Advanced Matrix Theory and Linear Algebra for Engineers

Prof. R. Vittal Rao

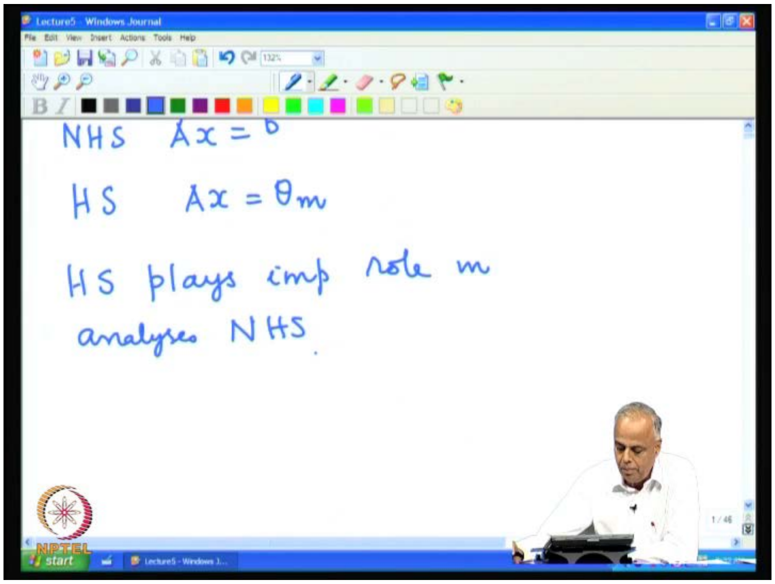
Centre for Electronics Design and Technology

Indian Institute of Science, Bangalore

Lecture No. # 05

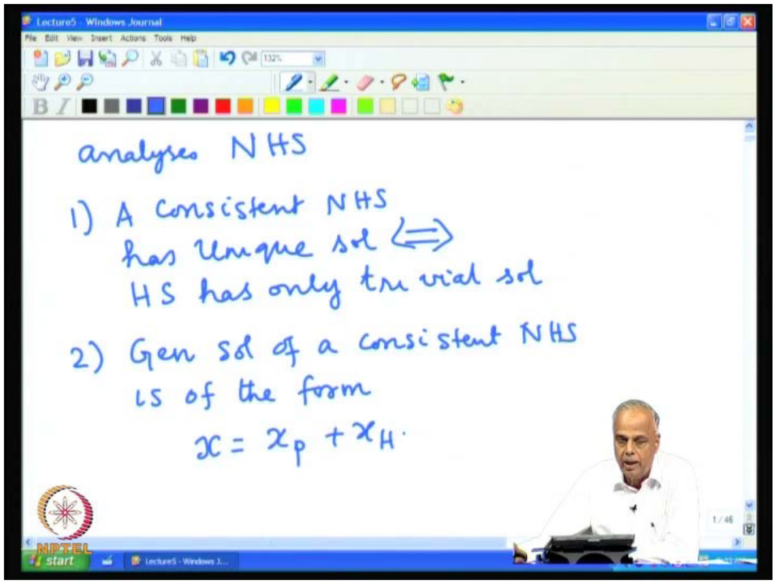
Linear System – part 2

(Refer Slide Time: 00:26)



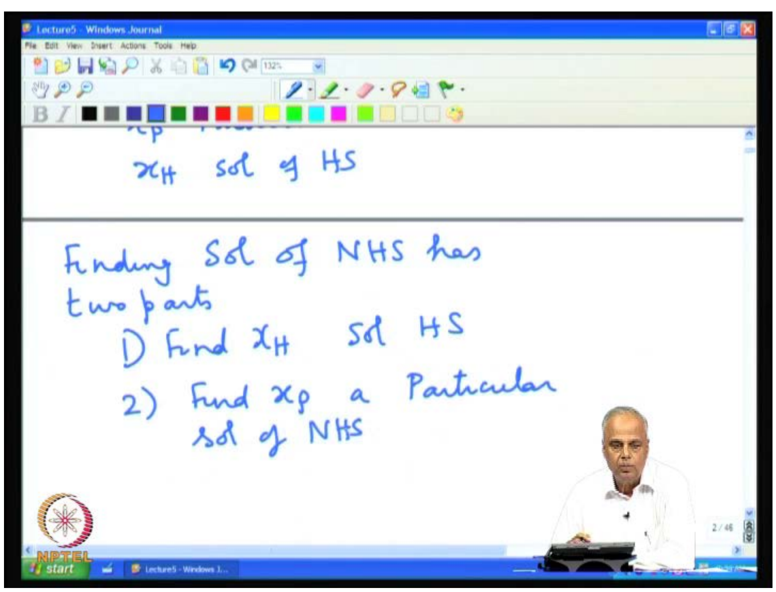
In the last lecture, we have been considering, a matrix A belonging to F m cross n, and a matrix B belonging to F m cross 1; and we were interested in the non homogenous system, A x equal to b. We consider the corresponding homogenous system, which was A x equal to theta m. We had observed that the homogenous system plays an important role in the analysis of the non homogenous system.

(Refer Slide Time: 01:08)



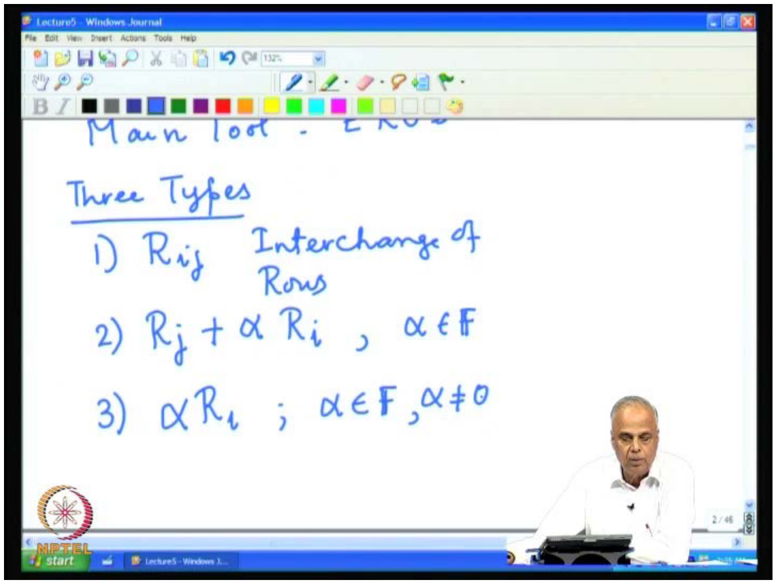
What we meant by this was the fact that 1 - a consistent non homogenous system has unique solution, if and only if the corresponding homogenous system has only trivial solution. Thus the homogenous system plays an important role in determining the uniqueness. Then the general solution of a consistent homogenous non homogenous system is of the form x equal to x p plus x H, where, x p is a particular solution of the non homogenous system and x H is solution of the homogenous system.

(Refer Slide Time: 02:01)



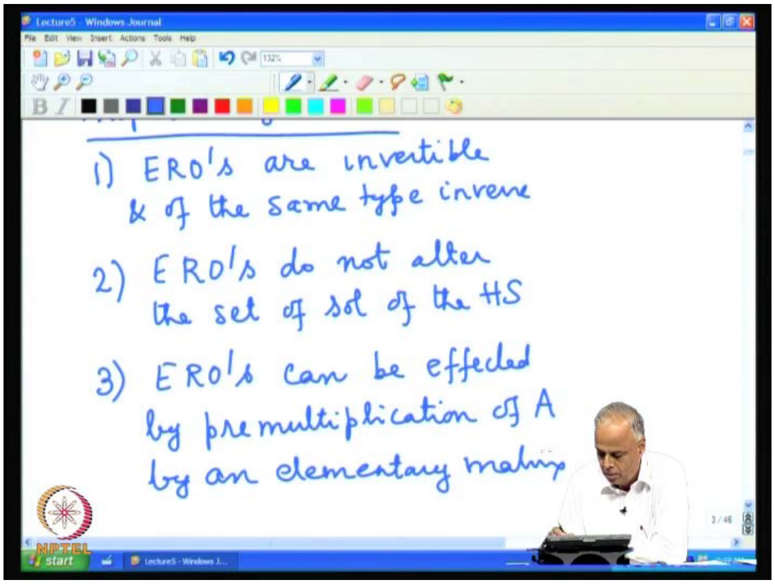
By varying x H by varying x H over will be solutions of the homogenous system, we speak all the solutions of the non homogenous system. Therefore, finding solution of non homogenous system has two parts; the first part, find x H, the solution of the homogenous system, and two find x p, a particular solution of the non homogenous system.

