Advanced Matrix Theory and Linear Algebra for Engineers

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Lecture No. # 09

Vector Spaces –Part 2

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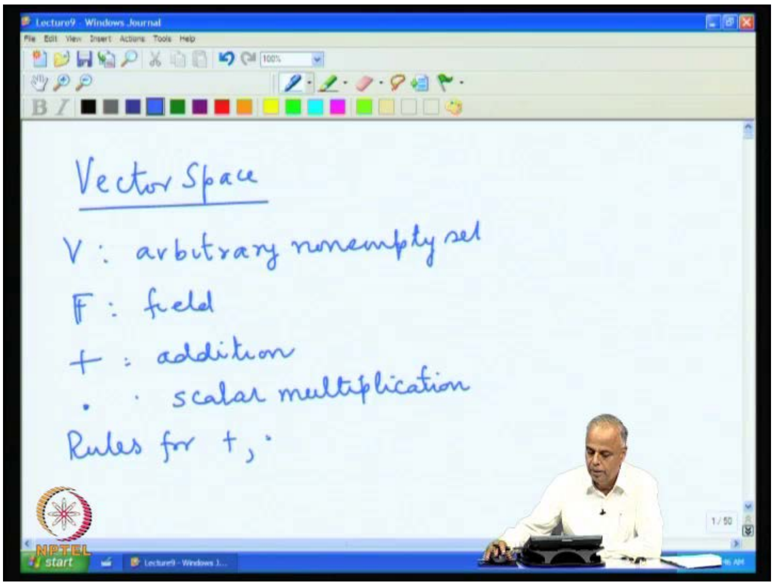

In the last lecture, we saw what is meant by a vector space. Let us recall that a vector space adds 4 ingredients one an arbitrary non empty set, then a field F and the addition or a rule of combining two elements in V and what is known as scalar multiplication. A rule of combining an F element with a V element and then we had the rules for plus and dot, together with this we get a vector space.

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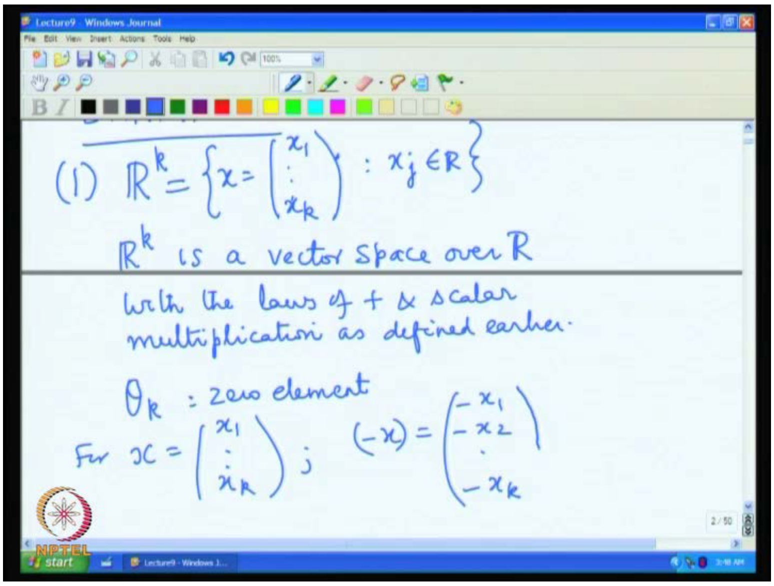

We shall now look at some examples of vector spaces. The first simple example with the following since, all our motivation for the definition of a vector space stems from our experience with R k, clearly R k is itself a vector space. So, R k which is the set of all (( )) call a column matrices such that, x j belong to R and then we have this usual laws of addition etcetera. So, R k is a vector space and the field is R over R, the scalars we consider R, with the laws of plus and scalar multiplication as defined earlier, how do we defined this scalar multiplications? We just, when we say add we shall simply add the corresponding entries, when we say multiply by a scalar we may multiply every entry by the scalar.

Now, what was the zero element of this vector space for the zero element, the zero vector or the zero matrix and for x - x 1, x 2, x k, minus x is a negative is just minus x 1, minus x 2 etcetera. This we have seen already these were the properties of R k which motivated us to give the abstract definition of R k.

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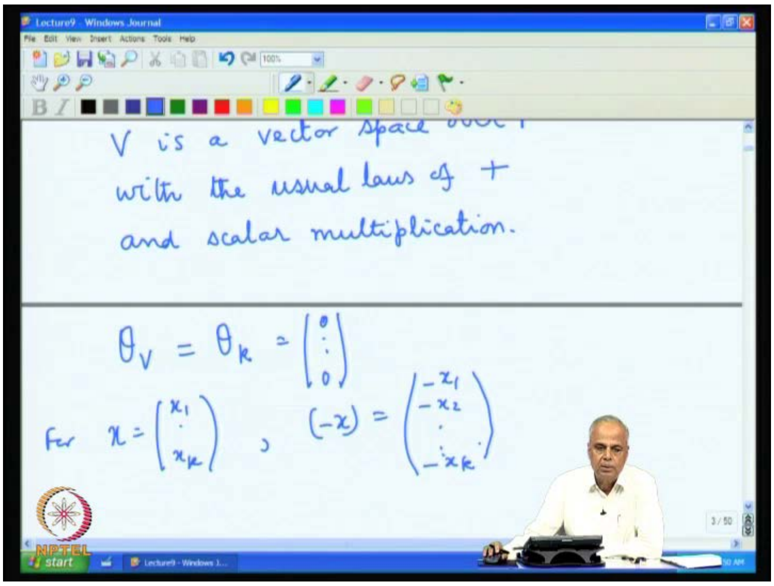

So, this is the first fundamental example of a vector space. The second example is, we replace the real numbers by the complex numbers. So, the real field is replaced by the complex. So, we take V to be C k which is the set of all the vectors or all the column matrices x 1, x 2, x k, where x j are complex numbers, again addition is entry wise, scalar multiplication is entry wise.

We call this entry wise addition and entry wise scalar multiplication as usual addition and scalar multiplication, then we take F to be C, then V is a vector space over F that is, C k is a vector space over C with the usual laws of addition and scalar multiplication again I repeat when we say usual law, it means entry wise addition and entry wise scalar multiplication. In this vector space the zero vector is the theta k again, which is 0, 0, 0 and for x - x 1, x 2, x k, then negative vector is minus x 1, minus x 2, minus x k, since, x 1, x 2, x k are complex numbers minus x 1, minus x 2, minus x k, is also a complex number and therefore, this vector we have define minus x also belongs to C k.

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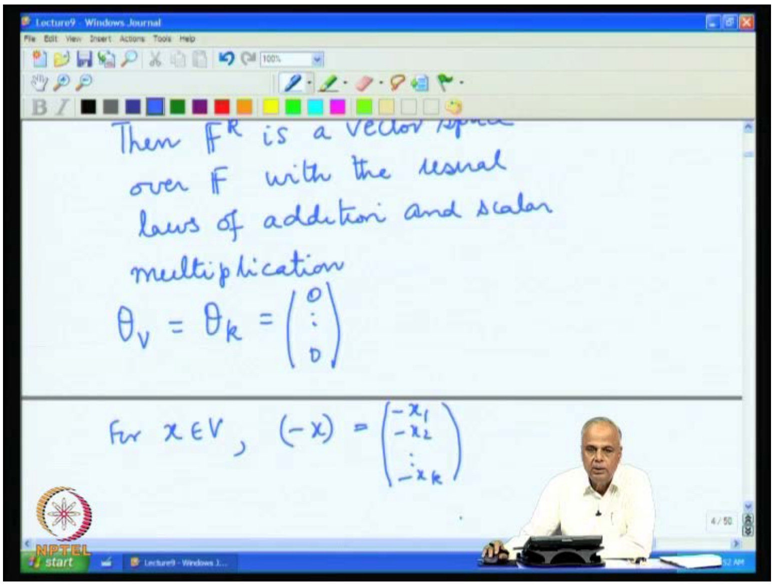

In general, we can start with any field F and look at F k so, F is any field and look at F k, which is the set of all x equal to x 1, x 2, x k where now x j's are from F. So, instead of F when we put F equal to R, we are looking at R k, when we put F equal to C; we are looking at a C k. In general, we can take any field F and look at F k, then again usual laws of addition. What does that mean? We add two elements in F k by adding the corresponding entries, usual laws scalar multiplication that means, you multiply alpha and x by multiplying every entry of x by alpha.

