Advanced Matrix Theory and Linear Algebra for Engineers

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Lecture No. # 06

Linear System – part 3

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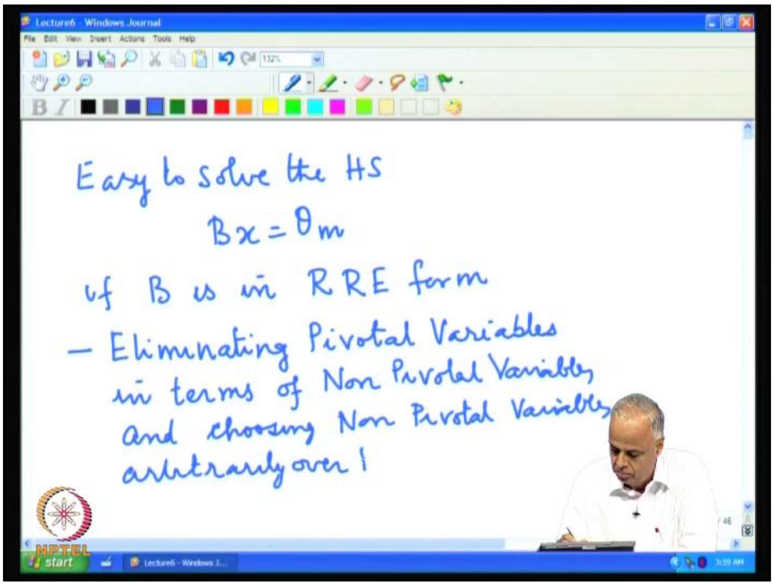

In the last lecture we saw that it is easy to solve the homogeneous system B x equal to theta m if B is in row reduced R R E form this involved eliminating pivotal variables in terms of non pivotal variables. And choosing non pivotal variables arbitrarily over f and therefore, I was suppose can we reduce A given f m by N matrix A to A row equivalent matrix B which is in R R E form this would enable as then solve B x equal to theta m instead of A x equal to theta m and since B x equal to theta m easy to solve being in row reduced echelon form we would that is solve the system A x equal to theta m. The main question was the reduction process to the reduction process.

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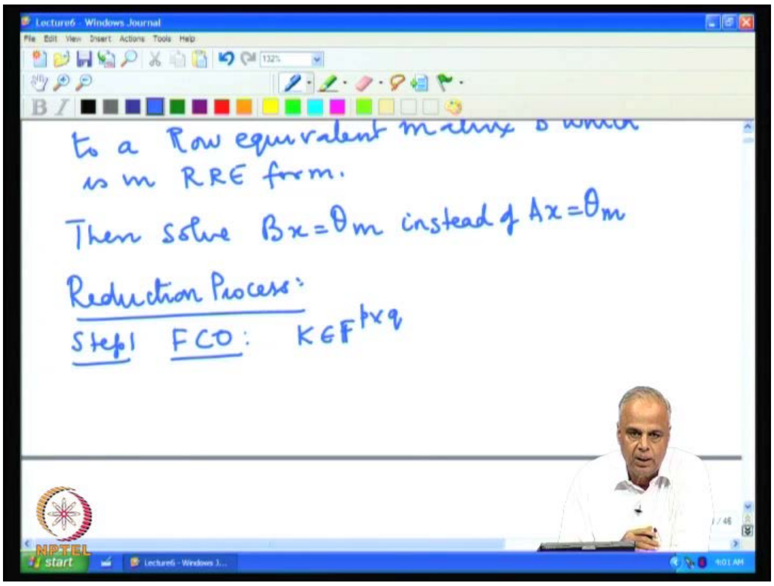

The first introduce the first column operation the first column operation was the following given any matrix K in f p q. You stand the first column bottom on in 0 entry to the top if there is 1 and made it 1 and made everything below that 0 and if the first column made no non 0 entry then we leave it alone at the end of the first at the end of the first column operation. We get either in this form 1 0 zero 0 and A sub matrix K 1 which is obtained by eliminating.

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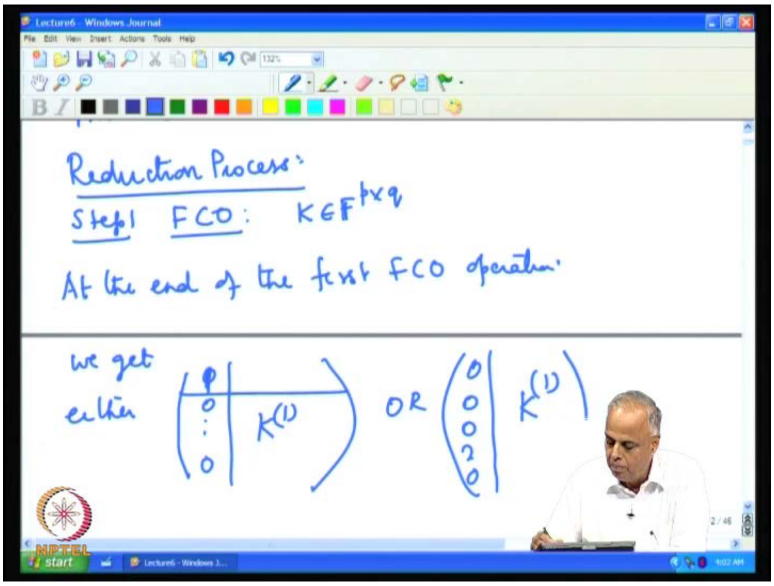

The first row to first column or we get all zeros in the first column in that case we will nothing and we look at the sub matrix look at by ignore in the first column. So at the end of this we get A matrix of the above type and we get K1 having m minus 1 columns now we shall go the general reduction the stage 1 if the following 1 you start with A given m by N matrix A is and m by N matrix and then you apply the first column operation on A you get A sub matrix a1 this may be either of this form or this form this call it as A 1. So we get A sub matrix a1 which is of this form and what we do is next apply the first column operation on a1 and then we get A sub matrix a2 which has now 2 columns less than the original matrix.

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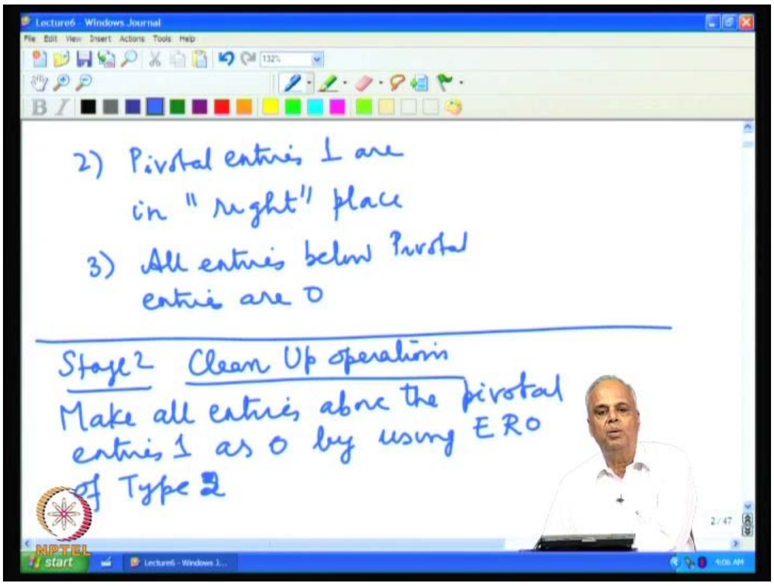

If we go out doing this at end of N minus 1 steps, if there are 1 columns at the end of let us say N steps at most we get A N step do will get A matrix which will be having its pivotal once in the way we want what we mean by in the way we want the pivotal once will be the first non 0 entry in the non 0 rows and they will be moving to the right as we go down to the rows and all the entries below the pivotal once will be 0 so that is all non 0 rows will be above all 0 rows 2 pivotal entry is pivotal entry is 1 are in right place will just put right place which means we have the first entries and then move to the right as.