(Refer Slide Time: 03:06)



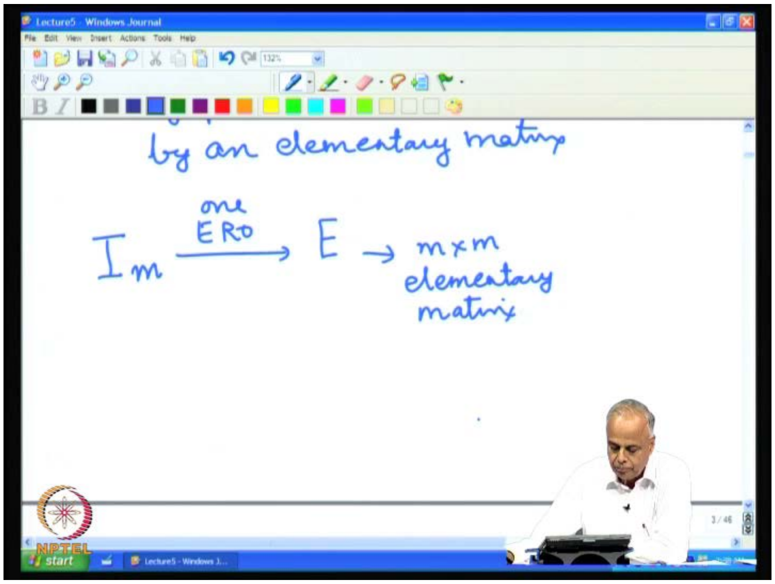
Thus we have two parts that are varying in to the solution of a non homogenous system and as the beginning; we looked at the solution of the homogenous system. For this, the main strategy or the main tool, we introduced was the, so called elementary row operations, which we denote by EROs. There are three types of elementary row operations that, we introduced. The first one which, what we denoted by R i j, it is the interchange of rows, the i th row and the j th row are interchange in the matrix A. The second row operation we did was, two a row j, let say R j the j th row, we add a multiple of another row R i, where alpha from F. Some row you add a multiple of another row, this is the second type of ERO. We introduced the third type of ERO, we introduced was take any row R i and multiplied by a nonzero number belonging to F.

(Refer Slide Time: 04:41)



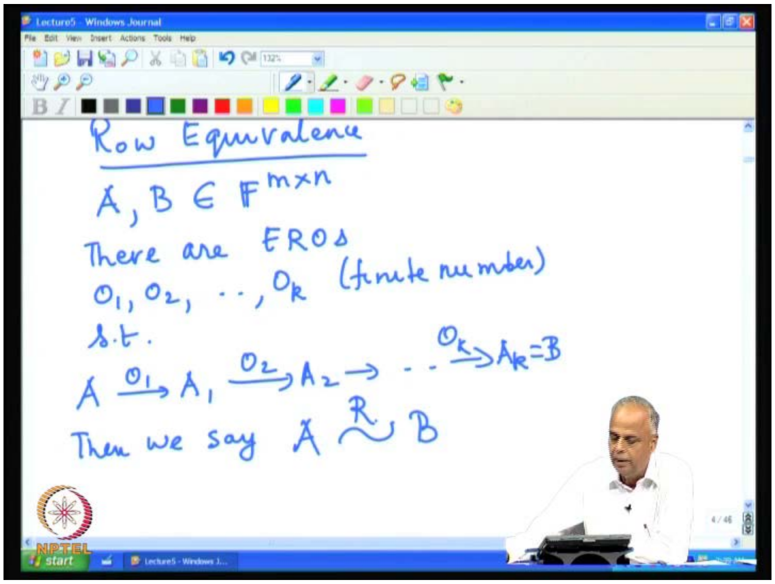
These, where the three types of elementary row operations that we introduced, the basic properties of the elementary row operations, that are going to be useful in our analysis, are the following; 1 the EROs are invertible and of the same type inverse. That is, if you interchange two rows, the inverse is again interchange of two rows, if you add a multiple of 1 row, the inverse is again adding a multiple of 1 row and if you multiply a row by a nonzero number. The inverse is again, multiply in the row by a nonzero number. The EROs are invertible and the inverse is at the same time. The second important property is that, the EROs does not alter the set of solutions of the homogenous system. This is the very crucial point in determining, the homogenous system solutions. The third, we shall be using later is that the EROs can be implemented or can be effected by pre multiplication of A by an elementary matrix. Recall by an elementary matrix, we mean a matrix obtain from the identity matrix by applying an elementary row operation.

(Refer Slide Time: 06:56)



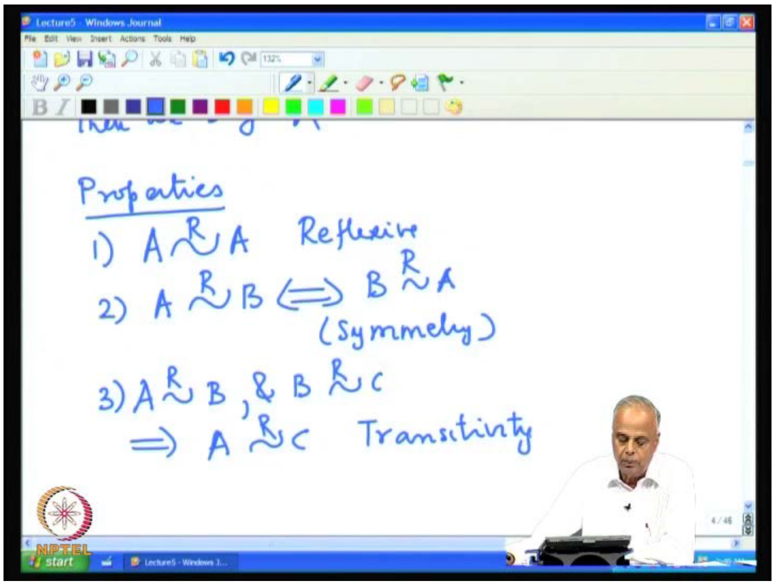
If you have the identity matrix I m and we apply an elementary row operation, one elementary row operation and we get a matrix A then, this is called an m by m elementary matrix. So, any matrix obtained from identity matrix, by a single elementary row operation is called an elementary matrix. These are the simple properties of elementary row operations.

(Refer Slide Time: 07:33)



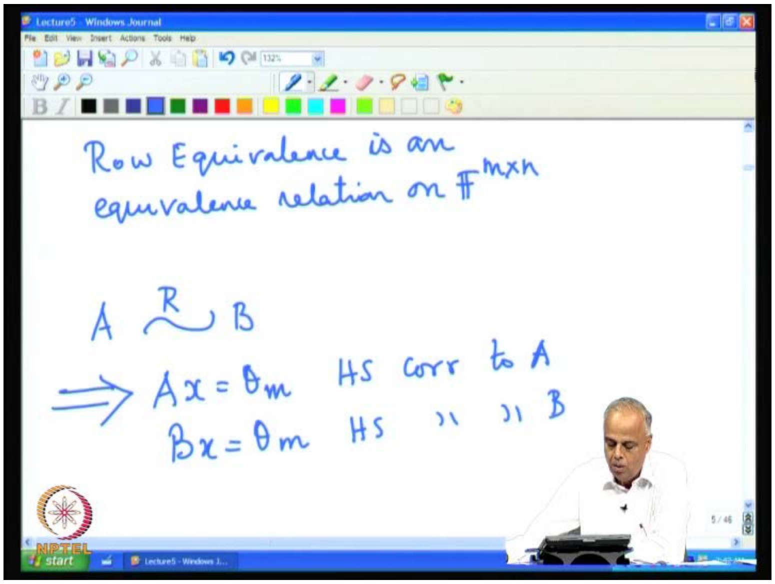
We introduced the notion of row equivalence. Supposing, we have two matrices A and B in F m cross n and there are EROs either or type 1 or type 2 or type 3. 1 type of are the, other of the EROs say o1, o2, ok finite number of them such, that you can start from A and apply the sequence of EROs to eventually in the last stage get A k which is equal to B, which means I can move from A to B by a finite sequence of EROs then, We say A is row equivalent to B and we write A or B.

(Refer Slide Time: 08:46)



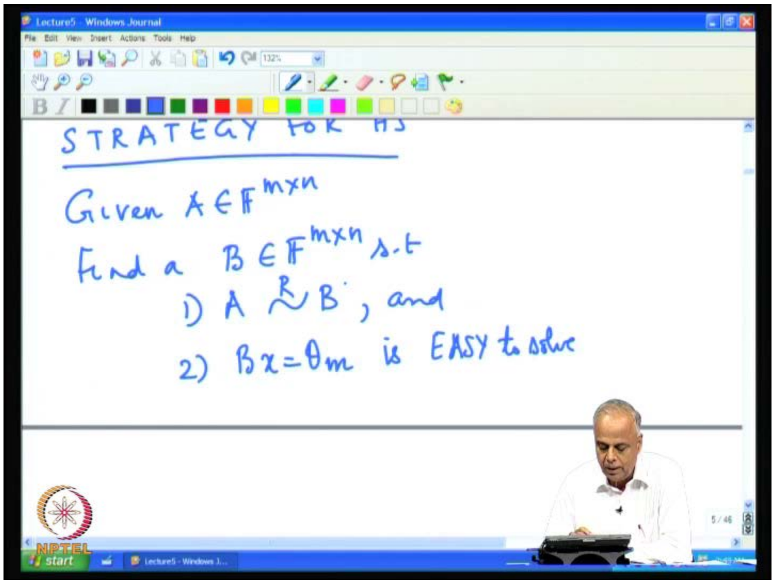
The properties of row equivalence are very useful. 1 every matrix is row equivalent to itself, this is called the reflexivity property. If a matrix A is row equivalent to a matrix B then, B must be row equivalent A. So, A is row equivalent to B if and only B is row equivalent to A. This is called the symmetric properties of row equivalence and the third if, A is row equivalent to B and B is row equivalent to C. Then, A must be row equivalent to C, this property is called Transitivity.

