Advanced Matrix Theory and Linear Algebra for Engineers

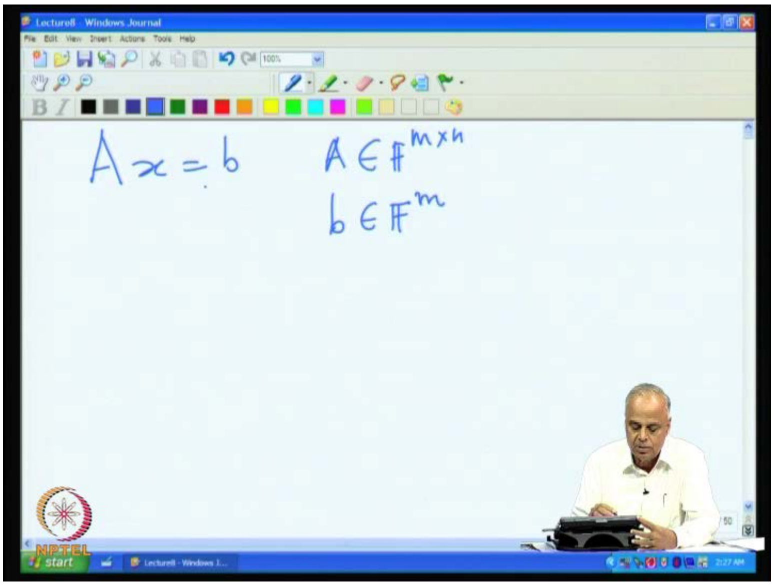
Prof. R. Vittal Rao
Centre for Electronics Design and Technology
Indian Institute of Science, Bangalore

Lecture No. # 08

Vector Spaces –Part 1

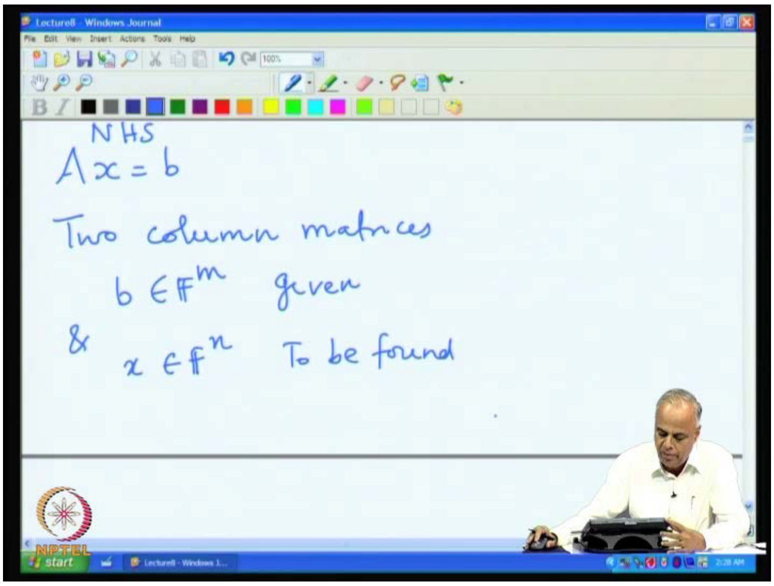
The last lecture, we learn how to **s**olve the non homogeneous system of equation A x equal to b. Where a is A F n cross m.

(Refer Slide Time: 00:25)



And b is a F m. We found that there are several coefficients connected with this problem which? we have still not able to arrive at what? we are able to do was whether then b satisfy, the consistency condition when you how to find the solutions. The other cases when we did not we did not even know how to how to proceed. In order to attain to such questions we want to develop certain mathematical framework.

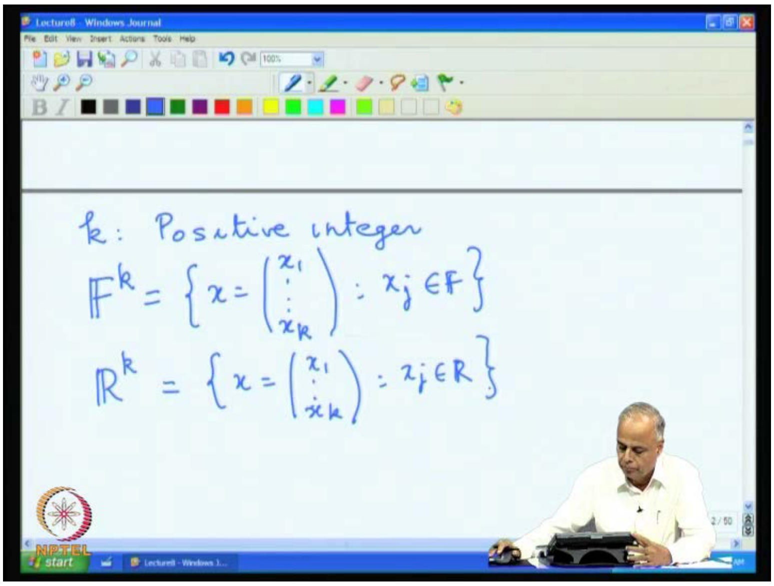
(Refer Slide Time: 00:57)



The main mathematical framework in which we shall work is what is known as a vector space. In this lecture we should look at what exactly a vector space mixes. While dealing with a non homogeneous system A x equal to b.

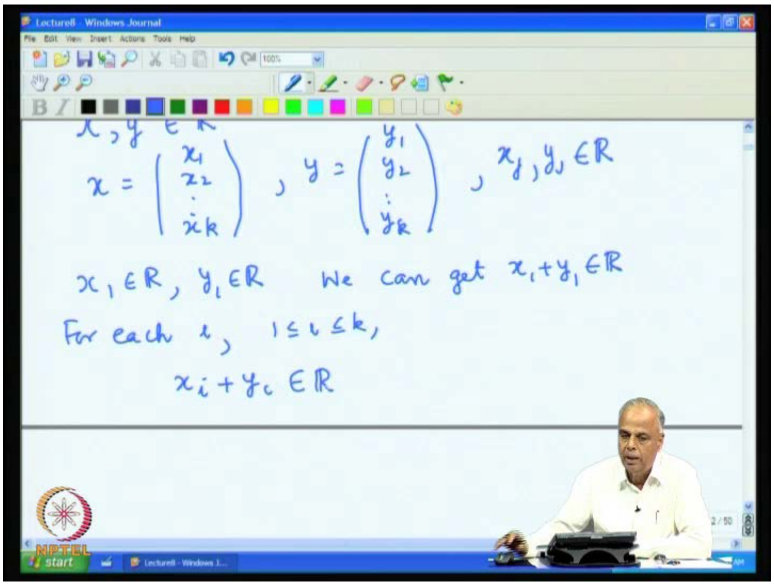
We have to encounter 2 column matrices namely b which is in F m which is given and x which is in F n and which is to found. The main fact is that we have to theme which such vectors or column matrices F m and F n. We shall now have look at this the structure of such column matrices so to look at this.

(Refer Slide Time: 01:59)



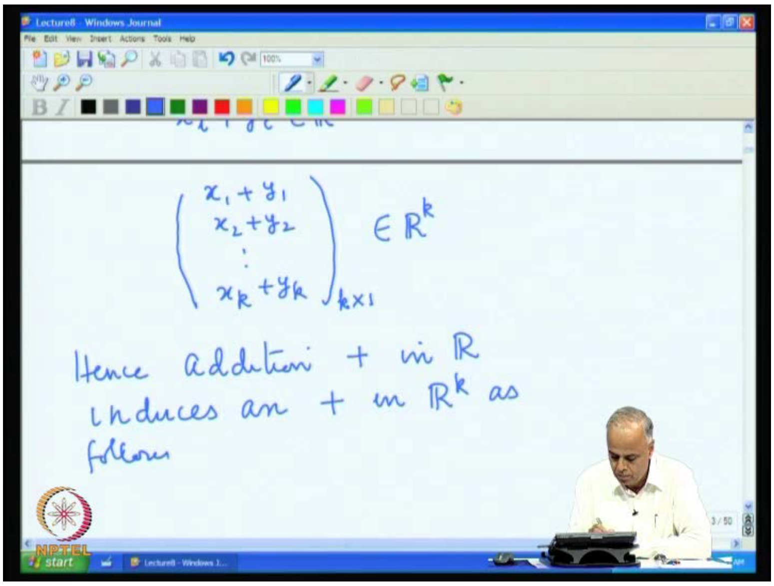
Let us look at a general k which is a positive integer is look at any positive integer k and then look at F k which is the set of all column matrices which have 1 column and k rows and with all the entries coming from F. Let us in particular take F equal to R so that we look at the set of all 1 column k row real matrices all the entry series are real. Now if we look at such matrices let us take 2 such elements x and y in R k. x will be x 1 x 2 and x k.

