Advanced Matrix Theory And Linear Algebra For Engineers

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Lecture No. # 07

Linear System –Part 4

You seen how to solve homogenous system of equations. The general idea following, we matrix a inch m cross n and we would like to solve the system a x equal theta m and general strategic is reduce a by elementary row operation to row reduce. If you want when we solve the system A R x equal to theta m solve this and general idea for solving.

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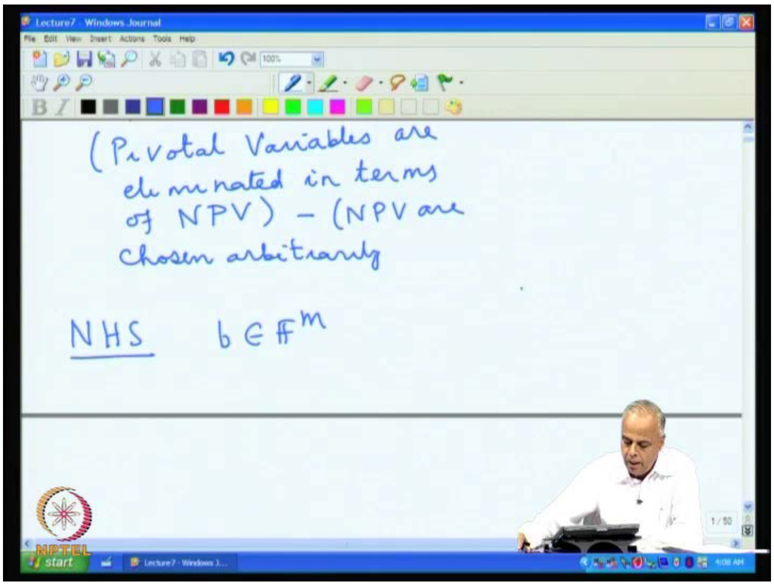

This is pivotal variables are eliminated in terms of non pivotal variables write NPV for pivotal variable and non pivotal variable are choose an orbital, but it general strategic for solving homogeneous system of operation. Now why does this work in homogeneous in system and why it does not working a homogeneous system? What is look at non homogenous system corresponding to a, then let b b in f m.

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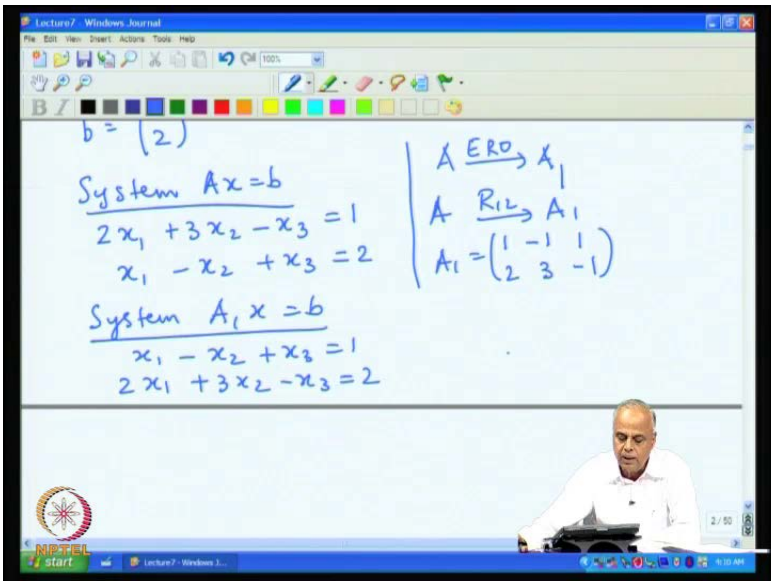

We look at a x equal to b, if you want to copy what we did for homogeneous system and again will reduce a to row reduce a form by ERO s and we would ask whether a x equal to b is equivalent to are having same solution to A R x equal to b. You see the question will ask is it true just i can homogenous system and b was theta m.

This is you are equivalent suppose, we take a b is equivalent not theta m will be take equal n, look at example, then consider the matrix a 2 3 minus 1 one minus 1. What is suppose, b 1 2 a is load that a 2 by 3 it has 2 rows and 3 columns m is 2 b must be in of 2, then let be the vector 1 2 what is the system a x equal to b it is now 2 x 1 plus 3 x 2 minus x 3 is 1 x 1 minus x 2 plus x 3 equal to 2. Now supposing we do elementary row operation A R not necessarily A R. Let us even take a very simple elementary row operation and same we make 1 elementary row operation and get the matrix a 1 for example, in this case let us say we are interchanging the first.

Second row so the a 1 will be 1 minus 1 one 2 3 minus 1 now, what is the system a 1 x equal to b the system a 1 x equal to b become x 1 minus x 2 plus x 3 equal to 1 2 x 1 plus 3 x 2 minus x 3 equal to 2.

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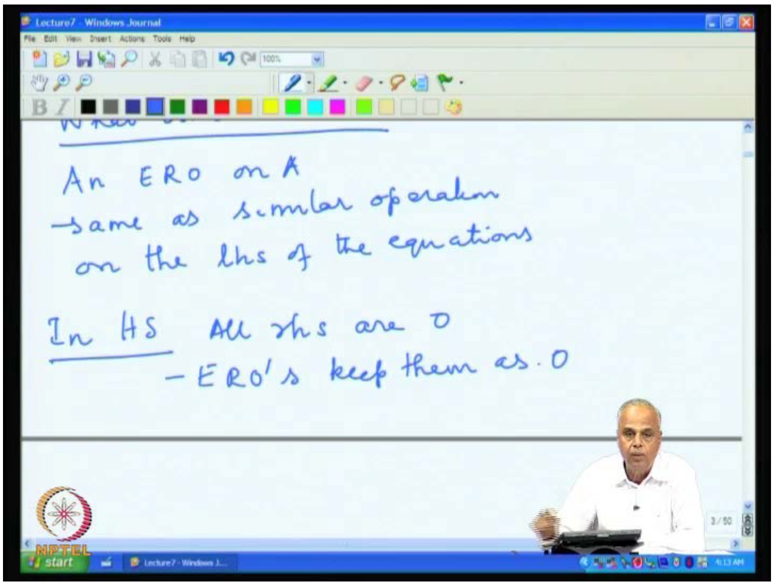

Now see this systems are not equivalent to this just observe but, x 1 equal to seven by 5 minus 3 by 5 x 3 equal to 0 is a solution of the system a x equal to b, but not the system a 1 x equal to b and then when you perform elementary row operation a x equal to b is not necessarily equal to a 1 x equal to b. What gone weather where as work for homogenous system is not working for non homogenous system, so what does ERO do so we just analysis why disturbing work what does ERO do and ERO an the matrix is same as per the as the system a x equal to b.

To consider similar operation on the left on side of the equation of the equation why only on the left on side because, the ray in the coefficient matrix it just as collected and the coefficient on side of the equation this are coefficient of the un known variable x 1 to 2 x n and therefore, any elementary row operation of the a is equivalent to the perform on the corresponding operation on the left hand side of this equation in the case of homogenous system if you want to balance this equation we have to perform the same operation on the R h s, but in the case of homogenous sequence all R h s 0 and this elementary operation live them the 0 you do any elementary operation and zero is and it is remain to the 0 so ERO is keep them as 0 and therefore, we do not have to worry about performing the operation on the right hand side of the sequence to maintain balance of the equation.

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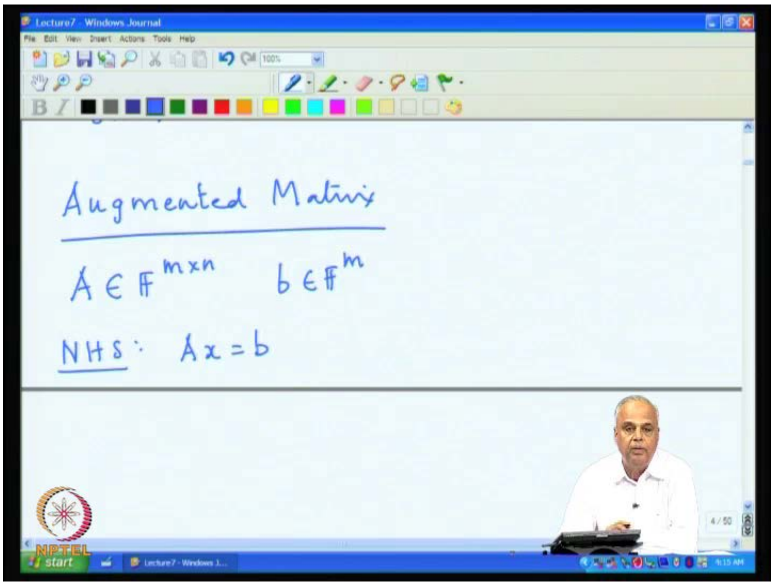

Hence, a x equal to theta m will be equivalent to A R x equivalent to theta m and say equivalent let be will have been same solution therefore, since the elementary row operation.