You down and all entries below pivotal entries are 0 this is the first stage of the operation this is got the pivotal entries the right place which we have make the pivotal entries as 1 make the move to the right as move to down the rows and made all the entries below the pivotal entries 0. Now the only thing that remains is the column supporting the pivotal entries must all this 0 accept the pivotal entry w e have already made everything below the pivotal entry is 0 we have to make everything above the pivotal entry 0 again use elementary row operation stage 2 will call it the clean up operation make all entries above the pivotal entries 1 as 0 by using e R o of type 2. So what we are done is in the stage will manipulated in such A manner that we got the pivotal once put them make the first entry in the non 0 rows put all 0 rows are the bottom and move the pivotal once to the right as we move down made all the entry below the pivotal entries is 0 in the second stage A flip and made all the entries above pivotal entries also 0 but cleans up and brings the matrix into this row reduced form what we get in the end is called end up with what is known as that final matrix is called row reduced echelon form of A. And we denote it by A R denoted by A R we start with A we got doing first column operations first and first A 1 and then A 2 and then A 3 something A n and then subsequently.

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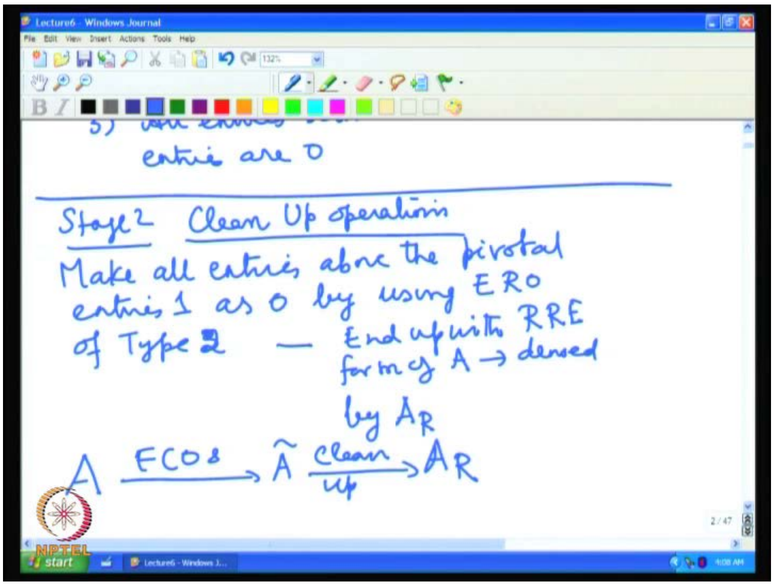

We may get some matrices A a tilde then we do the cleanup operation on this to get the row reduced. They are two stage in this reduction form of A given matrix to the row reduces echelon form first you adjust the pivotal once and the entries below the pivotal once and then you adjust the entries above the pivotal once in the cleanup operation let us look A very simple example consider the matrix 1 2 two 9 minus 1 one 2 3 thirteen minus 2 and on it take 4 rows and 5 columns this is obviously not in row reduce echelon form because take the first row which is A non 0 yes there is A first non 0 entry is 1 however everything below that is not 0. We have to now apply our algorithm our reduction process to this matrix to bring this matrix to the row reduced echelon form let us doing this process step by step. What we do in the first step the first step is state the matrix scan the first column the first column does it have non 0 entries yes there is A non 0 entry there is A non 0 entry all are non 0 entries in fact. We want to bring 1 of them to the top there will 1 already in the top that is the first stage is already done for us and next stage is to make that first non 0 entry as 1 it will already 1.

We have got do anything and then we have to clean up with everything below that non 0 entries the first think that we are first do. Here is now take this pivotal 1 and clean up in such A way that the entries 1 in the second row minus 1 in the third row and 1 in the fourth row in the first column all get naught naught to 0 this will achieve by the following elementary row operation to knock of this 1 in the second row. We have to separate the first row from the second row that we get A 1 minus 1 and this will become 0 similarly, to knock of this minus 1 in the third row. We have to add the first row that that 1 added to this minus 1 will give me as 0 here analogously to eliminate this 1 we have to separate the first row from the fourth row to get A 0.

Here the operation that we are looking for R r2 minus R 1 r3 plus R 1 R 4 minus R 1 and what we get we get 1 2 two 9 minus 1 now this 1 minus 1 R 2 minus R 1 is 1 minus 1 0 2 minus 2 0 3 minus 2 1 and minus 1 r3 plus r1 gives me 0 zero 1 4 0 and then r4 minus R 1 gives me 0 zero minus 2 minus 8 and 3. This is the effect of doing the first column operation on the first column of the given matrix A is complete full operation of the F C O on the first column of the matrix A at the end of edge. We get the first column A pivotal entry and everything below that 0 what we do next is we look at the sub matrix obtained by eliminating or ignoring.

The first column on the first row the sub matrix that we are looking for is this sub matrix consisting of 3 rows and 4 columns. Now what we do is we apply the first column operation to the second to this sub matrix that, we obtained if we look at this sub matrix the first column. It is all zeros there is A 0 there is A 0 there is 0 this both 2 here does not come because, we are going to concentrate only on this sub matrix. Now all the elements in the first column R 0 and our algorithm set when the first column is all zeros the F C O. algorithm say is do not do anything we just leave it as it is then we look at the sub matrix that we get by ignoring that first column of that sub matrix. Then we get A newer sub matrix which is not 3 rows and 3 columns we know how to apply the first column operation to this sub matrix our focusing only on these 3 entries in the first column because we are only focusing on this sub matrix now again if you look at this first column there are non 0 entries and we have already A 1. At the top our main aim is to bring A 1 to the top and it is already there now they act to do 1 clean up everything below that to be 0 to knock of this 1.

We have to separate this row from this row and to knock this minus 2 we have to 2 times this now will keep write the same number of the original row number so to behave from the third row separate the second row from the fourth row at 2 time the second row and what we get nothing happens to what we got here and nothing happen to this zeros nothing happen to this zeros so our operation do not affect what we are already obtained now the real action takes place from the third row we have to separate the second row so the second row also remains as it is from the third row if we subtract this second row we get 0 zero 1 then form the fourth row we have to add 2 times first second row we get 0 zero 1.

This complete the first column operation on the sub matrix that we were looking at end of which therefore, we now look at the sub matrix obtain by ignoring the first column and the first row of this sub matrix now this no sub matrix this time the first column again it all zeros and our algorithm says in A sub matrix when we look at the first column all the entries are 0 do nothing so we just leave it and we look at the remaining sub matrix that we are got which are 1 column matrix and we have all the entries non 0 .