(Refer Slide Time: 02:47)



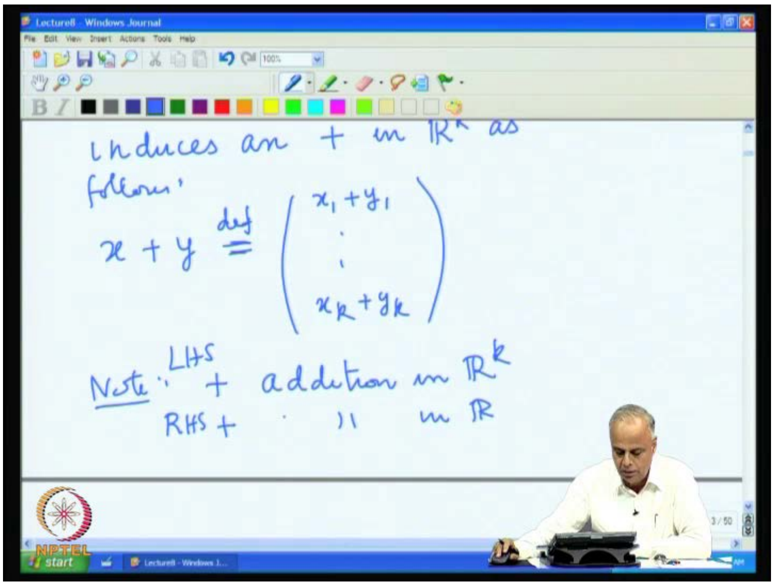
and Y will be y 1 y 2 and y k. Remember this x's and y's the x j and y j are all real numbers. We are looking at R k all the entries in matrix all real numbers. Now if we look at x 1 is a real number and y 1 is a real number. We know how to add real numbers, so we can get x 1 plus y 1 which is again real number. Analogously we can do this for every entry there so for each i 1 less than or equal to i less than or equal to k x i plus y i is well defined because x i is a real number y i is a real number and the sum of these real numbers is again be a real number. Therefore, we have got this k new real numbers namely x 1 plus y 1 x 2 plus y 2 and so on x k plus y k. Now we form a column matrix whose entries are this sums x 1 plus y 1 x 2 plus y 2 and so on x k plus y k.

(Refer Slide Time: 04:20)



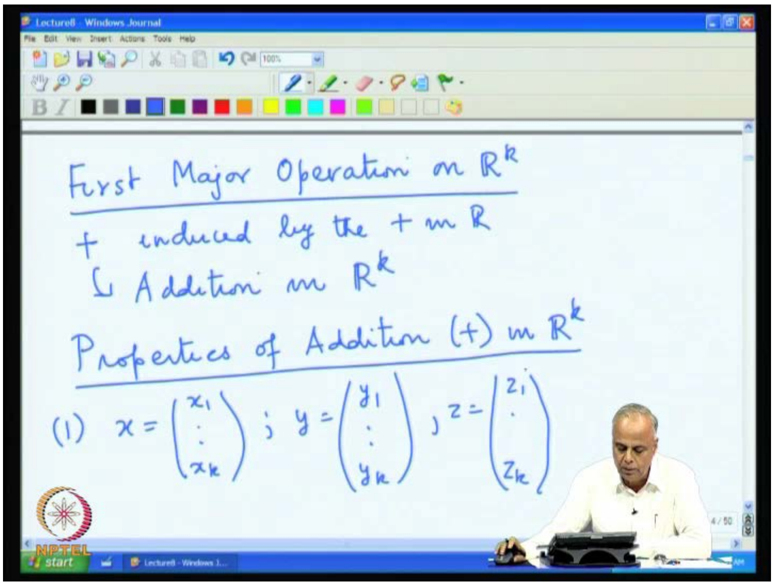
Now since this is a column metrics and it has k rows and 1 column and all the entries are real numbers this is an element of R k. Starting from 2 elements in R k we R now generated a new element in R k using the addition operation which is in R. Therefore, we have a new addition in R k induce by addition in R. Hence addition which is plus in R induces an addition which also a denote by plus in R k as follows.

(Refer Slide Time: 05:38)



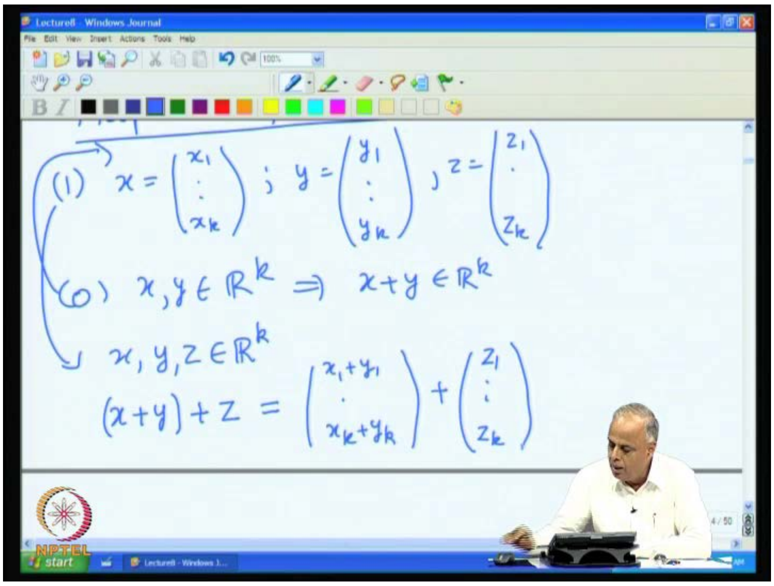
If x and y is R as above x plus y x is now defined as x 1 plus y 1 and x k and y k. This is the definition. It should be noted that the plus on the left hand side this plus refresh to addition in R k. Which is being defined and the plus on the right hand side and the RHS plus is addition in R which is known that is the known addition in R induces an addition in R k. Thus we are able to start with 2 elements in R k and define their addition and addition is in R k.

(Refer Slide Time: 06:36)



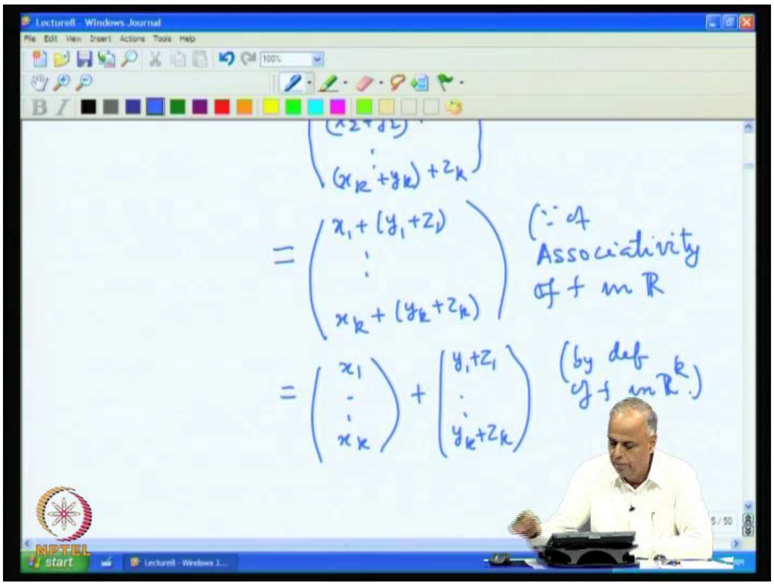
So thus we have an operation of addition on R k. The first major operation on R k we have is plus which is induced by the plus in R and this plus which have been induce we call as addition in R k. Let us look at some of the properties of this addition. We have the properties of addition which we denote by plus in R k. Let us look at three elements x 1 to x k y y 1 to y k and z z 1 to z k. One of the property that we should always keep in mind that then we add x and y the result is always again in R k. So the plus of 2 R k elements is again is on R k element. Let us note that as the property zero.

(Refer Slide Time: 08:16)



That is x and y belong to R k implies x plus y belongs to R k. Now let us take this property 1 we are taking three elements x y z in R k. Now let us look at x plus y we have seen the the sum of 2 elements in R k is again on R k. This x plus y is an element in R k any element in R k can be added with another element in R k. x plus y plus z is an element in R k and what is this x plus y is x 1 plus y 1 x k plus y k. This is the definition of the x plus y and plus z which is z 1 z k. Now again on the right hand side we are adding 2 elements in R k.