(Refer Slide Time: 10:00)



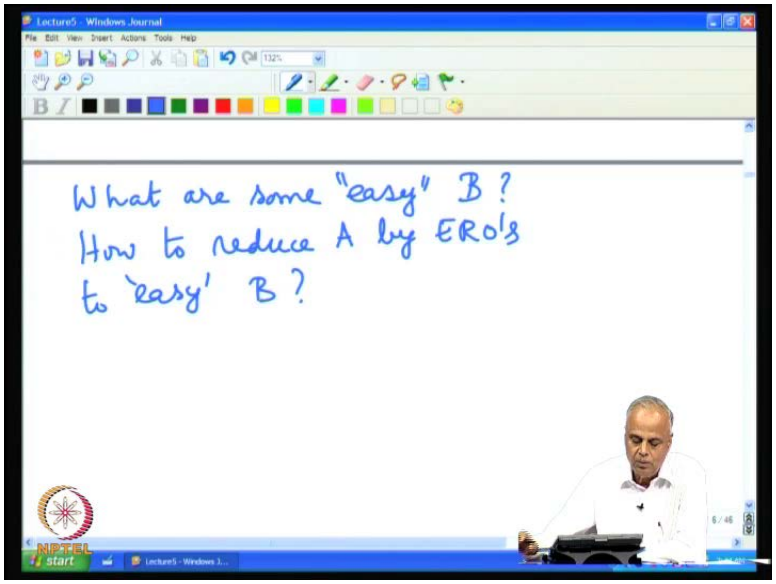
The row equivalence has these three simple properties, which makes it equivalent relations, row equivalence is and equivalence relation, on all m by n matrices and the set of m by n matrices. Now, if you have a matrix A. A is row equivalent to B, what means if you go from A to B by a sequence of elementary row operation. A finite sequence of elementary row operations, but we assume that the elementary row operations do not alter the set of solutions of the homogenous system. Therefore, what we obtain in the end B, we have the same set of solutions for the homogenous system as A. So, A row equivalent to B implies A x equal to theta m, the homogenous system corresponding to A and B x equal to theta m, the homogenous system corresponding to B. Both have the same set of solutions.

(Refer Slide Time: 11:15)



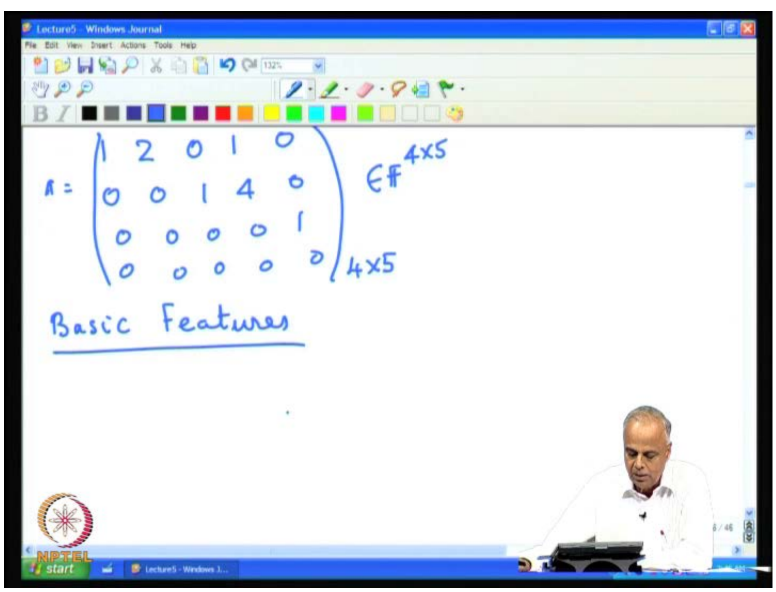
So, instead of solving the system A x equal to theta m, we could also solve the system B x equal to theta m, what is our main strategy for solving homogenous system? Given the matrix A, you find a matrix B, such that 1 a row equivalent to B, that A x equal to theta m and B x equal to theta m have the same solution and B x equal to theta m is easy to solve. If, we can do this, then we can solve the B x equal to theta m and since, the solution we have the same as the solutions of B x equal to theta m, we would have found the solutions of the given system A x equal to theta m.

(Refer Slide Time: 12:25)



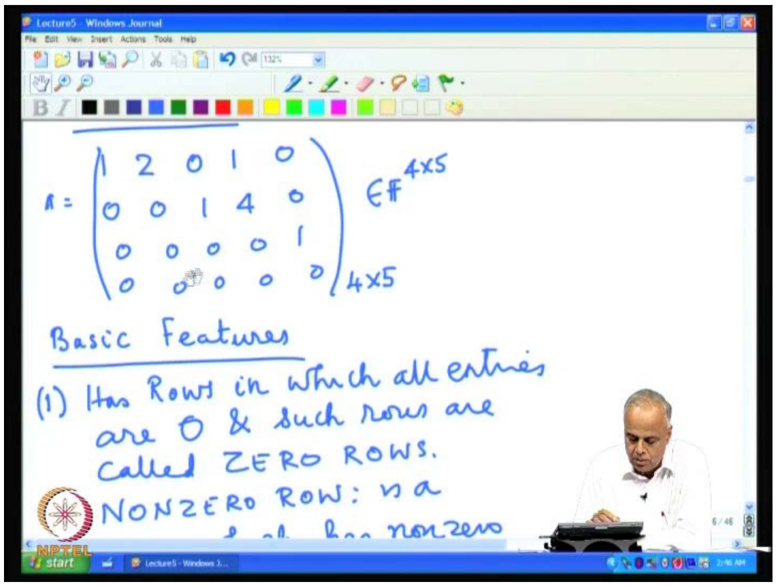
The question therefore is, what are some easy B? What are such B for, which the system easy to solve and then, how to reduce A by EROs to easy B, these are therefore do fundamental questions in solving homogenous system. We shall look at the first question, what are some using B, so let us look at a simple example.

(Refer Slide Time: 13:06)



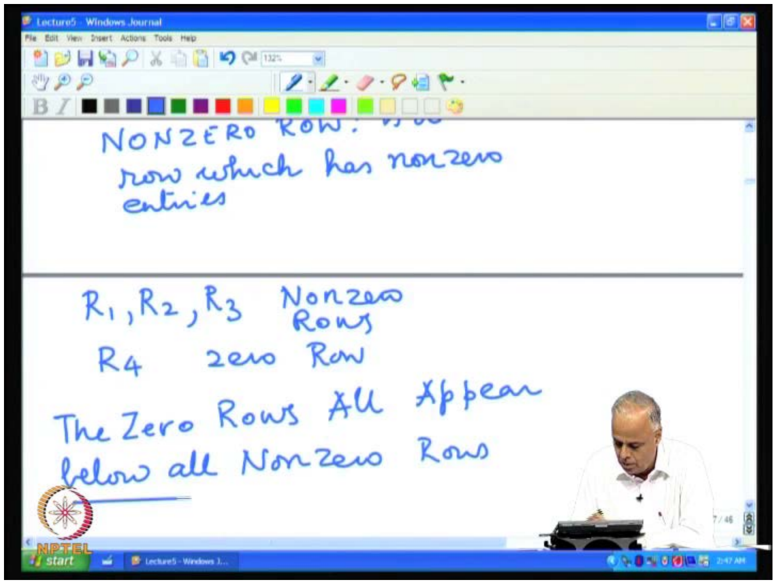
Consider, the matrix A 0 0 1 0 0 0 0, this is the matrix with 4 rows and 5 columns; it is in a 4 by 5. We have matrix which is 4 rows and 5 columns. Let us look at, some basic features of this matrix. This matrix has been written in such, a way that it has some special structure; some special features. Now, we are going to look at some of these special features, what are the special features?

(Refer Slide Time: 14:00)



The first thing that we observe is that the matrix A has some rows, which are all 0s and some rows which has nonzero entries. So, the matrix has rows in which all entries are 0 and such rows are called zero rows. Now obviously, therefore it is a matrix has non-zero entries, it is called a non-zero row. Nonzero row is a row, which has non-zero entries. The first thing we observe is that the matrix at zero rows and non-zero rows. In this example, the first row the second row and the third row are the non-zero rows and the last row the fourth row is the zero row.