Then F k is a vector space over F with the usual laws of addition and scalar multiplication and again what is the V in this case is F k. So, what is the theta V it is again theta k, in this case 0, 0, 0, where now the 0 refers to 0 elements of the field F and for x in v minus x, the negative if again minus x 1, minus x 2 minus x k, again since x 1, x 2, x k are on the field F, in the field every element has its negative and therefore, minus x 1, minus x 2, minus x k, belongs to F.

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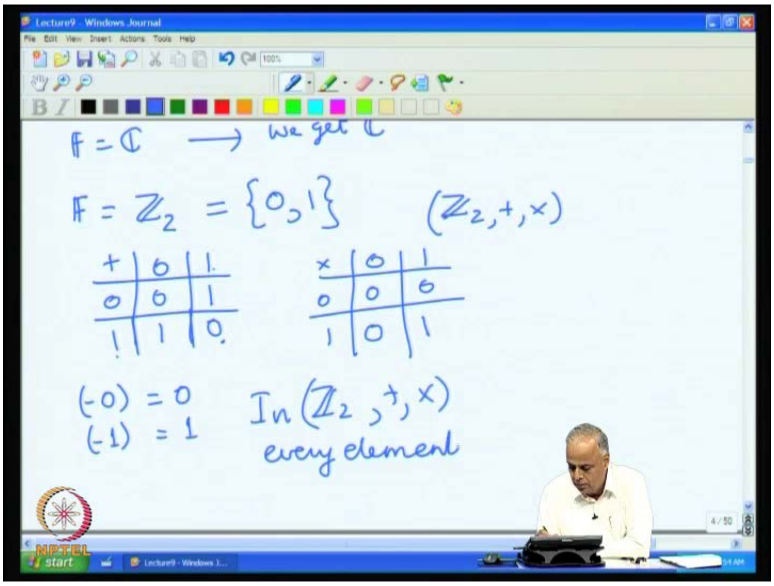

When we take F equal to R, so in this example if you take F equal to R, we get R k the vector space R k. when we get F equal to C, when we take F equal to C we get the vector space C k. Let us look at another typical example which is the field F which is different from all this let us take for example, F to be the binary field Z 2. What is this binary field Z 2? It consists of two elements 0, 1, where the addition is defined by the following table 0 plus 00, 0 plus 1 is 1, 1 plus 0 is 1, and 1 plus 1 is 0. And multiplication in this field is defined as 0 into 0 is 0, 0 into 1 is 0, 1 into 0 is 1, 1 into 1 is 1, this is simplest multiplication table that one can remember.

Now, with these operations the plus and into Z 2 is a field so, we are looking at the field Z 2 plus and into now, in this field for example, what is minus 0, minus 0 is 0 itself and minus 1 is 1 itself, because 1 plus 1 is 0 and therefore, 1 acts as the negative of 1Plus 1 is 0 here is 1, when add it with 1 I get 0. So, 1 acts as the negative of 1. So, in this field every element is its own inverse so, in Z 2 plus into every element is its own negative.

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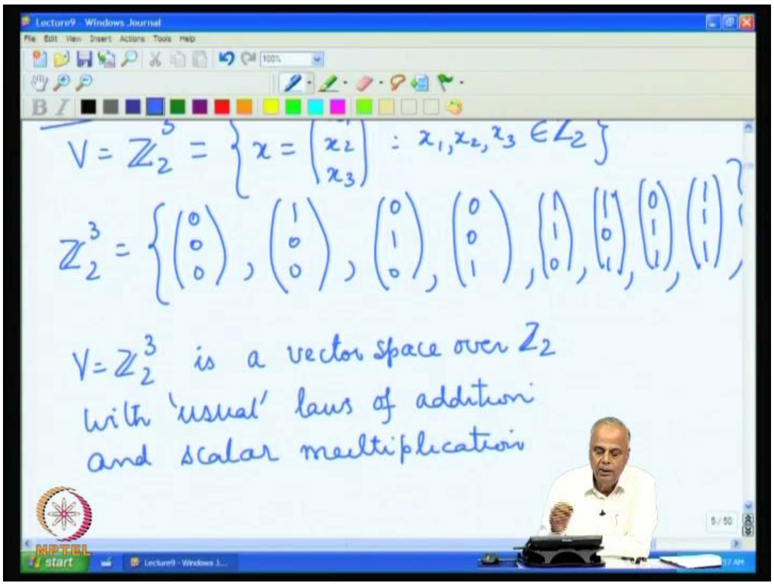

Now, let us look at V to be this field 3 take k equal to 3 so, F k is now 3 what is this? This is the collection of all x 1, x 2, x 3, 3 by 1 matrices, where x 1, x 2, x 3, are from the field Z 2 that means x 1, x 2, x 3, can be either 0 or 1. So, there are 2 choices for x 1 namely, 0 and 1 and for every choice of x 1. The 2 choices for x 2 namely 0 and 1 and for every choice of x 1 and x 2 there are 2 choices of x 3 namely 0 and 1. So, there are totally 2 into 2 into 2, 8 possibilities so, there are only 8 elements have 8 vectors in this field.

In this collection V what are these 8 vectors. So, actually Z 2 3 consists of these following vectors 000, 100, 010, 001 now, this is all 0 these 3 are only 1, Z1 and remaining 2,0 the 1 can be in the first place or in the second place or in the third place. Similarly, we can have two 1s and 1 0 and finally, we can have all ones so, 1, 2, 3, 4, 5, 6, 7, 8 elements in this field in this Z 2 3. Now, on this Z 2 3 just like the addition in R induce an addition in R k the addition in Z 2 induces, an addition in Z 2 3.

So, V equal to Z 2 3 is a vector space over Z 2, the field is Z 2 with usual laws of addition and scalar multiplication. Now, what do we mean by usual laws? We have to add entry wise, when we add entry wise each entry is a field element an element in the field Z 2. So, it has to be added according to the Z 2 addition laws.

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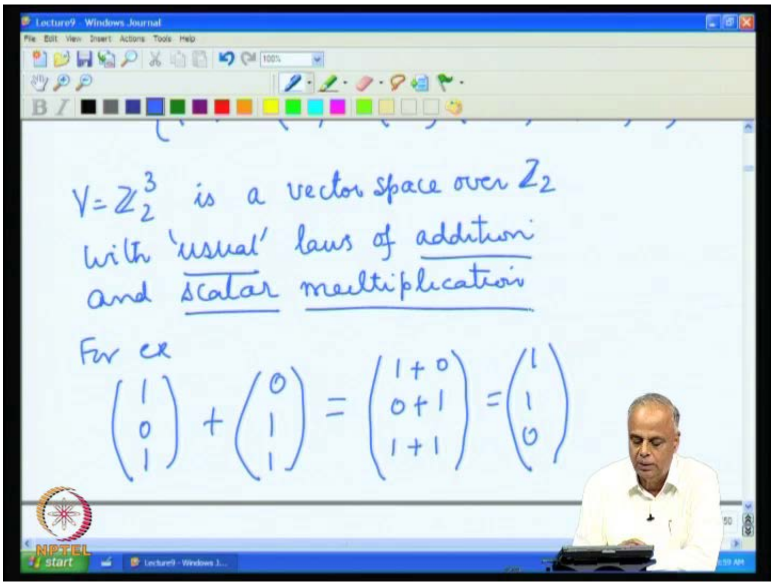

For example, if we add 1, 0, 1 and 0, 1, 1 according to these usual laws it should be 1 plus 0, 0 plus 1, 1 plus 1. Now, these pluses here refer to the plus in Z 2, but in Z 2, 1 plus 0 is 1, 0 plus 1 is 1, but 1 plus 1 is 0 .So, this is what we meant by usual laws, you have to add entry wise, but, the entry additions are done according to the field laws similarly, there are only 2 scalar multiplications possible, either multiplication by 0 or multiplication by 1, because the field has only two elements, when we multiply by 0 all the entries become 0, when we multiply by 1 all the entries remain the same, this is what is meant by saying that, we are looking at usual laws of addition and scalar multiplication.