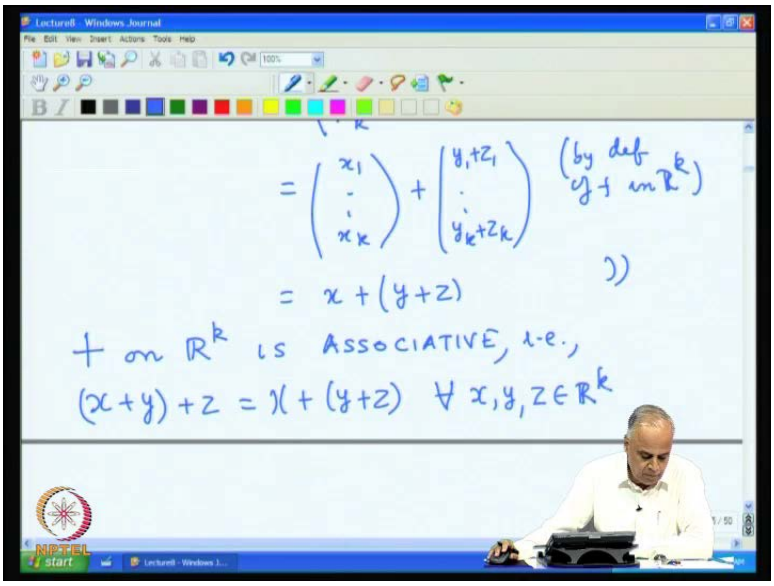
(Refer Slide Time: 09:20)



So the usual addition definition given above say is the this is equal to x 1 plus y 1 plus z 1 x 2 plus y 2 plus z 2 and so on x k plus y k plus z k. Now if we look at each entry they are all additions of real numbers I am adding x 1 y 1 z 1 these are real numbers x 2 y 2 z 2 these are real numbers and so on.

We know they are in the real numbers bracketing can be done anywhere. That is real number addition is associative and therefore, we can write this as x 1 plus y 1 plus z 1 and so on x k plus y k plus z k. This is because of associatively is called associatively law of addition in R. The addition among the real numbers follows a associative law. Now this can be re written again as x 1 to x k plus y 1 plus z 1 etcetera. y k plus z k. This is again by the definition of plus in R k. We are given above the definition.

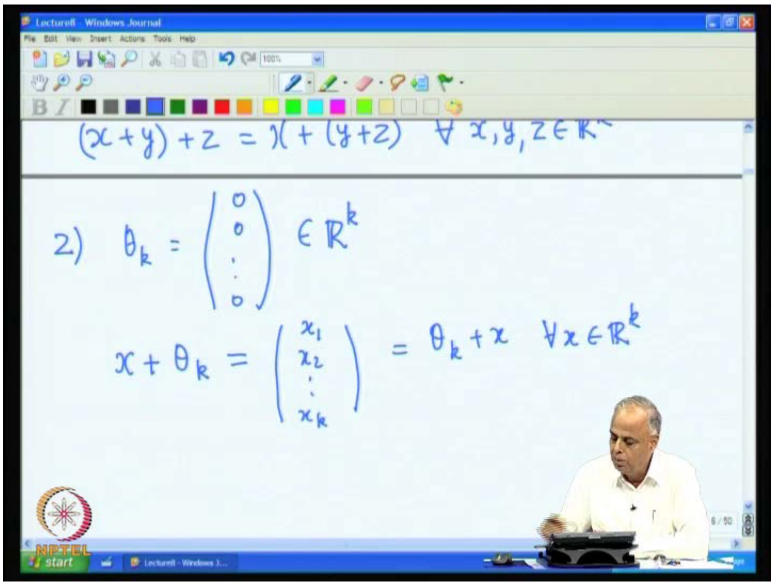
(Refer Slide Time: 11:01)



According to that definition this sum will be equal to this this is x plus and what is appearing here by the definition of addition. Again is equal to y plus z again the the definition definition of addition in R k.

Therefore, what we have observed is plus on R k the addition R k is associative. That is x plus y plus z is the same as x plus y plus z for every x y z in R k. First we have defined the notion of addition. Then we have observed that this addition is as that whenever x and y are in R k there is sum is also in R k. Then if the addition obeys the associative law.

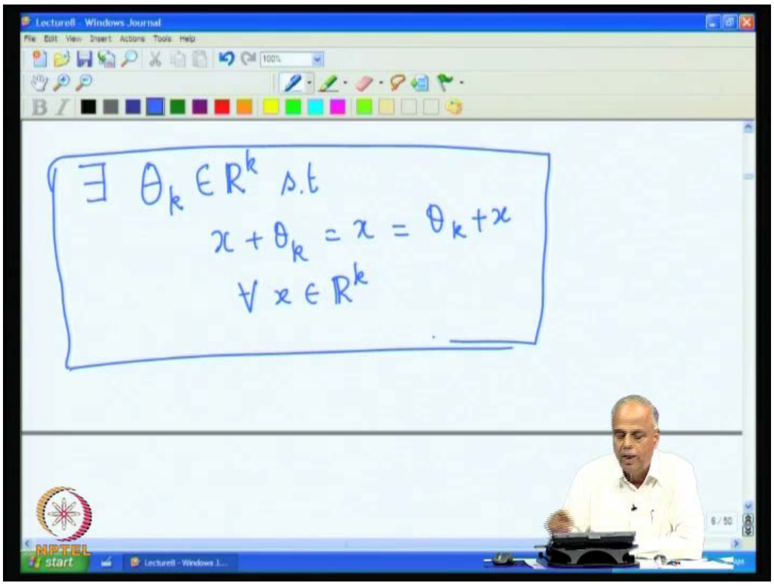
(Refer Slide Time: 12:06)



The next property we observe about addition is the following. Look at the vector R the metrics theta 0 0 0 0. Now the 0 on the right hand side is real number 0. R we have this real number 0 now we have formed a metrics column metrics all the entries are 0. Since 0 is a real number this belongs to R k. These are the property that if we now add x to theta k by our addition.

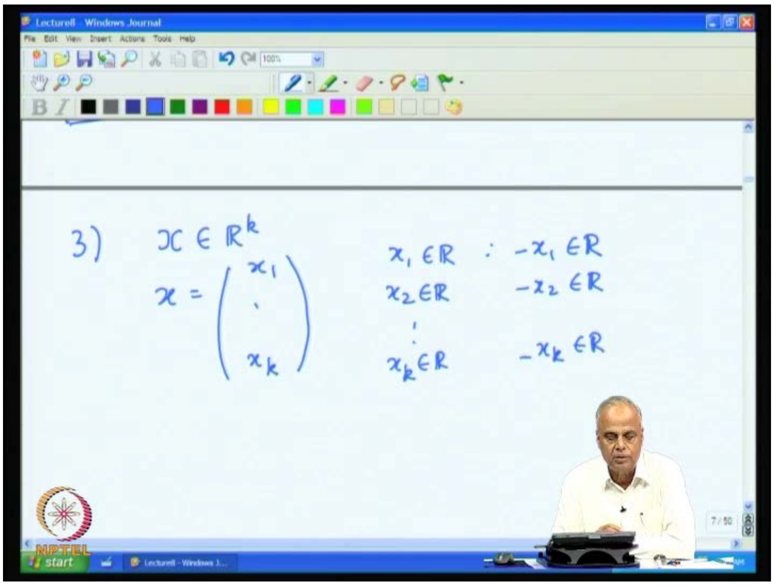
It will be x 1 plus 0 which is again x 1 x 2 plus 0 which is again x 2 and x k. It will be the same as theta k plus x because 0 plus x 1 is x 1 0 plus x 2 is x 2. This is true for every x in R k. T he property that we observe is the 0 in the real numbers induces a 0 in the R k the set of column matrices.

(Refer Slide Time: 13:28)



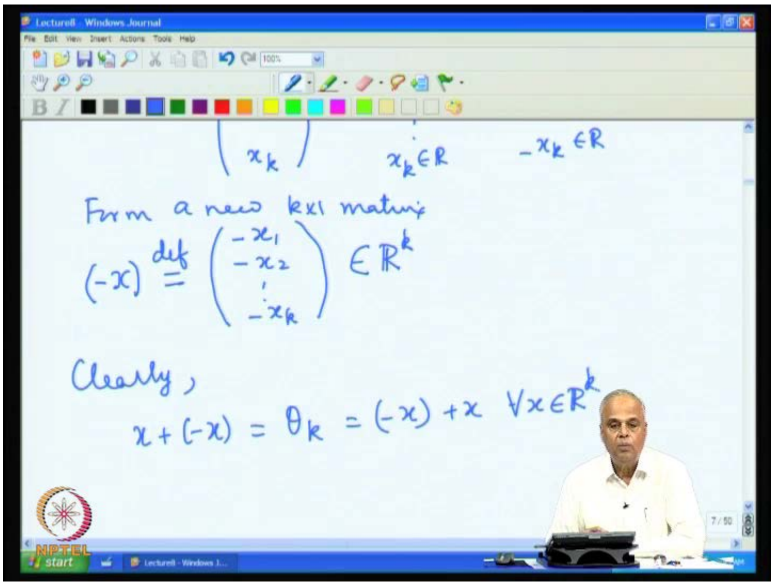
This is the conclusion we have that is there exist a vector a column matrices a theta k in R k such that x plus theta k is equal to x is equal to theta k plus x for every x in R k. This next property we see of addition in R k again. Let us recall addition is such that when x and y in R k x plus y is also in R k addition was associative. Now we see that there is a vector analogous to the number 0 in R there is vector theta k in R k. Such that x plus theta k is equal to x for every x. this is the next important property of addition on R k. The next property that we observe is the following.