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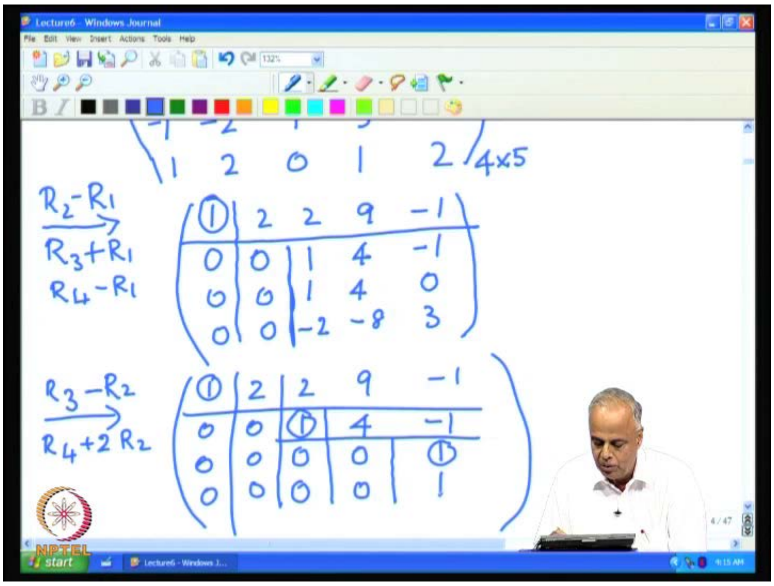

We have to bring it 1 to the top and we already have A 1 at the top it is all the privates we already at the 1 at the top. And now we have to clean up we have to finish up the 1 below that to be 0 to that we have to simply subtract the third row from the fourth row then we get 1 2 two 9 minus 1 0 zero 1 4 minus 1 0 zero 0 zero 1 0 zero 0 zero 0 these are the pivotal once. Which are now been brought into the position everything below the pivotal once all the entries below this pivotal 1 is 0, all the entries below this pivotal 1, is 0 all the entries below this pivotal 1 is 0.

So that completes the first stage of the reduction process this is apply 1 first column operation. You go on apply and the first column operation through 1 matrix after the other each time the size the number of columns getting reduced and finally, your left with 1 column you apply the F C O 2 that and then you are done with the first column operations. At the end of the first column operation. We will always end with the matrix of this form the 0 rows will be at the bottom non 0 rows will be at the top the first non 0 entry in each row will be 1 and as we move down. The non 0 pivotal once will be moving to the right and every entry below the pivotal once will be 0.

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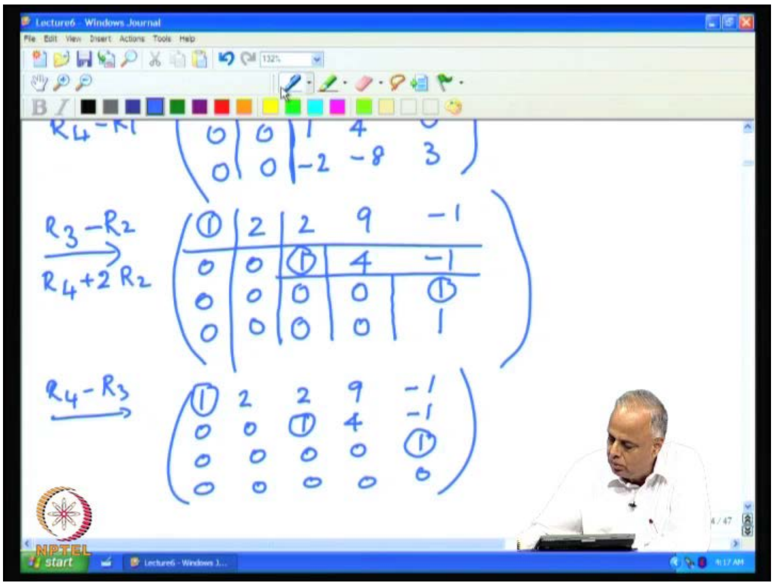

This is the young stage are apply at the first column operations then we go to the clean up operation in the cleanup operation. We are now to choose 1 by 1 the pivotal entry and make all the entry above the pivotal entireness 0. Now already I choosing this 1 and made everything below this 1 as 0 and there nothing above which the next pivotal 1 is in the second row and we want to make everything above with namely this 2 this pivotal 1 above this is 2 and this 2 has to be .

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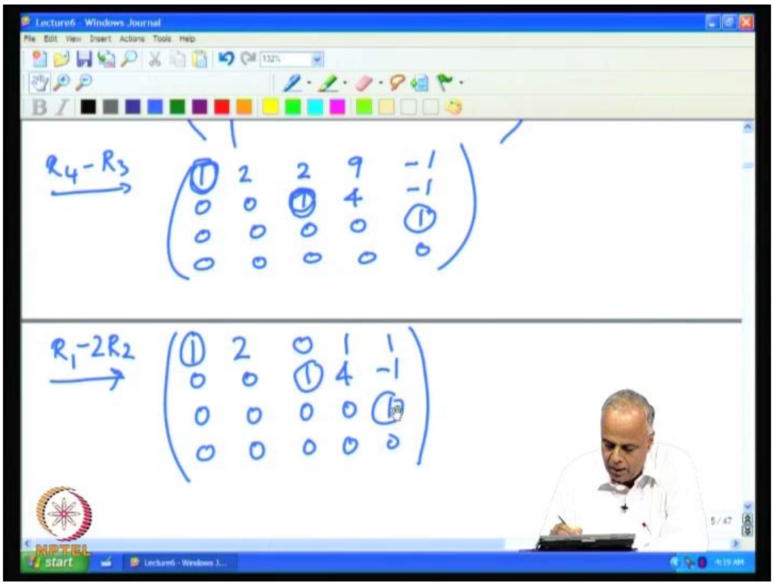

Now knocked out to achieve this we have to subtract try this 2 knock it. So we have to subtract twice the second row from the first row that will be our next operation R 1 minus 2 R 2 that we can may 1 2 0 1 one 0 zero 1 4 minus 1 0 zero 0 zero 1 0 zero 0 zero 0. Once again let me keep track of the pivotal once now we have made all the entries below this pivotal as 0 and then nothing above with we are made all the entry below this pivotal is 0 now omega above also 0.

Now we go to the last pivotal 1 you want you want to make everything above that 0 to knock this 1 we have to subtract the third row from the first row to knock this minus 1 of we have to add the third row to the second row now we do this operations R 1 minus R 3 R 2 plus R 3. We get 1 2 0 1 0 zero 0 zero 1 4 0 zero 0 zero 1 and 0 this matrix in the end. Now we are got after this cleanup operation is row reduces echelon form why this in row reduce echelon form we have non 0 rows on top and the 0 row on the bottom. Then if we look at the first non 0 row the first non 0 entry is 1 then thus non 0 entries 1 first non 0 entry is 1 and as we move down the non zero the pivotal once are may be into the right here is A pivotal 1 here is A pivotal 1 here is A pivotal 1 and as we move down this pivotal comes, now then we move down it is go to the right and on the pivotal once are moving to the right and the column.

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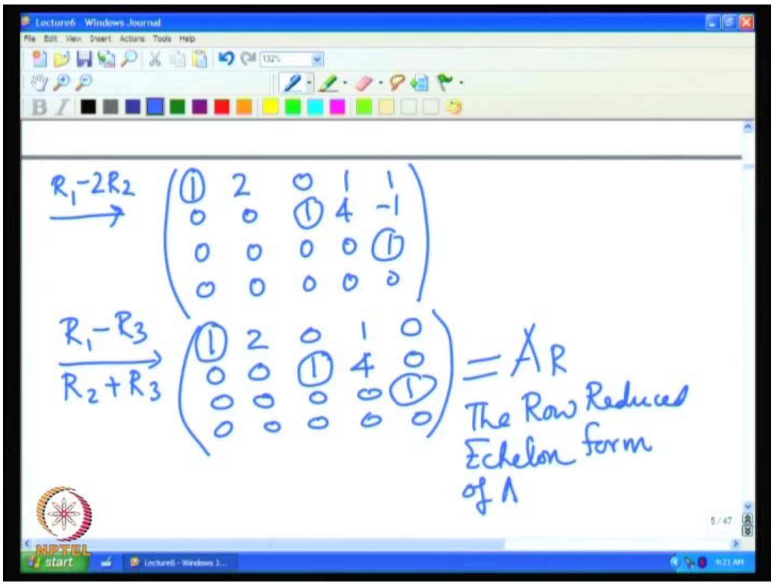

which supports A pivotal 1 other then the pivotal entry all are 0 other than the pivotal entry all are 0 therefore, this matrix using row reduce echelon form this is called the row reduced echelon form of A does our original matrix 3. Which was A 4 by 5 matrix which was not at all in row reduces echelon form as now been brought by A sequence of the elementary row operations into A row reduce echelon form that for instead of solving the original. A x equal to theta m the homogenous system corresponding to the original matrix has same set of solutions as A R x equal to theta m now let us recall how we solve this row reduce echelon form A R x equal to theta m we first the identify the pivotal variables that pivotal variables or the variables corresponding to the columns in which the pivotal once appears there is A 1 appearing in the first column.