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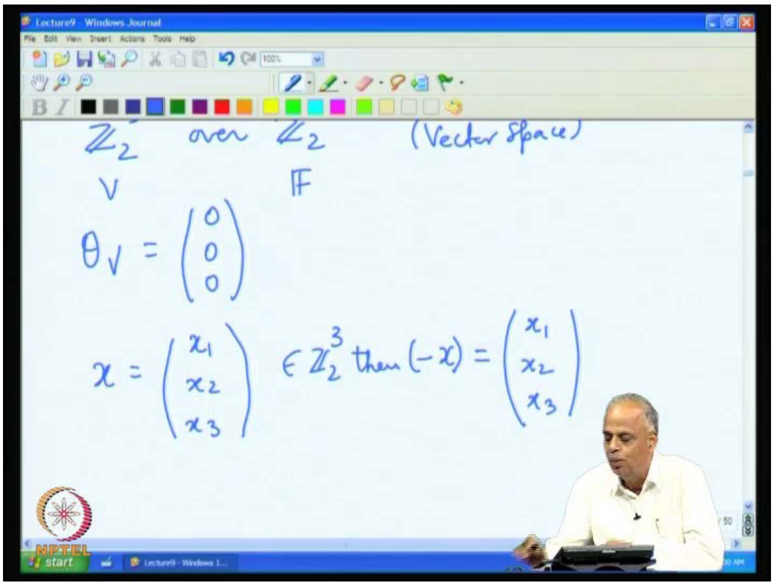

So, in this field so, in this vector space Z 2 3 over Z 2 this is a vector space this is our V this is the field F and the laws of addition and multiplication scalar multiplication, this is the vector space. What is the 0 element of this vector space? Is obviously 0, 0, 0. Now, something tricky happens here, if we take any x in Z 2 3 it is corresponding negative vector must be, I must take minus x 1. We have seen that in the field Z 2 every element has its own negative so, minus x 1 will be x 1 only, minus x 2 will be x 2, only minus x 3 will be x 3 only so, this is in this particular vector space every vector is its own negative.

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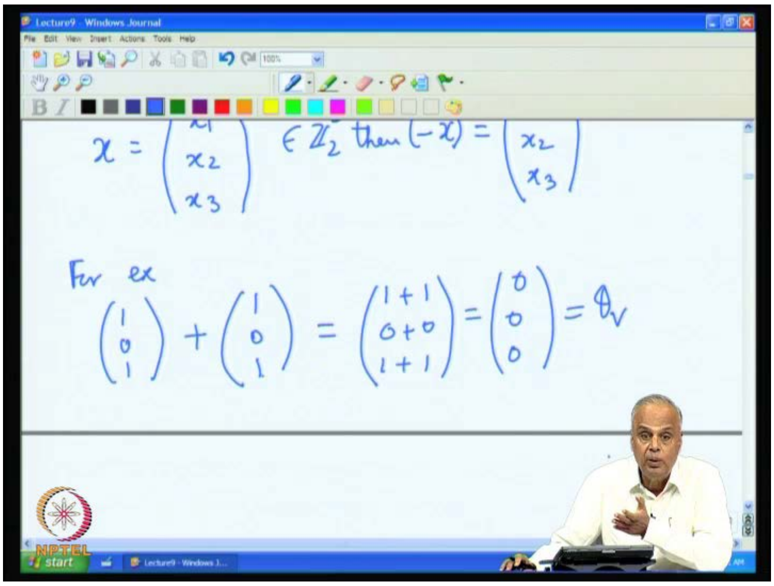

For example, 1, 0, 1 plus itself since, this should act as its negative you must get 0, but what is it 1 plus 1, 0 plus 0, 1 plus 1, but in this field 1 plus 1 is 0, 0 plus 0 is 0, 1 plus 1 is 0 and this is what theta V is. So, thus this is a vector space which is only a finite number of vectors in it, 8 vectors and in this vector space; every vector is its own negative. When we deal with vector spaces over finite fields, we get such very peculiar things which we are not familiar with, when we are talking about our usual notions of vectors; let us look at now some more examples.

So, in general we assume that F k is a vector space over F now, this take care of the basic extension of the idea of R k, from R k we got the general notion of a vector space, from R k we went to C k, from C k we went to F k and these were the examples of the various F k's.

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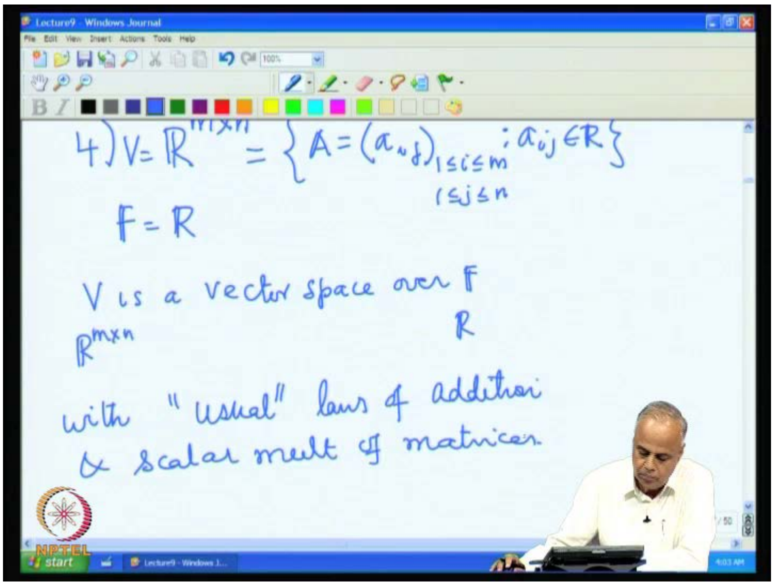

Now, we will go to the notion of matrices, the next example we look at is let us start with R and look at all m by n matrices entries are real numbers. So, these are all the matrix spaces, they are all matrices a i j where there are m rows and n columns and all the entries are real numbers. So, let us consider all the matrices real matrices with m rows and n columns and let us take F to the R.

Now, we know how to add matrices again entry wise addition, we know how to multiply a matrix by a scalar, every matrix is every entry in the matrix is multiplied by the scalar, we will call this the usual laws of addition and scalar multiplication V is a vector space, what is v? R m cross n is a vector space over F, what is F? R with usual laws of addition and scalar multiplication of matrices.

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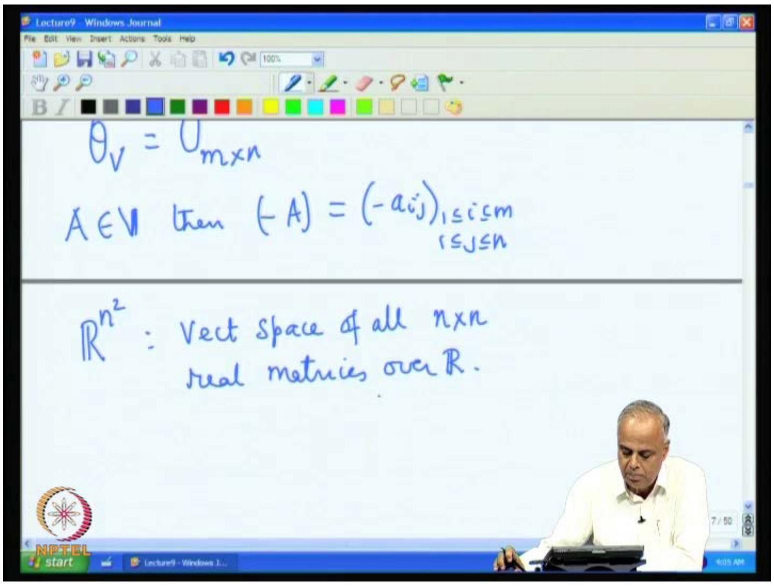

So, we have the vector space of real matrices, what is the 0 element of this vector space? It is the 0 matrix with m rows and m columns, we will use this. So, in this vector space the 0 element, the 0 matrix and if A is a matrix in this vector space, then what is minus A? Minus A is a matrix obtained from A by taking the negative of each element such minus A i j, 1 less than or equal to i less than or equal to m, 1 less than or equal to j less than or equal to k. So, we have the vector space of all m by n real matrices, in particular we take it m equal to n, that the vector space of all square matrices. So R n square is the vector space of all m by n real matrices over the field R, analogously just like we took R here we can take complex numbers.

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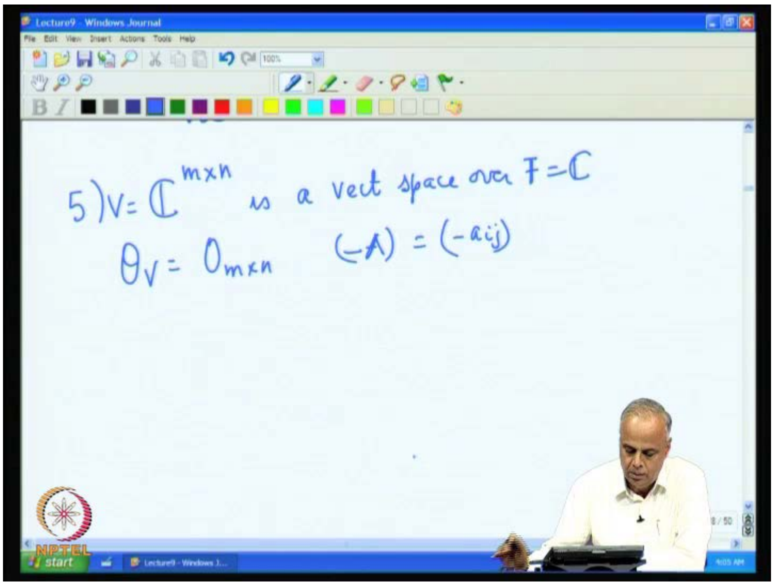

So, we have C m cross n, the set of all complex m by n matrices is a vector space over the field, the field is complex numbers. Again, what is the 0 element of this vector space? It is the 0 m by n matrix and the negative of a matrix is just minus a i j as before.