(Refer Slide Time: 14:31)



Suppose we have vector x in R k say x is x 1 x 2 x k. Now x 1 is the real number. Therefore, every real number has its negative so minus x 1 also belongs to R. x 2 is the real number and therefore, minus x 2 also belongs to R and so on. x k is the real number so minus x k also belongs to R. Take a vector or a column matrix x in R k look at each 1 of the entry that being a real number look at it is negative since every real number as is negative. We can find all these minus x 1 minus x 2 minus x k.

(Refer Slide Time: 15:30)

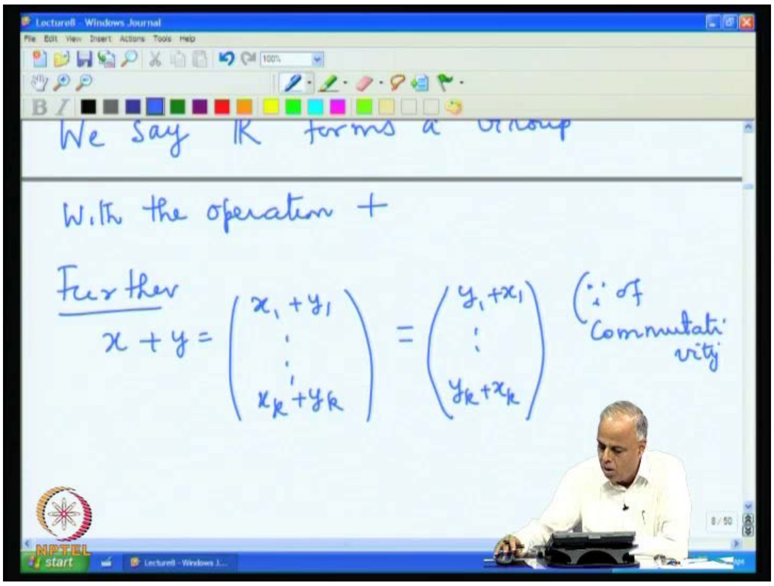


Now we form a new vector or a new matrix k by 1 matrix whose entries are now minus x 1 minus x 2 and minus x k. Since minus x 1 minus x 2 minus x k is all real numbers this is again in R k. Therefore, from x we have constructed a mu element in R k and we will denote this element by minus x. This is the definition if x is x 1 x 2 x k the corresponding minus x element in R k is defined as that matrix all of these entries are negative of the corresponding entries in x. Then clearly now by the addition rule of R k elements we see that x plus minus x is equal to theta k is equal to minus x plus x for every x in R k. What we see here is that the corresponding to the number 0.

It induces a matrix 0 or the vector theta k or the element theta k in R k. The corresponding to the number minus x we have a matrix minus x or the vector minus x or the column matrix minus x that has been define which behaves exactly like the 0. In real numbers we had a number plus its negative is 0. Now we have an element of R k plus it is negative is the 0 of the element of that space so this is the third important property.

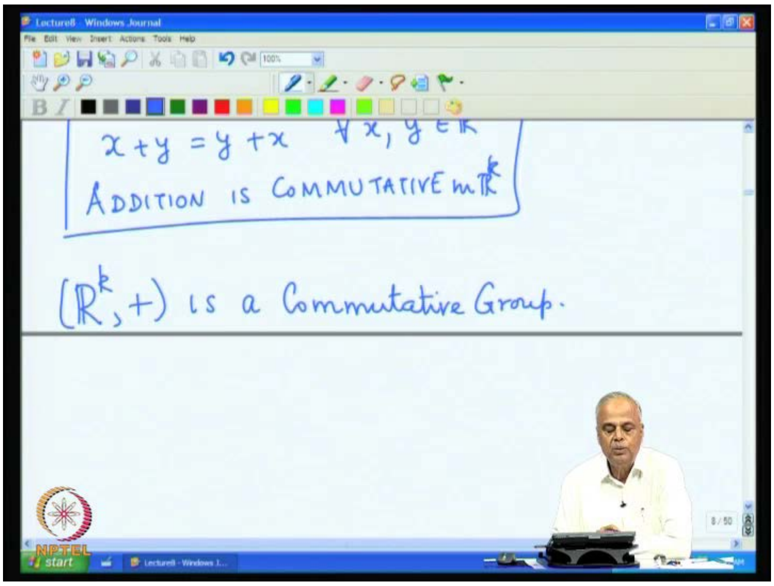
When we have these three properties together with the fact that x plus y belongs to R k for every x and y we have what is known as the notion of a group.

(Refer Slide Time: 17:44)



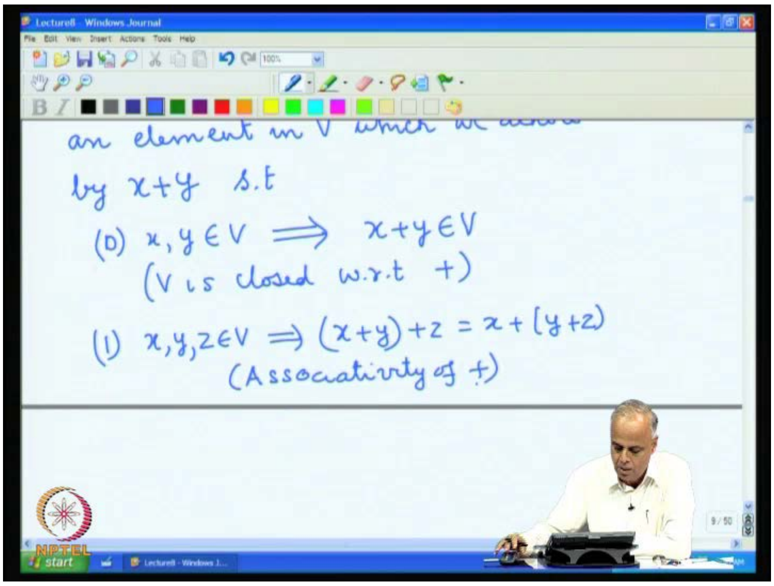
We say R k forms a group with the operation plus. That we have defined so basically the real numbers form a group the real numbers are such that the addition is present the addition of 2 real numbers is real number there is the 0 in real number. Every real number as in negative real number which is such that x plus minus x is 0. All these properties induce corresponding properties on R k. Therefore, the group structure of R induces the group structure on R k. Further if we have x and y we have x plus y is x 1 plus y 1 x k plus y k. Now x 1 and y 1 are real numbers and x 1 plus y 1 is the real number addition. Real number addition can be done in any order or in other words the real number addition is commutative. We can write this as y 1 plus x 1 y k plus x k because of commutatively commutatively of addition in R. But now by the laws of addition in R k this is the same as y plus x.

(Refer Slide Time: 19:29)



Therefore, the commutatively of addition in R again induces the commutatively of addition in R k. We have x plus y is equal to y plus x foe every x y in R k are addition is commutative it a group. If this rule of combination does not obey any order then we call it commutative group. The first thing that we structure that we observe about R k is that R k plus is a commutative group there also known as Abelian group.

