide Time: 37:24)



There is l n of y by x square is equal to c 0, or taking exponential on both sides, get y by x square is equal e to the power c 0. So, call this as c. Therefore, y is equal to c into x square is the solution. It is a general solution where c is an arbitrary parameter. For every value of c, y is equal to c x square is a solution, and in order to satisfy the initial condition, y at 0 is equal to 0, is obviously, satisfied; y x 0 is c into 0 square, which is 0, is obviously, satisfied for all values of c. Therefore, y is equal to c x square is a solution to the initial value problem. It is satisfying the differential equation. At the same time, it is also satisfying the initial condition for all values of c. So, y is equal to c x square is a solution to the initial value problem for every value of c.

What does it say? It says that the initial value problem has infinitely, many solutions. So, if you look at the solution, this is a parabola, y is equal to c x square for 1 value of c, and for another value of c, this another parabola, y is equal to c 1 x square, y is equal to c 2 x square, and for any value of c. So, this is also another solution. This is a family of parabolas. We get infinitely, many parabolas, happens to be solutions of the initial value problem. Therefore, for an initial value problem, there is a possibility that the solution is not unique; it may have infinitely, many solutions. Now, if you look at same problem, the same equation with a different initial condition; look at the same problem.

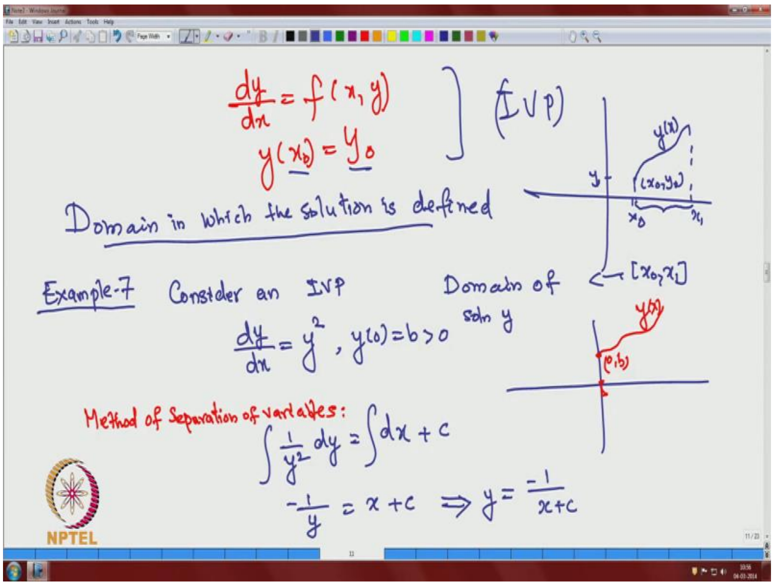
(Refer Slide Time: 40:40)



So, if the example is 5, then d y by d x is equal to 2 by x y. Now, the initial condition is changed; y at 0 is 1. As we have already noticed that all solution of this initial value, this differential equation, should pass through the origin. So, it cannot pass through the point 0 1. So, 0 1 is a point you are looking for. You want the solution to pass through the point 0 1, when x is equal to 0, and this is not possible. So, there is no solution; this is one, which is passing through the point 0 1, and is also, a solution to this differential equation. So, no solution satisfies the initial condition, y at 0 is equal to 1. Why, because y of x is c into x square, and you want y x 0 is equal to c into x is 1 0 square, and you want this to be 1, which is not possible.

Therefore, the moral of the story is no solution to the initial value problem, d y by d x is equal to 2 by x, y, the initial condition y at 0 is equal to 1. Then, one more thing we observe that if we change the initial condition for the same problem, say it is a example 6; the same differential equation d y by d x is equal to 2 by x y and y at 1 is 2, say for example. I am giving a condition at 1, which is a nonzero condition; the value is nonzero. So, it can be shown easily, that y of x, we have already seen the solution is of the form c x square, and you are looking for y at 1 is equal to c into 1 square, which you want this to be 2. So, this implies that c is equal to 2. Therefore, y is equal to 2 x square is a unique solution. So, this example, the same problem with a different initial condition, d y by d x is equal to 2 by x into y, with a different initial condition y at 1 is equal to 2, has a unique solution. So, same differential equation, which is giving rise to 3 different situations where, no solution and where, there is unique solution and also, there is infinitely, many solutions.