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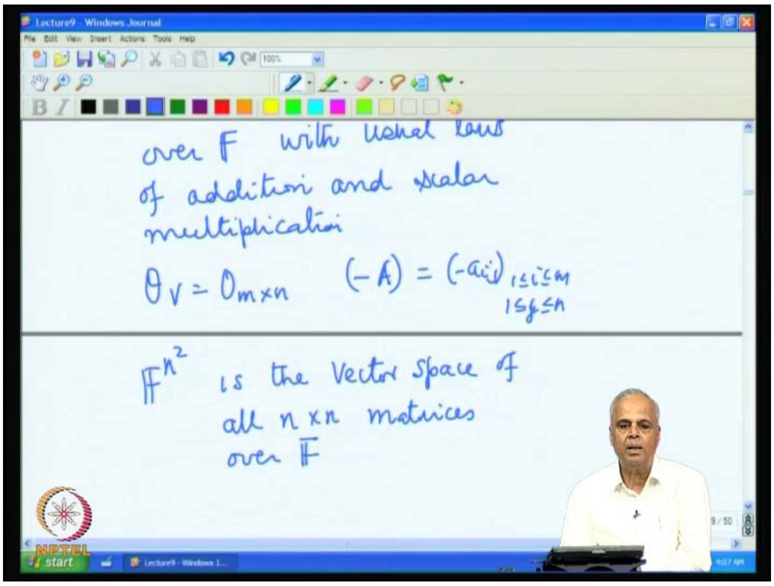

In general, we can take any F and look at all m by n matrices, whose entries are from the field F so, take a field F like we could have taken Z 2. So, it may talk about binary matrices, matrices with whose entries are only 0 and 1. So, we take any field F consider all m by n matrices over F and then this is a vector space over F, again when we say with usual laws that means, we must add the matrices by adding corresponding entries, when we add corresponding entry that must be according to the rule of addition in the field F so, the usual laws of addition and scalar multiplication.

Once again, in this vector space, the 0 vector is the 0 m by n matrix and minus A is minus a i j, where the minus a i j has to be taken as per the negative rule in the field F and in general, we can have the rectangular addition from the rectangular to the square matrices is the vector space of all m by n matrices over the field F. So, first we have the vector space of k by 1 all the entries from a field F, then we have the vector space of all matrices m by n where the entries are from a field. The field can be R set of real numbers, the field can be C field are complex numbers or the field F can be any general field now, we look at another set of examples useful examples.

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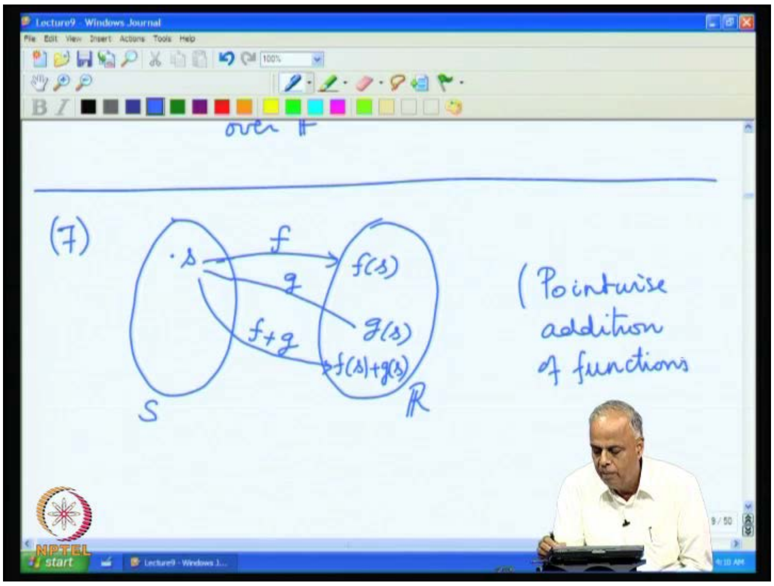

Let us now consider any set S so, S is an arbitrary set S and let us consider the set of real numbers. So, we have an arbitrary non empty set S and we have the set of real numbers. What is mean by a function from S to R, where function F from S to R, we mean a rule which associates with every number S an element which is a real number F S. So, with each element S, in S the real number F S is associated such a rule of association is called a function from S to R.

So, let us take a function from S to R and suppose I have another function which associates g s. So, in other words we have a function F which associates the value F of S with an element S and g of g is another function with associates the value g of s with S. Now, F S is in R note that the value of the function is in R the value of the function g is in R therefore, F S is a real number, g s is a real number. So, we can add F S and g s so, we will get when we add F S and g s we will get F S plus g s and that is also a real number. So, you have added the real number F S and the real number g s. Now, we can think of a new function which associates with S the value F plus F S plus g s we call that have the function F plus g.

So, for two functions from S and R, we have the same function and since, we are adding the value at each point to get the new function, we call this addition of functions as point wise addition, this is what is meant by point wise addition of function, that is at each point the value of the functions are added. So, we now know how to add two functions from S to R.,

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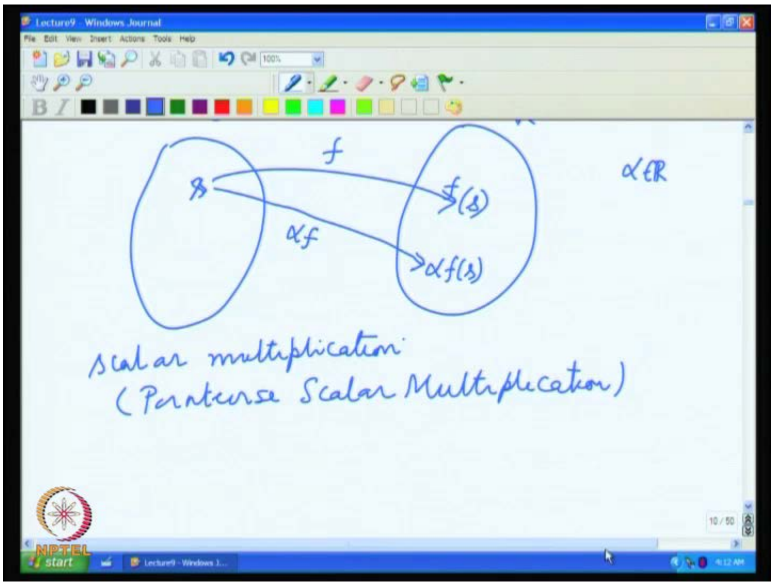

Similarly, let us take S and let us take R the real numbers, again suppose I have a function F which it each S associates the value F of S. Now, the F of S is a real number and suppose I choose any real number alpha, I can multiply F of S and alpha, because alpha is a real number and F of S is a real number and the result of the product is again a real number, because the product of two real numbers is real number. Now, we can think of a new function which associates with each S the value alpha times F S, we call it the function alpha F this gives us the scalar multiplication of functions, this is again point wise multiplication. So, addition of functions is defined as point wise addition and we have point wise scalar multiplication, at each point a value is multiplied by the scalar so, we know how to add functions and scalar multiplication.