Retain on the 0 and 0 the corresponding equation will be equivalent on the other hand, if you look at non homogenous system a x equal to b. As we say in the above example, the ERO is alter the R h s and therefore, in order to maintain the balance of the equation we must perform on b whatever ERO is we perform on a thus in order to and the non homogenous system.

Since right hand side is not 0 we have to keep track of the changes that are taking place of the R side of the equation as per performing changes on the left side of this equation, so in order to keep track of all the situation we will introduce some notation we paste introduce the notation of the augmented matrix let us matrix a with is as usual in f m cross n and we are the right side of the equation f m and therefore, we are non homogenous system a x equal to b therefore, this is the consent we want to find solution for this non homogenous system.

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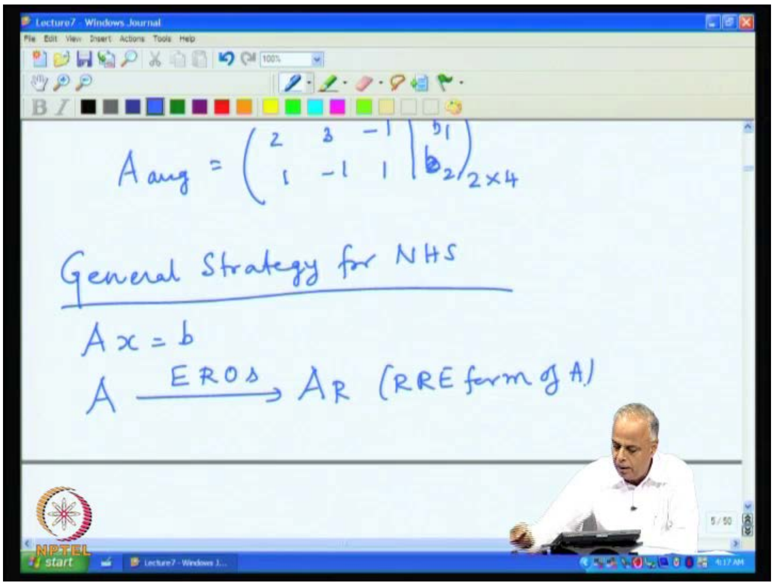

We define augmented matrix, which is denote by a sub aug this is the definition what we do is we take the matrix a and we obtain and addition column at the end. This is the column form by the vector b in order to keep in track of the last column was attended from the right hand side of the equation, then we write it as a with the bar separating a from the b now this will be the matrix.

Which is same number of row as a now it has the additional column this is will be a m by n plus 1 matrix for example, look at the previous example we had if you had a 2 3 minus 1 one minus 1 one and b had since, this is 2 by 3 matrix m is 2 in the square so b will be b b 1 b 2 at therefore, if you look at the system a x equal to b we would defined by the augmented matrix which is obtain by keeping the a in tag and just obtaining as 1 more column and b 1 and b 2 what again put that bar before the last column a long line 2 implies what is coming is from the right hand side of the equation now this is will be the 2 by 4 equation then what should be our general strategic non homogenous system for the general strategic for non homogenous system let us look at a x equal b and as usual. Suppose, we apply elementary row operation in order to get the row reduce form of a.

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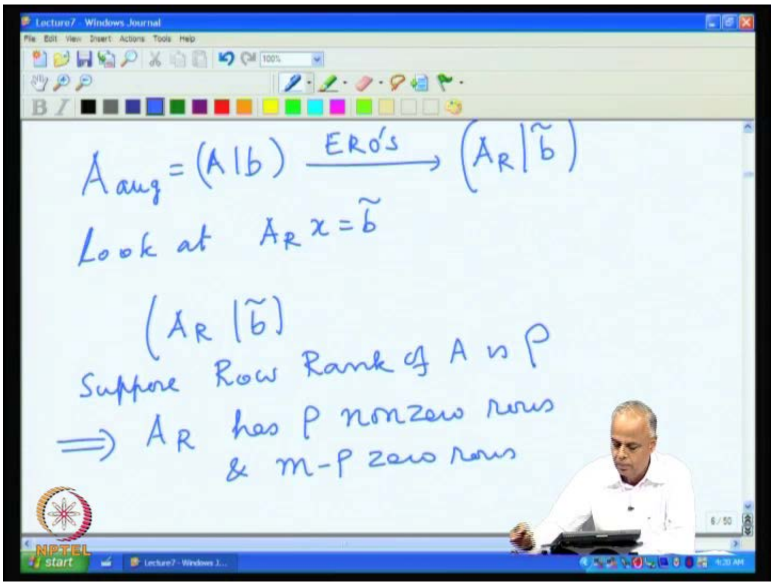

So apply elementary row operation on the matrix as usual we did in homogenous system and get we row reduce epsilon form of a. Now we have to apply the same e R o in the same sequence, what our order in which we apply elementary row operation to go from a to A R. We now apply the same elementary are operation in the same order on b save end of the b, then since we are no balance on the right hand side by performing the same operation the system A x equal to b the non homogenous system A x equal to b has same solution as non homogenous system A R x equal to b tiled, so it is not that a x equal to b equal to A R x equal to b, but a x equal to b is equal to A R x equal to b tiled the operation have to be perform on the right on side.

Therefore, this is equivalent to say if you start with a augmented matrix augmented matrix a b and then do the ERO is a part this becomes row reduce form the corresponding right hand side will become a b tiled now as we said a x equal to b will be equal A R x equal to b now we look at A R x equal to b tiled instantaneous of analyzing we system a x equal to b we can analysis the system A R x equal to b tiled since 2.

Systems are equal now look at the matrix A R x the system A R x equal to b tiled the matrix A R b tiled. Suppose the row rank of a is say row, then what is that mean this means the A R the row reduce epsilon form has row non 0 rows and m minus row 0 rows at the row reduce the epsilon form all the 0 rows comes at the bottom and non 0 come at the chart as we had seen earlier the first row rows of the matrix A R will be non 0 and importantly row plus 1 row plus 2 up to the m the row A R will be 0.

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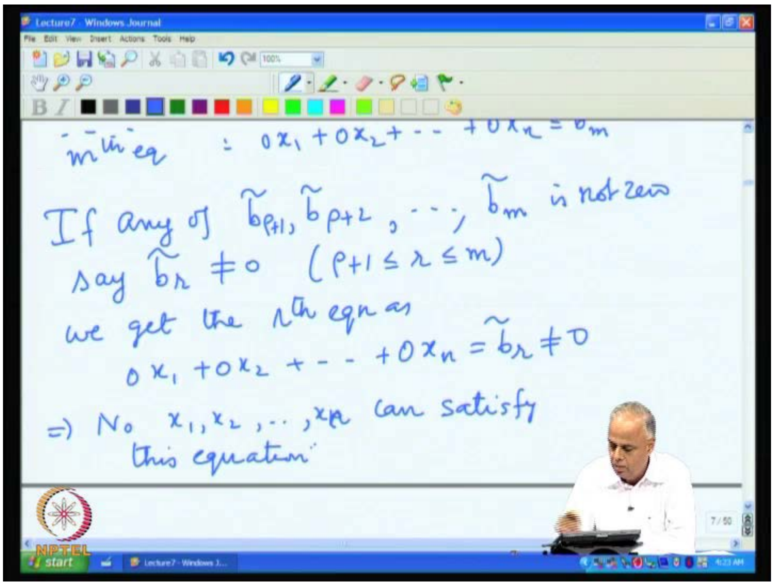

Then the row plus 1 row plus 2 etcetera m. The row of A R will be 0 that is the case look at A R x equal to b tilled the row plus 1 the equation we look like since, the left hand side all the enter row 0 it will look like 0 x 1 plus 0 x 2 plus 0 x m by the right hand side b p tilled and so on up to the m the equation the m the equation again like 0 x 1 0 x m equal to b m tilled.

Now, if any of the b tilled the from row down not going to be 0 will end of the quantization, if any of b row tiled p row plus 1 tiled p row plus 2 tiled and b m tiled is not 0 say b R tiled not 0. The off course R is between row plus 1 then 1 of the equations b R is not 0 we get the R th equation as 0 x 1 plus 0 x 2 plus extra 0 x n is equal to.