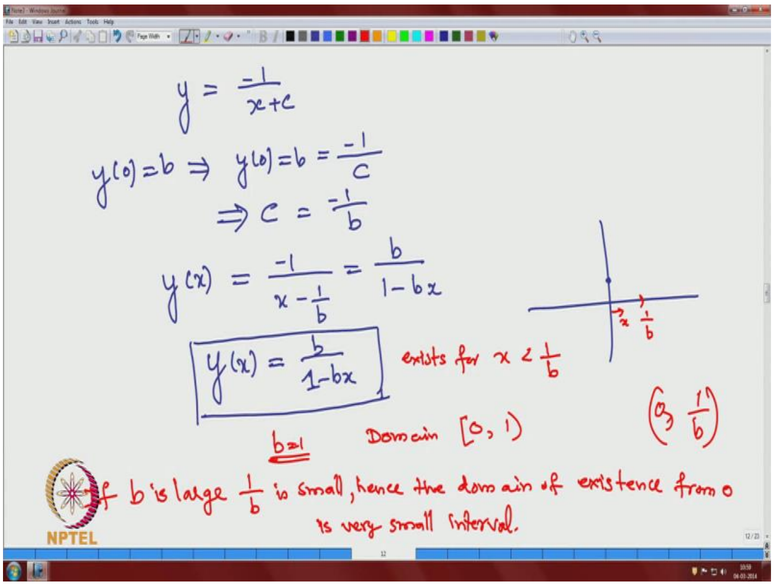
(Refer Slide Time: 44:56)



Now we look into the solution of a differential equation where, it is defined. So, our equation is d y by d x is equal to f of x y; initial conditions y at x 0 is y 0; the domain of in which the solution is defined. So, for an initial value problem of this form, there may be a solution starting from the point x 0 y 0 and that solution may exist only, for some interval on the x axis. So, this is a x 0, and this is y 0, and this is a point from where, we start; x 0 y 0; looking for a solution and the solution is y x, and you can in some many of the initial value problem, this interval, we call it x 1. The solution exists in a small interval x 0 x 1. This is called the domain of solution y. and it can happen that the solution is defined on the entire x axis and some problems; it is defined only on some small interval or only on some particular bounded set in the x axis. For that also, we look into some examples. So, example 7; consider an initial value problem.

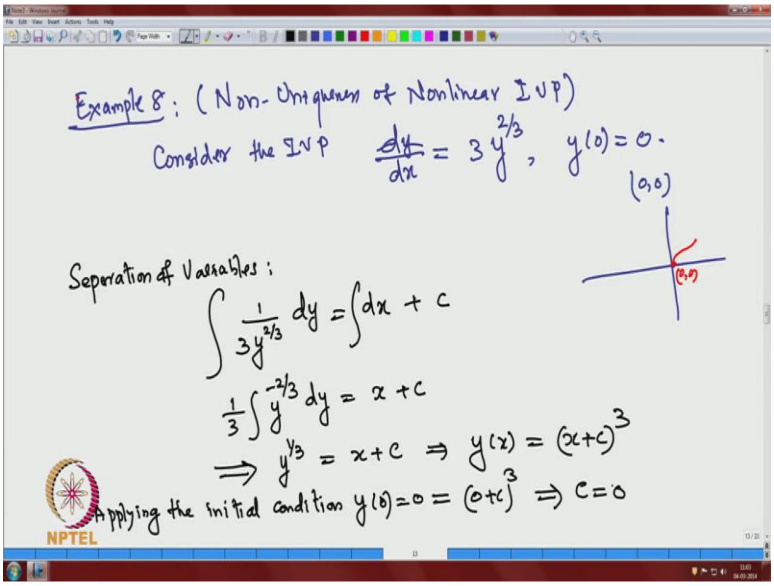
Consider an initial value problem given by d y by d x is equal to y square, and initial condition is y at 0, is equal to be a positive number. This is a non-linear differential equation. The initial condition is y at 0 is some number b, starting at this point 0 b. I am looking for a solution y, from the point x is equal to 0. Let us look at the solution again, by method of separation of variables. So, by the method of separation of variables, we get separating of the variables; this is 1 by y square, d y is equal to d x, and integrating, adding a constant c, and the integral of 1 by y square is minus 1 by y, is equal to x plus c. So, this just by simplifying, you get y is equal to minus 1 by x plus c.

(Refer Slide Time: 49:11)