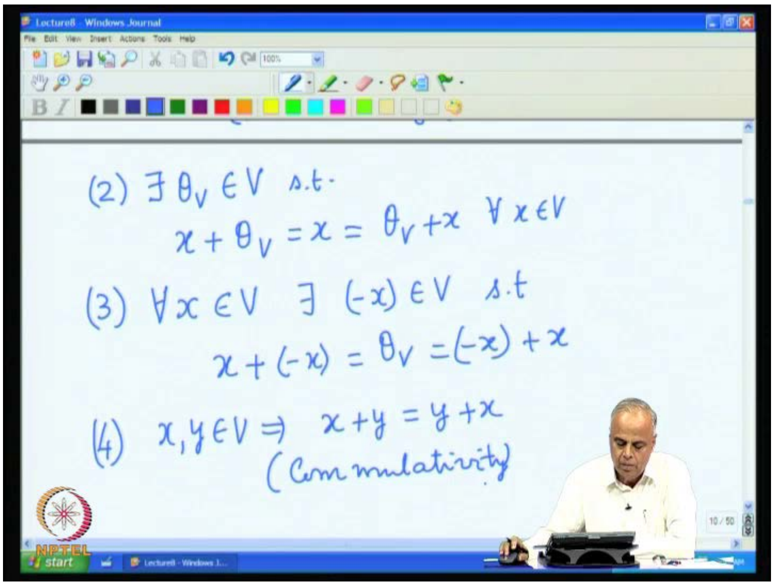
(Refer Slide Time: 21:19)



Now we can abstract this and generalize in place of R k we can take any set g. Let V be any nonempty set and in place of plus we will take some operation. Let plus we will also denote that plus see we are denoted by the same plus addition in R. We have use the same symbol plus for addition in R k. Now we use the same symbol plus in an arbitrary set V for an arbitrary rule. Let plus be a rule of combining an x in V with a y in V to produce an element in V which we denote by x plus y which we denote by x plus y. Symbolic the result outcome of this combination as x plus y. Such that we have all the we now demand all the rules at the plus obey the R and R k to be obeyed by this new rule of combination what are these rules.

We have the zero th rule x and y belong to V. Then the result of the combination must also be in V. We say V is closed with respect to plus V is closed with respect to plus this operation plus. Then we demanded associativity.

(Refer Slide Time: 23:53)

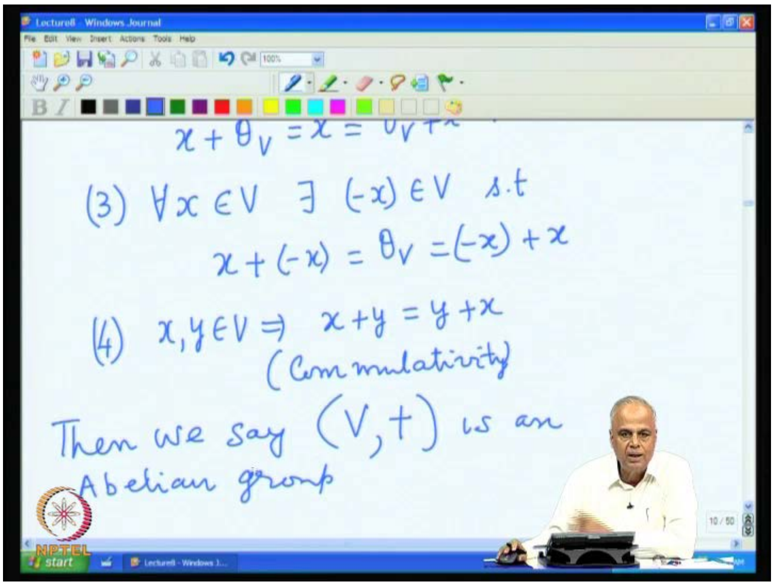


So x y z in V implies x plus y plus z must be equal to x plus y plus z. This is the associativity of plus. Then we wanted the existence of something at into 0. We had 0 in the real number theta k in R k. Now for V we shall call it a general theta V there exist an element theta V in V. Such that what are the property required of 0 when it combines with anything. Then the result is what you combined x plus theta V equal to x equal to theta V plus x for every x in V. That is the role played by 0 generalized.

Now the role played by the negative is going to be generalize. Now what is the role of negative for every real number x we had a negative real number for every x in R k. We had a negative x now for every x in V. We need a negative x which should also be in V the negative real number was a real the negative of R k element was again R k element. Now we want a negative of a next in V to be in V such that what are the role of the negative the element in the negative combined together to produce the 0. Now the role of 0 is played by theta V. We need this then we also had the commutative law that is x and y in V whether we combine x with V or y or y with x.

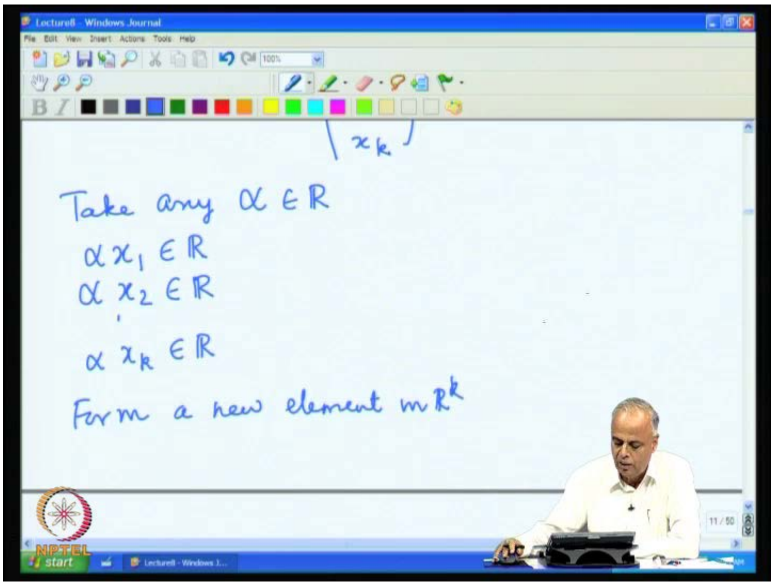
We must get the same result this is the commutativity of plus. When over we have an arbitrary set V we started with an arbitrary set V and we have a rule of combining 2 elements in that set. If that combinational rule obeys these properties namely V is closed with respect to this plus. That is the result of adding or combining 2 elements is again in that set the set is self contain with respect to this operation. This rule is associative and there is an element again to this 0 element in real numbers. There is the negative of every element and the rule of combining is commutative. Then we say V together with that operation plus is an abelian group.

(Refer Slide Time: 26:36)



This is the abstraction or generalisation of the ideas that we get from R k. We had first structure in R. We get R k by putting several copies of R. We put k copies of R first element is R second entry row entries R third row entries R the k th row entries R. We put k copies of R to get R k. Whatever properties that R had the R k carried over same thing addition properties were carried over group structure was carried over commutativity was carried over. Now we have generalized all these and we get the notion of abelian group. Before we see example we should also we will also look at another operation. The second major operation in R k second major operation in R k.

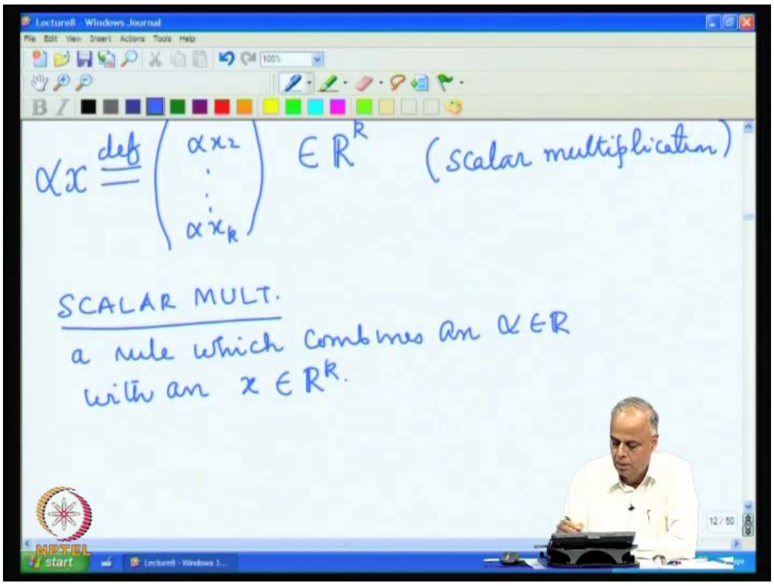
(Refer Slide Time: 27:39)



The first major operation was addition. Now we will look at the second major operation in R k. what is this operation. Now consider any element of R k it is of the form x 1 x 2 x k. There are k entries each entry is a real number. Now take any real number alpha we have x 1 is the real number alpha is the real numbers. Therefore, alpha x 1 the product of the real number alpha and the real number x 1 is well defined and that is also a real. Similarly, the product of alpha and real number x 2 is well defined and that is a real number and so on. Alpha x k is a real number therefore, starting with these real numbers x 1 x 2 x k multiplying each 1 of them by the real number alpha.

We got k new real numbers alpha x 1 alpha x 2 and alpha x k. We now form a new element in R k as follows.