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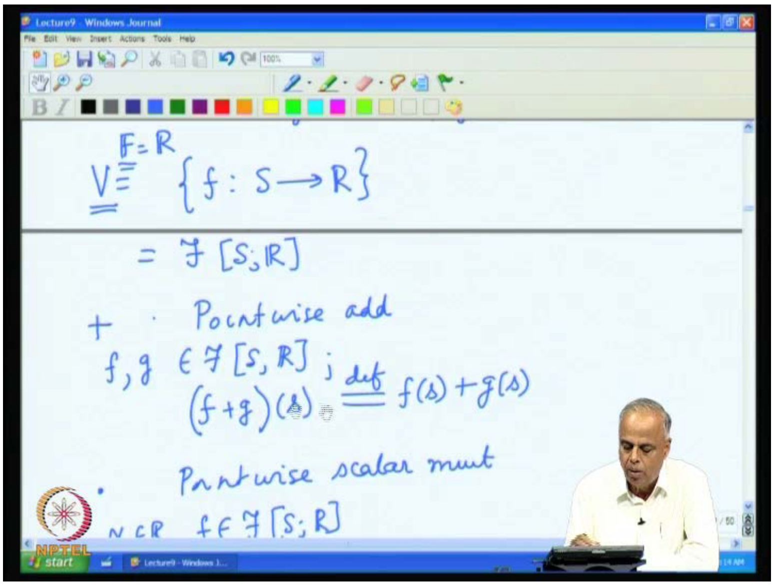

Now, we consider that S be any set any non empty set, let us denote and that is our V let us take F, the field which start with the real numbers and let us take the V to be the set of all functions which map S to R, we will actually denote this by F, S, R. So, this F says collection of all functions, what is the domain S and where be the end of a what is the value taken it is a real number. So, it is the collection of all functions from S to r so, we have now the set V we have the field, do we have the plus by of you take two functions, we have seen what is meant by addition point wise addition.

So, the point wise addition so, what is the definition F belongs g belongs to F, S, R, F plus g by definition is a function its value at any point is defined to be value of F at S plus value of g at S. So, we have the addition operation and we have the scalar multiplication point wise, point wise scalar multiplication what does that mean, we have alpha in R, F in g F in F, S, R, then alpha F is the function by definition is value at any point S is alpha times F of S. So, we have the set V now, we have the set V, the field F, the set V is the collection of all functions from S to R, real valued functions on the set S and the field, if the field have real numbers addition of functions is point wise addition, scalar multiplication means point wise scalar multiplication.

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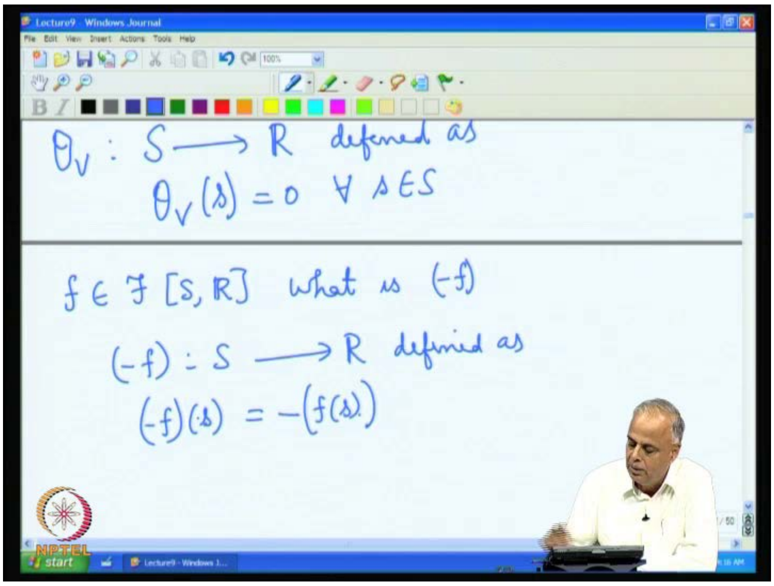

Now, with these laws of addition and scalar multiplication F, S, R is a vector space over R, what is the 0 element of this vector space? It must be an element of V therefore, it must be a function from S to R which function theta is that function from S to R defined as at each point S, it takes the value 0, rather the 0 element of this vector space or the 0 vector of this vector space. Now, if F belongs to F, S, R what is minus F? Now since, F belongs to F, S, R, we want minus F also to belong to F, S, R, because the negative of anything V in a vector space must be back in the vector space so, minus F must be a function.

So, minus F is that function, which is defined as minus F at any point S its value is take F's value and take the negative of that the value of F is given, because F is in given, we are trying to find the negative of F, the moment you know F at any point evaluates its value, that is a real number take the negative of that real number and therefore, we get another real number so, that real number is called the minus F of S it is called the value of the function minus F at the point S. So, we have the vector space of all functions from S to R.

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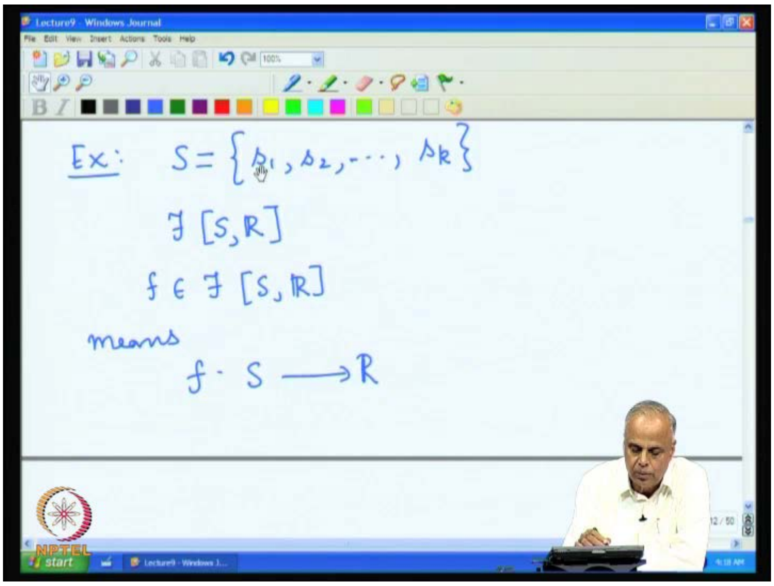

Let us take look at some very specific examples, in this suppose I take S to be a finite set S 1, S 2, say S k, suppose I take S to be a finite set and I look at F,S, R now, what do we mean by F, F in F,S,R. So, if I have a function F which is in S,R that means F is a function from S to R, if F is a function from S to R, F will assign a real number to each element in R therefore, F S will assign a real number to S 1, F will assign a real number to S 2 and F will assign a real number to S k.

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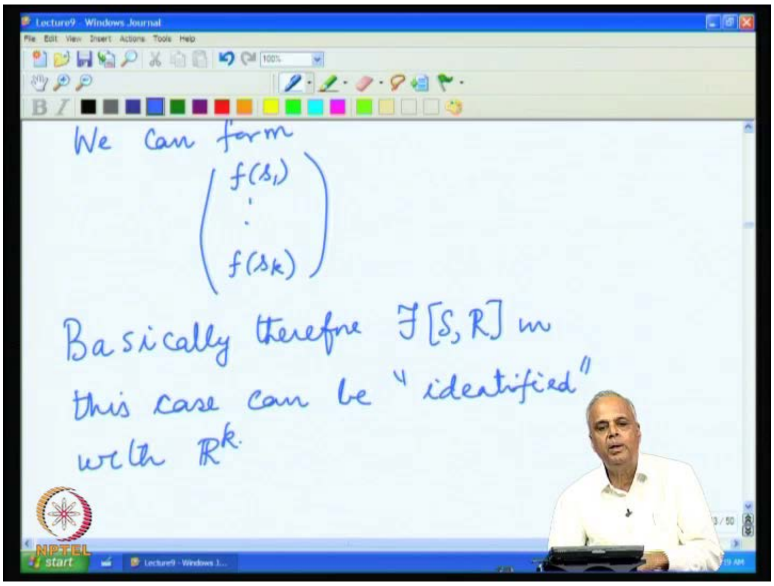

So, we will have F S 1, F S 2 etcetera, f, s, k will all be real numbers and therefore we have k real numbers. So, we can form the usual k row matrix 1 column f, s, k. So, essentially we see that, these are such a function can be identified with an element in R k. So, basically therefore, when we have S in as a finite set S 1, S 2, S k the F,S,R the collection of all functions from S to R is essentially the collection R k, we will see the more about more in detail about this, such sort of identifications in later time, but basically therefore, F,S,R in this case can be identified, we will talk about this identification in more detail later with R k, at least we see it is such a function is only an element of R k it disguise.