Here x 1 is A pivotal variable there is A 1 appearing in the third column, here x 3 is A pivotal variable its though it may come in the second row it appears in third column and therefore, x 3 the column in that tells you the pivotal variable x 3 is the pivotal variable the third variable appears in the fifth column x 5 is the pivotal variable in A. Recall we have seen this example earlier.

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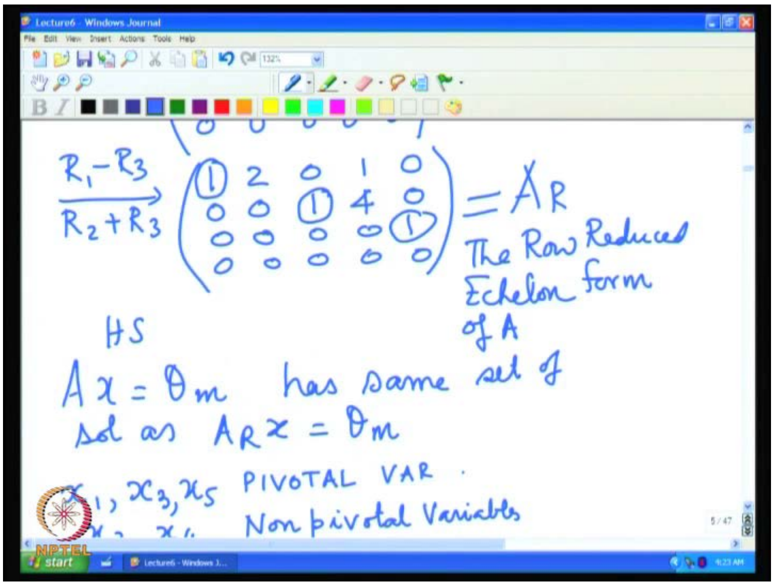

also, we have x 1 x 3 x 5 pivotal variables and x 2 x 4 non pivotal the remaining 2 variables are the non pivotal variables now what we the homogenous system A R x equal to theta m. If we look at the matrix the first equation is x 2 plus 2 x 2 plus x 4 equal to 0 the second equation is corresponding to the second row Dou x 1 Dou x 2 x 3 plus 4 x 4 is equal to 0. It is x 3 plus 4 x 4 equal to 0 and the last equation similarly, is x 5 equal to 0 again the pivotal variables of x 1 x 3 and x 5. And what we do is the first equation eliminate the pivotal variable x 1 and may be eliminate the pivotal variable x 1 we get only in terms of x 2 and x 4 and the other pivotal variables do not occur similarly, when we eliminate form this second equation the pivotal variable x 3 it is in terms of x 1 and not the other pivotal variable.

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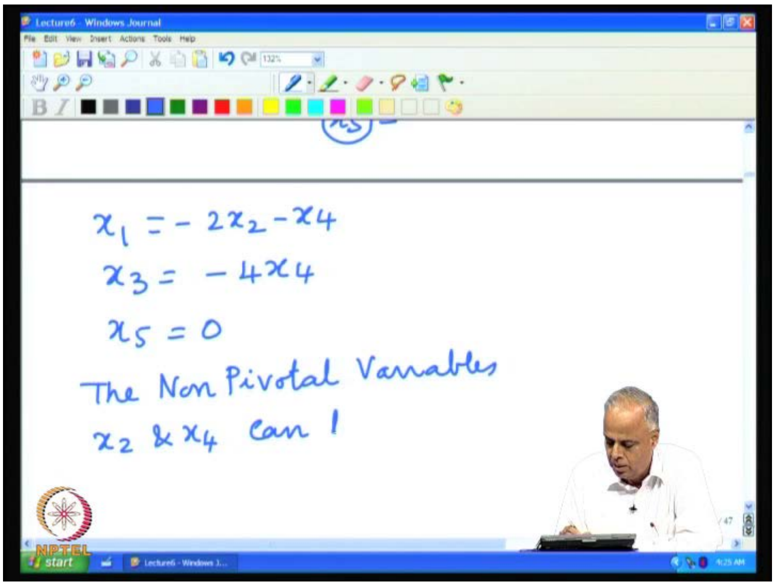

The last equation eliminates the pivotal variable x 5 and it does not involve the remaining 2 pivotal variables therefore, the eliminate the pivotal variable x 1 as minus 2 x 1 minus x 4 the pivotal variable x 3 as minus x 4 and the pivotal variable x 5 as 0 and the non pivotal variables x 2 and x 4 can be chosen arbitrarily so what will be general solution then any solution x is of the form x equal to remember x 2 has to be chosen as minus x 1 has to be chosen minus 2 x 2 minus x 4 x 2 .

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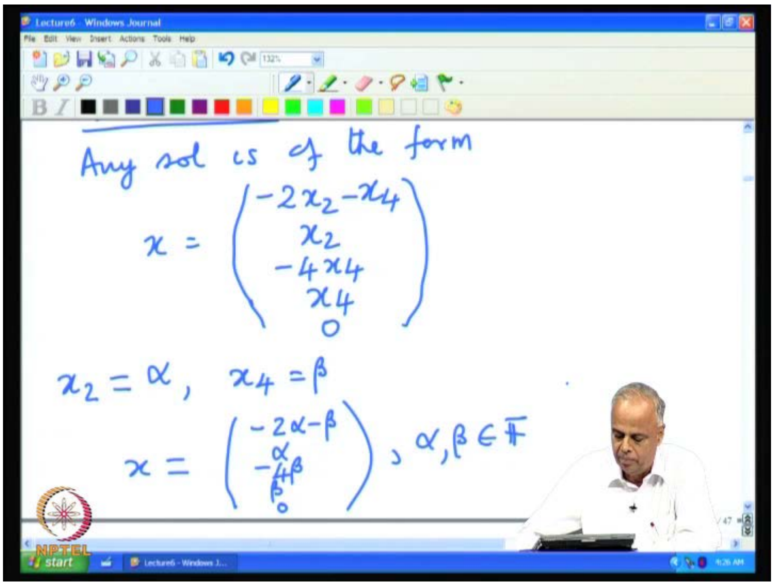

Can be chosen arbitrarily x 3 has to be chosen as minus 4 x 4 x 4 can be chosen arbitrarily and x 5 must be chosen. We denote the 2 arbitrarily follows the non pivotal variable x 2 as alpha and x 4 is beta then x equal to minus 2 alpha minus beta alpha 4 beta beta and 0 alpha beta choose an arbitrarily in f or we can separate the alpha or the beta 2 degrees of freedom we have we can write it as minus 2 1 0 zero 0 plus beta into minus 1 0 minus 4 1 0.

