Advanced Matrix Theory and Linear Algebra for Engineers

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Lecture No. # 04

Linear Systems-Part 1

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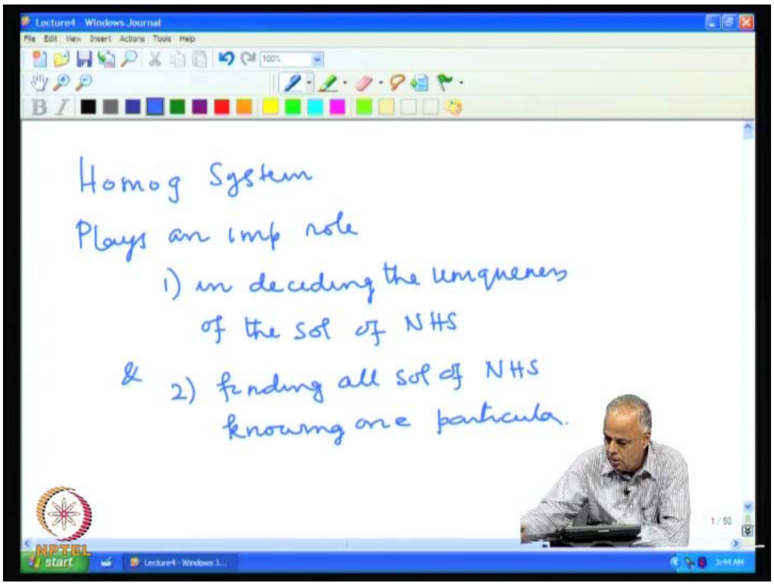

In the last lecture, we saw that the Homogeneous System plays an important role. 1 - in deciding the uniqueness of the solution of the Non Homogenous System; and finding all solutions of Non Homogeneous System knowing one particular solution of the Non Homogeneous System.

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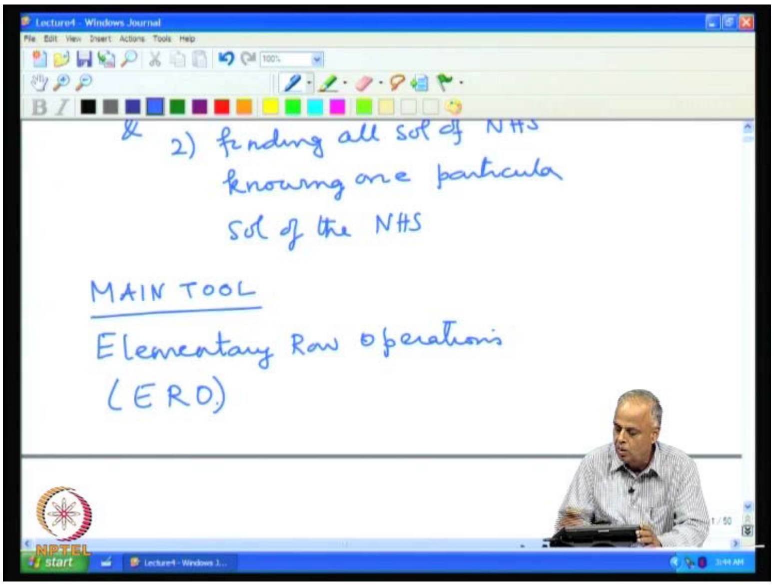

We said that the main tool for this, are the Elementary Row Operations, the so-called Elementary Row Operations, which we said we will denote by ERO for short. There were three types of EROs, which we introduce.

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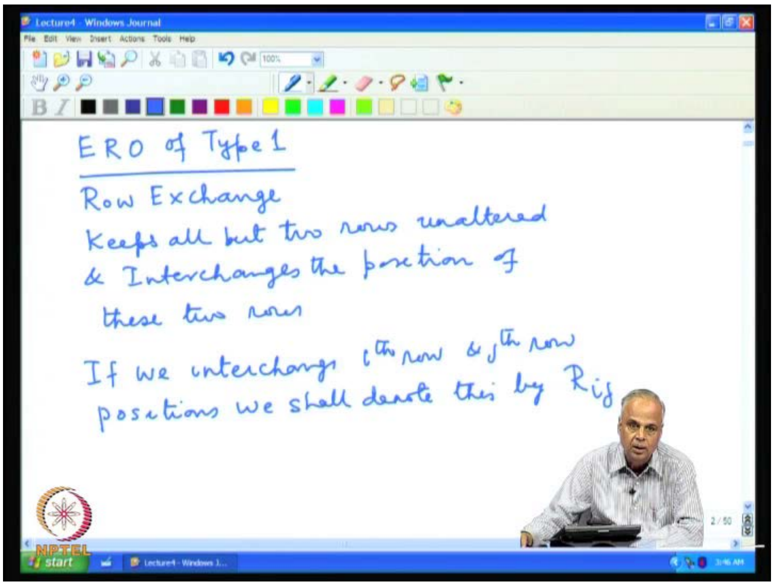

So, we shall discuss this one by one; the first type - ERO of type 1. This is called Row Exchange, what does this do? Keeps all, but two rows unaltered, then interchanges the position of these two rows. For example, if we interchange i th row and j th row positions, we shall denote this by R i j; R stands for row, and i and j are exchanging positions.