B R tilled, then which is not 0 no matter what values are x 1 x 2 x n b tiled the left hand side will be 0 and right hand side not 0 and therefore, x 1 x 2 x row x n can satisfy this equation what is the moral of the story.

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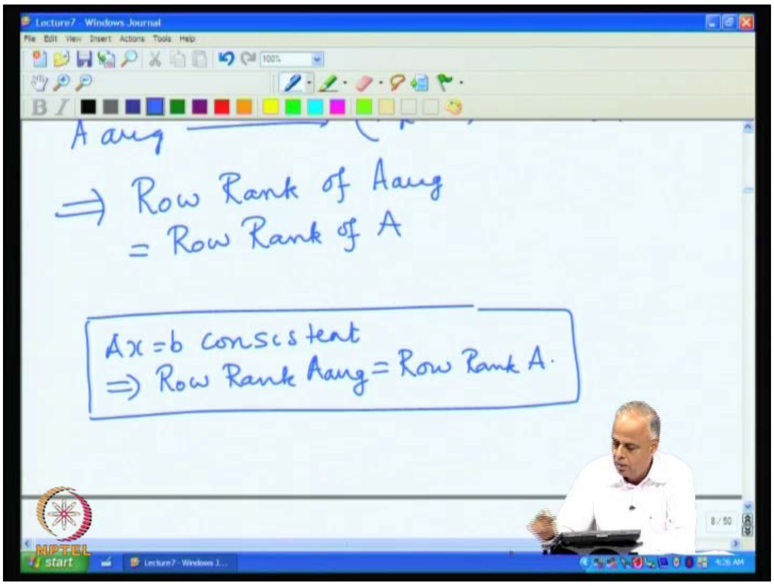

The moral of story is what is the conclusion will make? The necessary condition for the system to have the solution, but this equation from the row plus 1 th on wards like the row 1 th plus equation row from second equation the n th equation all of the balance in the first place and order the they are balance. We must have b row 1 plus tiled b row plus 2 tiled and b m must be 0 therefore, the first conclusion major conclusion is the a x equal to b is consistent if we ask to be consistent the minimum requirement is but b row plus tiled etcetera b m tiled the 0, then what does that mean if b row plus 1 tiled b m are all 0 it means in the matrix A R be tilled in the matrix A R will be tilled if we look at the row plus 1 through on wards the left handed 0 the right handed 0 the left hand side is 0 and right hand site is also 0.

Therefore, in A R b tiled the first row 0 will be not 0 and remaining will be 0 which is among to say but, the rank of the matrix a augmented went by w R o rows to A R b tiled and this Is the exactly the a augmented matrix is row reduce epsilon form and therefore, since this also same number of non 0 rows has the A R we have row rank of augmented equal to the row rank of the a thus we have this again will summit of a x equal to b consistent implies row rank of a augmented equal to row rank a we now fast get own of the necessary condition minimum this much will be require in order that a x equal to the salable and that requirement the row rank of the a augmented is equal to row rank of a conversely.

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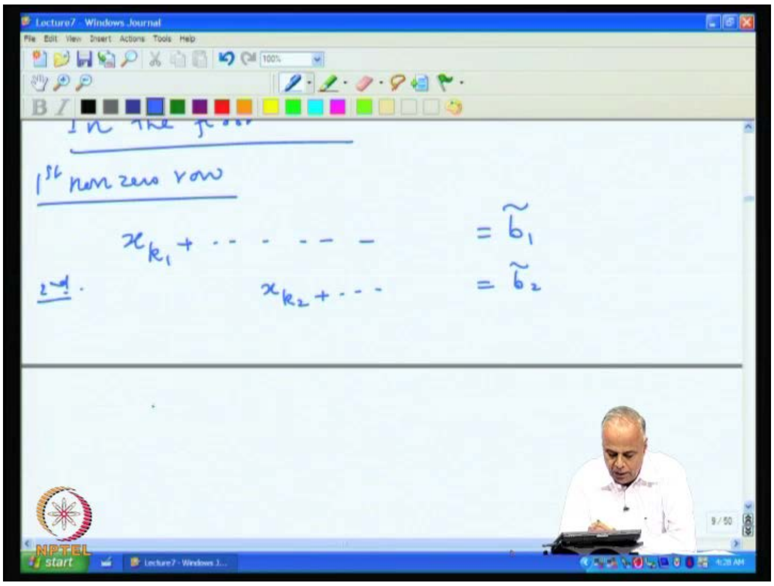

Suppose row rank a augmented equal to row rank the minimum requirement we are asking for matrix then the row plus 1 th equation on wards in A R x equal to b tiled will just be 0 equal to 0 zero equal to 0 and so an 0 equal to 0 . Then automatically take and care of above whatever value x 1 x 2 x n give the row plus 1 equation or on wards are automatically satisfy only thing then theory worry about at the first row equation.

Now he first row equation rows what we have vary about look the first non 0 row all it is look like the right hand obviously b 1 tiled now the left hand side it will have the first non 0 entry then we call that at the k 1 th column. It will start with the k 1 th variable and the coefficient will be 1, because we are in the row reduction epsilon form it will like x k 1 plus other variables equal to b 1 tiled how about the second now let us write all the other equations in the second 1 we will have x k 2 and because the row reduce form 2 variables move to the right we will have x k 2 1 words all the other previous values will not appear and so on and that will be equal to b tiled and will go on like this.

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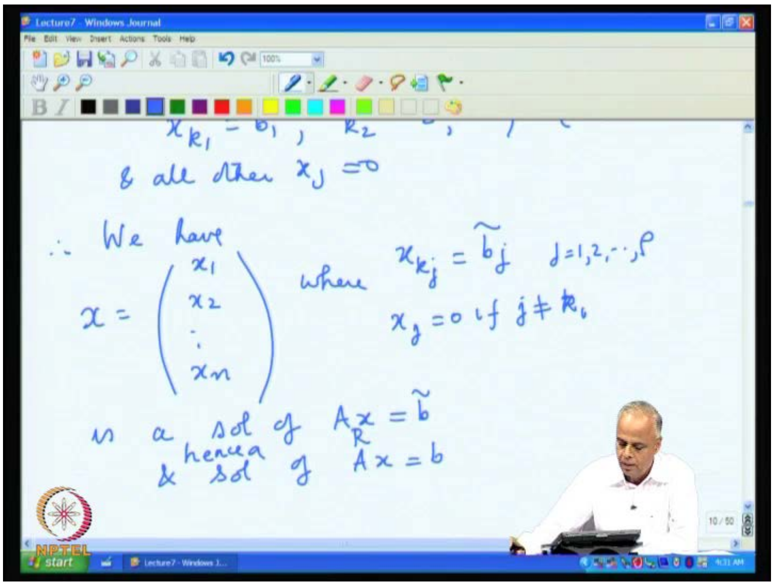

We will have finally, x k row tiled etcetera equal to b row tiled this stature of the first row equation which only matter, because later equation are equal 0 equal to 0 the first row equation which matter will look this now. Since any column that suppose the total variable is 0 is below the x 1 non of the other variable pivotal variable appear similarly, above and below x k none of the pivotal variable will apply and so on.

Here, he follows if we choose x 1 equal to b 1 tiled x k 2 equal to b 2 tiled and so on x k row equal to b p tiled and all other x i has 0 then other row equation are automatically satisfy. Because x k 2 does not appear here x k 3 does not appear here x k row does not appear here, but other things which appear all rows is to be 0 therefore, x k 1 equal to b 1 tiled satisfy this similarly, x k 2.

Equal to b 2 tiled satisfy, the second equation and therefore, we have the vector x equal to x 1 x 2 x n where x k g is equal to b j tiled for j equal to 1 2 row and x j equal to 0 if j is not equal to any of the k i is solution of a x. Actually solution of A R x equal to beta and b tiled and hence as solution of a x equal to b thus we seen the if you are assumed but, our condition the minimum condition is satisfied mainly the row rank of a augmented is equal to the row rank of a if a assumed minimum condition then we get, but a x equal to b has the solution.