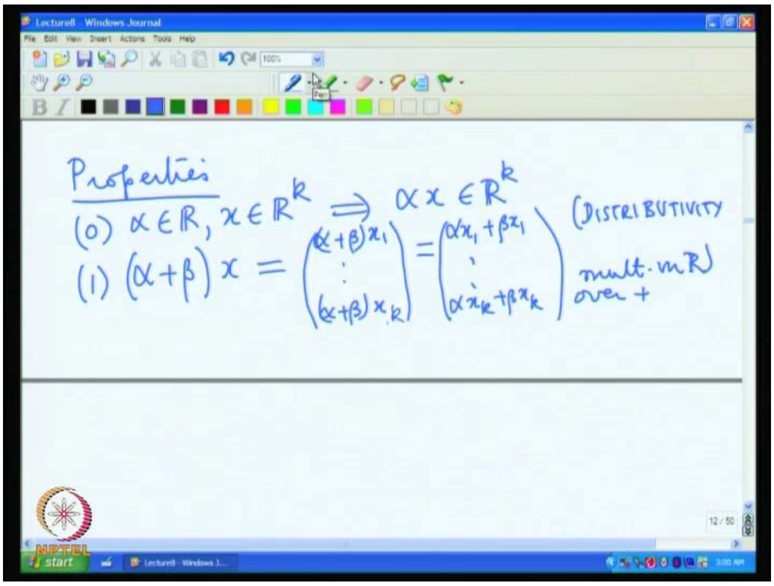
(Refer Slide Time: 29:30)



Take the first entry is alpha x 1 second entry as alpha x 2 and then third k th entry as alpha x k. Since each of these entries is the real numbers. This is R k this is in R k starting with the element x in R k. We started with an element x in R k. Then we took a real number alpha we combine the real number alpha and a element x in R k. So we are now combining a real number with in R k element. We are combined a real number alpha and on R k element and produce an element in R k. This process is what is called as scalar multiplication. We define this as alpha x this is a definition.

This is the rule of combining a real number of alpha and an element x in R k and this rule is called scalar multiplication. That is we are multiplying an element in R k by a scalar alpha. What is scalar multiplication? So scalar multiplication is a rule which combines an alpha in R with an x in R k. what is the important property or properties of scalar multiplication.

(Refer Slide Time: 31:33)

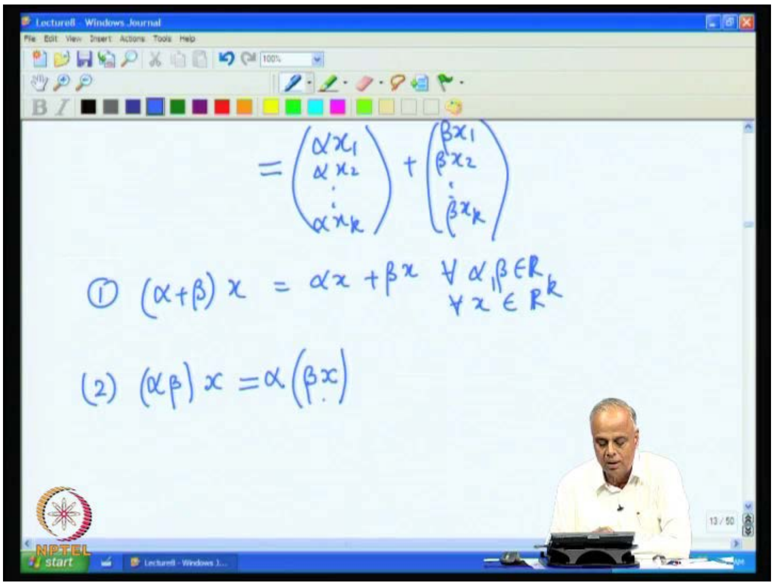


Once again we call of the 0 th property when we combined alpha real number and an element x in R k. We got an element in R k therefore, the rule combines alpha belongs to R x belongs to R k implies alpha x belongs to R k. The result is again in R k.

Now it is easy to verify that if I take 2 real numbers alpha and beta and then multiplied scalar multiplied with when I add alpha and beta are still that as scalar alpha is a real number beta is a real number. Alpha plus beta is a real number so combine this new real number alpha plus beta with the element x in R k. When we combine a real number. This is a real number; this is an element in R k. When you combine you must get an element in R k how do that look the rule of combination is alpha plus into x 1. Every entry must be multiplied by that scalar. Now alpha is the real number beta is the real number x 1 is the real number. The real number multiplication distributes over addition.

The first entries alpha x 1 plus beta x 1 alpha x k plus beta x k which is because of distributivity of plus over multiplication in R multiple distributivity of multiplication. Over plus put it that way some way appropriate we are saying that the multiplication distributes over addition.

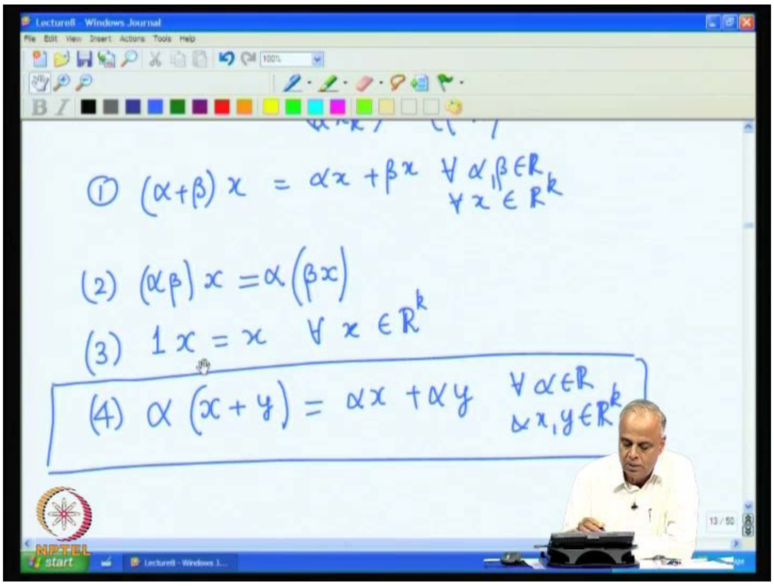
(Refer Slide Time: 34:00)



Now this by definition of vector the addition in R 1 it is alpha 1 x 1 alpha 2 x 2 alpha k alpha x k plus beta x 1 beta x 2 etcetera. Beta x k but, again by the rule of scalar multiplication. This is alpha x and that is beta x, the conclusion is alpha this is the first property alpha plus beta into x. This is true for every alpha beta in R and for every x in R k similarly.

If you take alpha any real number beta any real number and multiplied with the element x in R k. Alpha and beta are real numbers alpha time beta is a real number. The real number alpha beta is combined with the element in x in R k. The result must again be in R k. This is the same thing as first combine the real number x with beta you get an element in R k. Then combine the real number alpha with this element in R k you have obtained.

(Refer Slide Time: 35:20)

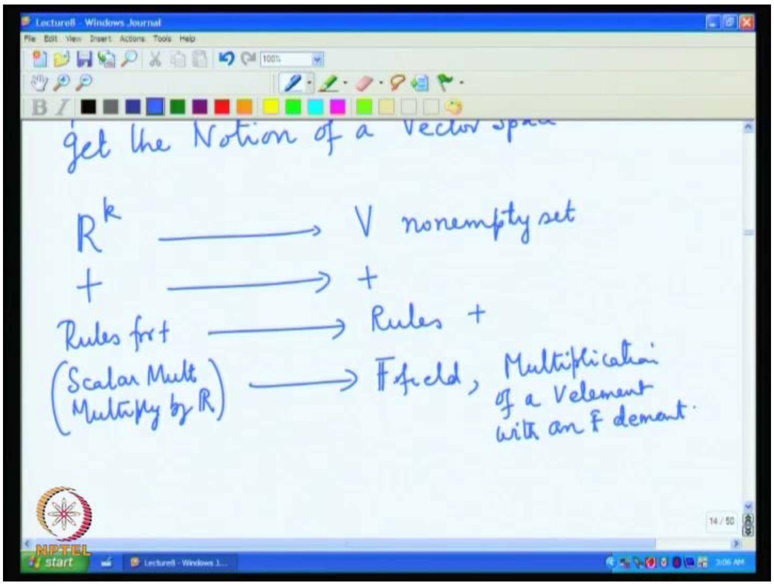


You have the real number 1 if you combine that real number 1 with the element in x k. What we get according to the raw of the combination. We must multiply every entry by alpha. Now alpha is 1 so I'll get 1 times x 1 but, 1 times x 1 is x 1 one times x 2 is x 2 and so on.

So I get back the element x for every x in R k. Then finally, supposing I have an element in R k. I take another element in R k. Now I can combine or add or plus 2 elements in R k and that will be again the element in R k. I have an element in R k. Here I can now combine an element in R k with a scalar. Suppose I do this and what would I get what I would normally expect. This should be alpha y this is true for whatever alpha real number and x and y in R k. Now we have 2 operations in R k 1 operation is about addition the other operation is about scalar multiplication.