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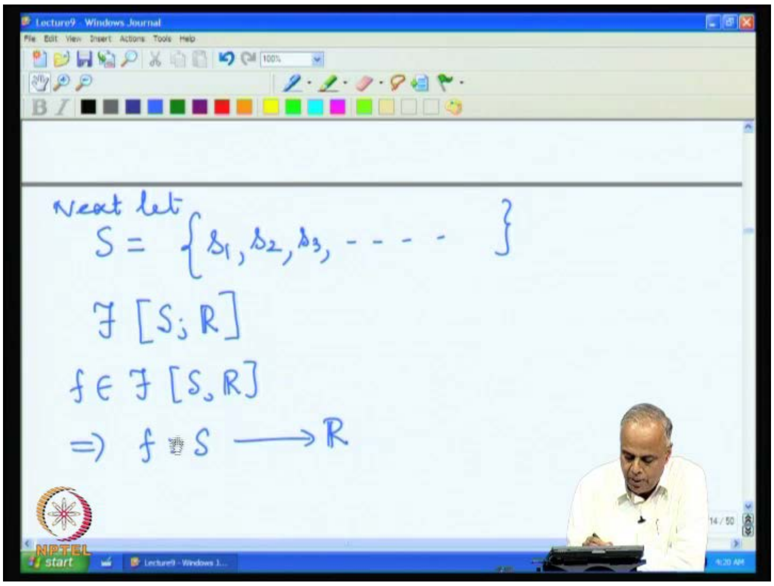

Next let us look at another simple version instead of the finite set, let us look at a discrete sequence S1, S2, S3 etcetera so, we have now an infinite sequence S1, S2, S3. Now, look at the set of all functions from S to R, what does this mean? This means if F belongs to F,S,R this means that F assigns, F is a function of mapping S to R, which means F assigns a real number with S 1, again F assigns a real number with S 2 real number with s 3 and so on.

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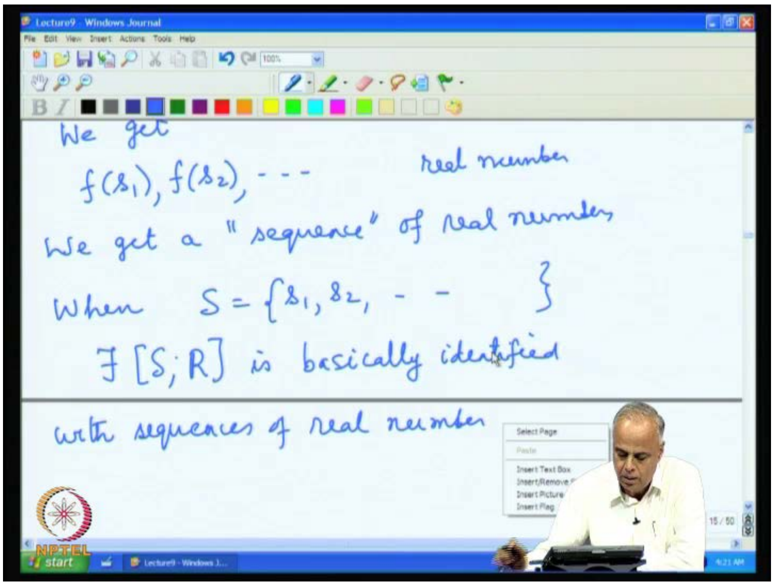

Therefore, we get f s 1, f s 2 etcetera real numbers, thus we get a sequence of real numbers therefore, we get a sequence infinite sequence of real numbers and therefore, in this case when s is equal to S 1, S 2 etcetera. The infinite sequence the F, S, R is basically identified with sequences of real numbers. So, thus when we have talking about R k, thus when we are talking about R k, we are really talking about in other words functions, from a finite set S 1, S 2, S k to real numbers and similarly, when we have talking about infinite sequences of real numbers, we are in other words talking about a functions from an infinite set infinite sequence S1, S2, S3 to the real numbers. So, these functions and this vectors column vectors can be identified with each other.

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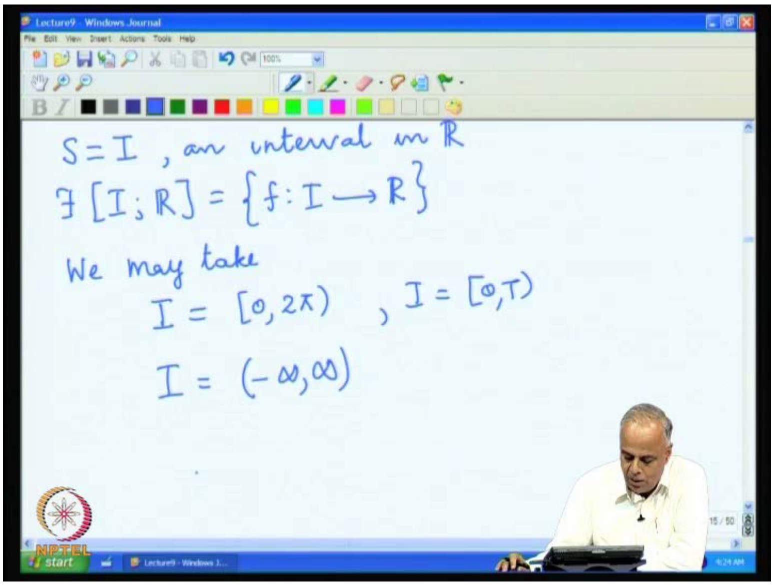

Now, let us take S to be an interval on the real length, this interval can be a finite interval or the full real line semi infinite, closed, open whatever type. So, let I be some interval on the real line and let us take the collection of all functions from I to R. So, this is the collection of all functions, which with each point in the interval I associate a real number, thus we have the collection of real valued functions in R. Now, for example, we may take I to be the interval 0 to pi or I to be 0 T.

Now, these are the types of functions we will be looking at we may talk about periodic functions in Fourier series, we may think of I as the minus infinity to infinity the whole real line, these are the type of functions which we will be talking about, we may talk about Fourier transforms and so different types of intervals introduce different types of situations, which will be useful to us in our applications. Now, among these functions which are mapping I to R there are some of particular entries in all the above examples we can replace R by S.

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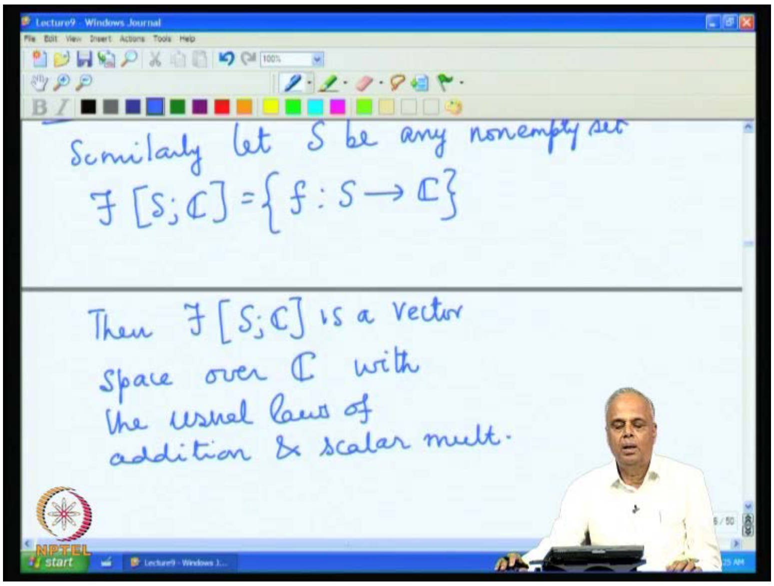

So, we can write similarly let S be any set, any non empty set then look at f, s, c the collection of all functions which map from S to C the collection of all functions from S to C, then laws of addition and scalar multiplication are usual again point wise, the values are added point wise. The collection of all these functions is a vector space over C with the usual laws of addition and scalar multiplication. So, we have the collection of all real valued functions, we have the all collection of complex valued functions, the domain S can be a finite set or in an infinite sequence or a like an interval on the real line. Now, among this class of function there are some special ones which are very useful.

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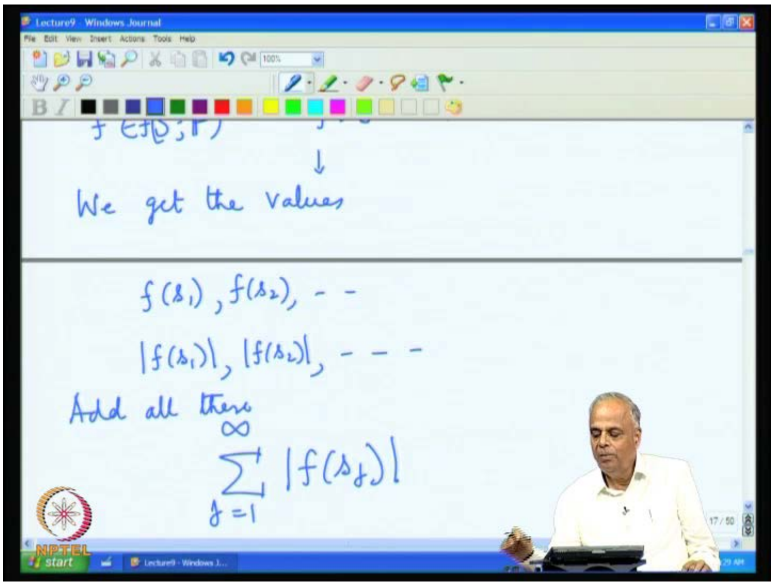

Now, let us look at the situation where we have S to be an infinite sequence S1, S2, S3 and then let us look at F, S, R. F, S, R is the set of all functions from S to R or C so, in order to keep both options available for us let us now write F, where F is equal to R or F is equal to S.