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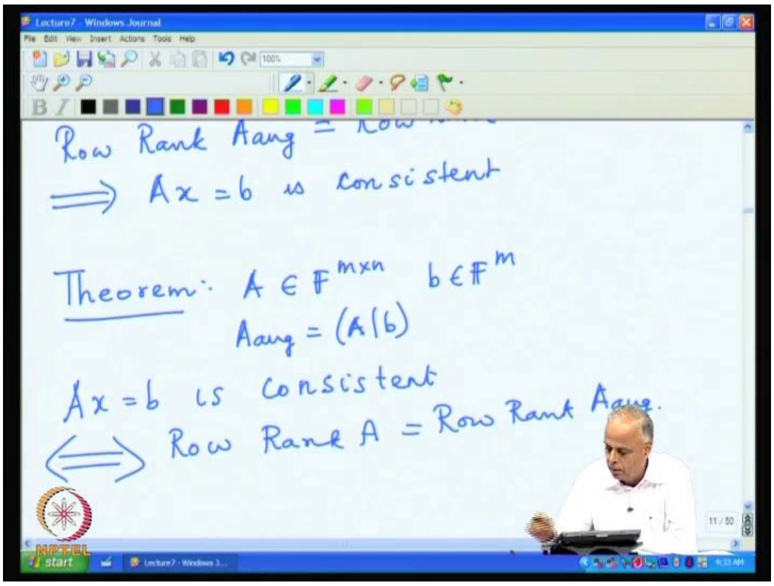

The conclusion row rank a augmented equal to row rank a implies a x equal to b is consistent the previous, we got the conclusion when it is consistent the rank must be equivalent. Now, we got the conclusion the row rank are equivalent that is consistent looking all t R his conclusion 2 conclusion are gather we get the theorem a b and f m b in f m a augmented matrix obtain the this column b the matrix a then this notation the theorem is a x equal to b is consistent. If and only a row rank a equal to row rank a augmented now if you re call the first lecture 1 of the fundamental question we raised was our question was what is the criterion for system ax equal b 2 have a solution now we are answer for the question to the theorem which says the criteria is require the row rank of the a must be equal to the row rank of the a augment.

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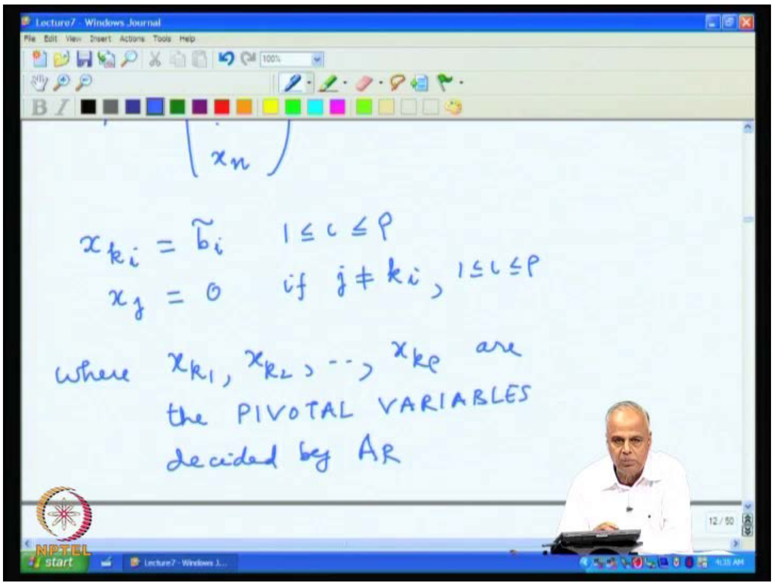

Now what happens, then when this criterion is met we found we love a what a solution as a subject before well will call the solution s p. Then now x p as x 1 x 2 x n where x b very pre size x k i is equal to b i tilled. Where b i tilled the i t the matrix are the vector are obtain the applying the same e R o applied to a to b you have to reach the row reduce the epsilon form for 1 less than or equal to i less than or equal to row.

The row i the row rank of the matrix and x j equal to 0 if j not equal to k i for 1 less than or equal to i less than or equal to row any 1 of the state where x k 1 x k 2 x k row are the pivotal variables decided by A R the row reduce the form what you mean by pivotal variable decide by the A R a R as the pivotal ones the leading non 0 entry each non 0 row in a particular column in the first row appear in the k 1 column the x k 1 in the pivotal variable in second row leading 1 appear in the k 2 column when x k 2 the pivotal variable and so an so we get a solution for a x equal to b now we recall but, we said the non homogenous system we solution consistent up to 2 parts 1 find the Particular solution of the homogenous non homogenous system and find all solution of homogenous system.

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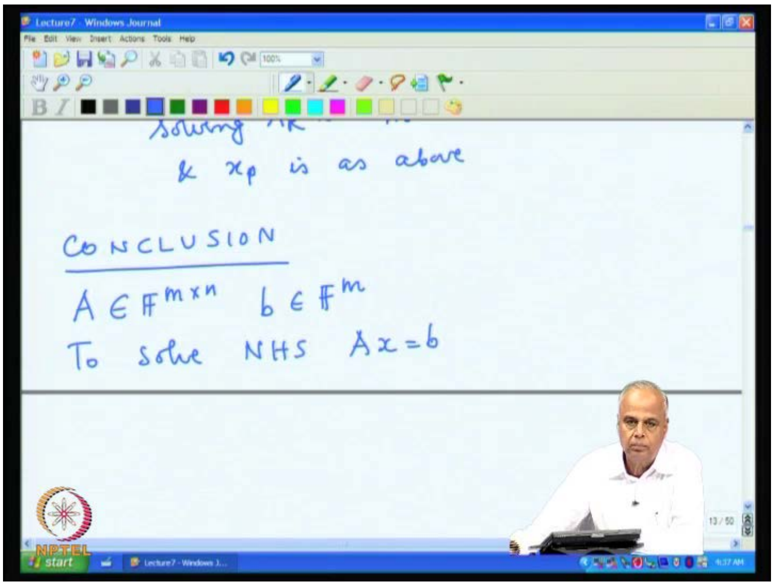

The general solution of the non homogenous system a x equal to the b, then will x equal to x h x p we had the little word a particular solution which we are not capture x equal to x h plus x p. Where x h found by solving A R x equal to theta m is what did in last lecture and x p is as above we just now seen know how to choose it? The pivotal variable are present to be b 1 tiled the b 2 tiled the and so on. The b 1 tiled and non pivotal variables are choosing to be 0 so thus we have complete picture of the solution for the non homogenous system from the consistent criterion is satisfy.

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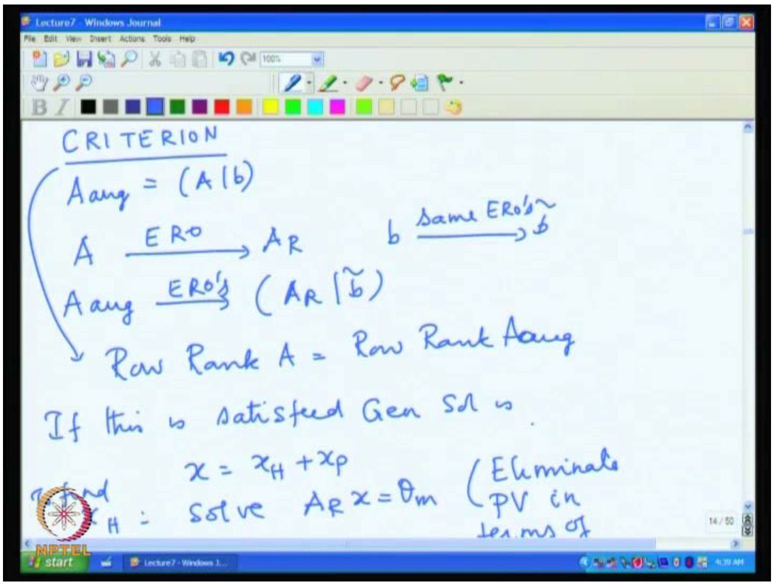

Let us look at the final conclusion, then how do we handle a non homogenous system the way to handle the non homogenous system is first. Again we have a matrix m cross n b is f m we want to solve the non homogenous system a x equal to b. Now, criterion to be check first so first you defined a augmented to be a b, then we produce a to A R the row reduce epsilon form apply the same e R o rows to be b. Then we get b tiled then a augmented goes to A R b tiled. We can straight away start with a augmented and applying this and now the criterion is we must have row rank a equal to row rank a aug then if this is satisfy general solution is x equal to x h plus x p x h solve to find x h solve A R x equal to theta m remember eliminate, how do we do this eliminate pivotal variables in terms of non pivotal variables.

Then find x p as above, we have seen how to choose x p equal we choose x p by choosing the pivotal variables b i tiled, then we choose the pivotal variable b tiled and all other non pivotal variable to be 0, then we choosing the x h and x p we have the complete picture of the solution of the non homogenous system what again repeat.