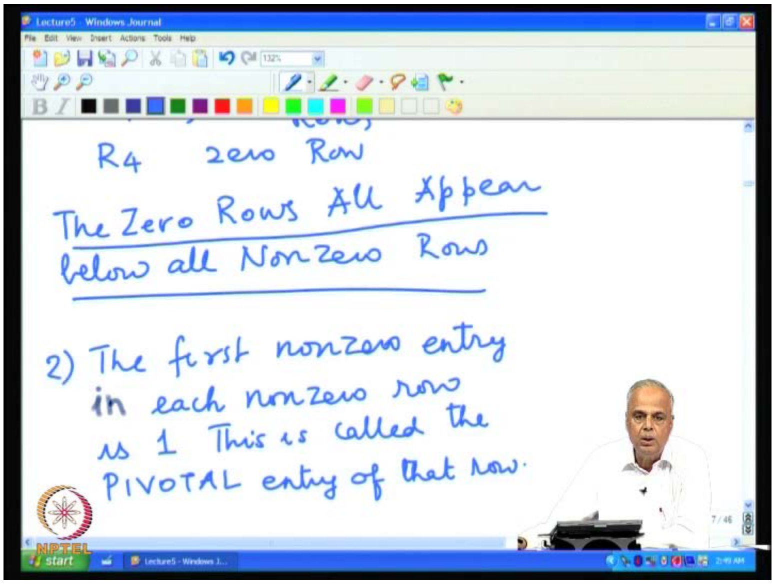
(Refer Slide Time: 15:30)



In this example, we have R 1, R 2, R 3 are non-zero rows and R 4 is zero row, there is the first simple thing that, we can have in a matrix. Some rows will have all 0s, some rows will have non-zero entries and the most important thing structure in this, the first starting point is that the row, which is all zero entries, comes after all the non zero rows. In non-zero rows appear on top and then, come the series of zero rows. So, the first important feature therefore, is the zero rows all appear below all non-zero rows. That is the first important feature that we observe. The second important feature is less get back to that matrix again.

We will see that, in every non-zero row take the nonzero row, there must be a non-zero entry for example, here in this row the nonzero entry is 1, in the second row the non-zero entry is this 1 and the third row the non-zero entry is this 1. Therefore, in each row there is a non-zero row in entry first non-zero entry and that times out to be 1, here it is 1, here it is 1 and here it is 1.

(Refer Slide Time: 17:44)



The important observation is that, the first non-zero entry we say first when, we read from the left at the matrix left entries from the first column on wards, the first non-zero entry in each non-zero row is 1. This is called the pivotal entry of that row. So, every non-zero row as a pivotal entry and that is 1 and that appears first from the left side at the first non-zero entry. Let us get back to that matrix again. We will see that the pivotal entry, here is 1 and if you go down to the next row and if you want to look at the pivotal entry. It has move to the right and if you want to go to the next non-zero row and look at the pivotal entry it has move to the right. That means, if you take 2 non-zero rows R i and R j then, if i is less than j then the pivotal entry, at the i th row appears, when left at the pivotal entry of the j th row.

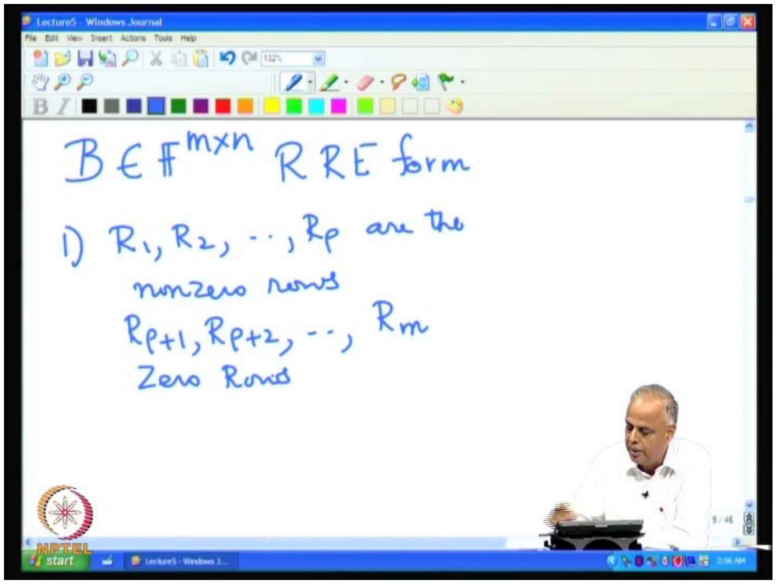
So, in other wards the important thing that, we have to observe is that if R i and R j are nonzero rows and the pivotal entry of R i appears in k i th column and pivotal entry of R j th row appears in k j th column then, i less than j that is the R i row appears above the R j row implies k i is less than k j that is, the pivotal entry corresponding to R i appears in a column to the left of the column in which the pivotal entry of R j appears.

This is the third important property then, the fourth important property that we observe in the following, pick up this pivotal entry, 1 all the entries below that R 0 look at the pivotal entry in the second column, all the entries in the second row corresponding to the pivotal entry. Which is in third column all other entries are 0? Similarly, the third pivotal entry, which is in the third row fifth column all other entries in that column are 0. So, we make that and we observe the fourth important property of this matrix is, if a column contains a pivotal entry, then all other entries in that column are 0.

So, the column which supports a pivotal entry would all the other entries is 0, there will be a one standing alone in that column, all others would have been similar 0. Similar for example, we had the first row, the pivotal entry was in the first column and all other entries in the first column were 0. In the second pivotal entry, which appears in the second row is in the third column and all other entries in the third column were 0. The third pivotal entry, which appears in the third nonzero row namely, the third row is in the fifth column and all other entries in the fifth column are 0. These are the basic features namely, there are nonzero rows and 0 rows, all the 0 rows appear after all the nonzero rows.

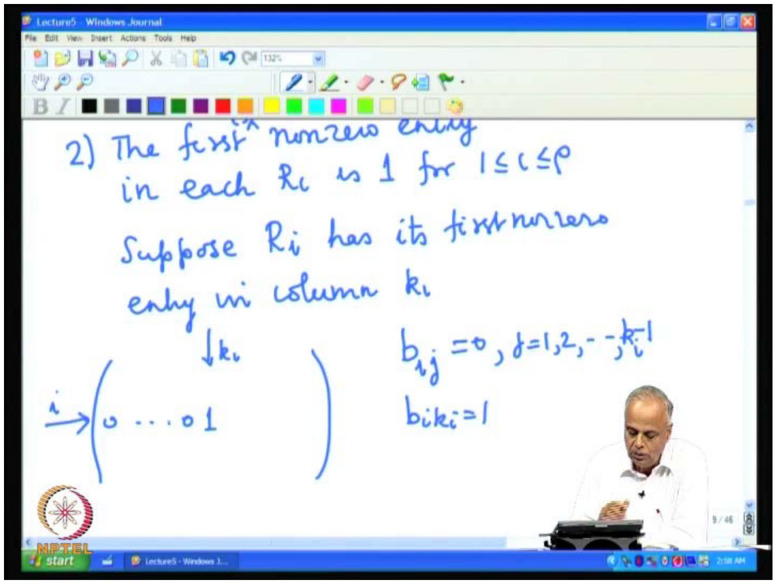
The first nonzero entry, in each nonzero row is 1 and these are call pivotal entries and as we go down the matrix, that pivotal entries keep moving to the right and any column which contains, that pivotal entry, all other entries of 0. Whenever, a matrix is in this form it is called, whenever a matrix is in this form is call a row reduced echelon matrix. Which has, these properties these 4 important properties is said to be in row reduced echelon form and such, a matrix is called a row reduced echelon matrix.

(Refer Slide Time: 24:00)



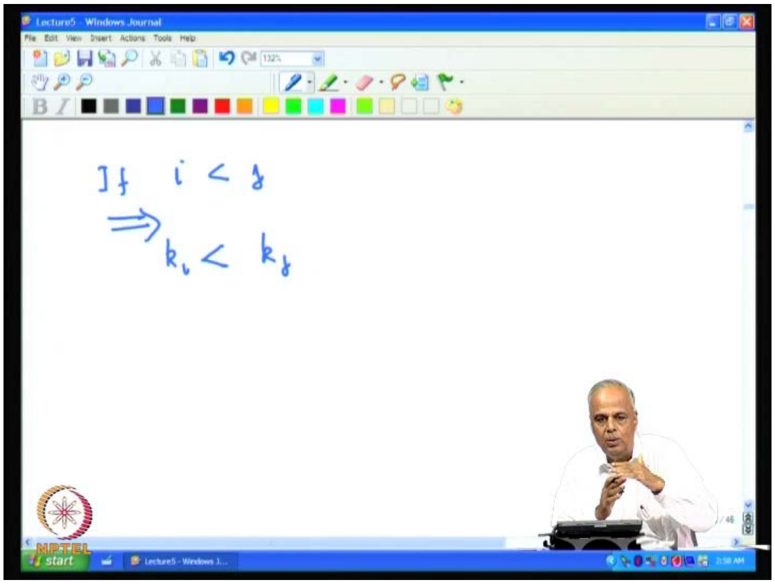
We will simply write RRE row reduced echelon matrix. In general if, we consider a matrix B, which is in F m n and if it is in RRE form, what are the features when, we mean that it is in RRE form let us say, R 1, R 2, R rho are the non-zero rows. Since, they appear always the non-zero rows appear on top, they will take all the first few indices for the rows, and then the remaining rows are, the 0 rows R rho plus 1, r rho plus 2 up to the R n th row are the 0 rows.