So, either we consider real valued functions on S or real valued complex valued functions on S, where S is an infinite sequence of points S1, S2, S3 an infinite set, but it is a sequence, the elements can be arrange in the form of a sequence for example, you can think of S has the set 1, 2, 3, 4, 5 the set of all positive integers. Now, consider that a set S and the set of all real valued or complex valued functions in this now, if you take any function F in S any function in F, S, R. So, we have F mapping S to F so, either a real valued function or a complex valued function take any such function corresponding to this we get the values f s 1, f s 2 etcetera, which is the sequence of real or complex numbers depending on whether we take F is equal to R or F is equal to C.

Now, what we do is we look at the absolute values of this now if F is a real valued function f s 1, f s 2, f s 3 are real numbers, but they can be positive or negative real numbers, but once we take modulus, we get only non negative real numbers. If F is a complex valued function f s 1, f s 2, f s 3 will be complex numbers, but when we take the modulus again we get to complex valued non negative real numbers.

Now, let us add all these we get summation say j equal to 1 to infinity mod F, Z, S, Z now this sum may be finite or infinite, because we have now a infinite series of non negative term the infinite series may or may not convert it, may diverge to plus infinity so, we would like to look at functions there the sum is under control.

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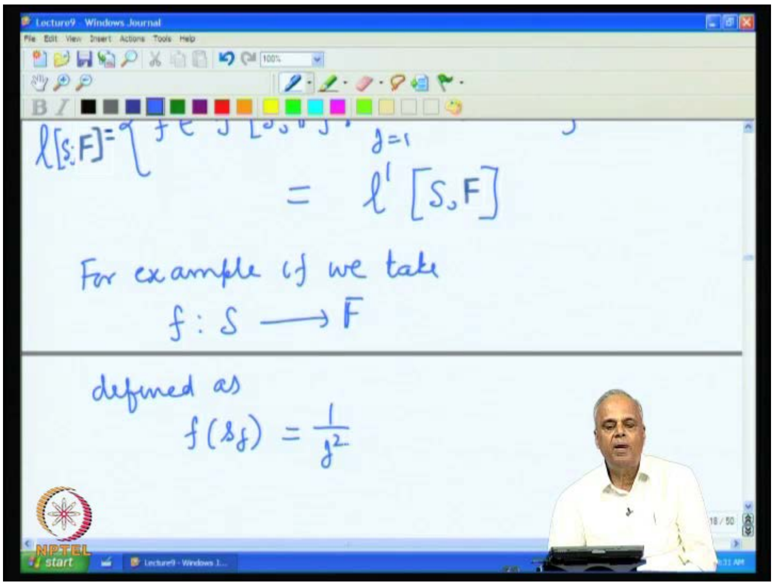

So, we are going to look at all those function so, this may be less than infinity or it may be equal to plus infinity. So, we are going to look at all those functions, which are under control that it is less than infinity. So, we are going to look at all those functions from S to F where again a remind that, F can be R, S, C such that, summation j equal to 1 to infinity mod f s j, is less than infinity the sum is the sum of the absolute values is under control, it is under finite it is finite.

So, consider the set of all functions this is called l, 1 we will use the little l, 1 let me write this carefully this is now denoted by l, 1 s to r l, 1 s to r so, this is the space l, 1 s to R. So, it is not all functions from S to R among the functions from S to R we only choose those functions for which the sum is finite for example, if we take F mapping S to R or S to F defined as f s j is equal to 1 by j square the value taken at the point s j is the square of the index j reciprocal of that.

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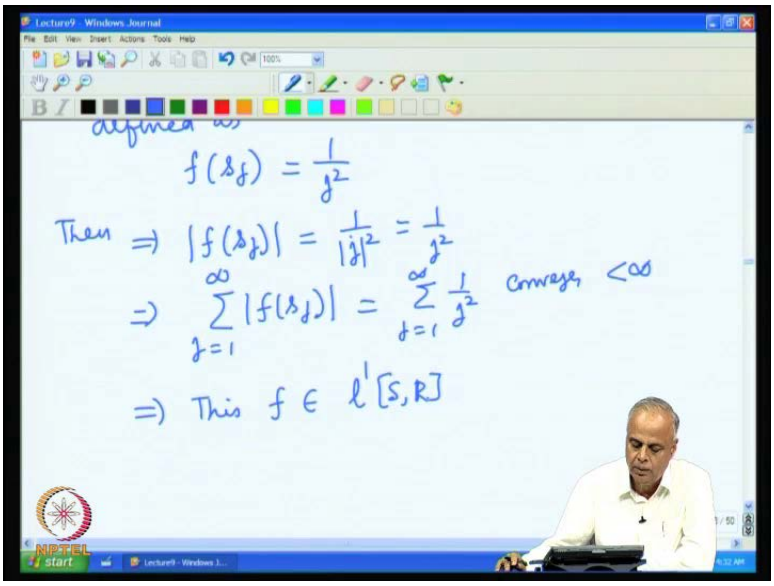

So, now what is mod f j, then this implies mod f j is just 1 by mod j square, j may be j is real so, it is just 1 by j square. So, what is mod f s j all it is this series and we know that this series converges and therefore, it is less than infinity and therefore, this function belongs to l 1, s 1.

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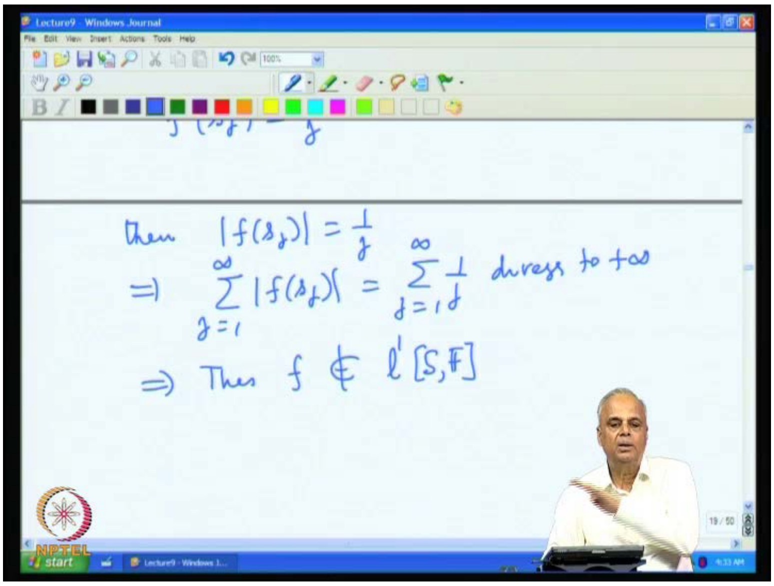

On the other hand, if we take f mapping S to R as F of s j is 1 by j then mod S f j is again 1 by j, j is an integer and therefore, summation mod f s j, j equal to 1 to infinity. The summation j equal to 1 to infinity 1 by j, which diverges to plus alpha and therefore, this F is naught or let us take a general F. F can be real or complex therefore, we have some F's in mapping from S to R, R in this l 1 some F which are not mapping from S to R naught, in this l 1, this l 1 consists of all those functions which are from S to R for which this total sum in absolute value is in control, it is finite with this notation in l 1, S F, we can verify is a vector space over F.

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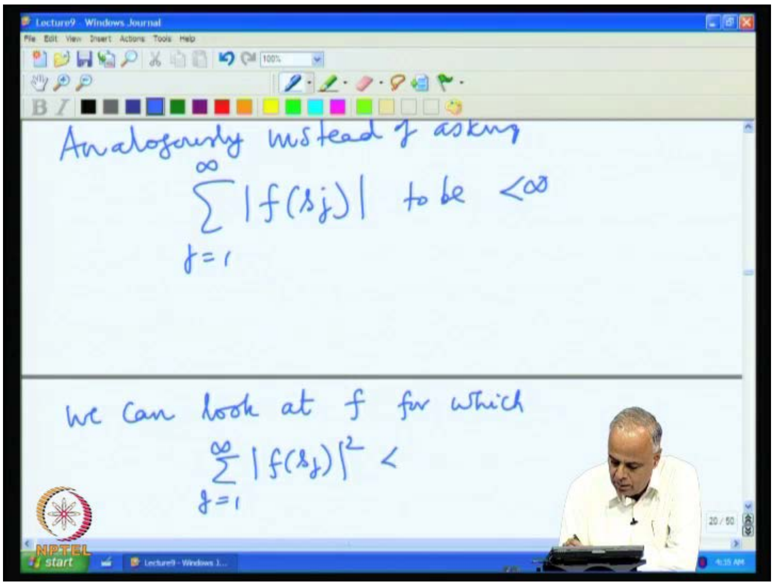

So, the set of all real valued functions on S for with the sum is finite, if a vector space over R the set of all complex valued functions for with this sum is finite, is a vector space over the field of complex number analogously. Instead of asking summation mod s f j, j equal to 1 to infinity to be finite, we can look at F for which summation j equal to 1 to infinity mod F s j square is finite.