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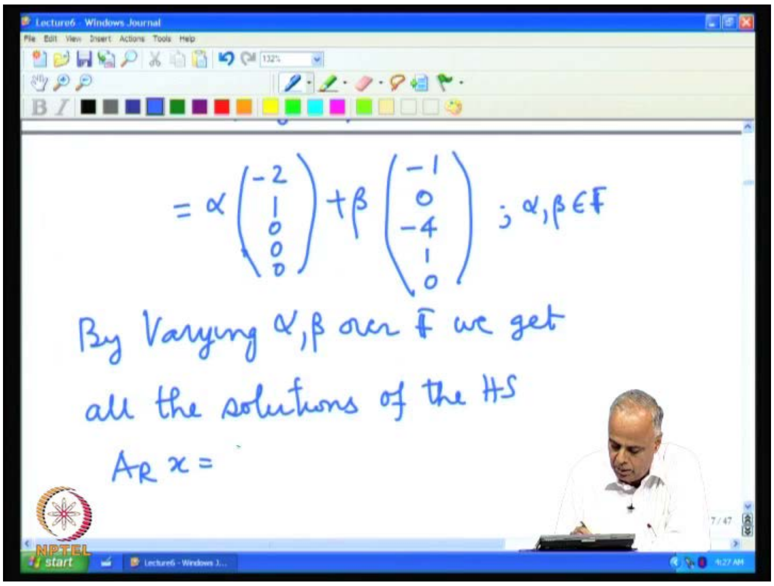

Alpha beta belonging to f. Now by varying alpha and beta over f we get all the solutions with all the solutions of the homogeneous system A R x equal to theta m and since, A R x equal to theta m at the same solution.

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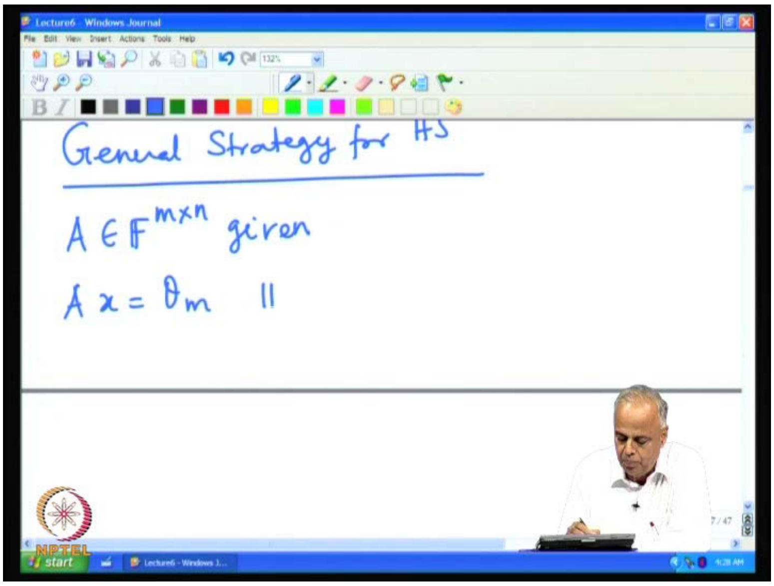

So A x equal to theta m we get also and hence all the solutions of A x equal to theta m therefore, we generate strategy for homogenous system terms out to be the following.

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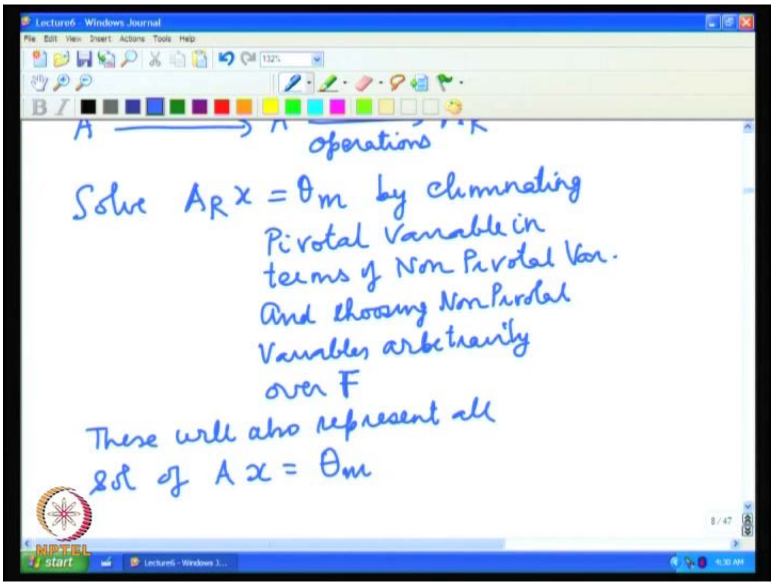

A is given we look at A x equal to theta m will be corresponding homogeneous system look at A. You apply the first column operations get A matrix A tilde then you do the cleanup operations which you have described and we get the row reduce echelon form solve A R x equal to theta m by eliminating pivotal variables in terms of non pivotal variables and choosing non pivotal variables arbitrarily we have to choose the non pivotal variables arbitrarily you are all the freedom over f and there will be general strategy.

If we solve and the same will be there will also represent all solutions or the original system we started with A x equal to theta m the solution of homogeneous system corresponding to the matrix A can be systematically handled in 2 stages 1 by using the first column operations and the clean up operations to reduce the given matrix to the row reduces echelon form and then instead of solving A. x equal to theta m solved easier system A R x equal to theta m it is easier because it is row reduces echelon form and once we solve A R x equal to theta m there will also be the solution of A x equal to theta m because both A and A R r row equivalent, now let us make some remarks let us look at A general matrix A which is m by N and then we reduce A to the row reduce echelon form of A row reduced echelon form of A by our usual first column operations and clean up operations you apply first column operations and the clean up operations as this described above you get the row reduced echelon matrix. Now once you look at A R A row reduce echelon form therefore, theta certain non 0 rows followed by A certain number of 0 rows there may not be any 0 rows or there may not be any non 0 rows but in general there will be certain number of non 0 rows sitting at top and all the 0 rows put A settle down at the bottom.

Now suppose say A as m rows some of them are non 0 and some of them are 0 let us say there are rho number of non 0 rows. Let A R have rho non 0 rows therefore, it has m minus rho 0 rows because the total number of rho system A rho of them are non 0 the remaining must be 0 A R will be of this form there will be this row 1 there will be this row 2 there will be this row R rho as call the rho 1 as R 1 rho 2 as R 2 R rho as R rho and then the R pho plus 1 the R rho plus 1 throw will all this 0 and it goes down and m th row will be all 0 so there will be m minus rho which are all zeros. .

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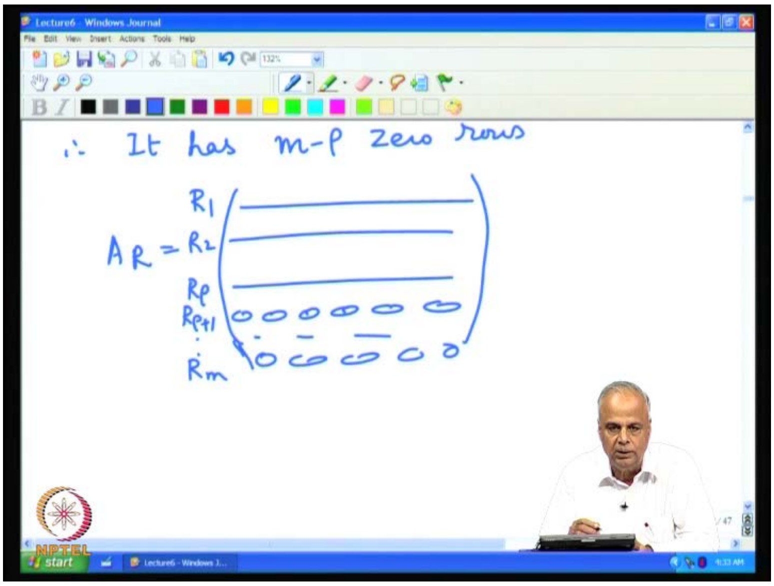

Which would have settle down at the bottom and at the top will have rho non 0. Now let us look at the first row it will have A first non 0 entry and we had the notation let us assume that the first non 0 entry of R 1 appears in K 1 column R 2 appears in K 2 column and R rho appears in K k columns let the first non 0 entry. Which we call at the pivotal entry of row R I appear in K I th column appear in K I th column so the matrix will now look like A R there is this zeros the R rho plus 1 down all the way to R m all these are zeros.