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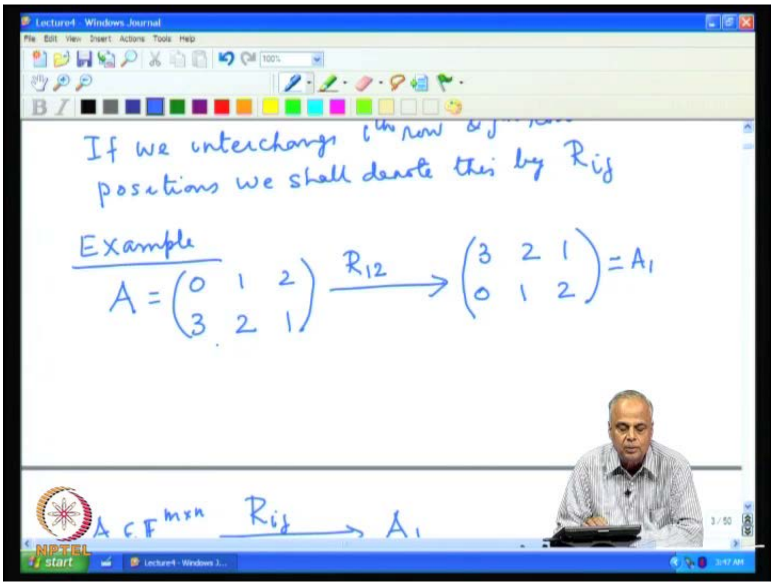

For example, if we take the matrix A 0 1 2 3 2 1 then, when we put the arrow and write R 1 2, what we mean is, we are applying an Elementary Row Operation in the matrix A. The operation is row exchange and it is 1 and 2 say the first and the second row are going to be interchange. This results in 3 2 1 0 1 2, the resulting matrix will denote by A 1. So, in general, if we have a matrix A belonging to F m plus n and we apply the i th row j th row interchange on that and what results to the matrix A 1? This is what is known as the Elementary Row Operation of type 1? Some simple properties we will observe.

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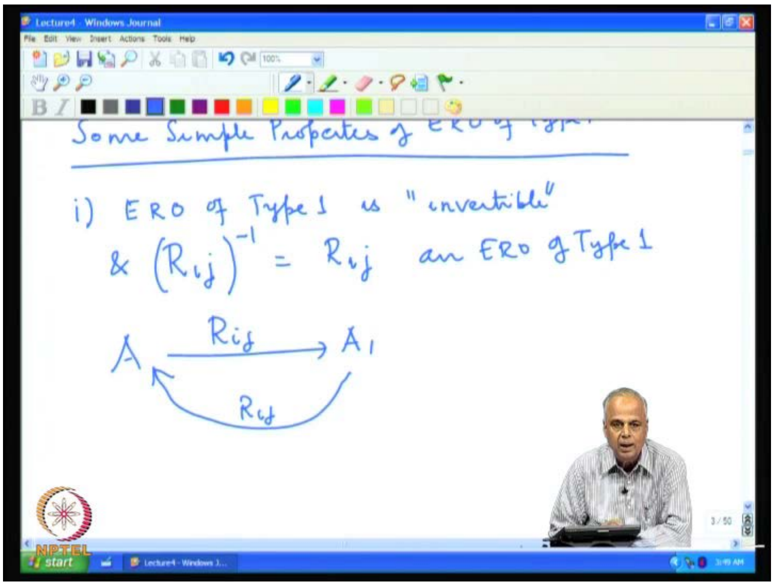

Some simple properties of ERO of type 1. Clearly, in this example, if we again interchange the first and the second row, we will get back to the original matrix. In other words, the interchange is a reversible process or an invertible process.

So, the first property we observe is, but the ERO of type one is invertible and if we have the transformation or the row operation as R i j, it is inverse is again R i j. If you want to anal the exchange of the i th and j th row, you have to again exchange the i th and the j th row. So, that they get back into the original position. So, the ERO of type 1 is invertible and the inverse is also an ERO of type one. So, the inverse of an ERO of type one is an ERO of type one. So, if you have A going to A 1 under the ERO of type 1 of exchange of the i th and the j th row and then, A 1 will go to A again by an interchange the i th and the j th row.