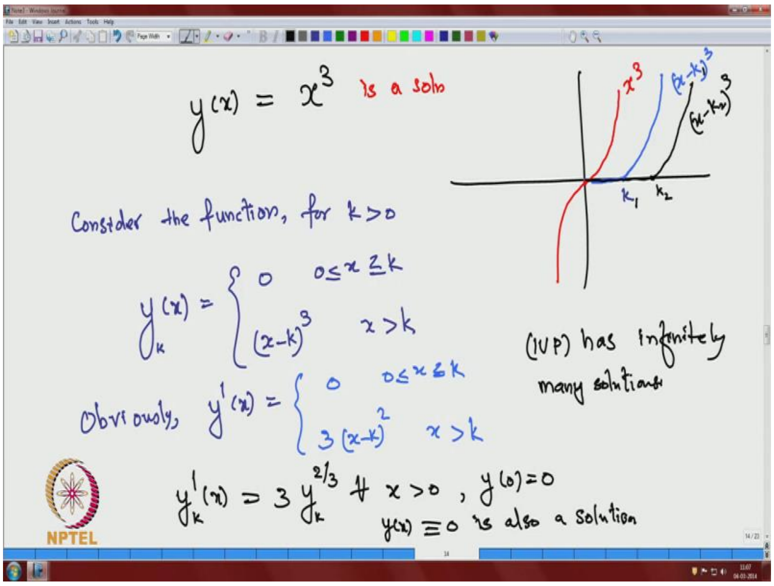
So, the solution is y is equal to minus 1 by x plus c. Now, the initial condition y at 0 is b. So, this implies that y at 0 is equal to b, which is equal to minus 1 by c. This implies that c is equal to minus 1 by b. So, plugging this value to the solution, y of x is equal to minus 1 by x minus 1 by b, which is equal to b by 1 minus b x. So, this is the solution to the initial value problem. So, the solution is y of x is equal to b by 1 minus b x. Look at the solution, starting from 0 b. As you increase x, if x is less than 1 by b, see if x is equal to 1 by b, then the solution blows up. Therefore, solution exists. So, this exists for x strictly, less than 1 by b. Therefore, the solution exists; 1 by b. If b is 1, then 1 by b is 1. So, this is 0 to 1. For example, b is equal to 1, then solution exists; the domain of solution is 0 to 1, but if b is a very large number, if b is large, 1 by b is small, and hence, the domain of existence is from 0, is very small interval. Note that although, the equation looks to be a very nice equation, the initial value problem looks nice on the entire real line, starting from 0 b, the solution y x exists in a small interval 0 1 by b. So, 0 1 by b, if b is large, is a small interval. So, solution exists, if b is large, then the solution exists on a small domain to the right of x is equal to 0.

(Refer Slide Time: 53:03)



Now, looking at another problem, example; I call it example 8 here, non-uniqueness; non-uniqueness of non-linear initial value problem. Consider the initial value problem, given by d y by d x is equal to 3, y to the power 2 by 3 and the initial condition is y at 0 is 0. So, this is also a non-linear differential equation with a given initial point 0 0; initial condition is 0 0. So, you are looking for a solution that starts from 0 0, and again, by the method of separation of variables, this is a d y 1 by 2 to the power 3 by 2. So, 3 is equal to d x and integrating, and adding a constant c, which is integral 1 by 3, y to the power minus 2 by 3 d y, which is equal to integral d x is x plus c. So, integral of this one, y to the power minus 2 by 3 plus 1 or divided by minus 2 by 3 plus 1, which is y to the power 1 by 3, is equal to x plus c. So, this implies that the solution y satisfies, solution is given by y of x is equal to x plus c the whole cube. Now, applying the initial condition, y at 0 is 0, which implies that 0 plus 3 whole cube. So, implying that c itself is 0.

(Refer Slide Time: 56:38)



That means, we get a solution, y x is equal to x cube for the initial value problem. So, the solution looks like function y is equal to x cube. So, this is a solution and we can also see that. Consider some other functions. Consider the function for k is a positive constant y x defined by y x is equal to 0; for x is less than equal to k and greater than equal to 0, and x minus k the whole cube for x strictly, greater than k. So, if you consider these functions, for every value of k positive, this gives a family of functions y k. So, for every k, you have this function, and we can show that or you call it y k, and obviously, the derivative, if you differentiate it, the derivative of this one; this looks like up to k; say, this is k; up to k, the value of the function is 0; from k, the function is a smooth. So, x minus k cube and it is a smooth function. If you differentiate it, y prime x is 0 for x between 0 and k, and its value is 3 times x minus k square for x greater than k. For a different k; this is another k; let us call it k 1; and another k. So, the value of the function up to k 2 is 0. Then, this x minus k 2 cube, and we can see that the derivative y prime x satisfies, because of this form.

This satisfies a given differential equation; this is 3 times y to the power 2 by 3 for all x greater than 0, and it satisfies y at 0 is 0. So, you can say this is, you can show that this is a family of curves. Therefore, and also, we can see that y x is equal to 0, is also a solution. So, y x is equal to 0; zero function is also a solution, which is a singular solution; otherwise, you get a family of solution. The conclusion is the given initial value problem has infinitely, many solutions. In this lecture, what we have discussed is we have taken a few initial value problems, and we posed the well-posedness of Hadamard and have seen that there are problems, which the initial value problems has no solution, and there are problems in which the initial value problem has unique solution, and there are initial value problems having unique solution. Now, how to characterize these solutions, these properties? That we will see in the next lecture.

Bye.