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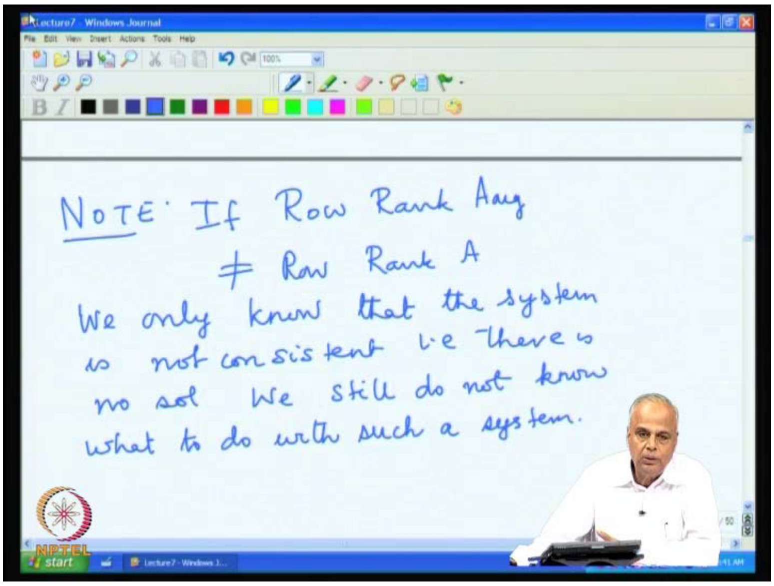

If row rank a augmented and row rank a where equal we could all this suppose, this is not equal we still we only know that the system is not consistent that is there is no solution. But we do not know what do that, then still do not know what to do with such a system is a major part of the problem still and handled, but what we days among the several question that we are asked we are answer to the question what is the consistent criterion are what are the condition we should satisfy the order the system a x equal to b have a solution.

We have answer to that the question that row rank a equal to row rank augmented and my question is if be satisfy by b satisfy the criterion. Suppose the answer is yes then we know how to point to the solution we know when it is unique we already seeing that the system is unique solution he said only a the homogenous system as only the previous solution and it has infinite number of solution, then the homogenous system has no previous solution.

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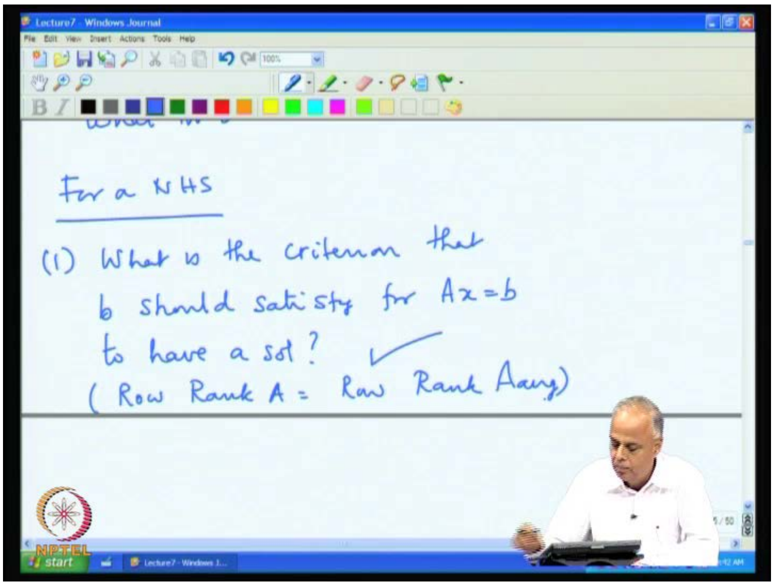

So let us therefore, put all question to be for a non homogenous system the question was what is the criterion? That b should satisfy for a x equal to b to have solution this is the first question that we raised on the first lecture. We have now answered to the question namely the row rank a equal to row rank a augmented. This is should seeing the same answer different format later at least we are know one answer.

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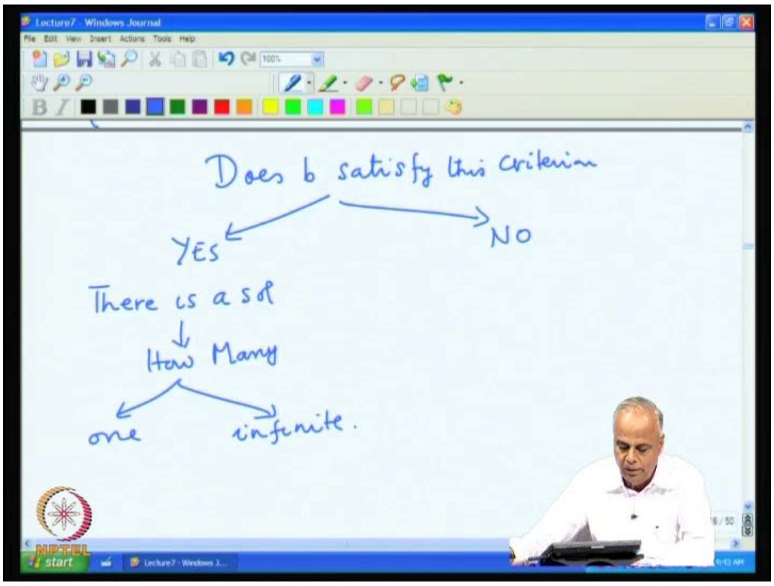

Then the ask the question does b satisfy. This criterion is said 2 possible answer yes and no in the case yes the conclusion is there is a solution. Because the condition is the consistent is satisfy, then bound by the solution then we ask how many we said 1 and infinite.

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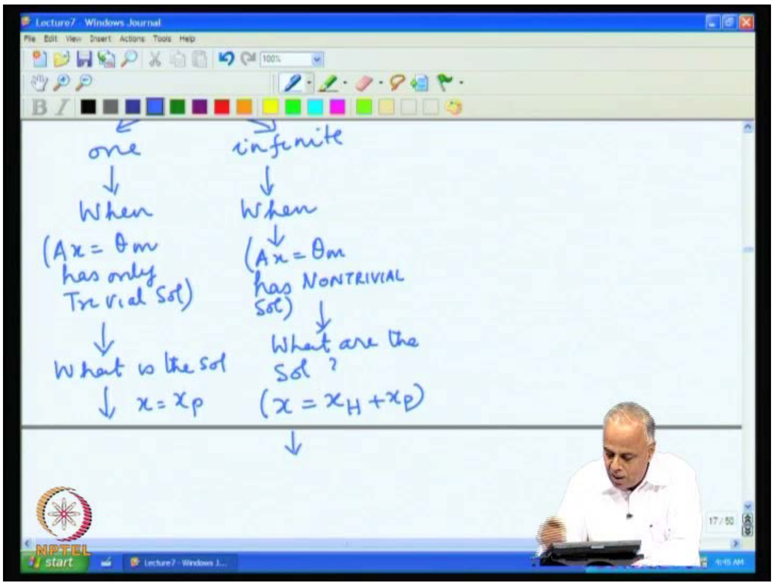

And if recall we ask the question, then when is it 1, when it is infinite? Now we are answer to the question, the answer is 1 when a x equal to theta m has only trivial solution this is 1 form of the answer. Again again say the answer in different format latter again when it is infinite when a x equal to theta m has non trivial solution then it is only in the situation. There is only 1 solution, what is the solution remember we are seeing that any solution form s p equal to x h plus x p but, in case x h only theta m but, only solution that a x that theta m has and therefore, the solution the answer is x p that we got.

But only the solution because plus x h will give you nothing now in the case of infinite solution we as what are all the solution now to answer to this. All solution or x form x equal to x h plus x p where x h has explain above of the solution of the homogenous system and x p in the particular solution that we obtain subsequently we ask another question namely any criterion.