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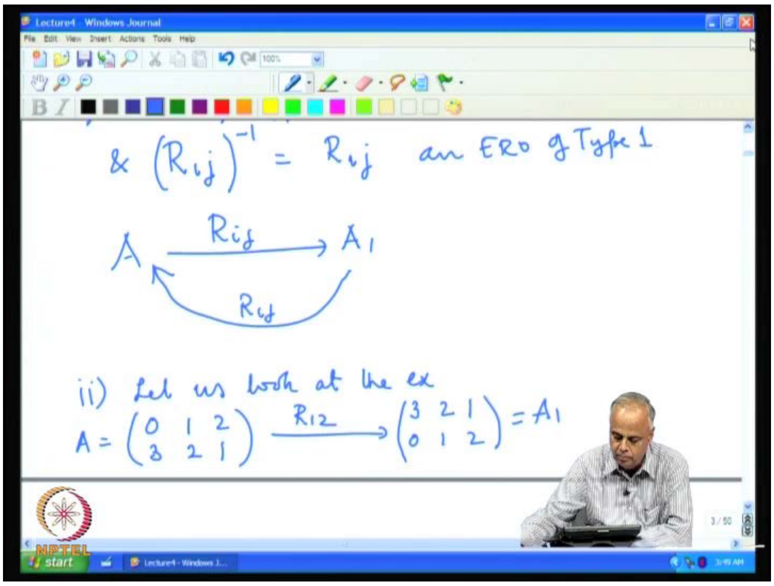

So, this is the first observation we make. Now the second one, we will first look at the example we had. We had 0 1 2 3 2 1 was our matrix and we applied R 1 2 a and what we get was 3 2 1 and 0 1 2. This we called as A 1.

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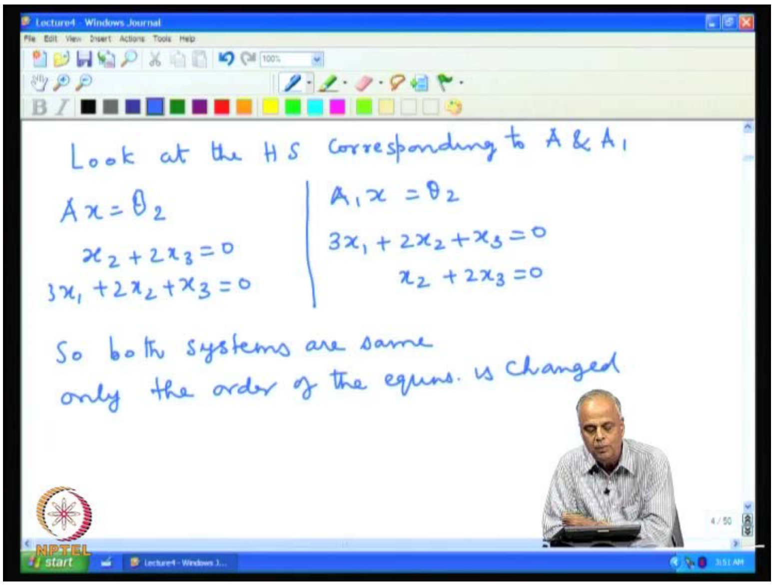

Now let us look at, the Homogenous System look at the Homogeneous System corresponding to A the original matrix and A 1 the matrix, that has been obtained from A by applying an ERO of type 1. So, A x equal to now m. This is 2 by 3 matrices. So, m is 2, so A x equal to theta 2 corresponds to x 2 plus 2 x 3 is equal to 0, 3 x 1 plus 2 x 2 plus x 3 equal to 0. The Homogeneous System corresponding to A 1 is 3 x 1 plus 2 x 2 plus x 3 equal to 0, x 2 plus 2 x 3 equal to 0. We observe, that both are the same systems by the same equations appear in this system, only the order in which we write this equations are different. So, both systems are same, only the order of the equations is changed. What does this imply? This implies that, whenever something is a solution for the system A x equal to theta 2, we will also be a solution for the system A 1 equal to theta 2 and vice versa.

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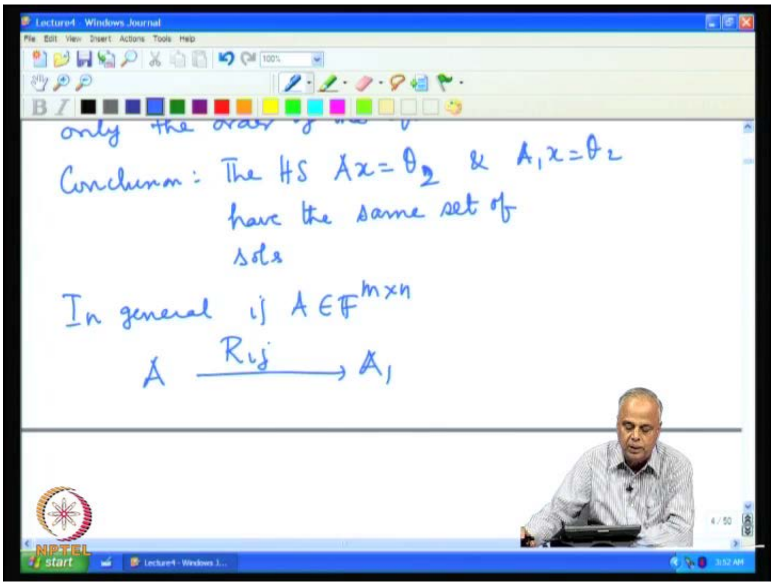