(Refer Slide Time: 26:45)



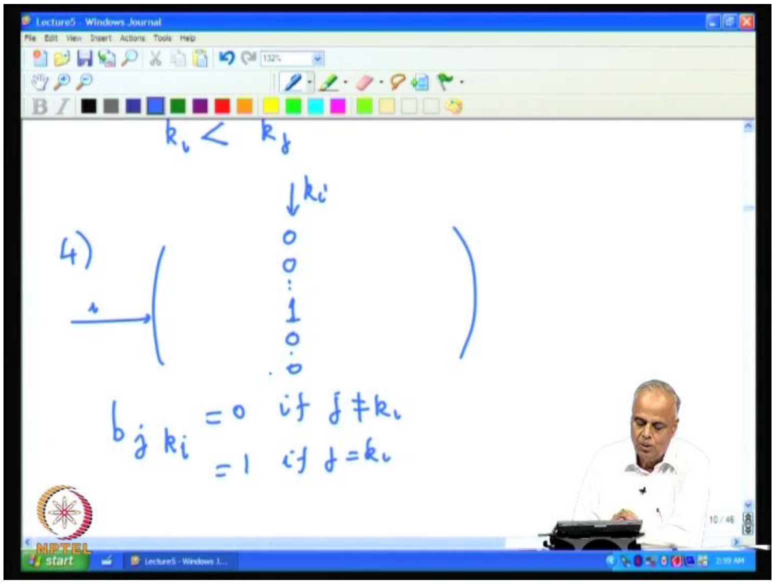
It will have say, row non-zero rows, they will all appear on top and it will have m minus rows 0 rows, they we will all appear at the bottom that is after the 0 rows. The next important property is the first non-zero entry from, the left the first means from the left the first nonzero entry in each R i is 1 for 1 less than or equal to i less than or equal to row; the first rows are the non-zero rows and the first entry is always 1, in this non-zero rows appear. Suppose, R i has it is first nonzero entry in column k i, the column index is k i. What does that means, if we take the matrix this is the it h row that is non-zero and this is the k i th column and that must be one and all the entry before that must be 0 that means B i j must be 0 for j equal to 1 2 up to k i minus 1and when I came to k i i get a 1.

(Refer Slide Time: 27:20)



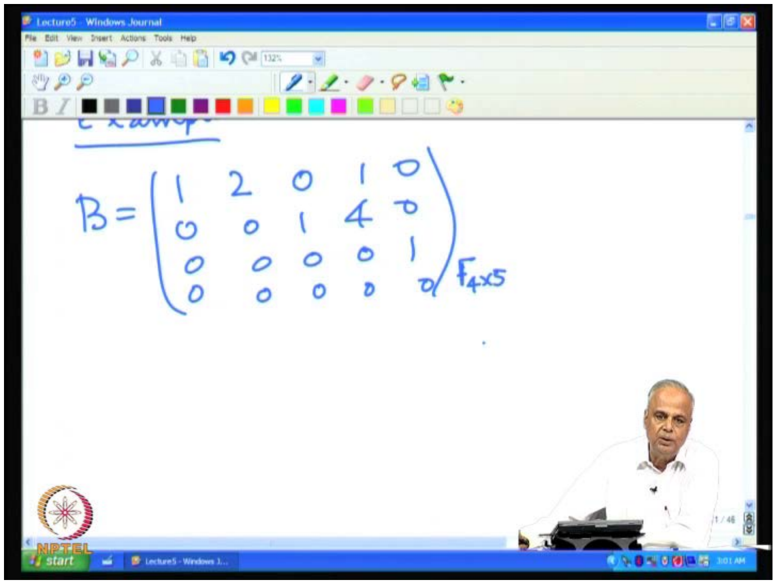
This is what we mean by say the R i has, it is first non trivial entry in the k i column then we observe that the nonzero entry 1, that pivotal 1 keeps moving to the right, if i is less than j that column in which that first non trivial entry corresponding to row i appears in k i, the first non-zero entry corresponding to row j appears in column k j. Since, they keep moving to the right k i must be less than k j, the pivotal once as we go down the matrix the pivotal, once keep moving towards the right.

(Refer Slide Time: 28:35)



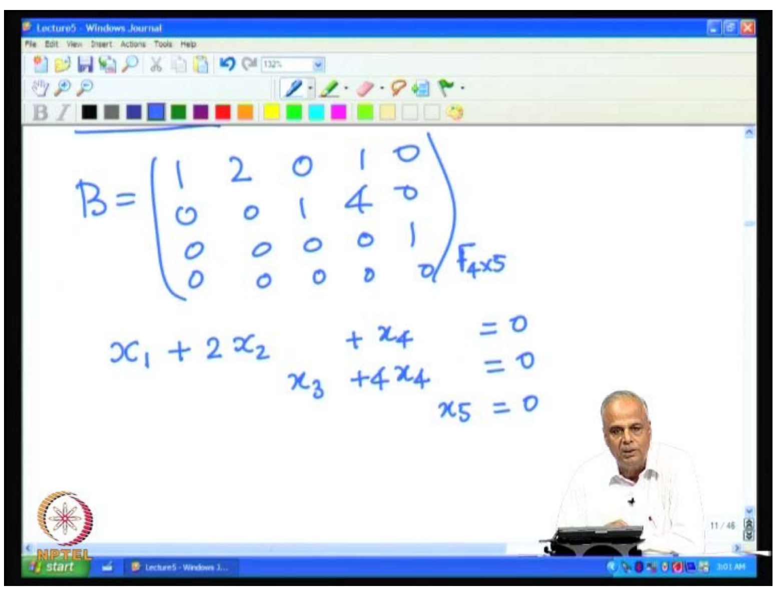
We have the i th row and we have the pivotal 1 in the k i th column, we should have all the entries in the pivotal column other than, this 1 to be 0. What are that mean, if we look at the k i th column and if we look at any row j, that must be 0. If, j is not equal to k i that is, this must be 0 that is, b 1 k i, b 2 k i must be 0, when you k i we must get 1 and again, when you go out of k i range you must get 0, the column supporting, the pivotal 1 must have all other entries. This is the general structure of a row reduced echelon matrix. The question is, what such a row reduced echelon matrix, it is easy to solve the homogenous system g x equal to theta m, if B is in row reduced echelon.

(Refer Slide Time: 30:00)



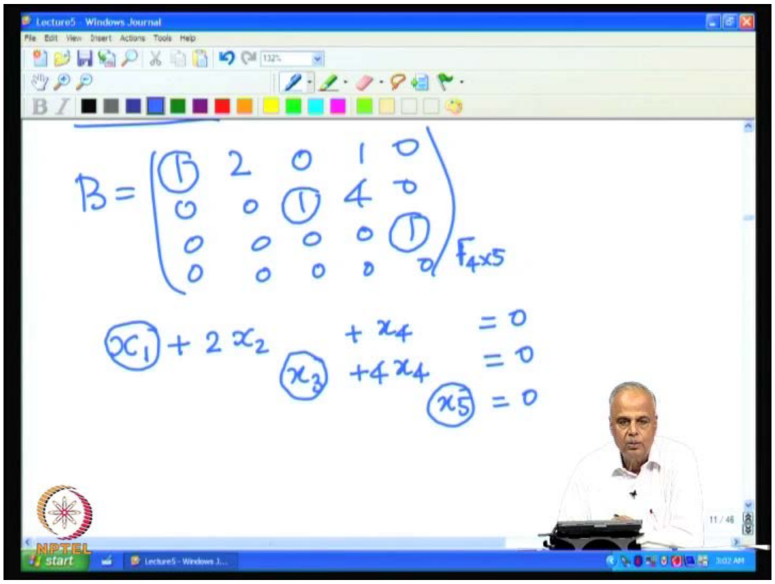
We shall see this part, let us first illustrate this within example, this is the same example. We started with this is again, a matrix which has 4 rows and 5 columns as, we had already observed.

(Refer Slide Time: 30:40)



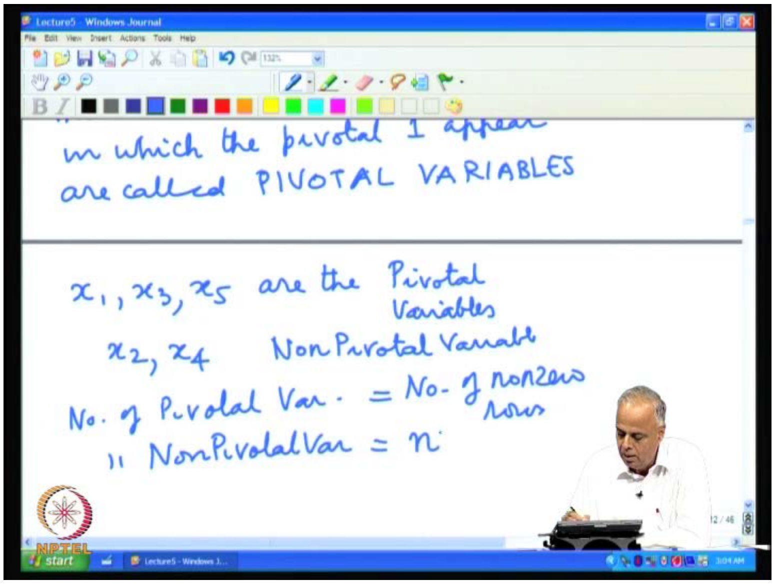
This matrix is in row reduced echelon form, let us look at how the homogenous system looks like, in this case if, we look at the homogenous system corresponding to this, it is x 1 plus 2 x 2 plus x 4 equal to 0. That equation corresponding to, the first row the second row x 3 plus 4 x 4 equal to 0. The third row is x 5 equal to 0. The last row, these just 0 equal to 0 effectively therefore, there are only three equation the number of effective equations is equal to the number of nonzero rows in that row reduced echelon form.