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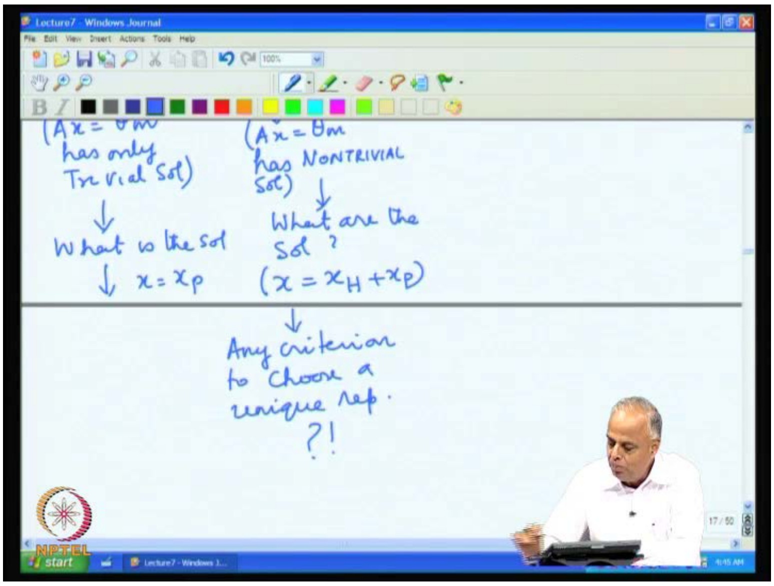

To choose a unique representative among the infinite number of solution, we have not got answer to this that left to the answer and that any question we raise following the obviously remain and answer this is 1 portion of the serious of the question. That raise we not get answer and then answer would that not satisfy, we consistent the condition.

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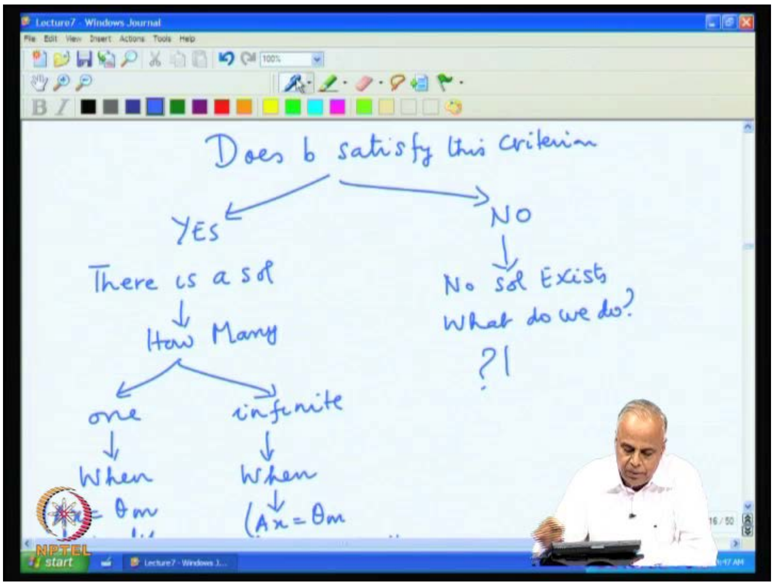

No solution excites, then what we do in such a situation again we do not know serious of the question so among the various question raised about non homogenous system of equation. We have answer to some of them on this site, but even when the satisfy the consistent condition in the case when the there is infinite number of solution we really do not know how to pick 1 of them we know all the solution a problem of lengthy the minutes solution. But we do not know which 1 to the right representation we do not have the answer to that question and in the case of when we does not satisfy the consistent criterion to simply denote know how to proceed forget anything meaning full for that system these are the question which will be remaining question which will be addressing as we move along.

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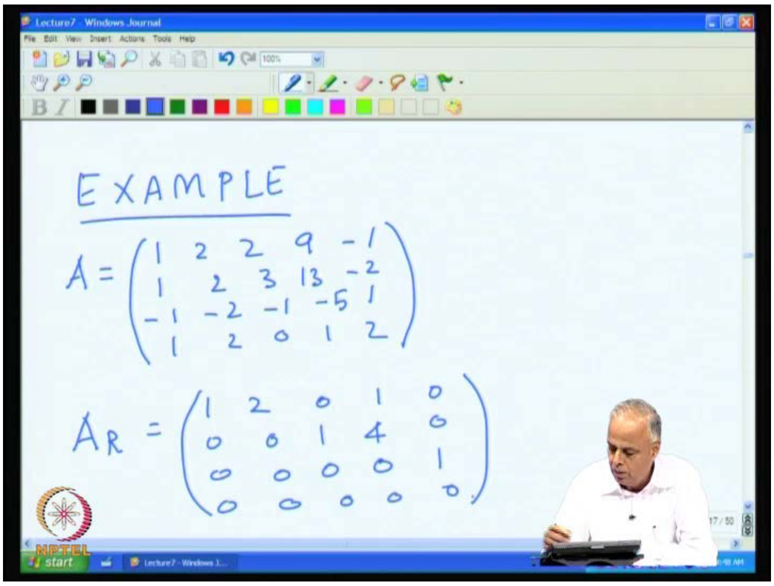

Now look at 1 simple example, to illustrate all the methods of solving the non homogenous system you may recall we look at the matrix in the last lecture, then which are repeat 1 2 3 thirteen minus 2 minus 1 minus 2 minus 1 minus 5 1 one 2 0 1 2. We are consider this matrix in the last lecture and we found, but the row reduce epsilon form of the matrix was following 0 zero 1 4 0 zero 0 zero 0 1 0 zero 0 zero 0 we are found row reduce epsilon form of the matrix by applying trivalent e R o recall the 2 stages.

The first stage called first column of the operation there we abroad the first column sun sequence 1 by 1 smaller and smaller sub matrix is and then we had a cleaning of operation at the end of edge we got a row reduce form of a thus part now if you want.

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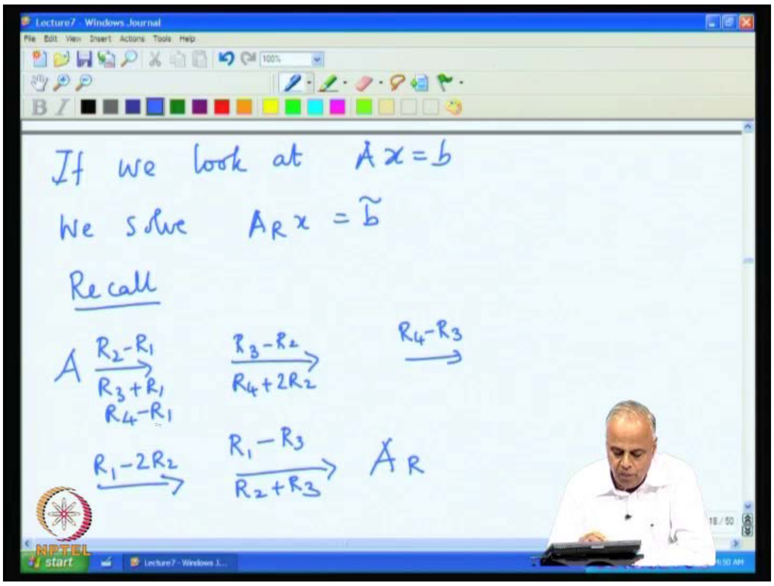

To solve, if we look at a x equal to b the non homogenous system, then we solve instant of a x equal to b. We solve A R x equal to b tiled not A R x equal to b now b tiled has to be obtain from b using the same operation that we used go from a to A R.

So let us recall what were the operation that we perform go from a to A R we started with a then we applied the first column operation on the first column of the first matrix a and these were the following, then we look at the second sub matrix by what by ignoring after the first column of the operation of the first column of a then we smaller sun matrix we applied R 3 minus R 2 R 4 plus 2 R then went a further smaller sun matrix and we applied R 4 minus R 3 this were this sequence of the operation first column operation that perform as smaller and smaller smaller matrix then we follow the clean of operation the clean up that we had 2 R 1 minus 2 R 2 and that was followed by R 1 minus R 3 and R 2 plus R 3 therefore, at the end of edge we got A R so in order to solve a x equal to b and we must solve A R x equal to b tiled for but we must apply on the all the sequence of the operation.

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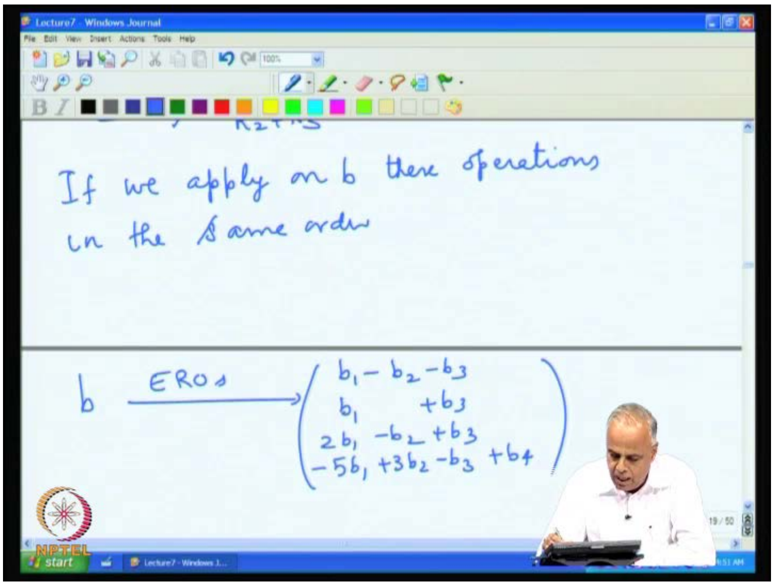

If we apply on b these operation in the same order in the same order we get b at end of the ERO reduce to following from b 1 minus b 2 minus b 3 b 1 plus b 3 2 b 1 minus b 2 plus b 3 minus b 1 plus 3 b 2 minus b 3 plus b 4 believe at an exercise to perform.