So, the conclusion, we can draw is that, the Homogeneous Systems A x equal to theta m theta 2 and A 1 x equal to theta 2 have the same set of solutions. the same set of solutions Now, it is easily seen that, the same thing happens in the general situation of an m by n matrix, because all equations except two are put in the same positions and two equations are interchange in that positions and therefore, both of them will have the same set of equations. So in general, if A is an m by n matrix and we get A 1 from A by an Elementary Row Operation of type one.

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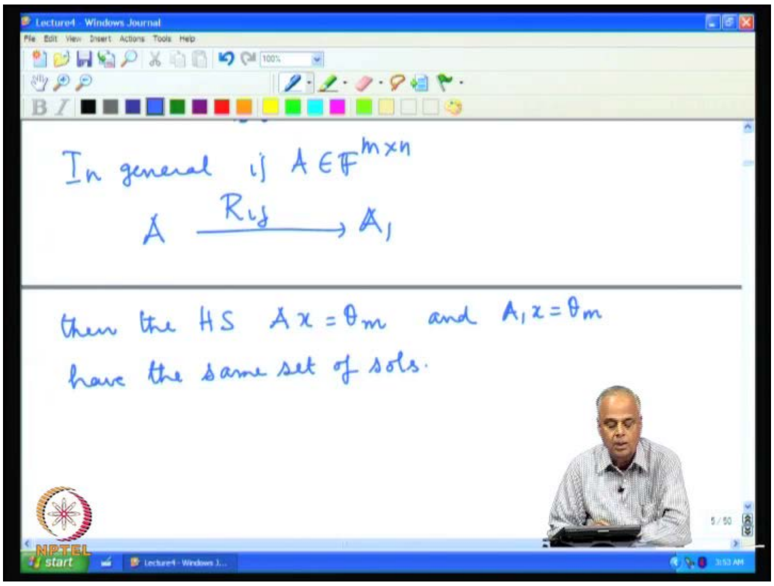

Then, the Homogeneous System A x equal to theta m and A 1 x equal to theta m have the same set of solutions the same set of solutions and therefore, we can solve the system A x equal to theta m or the system A 1 x equal to theta m. This is the second simple property of the Elementary Row Operation of type one. What it says is? By applying Elementary Row Operation on the matrix A, you are not destroying the set of solutions; you are not removing any solution; you are not creating any new solution; the same set of solutions remain for the resultant matrix also.

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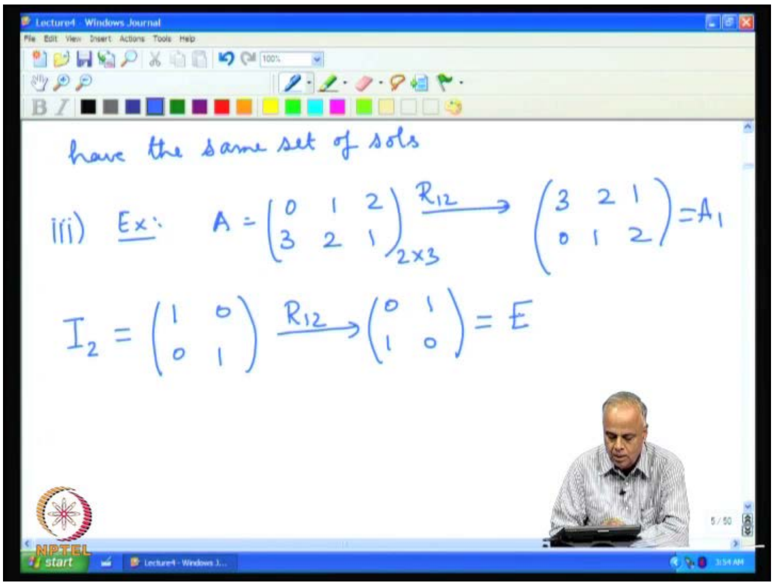

The third observation we make is, let us get back to our example, we had A equal to 0 1 2 3 2 1, which is a 2 by 3 matrix and when we applied R 1 2? We got 3 2 1 and 0 1 2 we call this matrix A 1. Now, let us look at the value of m in this case it is 2. So, we look at the 2 by 2 identity matrix, which is this and we apply the same Elementary Row Operation which we apply to A to this matrix and we get this matrix 0 1 1 0 which we will call as E.