If you look at this fourth operation this is an operation which combines the addition operation. It is scalar multiplication operation. This is the rule which relates or combines and make these 2 interact these 2 operations interact the scalar multiplication. The addition operation their interaction rule is provided by this. Otherwise the first three properties or four properties 0 1 2 3. That we wrote of scalar multiplication and the other properties of plus they leave independently. They do not involve in mixture. But, now this property involves both scalar multiplication and addition in R k. This is the very important rule it combines the 2 basic operations that we have studied.

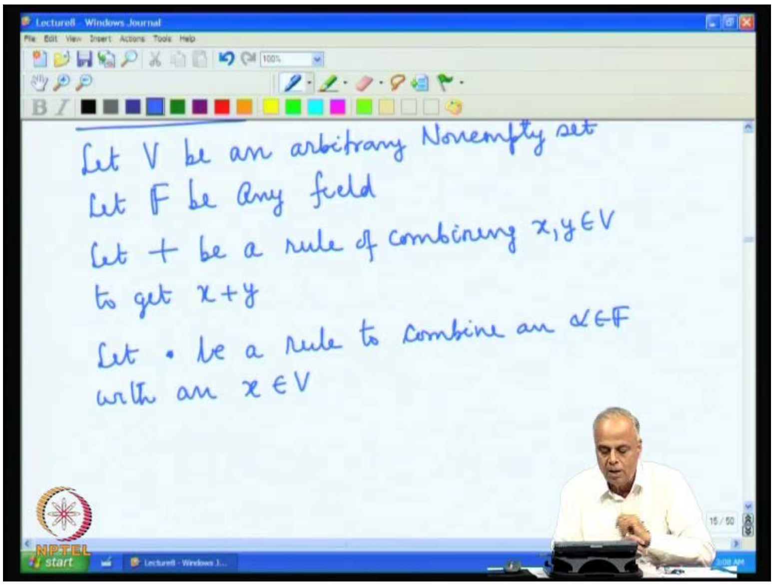
(Refer Slide Time: 38:04)



So for when we generalize all these we get the notion of a vector space. This generalizes all these we get the notion of a vector space. What is we had R k we have to replace this by an arbitrary nonempty set. Then on R k we had an addition that will have to be generalizing to an operation plus on V. Then we have the rules for plus and they will have to be generalize for as rules for plus. Then we add scalar multiplication here what is meant by scalar multiplication. Multiply by R now this has to be generalizing this R is replace by what is known as a general field. F is a field we will see more examples as when we deal with examples of vector spaces. This is the generalize form of the generalization field at the notion of a field is again to the real numbers with both addition and multiplication.

Take all those properties that addition and multiplication have nor and put it as an abstract form we get the notion of a field. Therefore, this has to be generalized as multiplication of a V element with an F element. This is the generalized version of a scalar multiplication. Then we add the rules for scalar multiplication and it has to be generalize for rules for scalar multiplication. When we do all these generalization we get the notion of an abstract vector space.

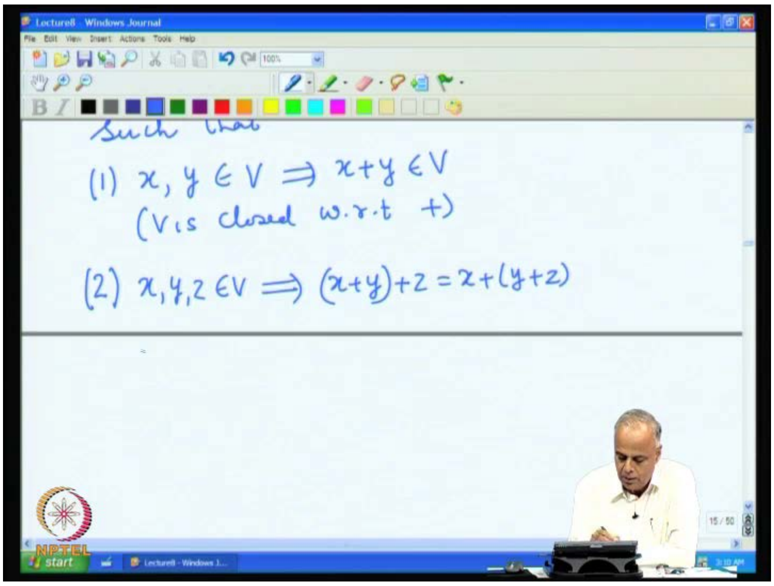
(Refer Slide Time: 40:37)



We shall now give this formal definition of a vector space. Let V be an arbitrary nonempty set. This is the set which plays the role of R k. Then let F be any field this F plays the role of real numbers.

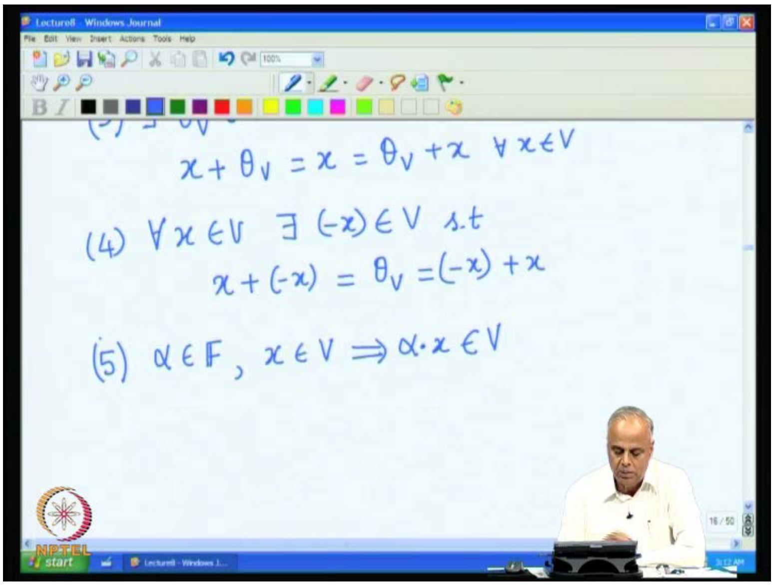
The role of R k is played by this arbitrary set V the role of R is played by the field F. Then we have 2 operations addition and scalar multiplication let plus be a rule of combining x y in V to get x plus y. What we get the result will denote by x plus y this such a thing is called binary operation. Then let dot the scalar multiplication will denote by dot after some time will stop writing the dot also dot be on up a rule to combine an alpha in F with an x in V. This now the role of V plays the role of R k. F plays the role of R addition in R k is now played by this operation or the rule of combining 2 elements in V and multiplying as R k. Element by real number R is now replace by multiplying an element x in V by field F by the field element f.

(Refer Slide Time: 42:49)



This now what are the rules such that let us list the rules of addition. First now list them on continuous numbers the first 1 was when we add 2 R k elements. We got on R k element that is R k was closed with respect to plus. Analogously when we combine 2 elements in x y the resultant is also in x y. This is what is V is closed with respect to this rule plus is also said that plus is the binary operation on V. Then the plus on R k was associative that is what we are going to demand x y z belongs to V implies x plus y plus z is equal to x plus y plus z. That is plus is associative on V the addition. This is call the addition on V the addition on V is associative next we had the role of 0 in real numbers theta k in R k.

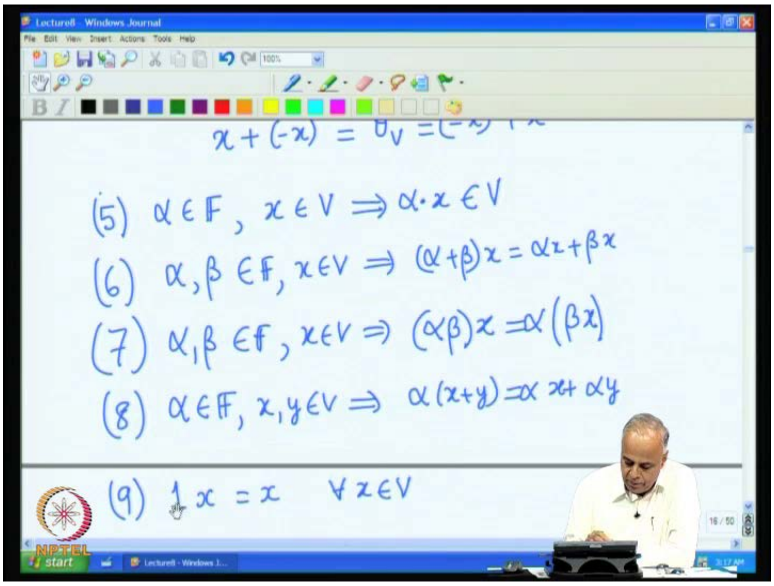
(Refer Slide Time: 43:52)



Now we want the theta V there exist an element theta V in V. Such that what is the role of the theta V when it combined with an element. It produces the same element that is the role of theta V then we had the notion of power. Every element there must be the negative role that is for every x in V there must element exist an element in V. Remember in negative of a real number was a real number in negative of an element in R k was in R k. Now the negative of an element V we want it to be in V such that we got next in negative. When it combines with its when an element combines with its negative this produces the 0 the 0 is now the theta V the role of 0 is played by theta V is equal to theta V is equal to minus x plus x is minus x plus x. These are the properties of plus that we demanded. Now we list the properties of scalar multiplication we had alpha belongs to F whenever in R k.