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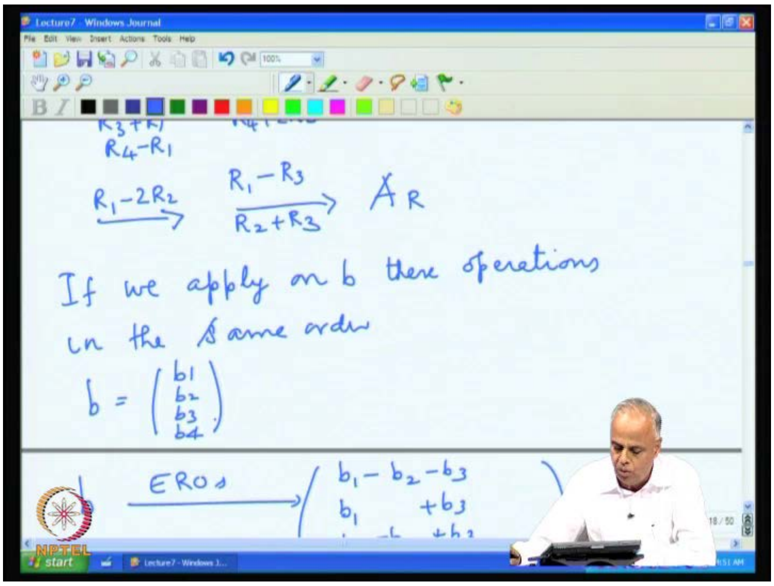

You start with the b which is b 1 b 2 b 3 b 4 four column, because our matrix now is 4 by 3 matrix 4 by 5 matrix, then equal to 4 take of b which is b 1 b general b 1 b 2 b 3 b 4 and apply this sequence of operation in the same order 1. After the other to the b and then we do that to check that you get the matrix this is our b tiled.

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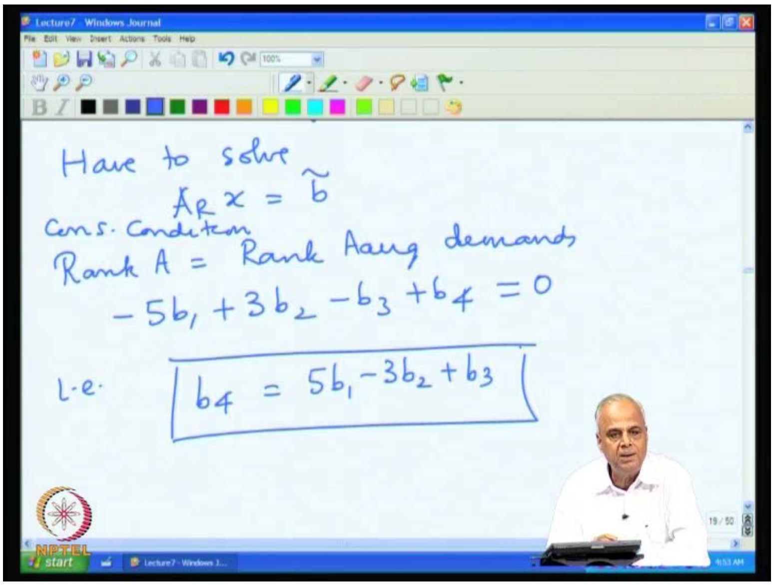

We have to solve A R x equal to b tiled now look at what is happening, then we have this matrix A R which are the fourth row as 0. Whenever we add the 0 they corresponding row in the b tiled must be 0 in order to get rank a equal to the a of a augmented. So the rank a equal to rank a augmented this is consistent condition demands minus 5 b 1 that fourth row entry, then the b tiled must be equal to 0 that is b 4 must be equal to 5 b 1 minus 3 b 2 plus b 3 this is the consistent condition, but b must satisfy in order that system.

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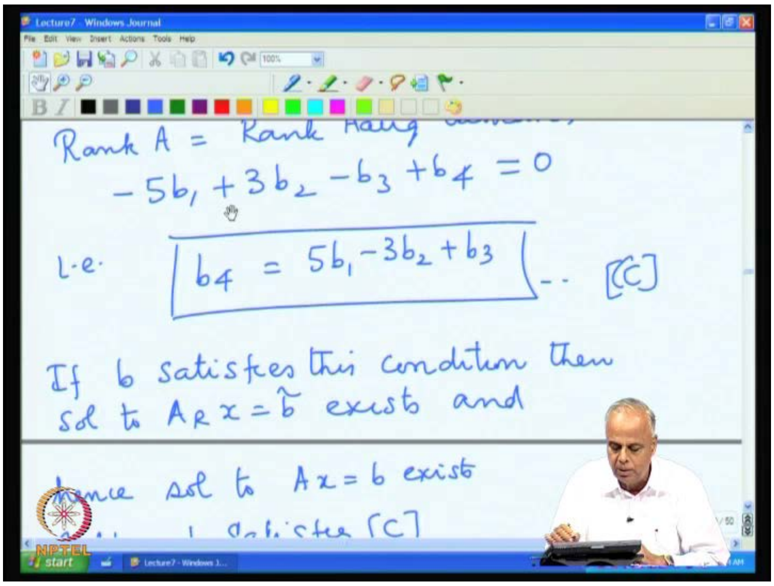

If b satisfy this condition, then solution to A R x equal to b tiled exist and hence solution a x equal to b exist. Suppose b satisfy this condition c that called the condition has c suppose, b satisfy this condition you are given b 1 b which has the component b 1 b 2 b 3 b 4 where the forth component b 4 is related to the first second and third component by this relationship. If the 4 component related by mean by the this relationship when the system has the solution at what is this solution.

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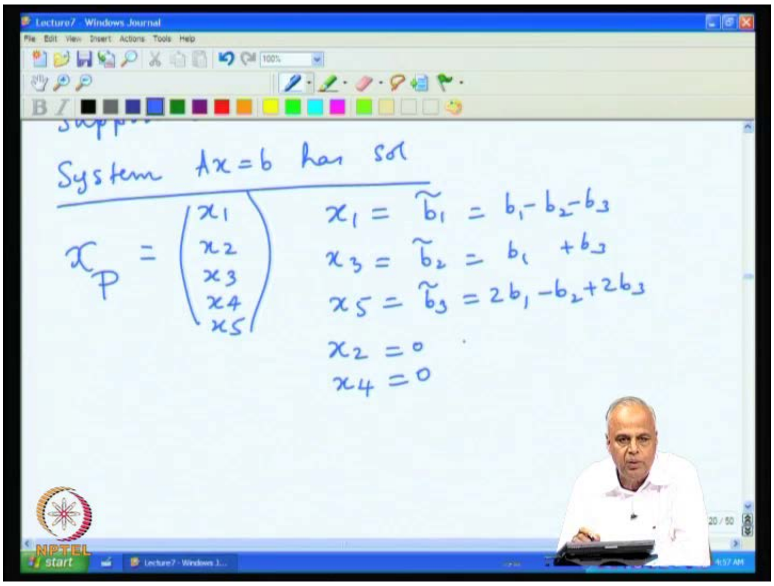

System a x equal to b as a solution, now what is the solution that will have to point first of all remember what x p? How do you get the x p? The x 1 x 2 x 3 x 4 x 5 there are 5 unknown because, the matrix as 5 column there are 5 column first variable second variable third variable fourth variable and fifth variable.

So each column corresponds to 1 variable so the matrix a x 5 column there are 5 unknowns x 1 x 2 x 3 x 5 m is the number of variable that is 5 so we are 5 variable must be found how do be this is the x n so it is call it us x p particular solution now how do we found out x 1 x 2 x 3 x 4 first we takes the pivotal variables what are the pivotal variables in this case they corresponds in the first row the pivotal variable the 1 appear in the first column so x is the pivotal variable in the second row the pivotal appear in the third column so x 3 pivotal variable.