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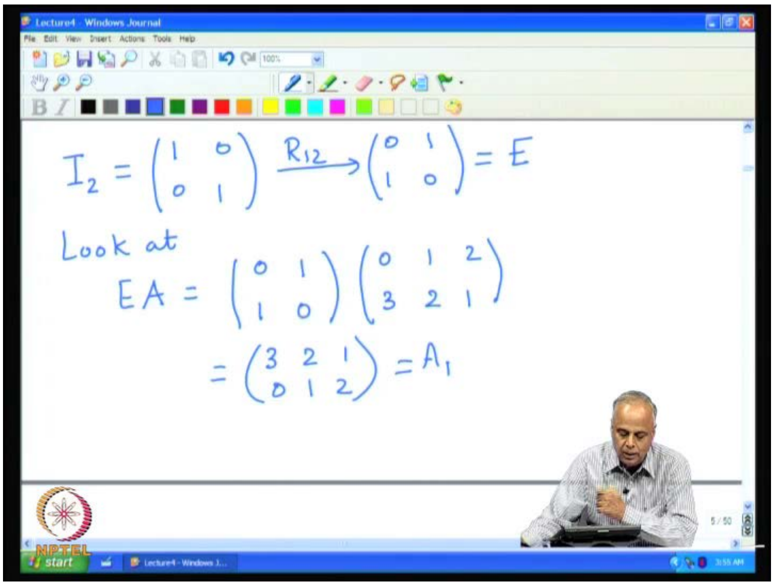

Now look at, E times A that is pre multiply the matrix A by E which is 0 1 1 0 times 0 1 2 3 2 1. When we carry out the multiplication? We get this, which is precisely A 1. In other words, the effect of operating with ERO type 1 on A, if the same as pre multiplying the matrix A by E, where E is obtained by effecting in the same transformation on the 2 by 2 matrix.

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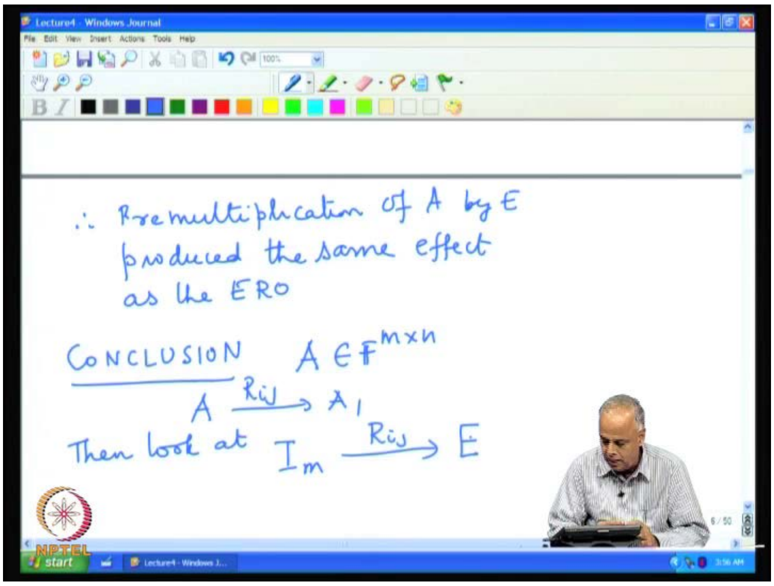

So, pre multiplication by E; pre multiplication of A by E produced the same effect as the ERO . Now, this is a now we consider this is in general through. So, the conclusion is, if we have an m by n matrix analogously in the general situation and if A 1 is applied from A by R i j then, look at I m the m by m identity matrix apply R i j the same Elementary Row Operation that, we did on A to the identity matrix I m to get A matrix E.

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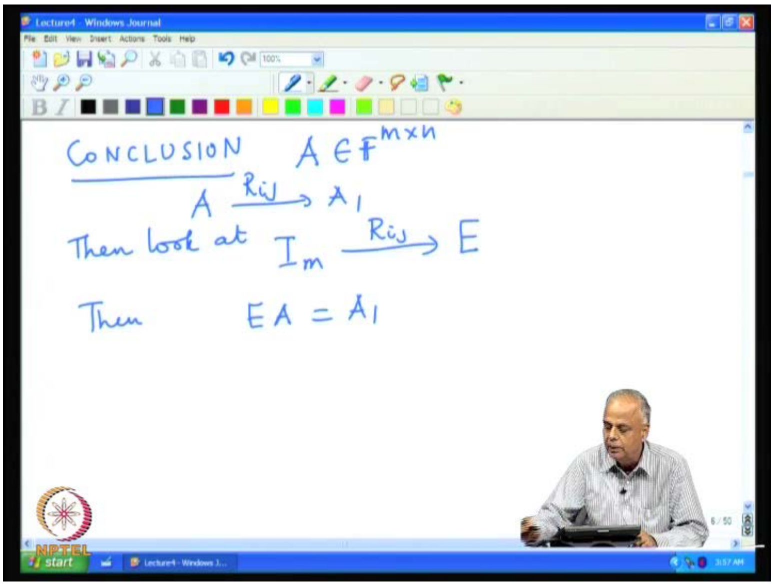

Then, E A will be equal to A 1. So, pre multiplication of this type of matrix effects the same change as applying the Elementary Row Operation. So, this is the first type of Elementary Row Operation that we consider.