(Refer Slide Time: 31:18)



The system the variable corresponding to the first pivotal entry is x 1, the variable corresponding to the second pivotal entry it appears in third column and therefore, it is x 3. The third pivotal entry appears in the fifth column and the variable corresponding to that is x 5, the column index of the pivotal entries give us. What are known as the pivotal variables, the variables corresponding to the columns in which the pivotal variables, the pivotal 1 is appear are called pivotal variables.

(Refer Slide Time: 32:43)

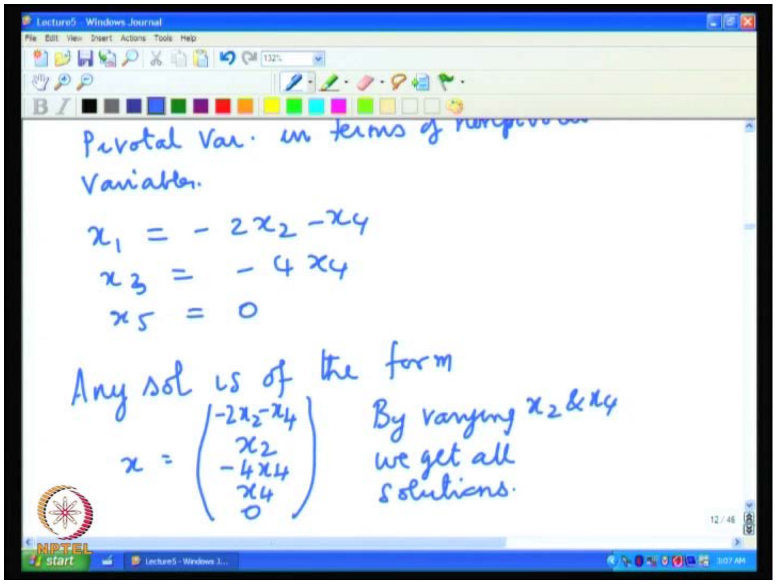


This example the pivotal variables are x 1, x 3, x 5 are the pivotal variables. Obviously, the other variables, there are totally 5 the remaining variables are x 2 and x 4. They are called non pivotal variables. The number of pivotal variables is equal to the number of nonzero rows and the number of non pivotal variables.

It is n minus number of nonzero rows let us get back to the system. We have identified these 3 pivotal variables, the important part is each 1 in this pivotal variables appears exactly, in one equation the pivotal variable x 1 appears only in the first equation; the pivotal variable x 3 appears only in the second equation; the pivotal variable x 5 appears only in the third equation, therefore we can eliminate in the pivotal variable x 1 from the first equation and when, we eliminate the remaining pivotal variables will not appear. Because each equation is only one pivotal variable that x 1 can be eliminated in sums of x 2 and x 4 get to non pivotal variables.

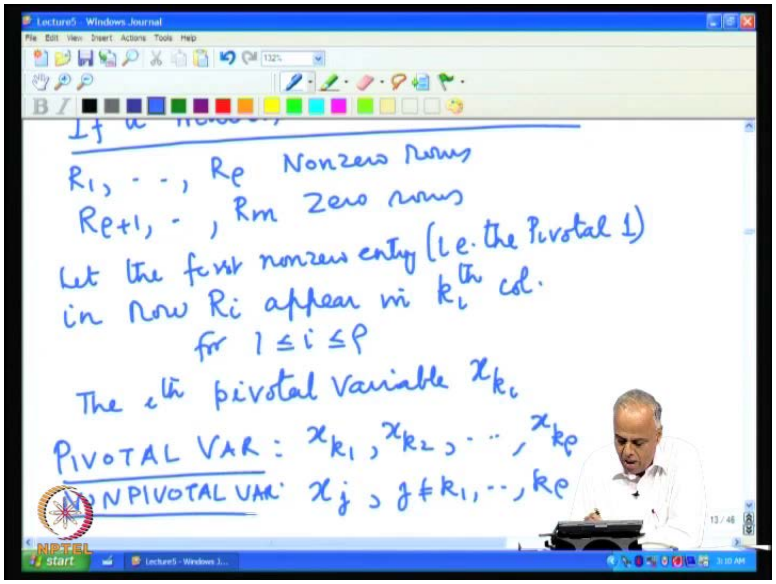
In the second equation x 3 helps you, the second equation helps you to x 3 in terms of non pivotal variables and the third equation eliminates the pivotal variable x 5, the i th equation eliminates, the i th pivotal variable k i th pivotal variable. We should called pivotal variable in terms of non pivotal variables therefore, in this example we have x 1 is eliminated in terms of x 2 and x 4 x 3 eliminated in terms of x 2, x 4 and x 5 is eliminated in terms of x 2 and x 4.

(Refer Slide Time: 35:00)



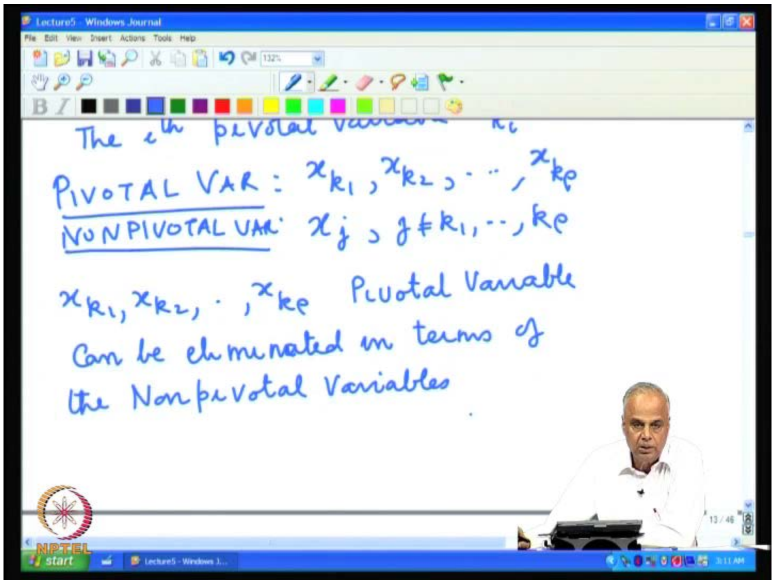
The first equation gives us the x 1 has minus 2 x 2 minus x 4; x 3 has minus 4 x 4 and the x 5 is 0. We have x 1 equal to minus 2 x 2 minus x 4 x 2 equal to minus 4 x 4 x 3 x 5 equal to 0. Whatever, values we choose for the non pivotal variables x 2 and x 4 are longer, we choose x 1 and x 3 and x 5 as per this rule then, the system is going to be satisfy, any solution is of the form x equal to we have to choose x 1 is minus 2 x 2 minus x 4 x 2 any value you can choose x 3 should be minus x 4 x 4r can be any value x 5. Thus the solution by varying the x 2 and x 4, we get all solutions.

(Refer Slide Time: 37:20)



The moral of the story, is if a matrix B is in RRE form suppose, we have a matrix which is in row reduced echelon form let us, describe it R 1 R 2 R rho are the nonzero rows R row plus 1 etc. R m is the 0 rows. Suppose, we have a matrix in row reduced echelon form and we have row nonzero rows and n minus m minus row 1 rows each nonzero row produces, a pivotal variable where, does it come from it locates the column in which the first nonzero entry, that is let the first nonzero entry that is, the pivotal one in row R i appear in k i th column or one less than or equal to i less than or equal to rho, for rows beyond row there is, nothing like nonzero entry because all the entries are 0. The R i th row has the nonzero pivotal one in the k i th column corresponding to this the i th pivotal variable will be x k i. We get all the pivotal variables has x k 1 x k 2 and on x k rho and x j for j not equal to k 1 k 2 k rho are the non pivotal variables.

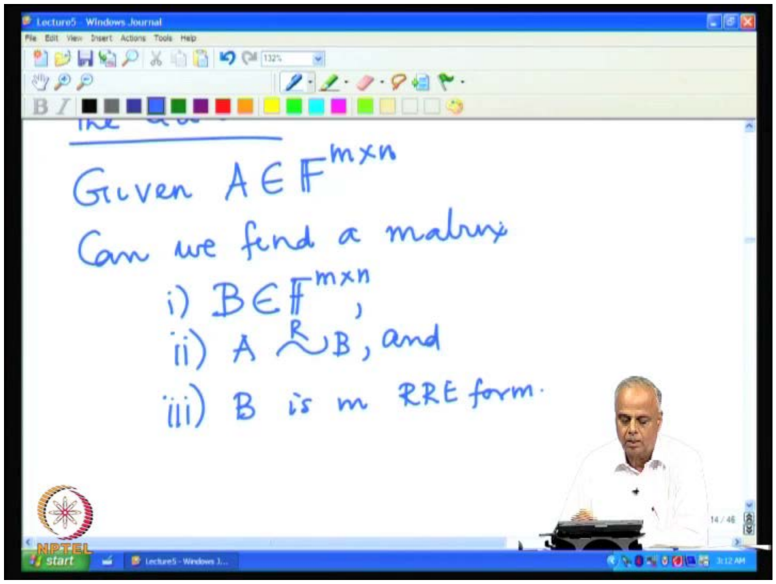
(Refer Slide Time: 40:00)



If we have a matrix B, which is m rows and m columns the out of the m pivotal variables row R out of the n given variables row R pivotal variables. Where, row is the number of nonzero columns in that row reduced matrix and the remaining are all non pivotal variables and x k 1 x k 2 x k rho. The pivotal variables can be eliminated in terms of the non pivotal variables.