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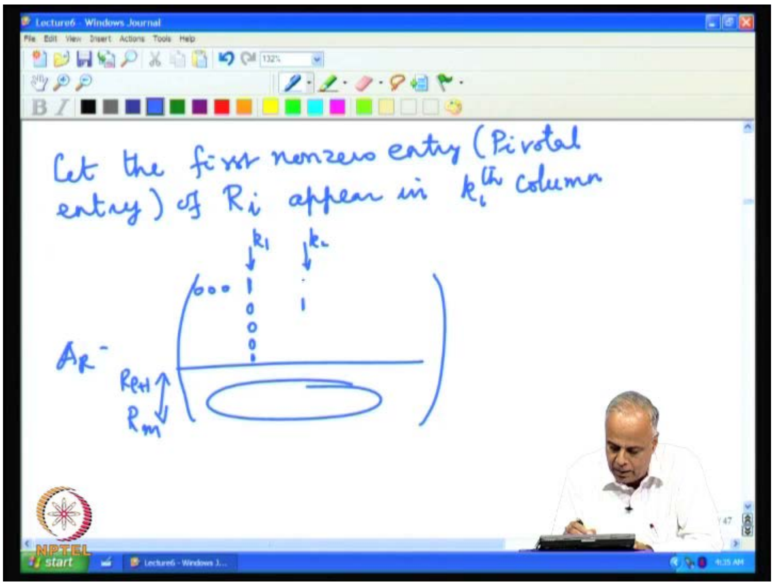

Now in the first 0 in the K1 column there is 1 all the others are 0 and since everything below and above A pivotal 1 must be 0 like this and then row 2 there will be A column K 2 to the right and 1 will appear there and all others will be 0 and it will go on like this up to K row therefore, we have the non 0 entries in the R I th row appearing in the KI th column. Now we know that the variable corresponding to the pivotal once or the pivotal variables the pivotal variables will be x 1 x K2 and x K rho the non pivotal variables will be x j j not equal to K1 K2 K rho these are eliminated in terms of the non pivotal variables and the surface in arbitrarily now the number of pivotal variables if equal to rho which is the number of non 0 rows in A R the number of non 0 rows in the row reduce echelon form at the matrix if the same add the number of pivotal variables and that is denoted by rho and

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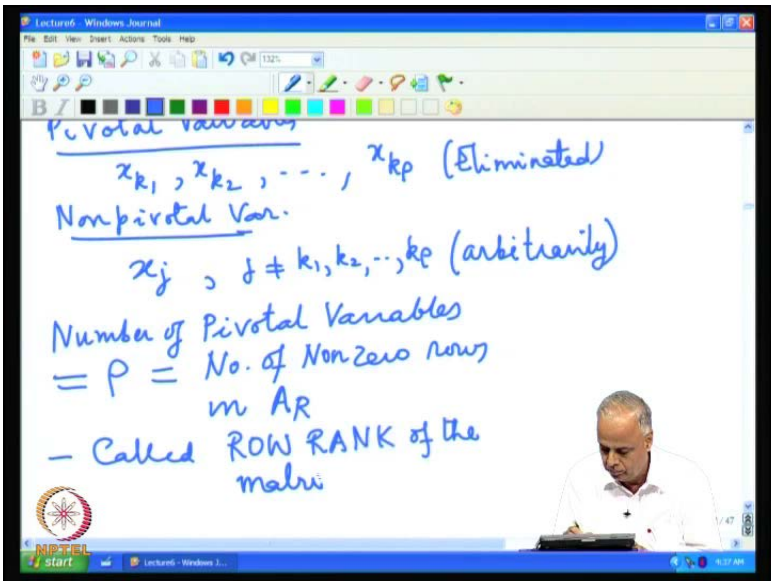

This number is called the row rank of the matrix A called row rank of the matrix A what is the row rank of the matrix A .There definition is the row rank of A is the number of it is the number which are whole integer.

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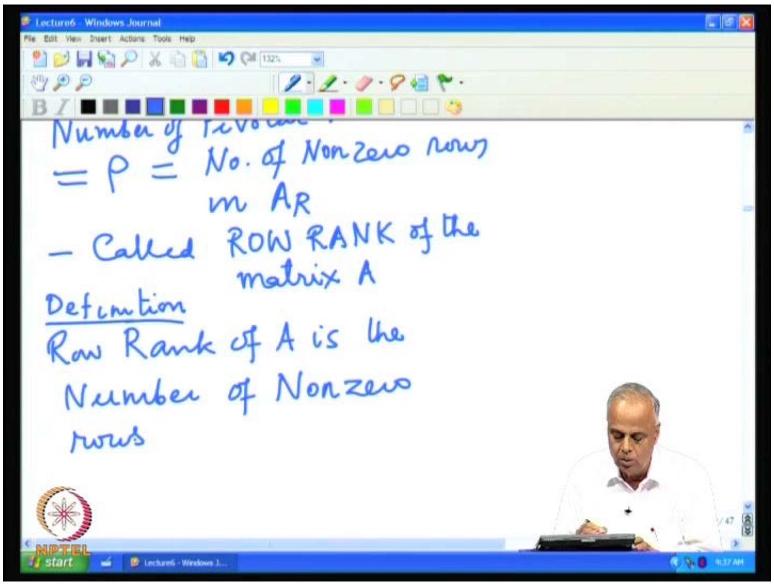

The number of non 0 rows in the R r e form A R of A let us look at example in our example, we saw we reduce the matrix to this following row reduces echelon form if we look at this matrix this is already I row reduce echelon form.

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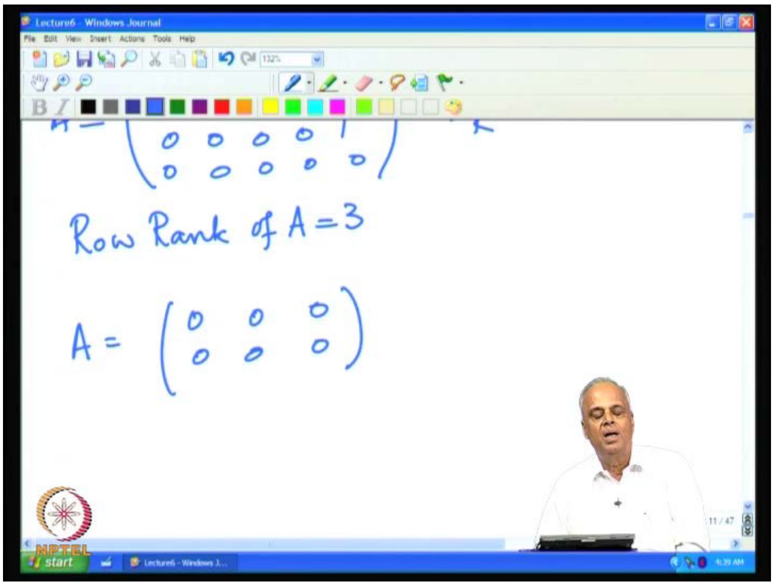

So we do not have to do any reductions that itself is where reduction echelon form how many non 0 rows are there are preciously 3 non 0 rows row rank of A is equal to 3 consider this matrix which is the 0 matrix 2 rows and 3 columns is it row reduce echelon form yes because what is the structure all 0 rows must settle down and if there are non 0 rows they must come on top there are non zero no non 0 rows all are 0 rows. So the row rank which is the number of non 0 rows in the cases 0 the row rank of A is equal to the number of non 0 rows. This is also equal to the number of pivotal variables number of pivotal variables.