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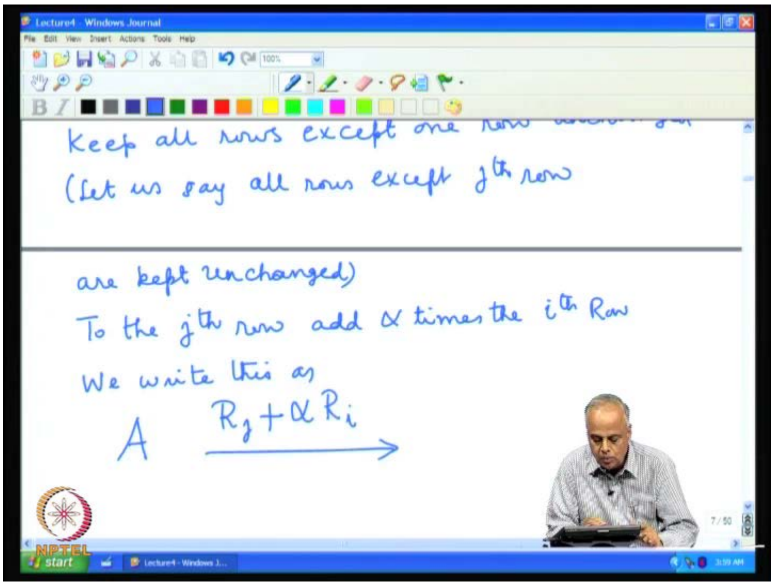

The second type of Elementary Row Operation here, what we do is? Keep all rows except one row unchanged. We keep all rows except one row unchanged then, let us say all rows except j th row are kept unchanged. Then, we are going to effect a change only in the i th row. To the i th row to the j th row add alpha times the i th row. We write this as R j to the j th row, we are going to add alpha times the i th row. So, we take a matrix A and we apply this transformation, the j th row is replaced by the j th row plus alpha times the i th row to get a pure matrix. This is the Elementary Row Operation of type two and the notation, we use is R j plus alpha R i.

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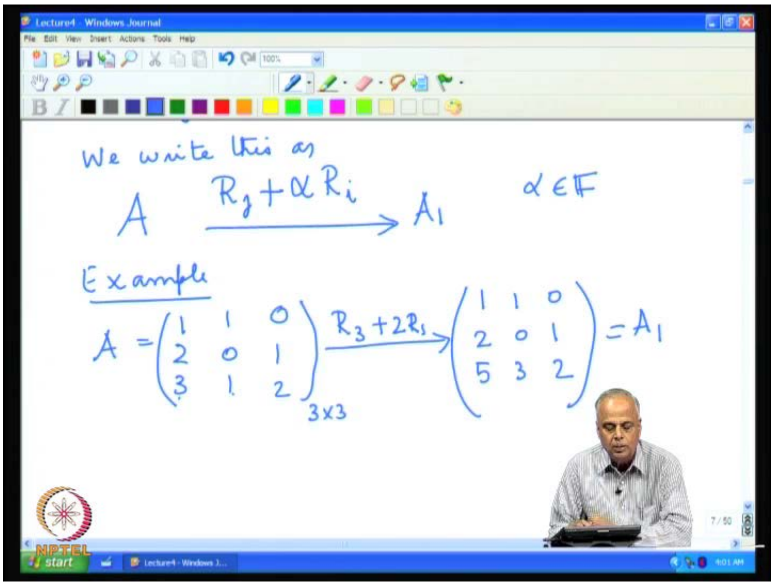

Let us look look at an example, let us say A is equal to 1 2 3 1 0 1 0 1 2 this is our matrix. Now, is a square matrix any square or rectangular matrix you can perform this. So, now let us look at the transformation then, we write R 3 plus 2 R 1. What we mean here is it will be third row alone that is undergoing a change, because R 3 plus so R 3 is going to change R 1 and R 2 are going to be the same. So, R 1 and R 2 are unaltered and R 3 is going to change and how is this going to change to the third row, we are going to add twice the first row. So, you get 3 plus 2 5, 1 plus 2 3 and 2 plus 0 2 and this matrix we call as A 1. So, here is a example of Elementary Row Operation of type 2 in general, we denote it by R j plus alpha times R i. If the j th row undergoes the change by adding alpha times the i th row to that alpha can be any number in F. It can be positive or if. In fact, is real alpha can be positive or negative or 0, if F is complex it can be positive negative real or it could be pure imaginary or it could be any complex number.