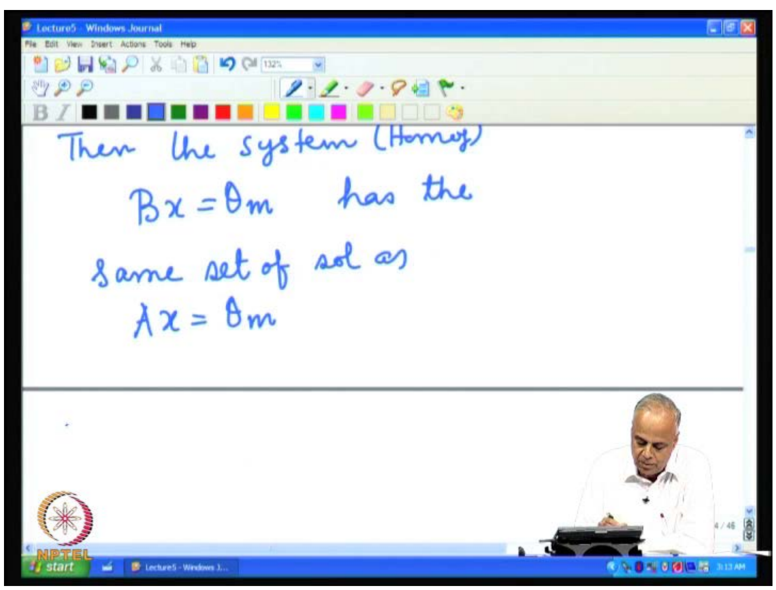
This is the general strategy, we are solving systems of equations homogenous systems of equations from a matrix, using row reduced echelon form and then eliminating terms are non pivotal variables the non pivotal variables from the chosen arbitrarily in F by varying all the possible values did, we can choose our F with that all the solutions of the homogenous system, the general strategy therefore where a matrix when, it is in row reduced echelon form these fairly.

(Refer Slide Time: 41:42)



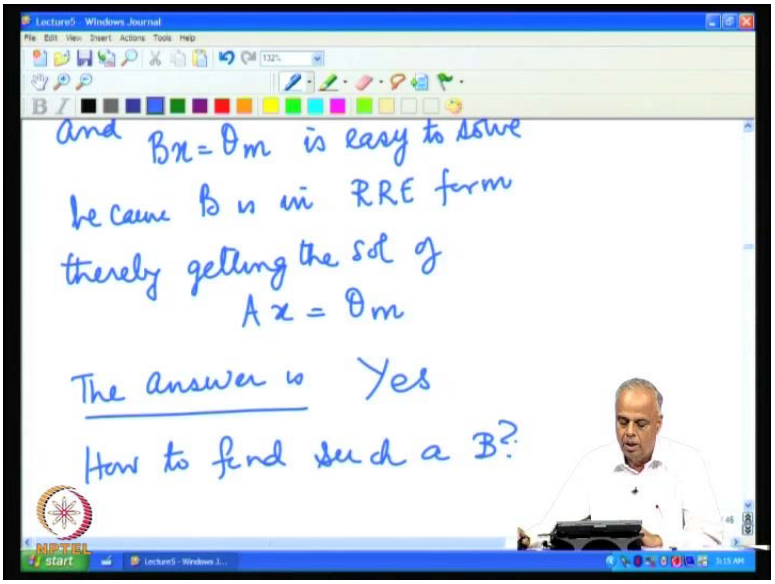
The question is given any matrix A, which is m by n can we find a matrix B which is in also an m cross n 1 2. It is row equivalent to B and C. B is a row reduce echelon form given a matrix A, we want to find out matrix B, which is have the same size; which is row equivalent to A and which is in row reduced echelon form.

(Refer Slide Time: 42:42)



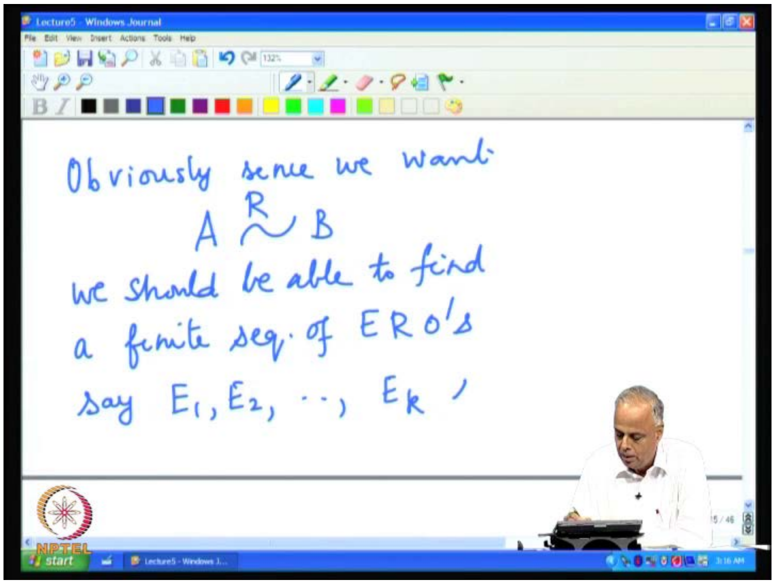
The system homogenous system B x equal to theta m, has the same set of solutions as A x theta 1 the reason is they R row equivalent and we have seen that when, you have 2 matrices, which are row equivalent they corresponding homogenous system will have the same set of solutions; B x equal to theta m has the same set of solutions is A x equal to theta m and B x equal to theta m is easy to solve because b is in row reduced echelon form.

(Refer Slide Time: 44:00)



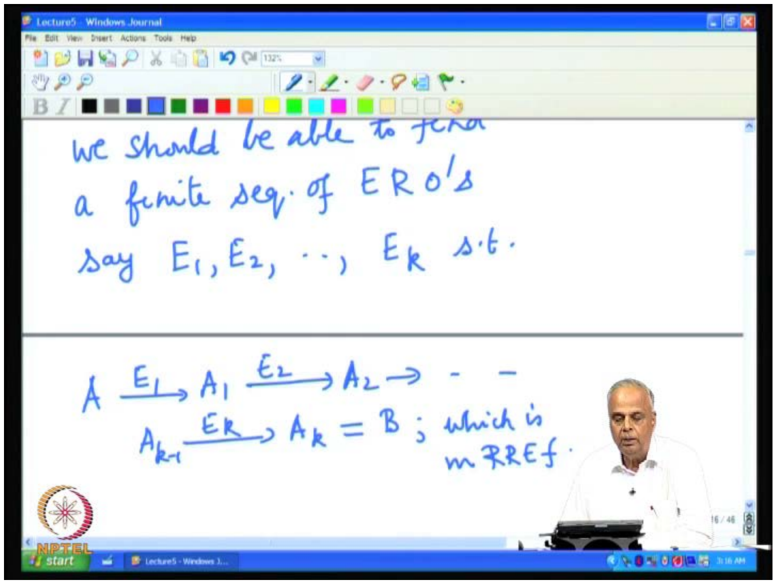
We have just said that, if a matrix in row reduced echelon form we have a systematic way of eliminating the pivotal variables in terms of the non pivotal variables and choosing the arbitrary values for the non pivotal variables over the entire F. We can recover all the solutions for the homogenous system and there by getting the solutions of A x equal to theta because now, as observed above the A x equal to theta m system and B x equal to theta m system, both have the same set of solution therefore, the general strategy would be to get such A B can may find such A B and the answer is yes.

(Refer Slide Time: 45:00)



We shall see how to find such, A B obviously since, we want A to B row equivalent to B some over that there, and we must be able to move from A to B by elementary row operations. We should be able to find yes finite sequence of elementary row operations because we want to move from A to B in a finite number of steps; a finite sequence of elementary row operations say E1, E 2, E k.

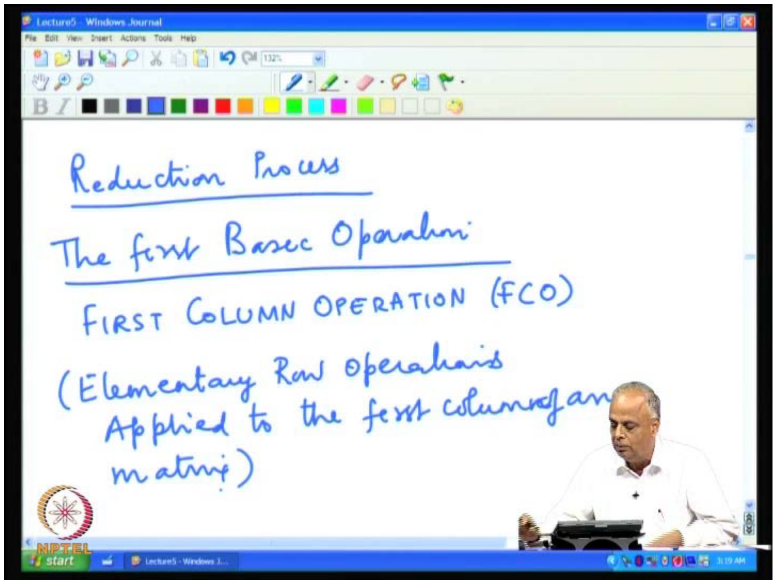
(Refer Slide Time: 46:00)



When, we go on operating this plus, the operate the row operation E 1 to go to A 1 then, operate E 2 to go to A 2 and finally, operate E k to get A k and A k must be equal to B. Which is in row reduced echelon form? Which means, we will start with the matrix A, develop a sequence of elementary row operations step by step in such, a way that after a finite number of steps. We end up with a matrix which is in row reduced echelon form.