We had alpha was in R the role of R is played by the field F and then we took an element in R k. Now we taken element in V and there we form the scalar multiplication. Now it write alpha dot x after to sometimes we stop writing the dot we simply write alpha x. This must be in V when we scalar multiply an element in R k. We got an element in R k so when we field multiply an element in V.

(Refer Slide Time: 46:17)



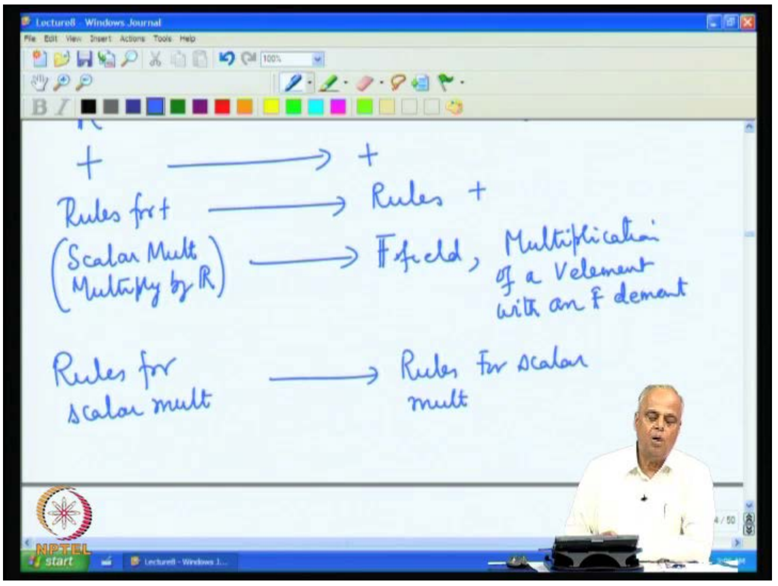
We must get an element in V then we had this property is alpha beta in F x in V implies alpha plus beta into x with this same as alpha x plus beta x. This is the the what we see here the addition there you see a addition symbol. Here this addition refers to addition of alpha and beta alpha.

Beta are field elements this is the addition in the field here we see a addition on the right hand side this is adding alpha x and beta x alpha. x is an element in V beta x is an element in V. The right hand side addition is plus in V. Now the plus in F has been converted to a plus in V. This is a relation which combines the addition operation of the field with the addition operation of the V elements next. Similarly, if alpha beta in F x is in V then alpha is F element beta is an F element the product of the 2 F elements is in F element. That can be combine with a V element the result must be a V element what is that for this combine the F element beta with the V element x.

The result is again a V element that can be combine with the F element that is alpha of beta of x and we had that important property which combines scalar multiplication and addition in V. We had alpha in F x and y in V imply alpha x plus y now x is in V y is in V. Therefore, this is the addition in V when we have 2 elements in V we get an V element. This V element and this F element are combine we will get a V element what is the V element first combine the F element alpha with x combine the F element alpha with y we have 2 V elements. We add them and finally, we had the 1 in the field and if the scalar multiply an element in V you must get x for every x in V.

Let us now go back what are all the ingredients it V required V required an arbitrary set V which played the role of R k V required an arbitrary field which played the role of R, then we had this V playing the role of R k we have F playing the role of R, then we have the operation of combining 2 elements in V which plays the role of addition in R k.

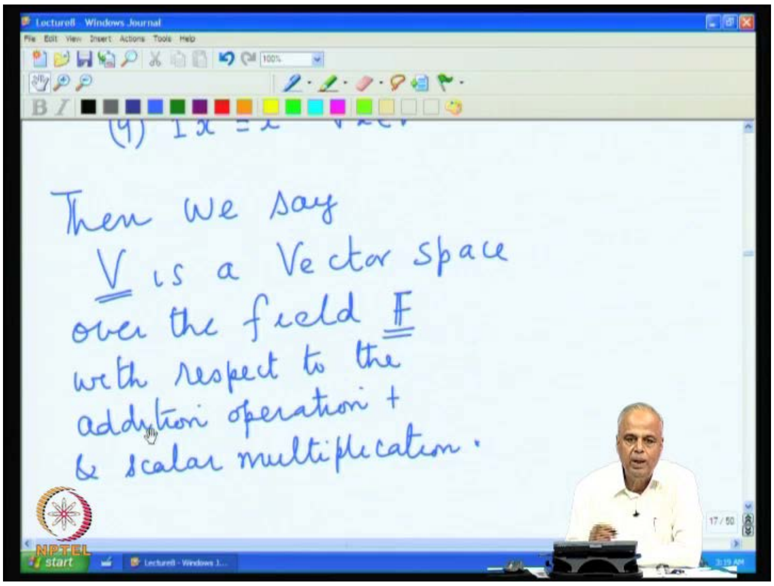
(Slide Refer Time: 40:12)



Then we have this rule of combining a scalar alpha in F with an element in V this plays the role of scalar multiplication in R k, then these this plays the role of the fact that addition of R k element with R k element is R k element. Now, we want additional V element with a element is V this plays the role of associative law of addition in R k this plays.

The role of the 0 metrics in R k or 0 column metrics in R k. This plays the role of the negative vector or negative metrics or negative column metrics in R k, then this rule five plays the role of the fact that when you multiply an R k element with the scalar you get an R k element. Now we say multiply a V element with the scalar in F we get a V element this plays the role of addition, multiplication and addition scalar multiplication with respect addition of scalars and multiplication of scalar and scalar multiplications are expended here and addition of V elements and scalar multiplication, then how they combined this is again the role where we saw the 2 basic operations are scalar multiplication and addition how they interact with each other then we have this 1 into x equal to x.

(Refer Slide Time: 51:26)

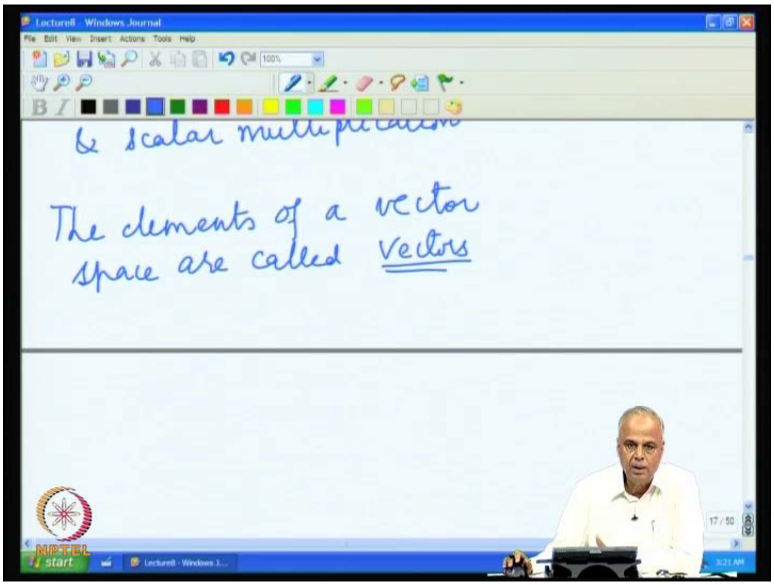


Then we have all these then we say then we say V is a vector space over the field F with respect to the addition operation plus and scalar multiplication operation dot all the four things play an important role. First of all, the set V plays an important role the field F, because these are the people combine the addition combines V elements scalar multiplication combines F element with V and the result is always going to be in V. These 2 are going to play important roles the V and the F and then be the without these 2 rules of addition and scalar multiplication there is no structure.

This addition and scalar multiplication play an important role therefore, we have to say it is a vector space over the field F with respect to this operations plus and scalar multiplication. All the four play an important role in the structure of the vector space, then all these if you for example, if we change if you retain the same F if you retain the same V if you retain the same set V we retain the same field F suppose we change the operation plus and still have the same properties.

Then we will have a vector space, but this will be different this is still be a vector space over F, but it will be a vector space over F with the new operation so in a vector space there are four important ingredients 1 is the set V 1 is the field F and the operation plus addition and the scalar multiplication operation dot together with all these.

(Refer Slide Time: 54:05)



We get the notion of a vector space we will just call the elements of a vector space are called vectors, then we shall see in the next lecture and number of examples of vector spaces many of these examples, we will encounter in some problem or the other in physics or engineering or in mathematics, then we shall be looking at the example in the next class.