Similarly, third row the 1 appear in the fifth column and therefore, x 5 is the pivotal variable we first fix the to get the particular solution we fix the pivotal variable x 1 x 3 and x 5 x 1 the 3 variable are b 1 tiled and b 2 tiled and b 3 tiled in the same order and what are the b tiled and b 2 tiled as we found about this is b 1 minus b 2 minus b 3 this is b 1 b 3 and this is 2 b 1 minus b 2 plus 2 b 3 we fix the pivotal variables which are the non pivotal variables is x 2 and x 4. They are chosen to the 0 so first we get the particular solution which is obtain in this manner the pivotal variables successfully taken in to success values b 1 tiled b 2 tiles and b 3 tiled row in the cases 3 and non pivotal variable chosen to be zero.

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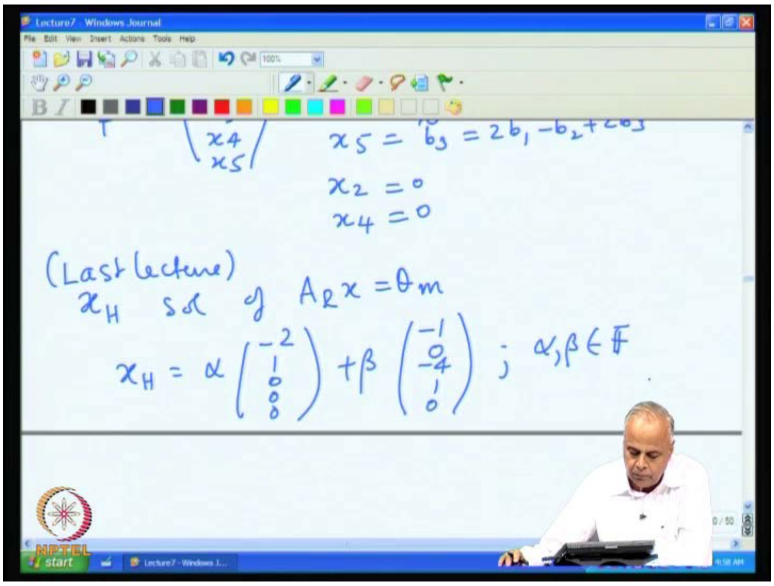

Now, what is x h s h is the solution of the homogenous system? Which we found refer the last lecture x h is the solution of A R x equal to theta m and found at to be of the form alpha minus 2 1 0 zero 0 plus beta in to minus 1 0 minus 4 1 0 there alpha and beta and arbitrary chosen in m we are got this.

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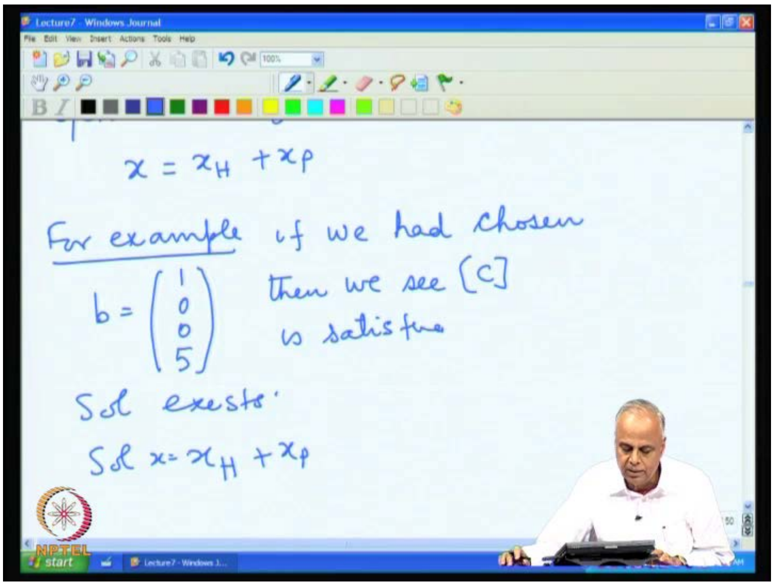

Therefore, the general solution a x equal to b is x equal to x h plus x p where x h is as shown above and x p as be choose here and x h this here, then we got the complete solution of the system for example, if we had chosen b equal to 1 0 zero 5 then we see the consistent.

The condition and c is satisfy what is the consistent condition the consistent the condition b 4 equal to 5 b 1 minus 3 b 2 plus b 3 in the situation b 1 is 1 b 2 as 0 b 3 was 0. The right hand side just 5 minus 0 plus 5 and left hand side is b 4 we chosen b 4 the exactly 5. Then consistent condition is satisfy and therefore, solution excites the solution excites and solution given by x h plus x 1 and what is x h we have this x h in the format already a am all that word find in x p and x 2 format is here now we are choose to x 1 equal to this now b 2 and b 3 are 0 we are chosen x 1 b 1 as 1 x 3 as b 1 plus b 3 b 3 0 so again x 2 chosen b 1 which is 1 x 5 as be chosen 2 b 1 which is 2 and these 2 terms are 0 and x 2 and x 4 are 0.

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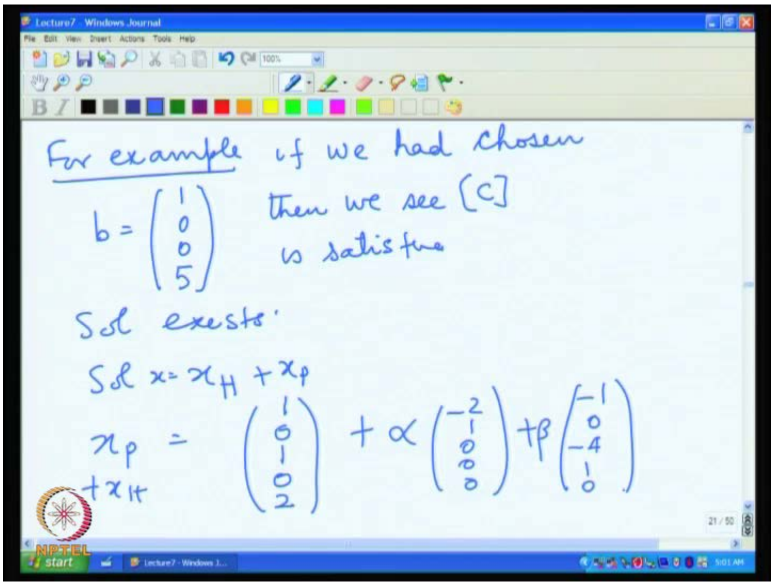

The x p tends out the x 1 is 1 x 2 is 0 x 3 is 1 x 4 is 0 x 5 is 2 and plus x h will be plus alpha in to minus 2 1 0 0 0 plus beta into minus 1 0 4 1 0.

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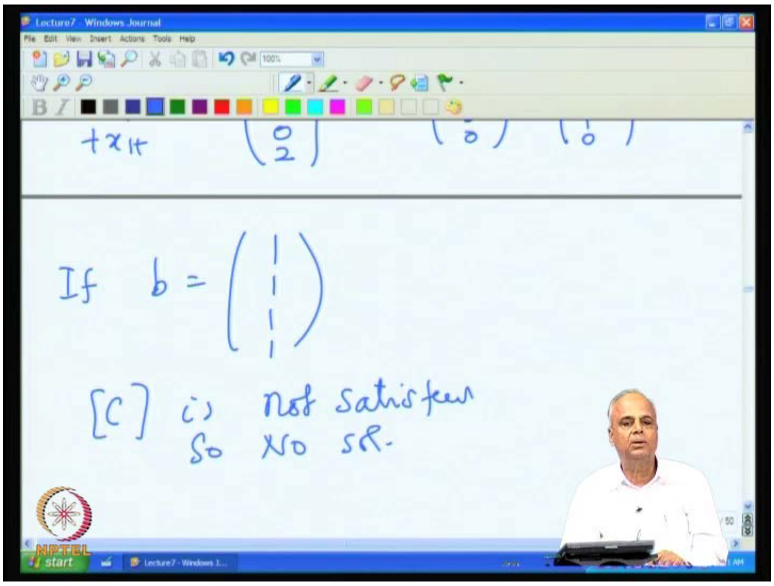

We have complete solution, if we choose b equal 1 one 1 one the consistent the condition is not satisfy. The solution is could not know what to do after this, then we are partial answer to some of the question that we raised in order to get the other question answer to the other question. We need to develop more machinery the next lecture, we look at the fundamental basic mathematical stature that we need let me the vector spaces.