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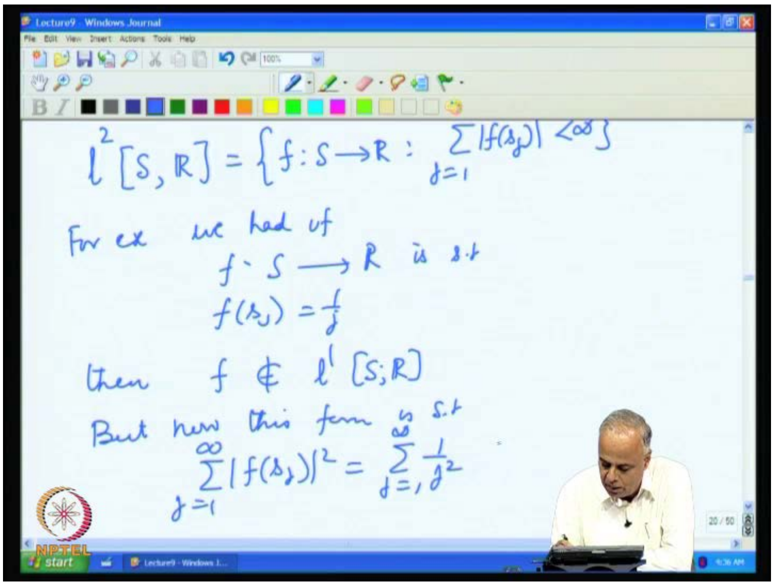

So, we denote this by l 2, S R where S is our discrete infinite set S1, S2, S3 etcetera to be all those functions mapping from S to R, for which the sum mod f s j square is finite for example, we had if f mapping S to R is such that f s j equal to 1 by j, then F was not in l 1, S R, but now this function is such that summation mod f s j square j equal to 1 to infinity, a summation j equal to 1 to infinity 1 by j square, because f s j is 1 by s j. .

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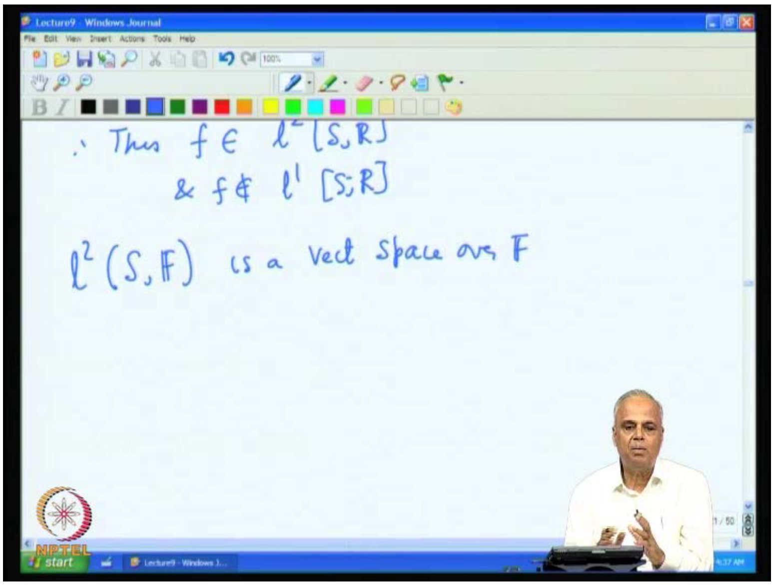

We know is a convergence series therefore, this F belongs to l 2, S R whereas, and F does not belong to so, these are two different objects l 1, S R and l 2, S R 2 different object similarly, you can replace R by C you get the complex functions therefore, l 2, S F where F is R, e C is a vector space over l 2 is a vector space over F will be usual laws of addition, because you know how to add functions, we know how to multiply a function by a scalar, these are all point wise and with that these.

So, these are functions from S to R which have special properties first, we considered all functions from S to F, that was a vector space over F then we considered all functions over S, from S to F for with the summation mod f s j is finite and we got l 1, S F, then we considered all functions from S to F, for which summation mod f s j square is finite and we got the L 2 S.

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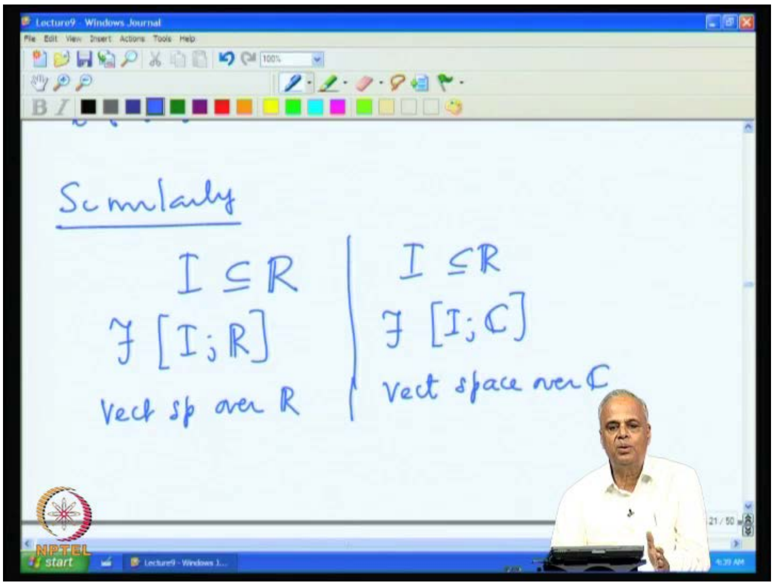

Similarly, let us look at l 1 similarly, first thing is look at any interval I in R and then look at all functions from I to R or you look at any interval in R and look at all functions from I to C. So, either real valued functions on the interval I or the complex valued functions on the interval, then this is a vector space over R this is a vector space over C, just like we considered special functions above where the sum of finite or the sum of the square as finite, here we can consider functions for with integral value of the F is finite and the integral value of absolute value of F square is finite.

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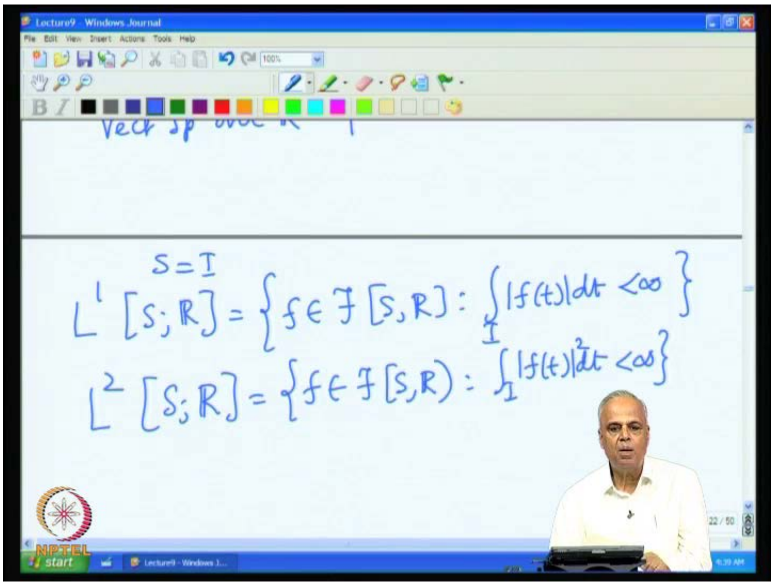

So, we get l, 1 s to R to be the set of all functions from S to R, where S is now the interval I such that, integral mod of f t, d t is well defined and is finite of course, we must make sure that the integral is finite, we will not worry about given formal definitions of the integral at this stage similarly, l 2, S R when s is the interval if the set of all F, from S to R such that integral I mod f t square d t is finite these are some of the important vector spaces of functions. We shall see some more vector spaces of functions as examples in the next class.