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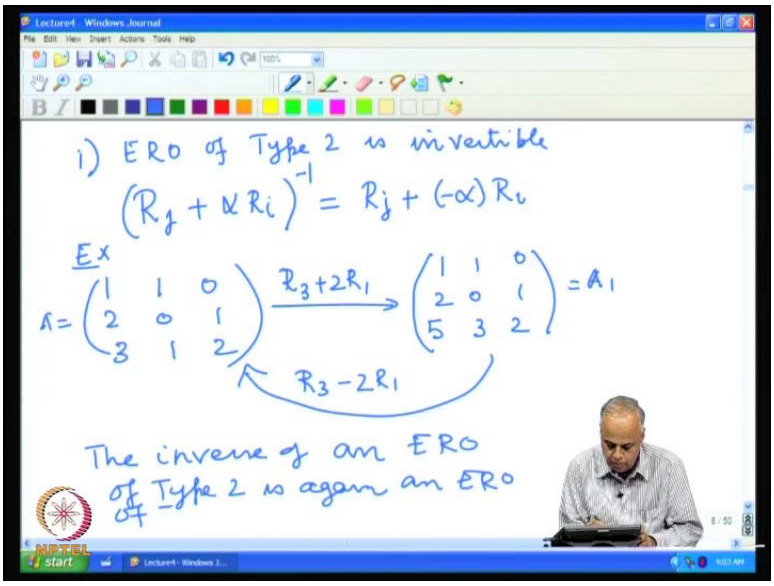

Now, let us look at simple properties of ERO of type two. Once again, we observe that if we change the j th row by adding alpha times the i th row to anal this change to get back to the original position, we have to remove whatever we had added. So, we have to subtract alpha times the i th row. So, we have that first property that ERO of type 2 is invertible. That is if we have R j plus alpha R i to reverse that process from the j th row whatever was added must be removed now.

So, it should be minus alpha times R i for example, in the above example we had 1 1 0 2 0 1 3 1 2 and this was the matrix A and we had R 3 plus 2 R 1 and we got 1 1 0 2 0 1 5 3 2. Now, to anal this if we do R 3 minus 2 R 1 will get back to A. This is what we mean by saying that the ERO of type two is invertible notice that the inverse R j plus minus alpha times R r i is also an elementary operation of the type 2, because again we are adding a multiple of some row to another row. So, the inverse of an ERO of type 2 is again an ERO of type two. That is the first fundamental property of ERO s of type 2.

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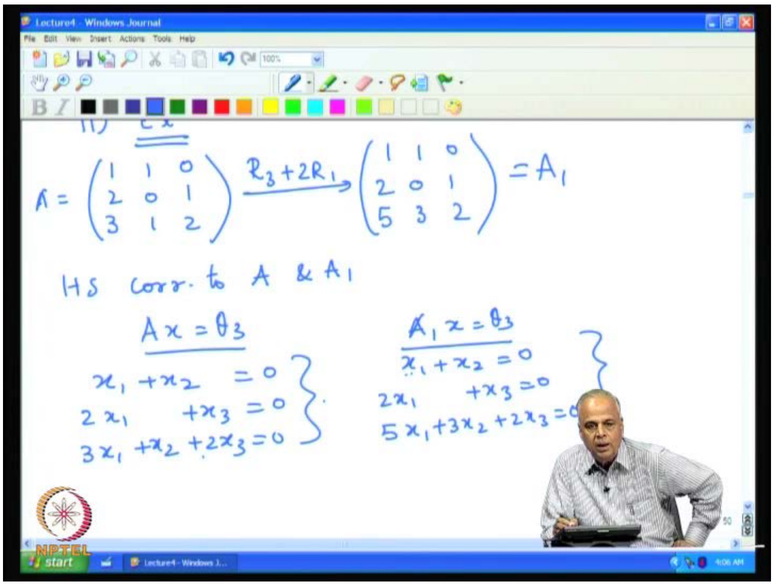

Let us look at previously, when we discuss with ERO of type 1, we have found that it as invertible and inverse of ERO of type 1, again we have found that when you doing with ERO of type 2, it is invertible and inverse is again ERO of type 2 and when we discussed ERO of type 1. We said that it does not alter the solutions of the Homogeneous System. Thus, we would like to know whether the same thing is true here also. Let us again, we look at our example, we had 1 1 0 2 0 1 3 1 2 this was the matrix A and we did R 3 plus 2 R 1 to get 5 3 2 which is A 1. Now, look at the Homogeneous Systems corresponding to A and A 1.

So, we have A x equal to theta 3 now m is 3. In this case the equations are x 1 plus x 2 equal to 0, 2 x 1 plus x 3 equal to 0, 3 x 1 plus x 2 plus 2 x 3 equal to 0. The system corresponding to A 1 is x 1 plus x 2 equal to 0, 2 x 1 plus x 3 equal to 0, 5 x 1 plus 3 x 2 plus 2 x 3 equal to 0. Now, we see that if any x 1 x 2 x 3 satisfies this set of equations, it will certainly satisfy the first equation in the A 1 x, because they are same, it certainly satisfies the second equation in A 1 x, because they are the same, it will also satisfy the third equation because the third equation is nothing, but the third equation of the first set, plus twice the first equation of the first set and say the first equation is satisfied they will also be satisfied. similarly in the reverse manner.