Now if there are row pivotal variables how many non pivotal variables are there the total number of variables are unknowns or whatever we want call them there where if it is m by N matrix the unknowns are x 1 x 2 x 3 x m so the total number of variable is N number of pivotal variables number of pivotal variables is equal to rho which is the row rank of the matrix therefore, the number of non pivotal variables to be equal to 1 minus rho there where N rows N the pivotal variables in all row of them but, pivotal so N minus rho will be non pivotal this is called the nullity of the matrix A called nullity of the matrix A so the nullity of the matrix A is equal to N minus rho or we have rho is the row rank row rank of A plus nullity of A is equal to the.

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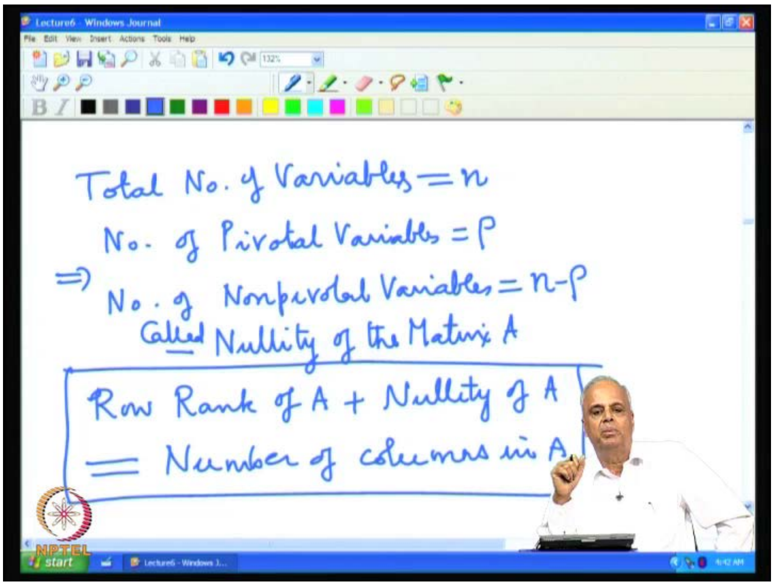

number of columns in A there will be very useful result which will be seen in the different forms later but, this is the same thing as saying that number of variables are pivotal that number of variables pivotal of non pivotal look at form the homogeneous system of equations point of view. We talk with variables language the number of pivotal variable plus the number of non pivotal variable is equal to the total number of variables purely from the matrix point of view. We say that the row rank of the matrix A plus the nullity of the matrix A is equal to the total number of columns in the matrix for let us summarize we know or we have learnt for we have learnt that how to solve homogeneous systems how to solve homogeneous systems the idea to we have seen reduce it to. A reduce the given homogeneous system by reducing the corresponding matrix to the corresponding row reduce form and solve the row reduce echelon form equation and thereby get the solutions we have A clean picture A clean method A clean algorithm to handled homogeneous systems. If we recall that then we solved non homogeneous systems the solution for non homogeneous system involved 2 parts 1 was finding the solution of homogeneous system. Which we are now clearly handled the reduction to row reduce echelon form and we also add to find A particular solution of the non homogenous system to put together A complete picture of the solution of the non homogeneous system.

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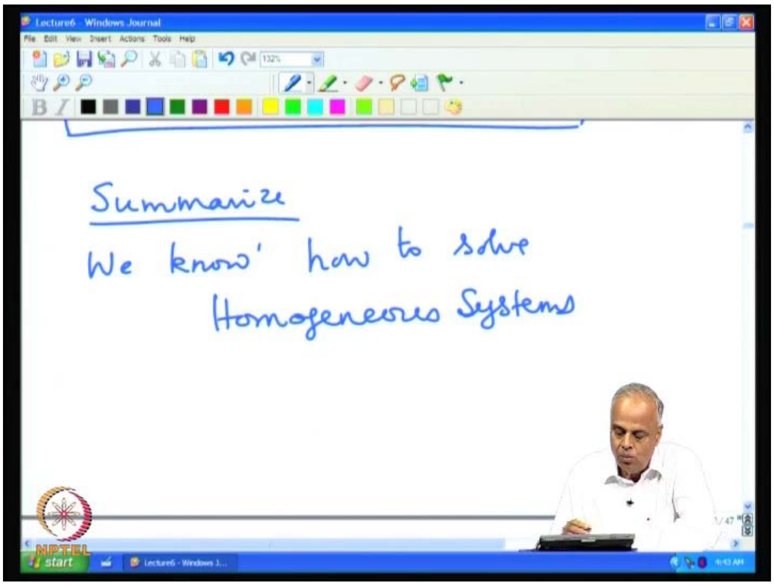

Now let us proceed towards handling N H S the non homogeneous system it remains to find A particular solution x p of the non homogeneous system. Because we say that any solution of the non homogeneous system can be obtained as x p which is A particular solution by adding to A a solution of the homogeneous system and simply already have all the homogeneous solution system.

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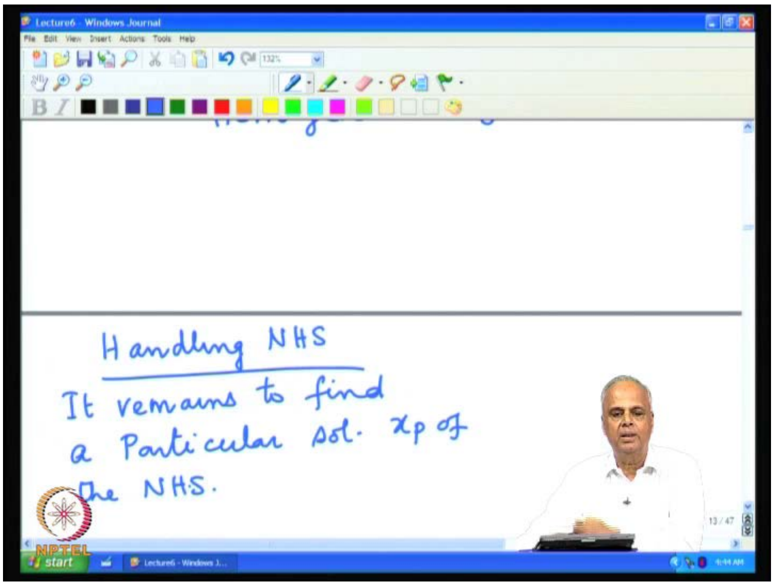

We have seen how to get them all way need to find is 1 particular solution of the non homogeneous system. Now how do we go about doing this we shall look at the first idea towards this now why does not the same thing that we did for homogeneous system work in this situation also, further we shall see what was the strategy for homogeneous system we used E R O S basic idea was used E R O S that the E R O S was 1 that were used to reduce the given matrix to the row reduced echelon form now what exactly if the effect of E R O S that we are doing on the matrix A.

We may comes to this homogeneous system corresponding to them then we are doing some operations on the rows of matrix which that we are trying to do this same type of operation on the corresponding equation left hand side for example, if we had A matrix A and this is the I th row and if we are performed in e R o or R I. So we are going to performing e R o or R I then it like for example, to this I th row I add some multiple of j th row R I multiply the I th row by A constant in terms of this system of equation what we are doing is we have looking at the I th equation equal to 0 and we are doing the same operation on the left hand side in other words. We are multiplying this row that means we are multiplying the left hand side of this equation by alpha if we are adding 4 time this second row to this equation it means we are adding 4 times the second equation left hand side to this I th equation similarly, if we are interchanging 2 row here it means we are interchanging the 2 left hand side of this equation e R o S on matrix A is same as the from the point of view A homogeneous system similar operation on the l H S of corresponding homogeneous system.