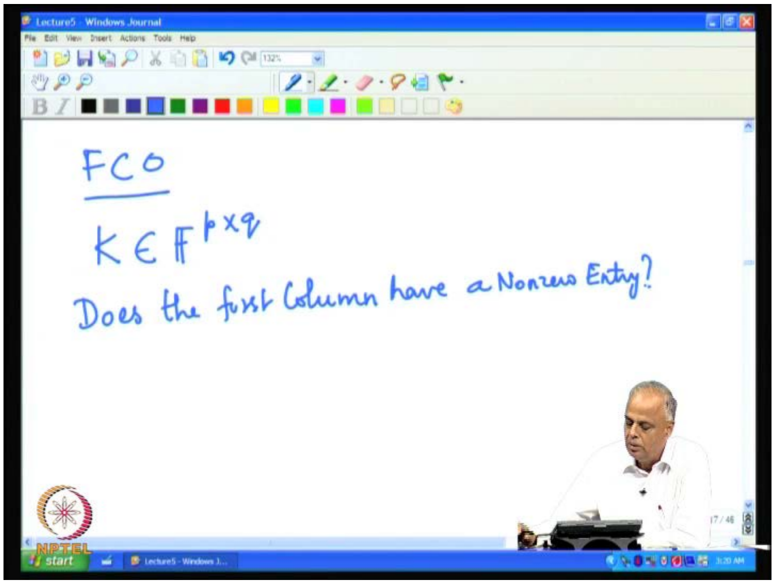
We are move from A to this by only elementary row operations, this row reduced be row equivalent to A and therefore, instead of solving A x equal to theta m, we can solve B x equal to theta m. How do we do this, we shall now look at this reduction process that if the process of finding these elementary row operations, which will eventually take us from a to a row reduced echelon form in a finite number of steps after a finite sequence of elementary row operation.

(Refer Slide Time: 48:00)



We shall see, it is a very simple operation essentially, we will be repeatedly applying one particular idea and we shall describe this to the first basic operation. We shall called the first column operation FCO is a what, we mean is obviously, it is an operation which, we are going to perform an column, which column the first column and obviously we will be talking only about row operations. It will be an elementary row operation. It is elementary row operation applied, it may be a number of them applied to the first column of any matrix given any matrix, we applied this operation on it is a set of operations all of them are elementary operation.

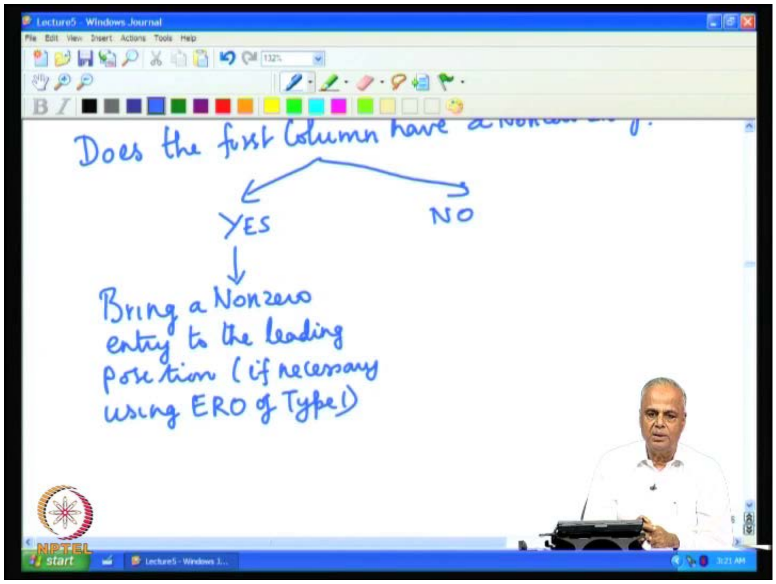
(Refer Slide Time: 49:00)



What is this operation, we will call it as FCO, the first column operation start with k any matrix k, which has p rows and q columns any general matrix p by q once a given this matrix, we first ask the following, since is going to be operating on the first column, we look at the first column and ask this question does the first column have a nonzero entry.

We ask the following the question, does the first column have a nonzero entry, we are gave any matrix. We just focus on the first column and see whether, A for somewhere, some entry is nonzero obviously, and we will get 2 types of answers possible for this yes and no. We shall described, what we shall be doing if, the answer is yes and what we shall be doing if, the answer is no; if the answer is yes what it means is, there is a nonzero entry in the first column. It may be on top or the top entry may be 0 the next may be also 0.

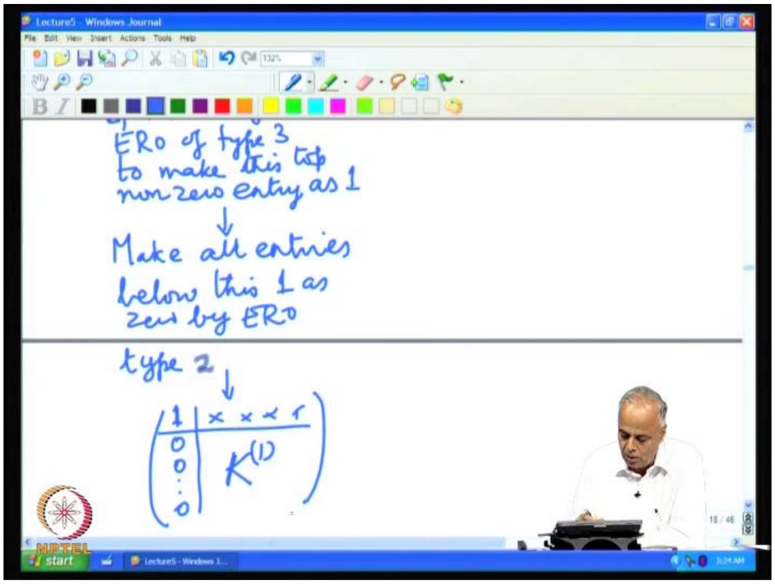
(Refer Slide Time: 50:45)



Somewhere, there will be a nonzero entry if, it is already on top do not worry. If, it is not there will be somewhere, else a nonzero entry bring it to the top how do I bring it you have to do only a row exchange bring a nonzero entry. When, we say top we mean the leading position first column, first row to the leading position. We may have to use an elementary row operation of interchange of rows to achieve this, if necessary using ERO of type one which is the row exchange the first top in this operation scan the first column see if, there is a nonzero entry bring one of them to the top position.

If necessary by using row exchanges, we may ask then there are many nonzero entries which one should have break theoretically, it is which nonzero entry will bring to the top but in computational methods from the point of a of error control. It is always better to bring that entry which has the largest magnitude to the top position hamming brought that, entry to the top position. It may be 46; it may be 32; it may be a minus 5; it may be root 2. Some, nonzero entry as come to the top we would like to have a standardization there, we shall make that one how do I make that one if, it is already one at one that do anything. If it is 15 in order to make it one I will divide it by 15 what are that mean, you have to multiply that top row by 1 over 15 and that is an elementary row operation of type 3.

(Refer Slide Time: 52:00)



If necessary, use ERO of type 3 to make this top non-zero entry as 1. So, what we have achieved is, we have look at the first column. We are brought a one to the top most position below that 1 there may be 0; there may be non-zero entries. We are standardize will clean up and make all the entries below; this 1 as 0 in order to do that all we have to do is multiplying a suitable multiple of the first row and subtract from here suppose, we had a four in the fourth row.

We multiply, the first row by 4 and subtract we get a 0 make all entries below; this 1 as 0 by ERO type 2. When we do that the first row first column, we had been come like this we have brought a 1, we have brought some non-zero entry to the top made it as one and brought everything else EROs and then, there will be all kinds of entries here and this sub matrix we will call as k 1 this is, what we do in the first column operation. If the answer is no if the first column did not have any 0 entries. We do nothing matrix will be of the form 0 0 0 0 0 first column and what is remain there we call as k one in both the cases the matrix k 1, then we got in this situation and the matrix k 1. We got here both are 1 column less than what we started with we are an n by matrix.

These people will have q minus one columns the first stage of this reduction operation if, this first column operation which brings the first column to a standard format, where all the entries are 0 as is shown here or all the entry are 0 below the one under top most position form why like these two formats if this standard format that will first column operation brings the matrix 2, what we do from this stage on that is the next stage of the reduction process to vary this form which, we will look at in the next lecture.