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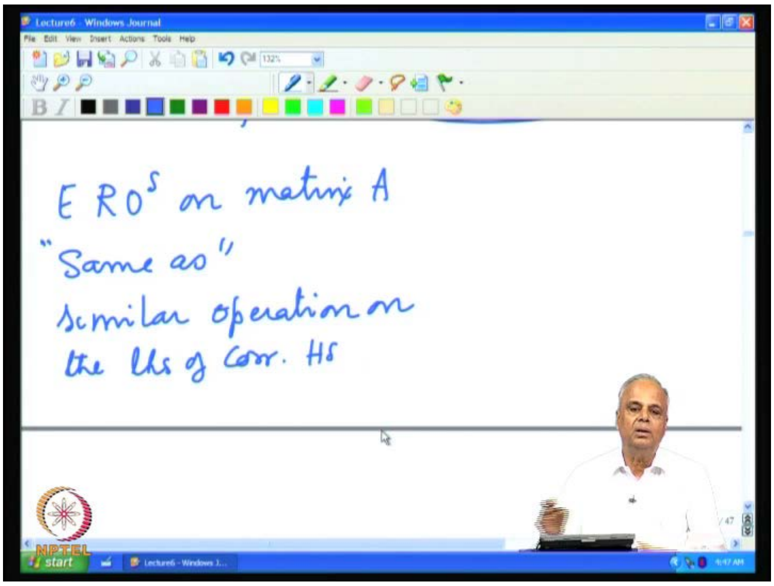

Now an equation has both left hand side and the right hand side if you want do something on the left hand side to maintain the equality. We must do the same thing on the right hand side why are we not doing in the right hand side the reason is the right hand sides are all 0 in A homogeneous system and therefore, if you do any of this elementary row operations the right hand sides are not going to be affected but, remain to the zeros so we notice that none of these E R O's affect the R H S. Because they are all 0 and they remain to be 0 after the E R O's and therefore.

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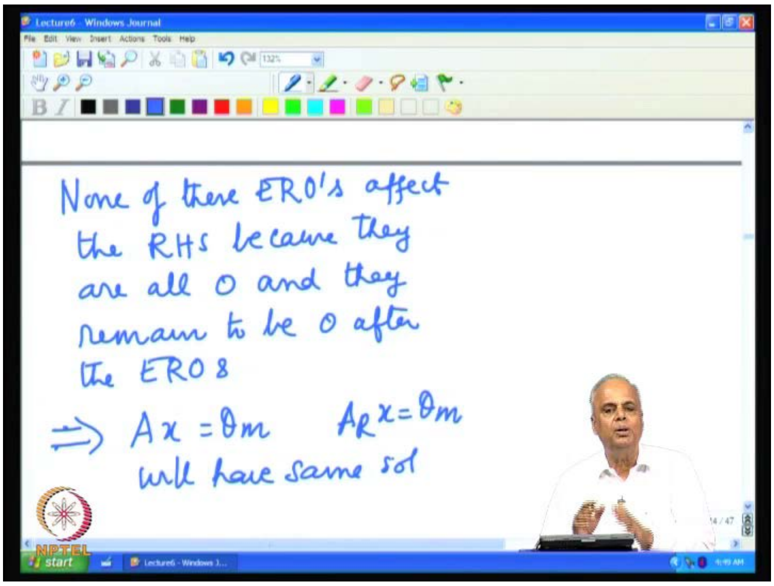

We do not even bother to do this E R O's some the right hand side and hence the system becomes and therefore, this implies A x equal to theta m and A R x equal to theta m will have same solution because both the right hand side are going to the still theta m only why thus this not work in non homogeneous this will not work in N H S. What do you mean what do you mean is suppose I go from A to A R by elementary row operations .

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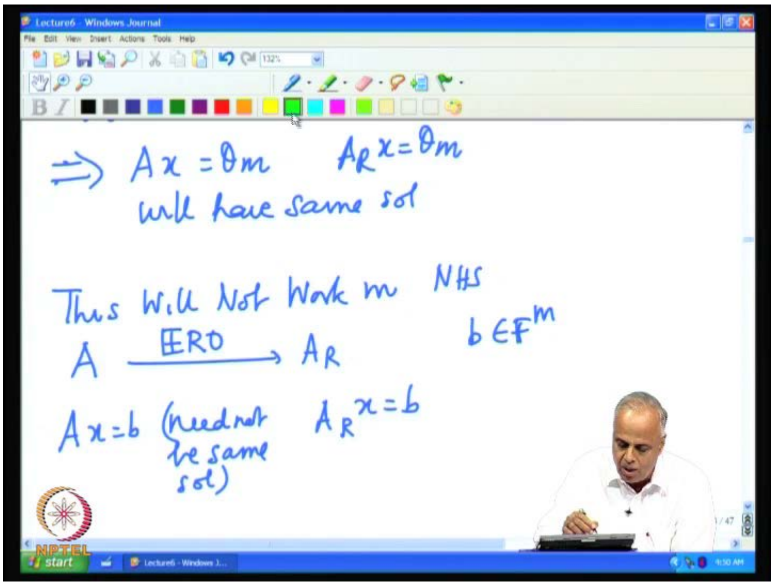

I had A B in f m then I look at A x equal to B and A R x equal to B the 2 non homogeneous system. They need not be same solution they both need not be having the same solution now previously when we retain the same right hand sides when A homogeneous system we retain the same right hand side both sides but now if you try to retain the same right hand sides the systems are not going to the equivalent because let us the right hand sides. We will change let us look at an example to illustrate this let us consider the matrix A equal to 2 3 minus 1 one minus 1 one and B equal to 2.

Now if you look at this matrix and do the following row operation there are only 2 rows so let us interchange the first and the second row 1 minus 1 one 2 3 minus 1 one now this is let us call it as A 1 I have A system generated by this matrix A. And I have now made A elementary row operation on this matrix now look at the non homogeneous systems A x equal to B and A 1 x equal to B these are the systems A is 2 x 1 plus 3 x 2 minus x 3 equal to 2 and x 1 minus x 2 plus x 3 is equal to 1 and this is the system x 1 minus x 2 plus x 3 equal to 2 x 1 minus 2 x 1 plus 3 x 2 minus x 3 equal to 1 notice that both this systems are different and they will not have the same solution we leave it as an exercise for you to find 1 solution.

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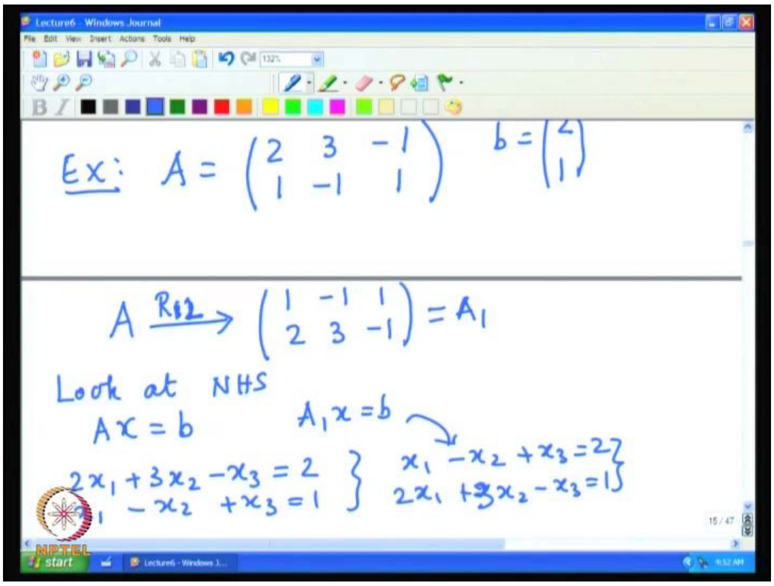

which is the solution of A x equal to B and which is not A solution of A 1 x equal to B therefore, elementary row operations affect the solutions with do not change A right hand side how do we handle this we shall see that in the next class.