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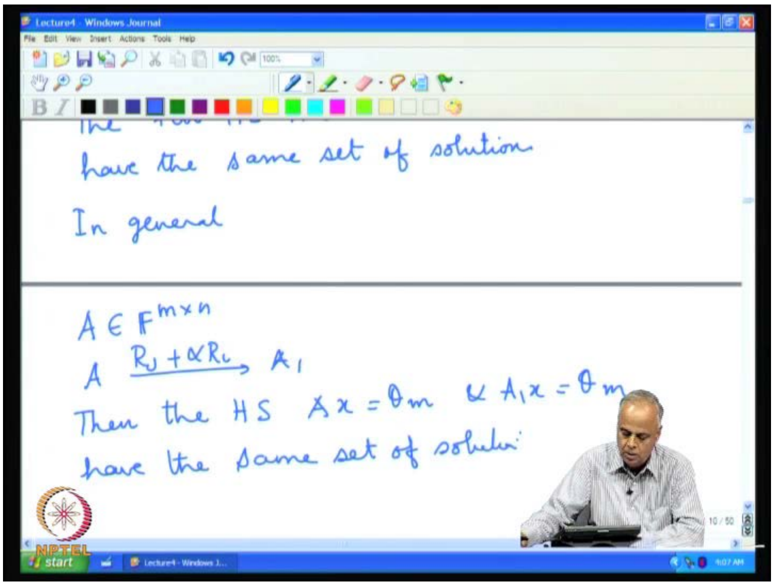

So, what we observe is that the two Homogeneous Systems, A x equal to theta 3 and A 1 x equal to theta 3, have the same set of solutions. Therefore, whether we solve A x equal to theta 3 or A 1 x equal to theta 3. So, in general analogously for any m by n matrix we see that this is true. So, in general if A is an m by n matrix and A goes to A 1 under the transformation R j plus alpha R i then, the Homogeneous Systems A x equal to theta m and A 1 x equal to theta m have the same set of solutions. Therefore, whether we solve the system A x equal to theta m or whether we solve the system A 1 equal to theta m both are equal and because you are going to get the same set of solutions.

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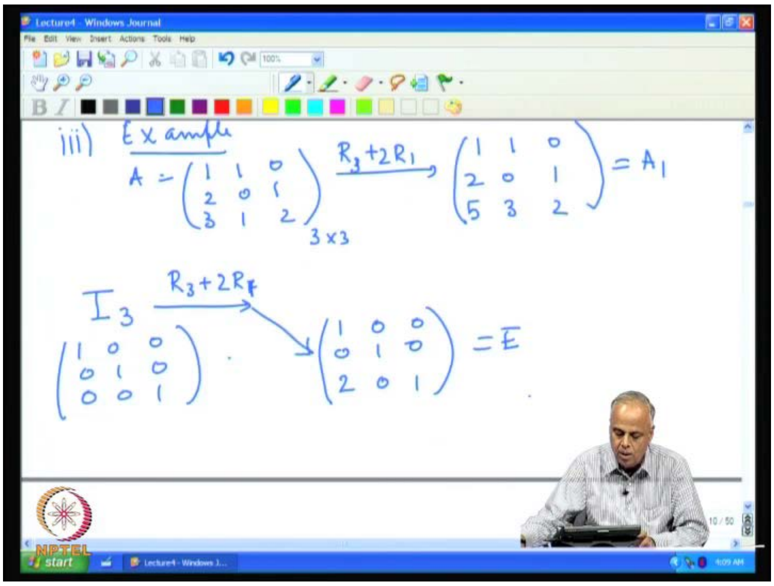

The third again, when we look at ERO of type 1, we found that the ERO of type 1 can be implemented or effected by pre multiplying the matrix by a new matrix obtained by applying the same transformation to the identity matrix. Let us, look whether that is true in this case again, let us look at the example we had we had A equal to 1 1 0 2 0 1 3 1 2 we did the operation R 1 R 3 plus 2 R 1 to get a new matrix 1 1 0 2 0 1 5 3 2. Which we called as A 1. Now, we have a 3 by 3 matrix m is 3 in this case. So, we consider the 3 by 3 identity matrix and now on this we apply the same transformation we applied as above R 3 plus 2 R 1, what do we get? I is 1 0 0 0 1 0 0 0 1. So, when I apply this transformation, I get 1 0 0 0 1 0 the first 2 rows are not a vector, the third row is going to change by adding twice to first row to it. So, we get this is what I call as E. So, when I apply on I this transformation.

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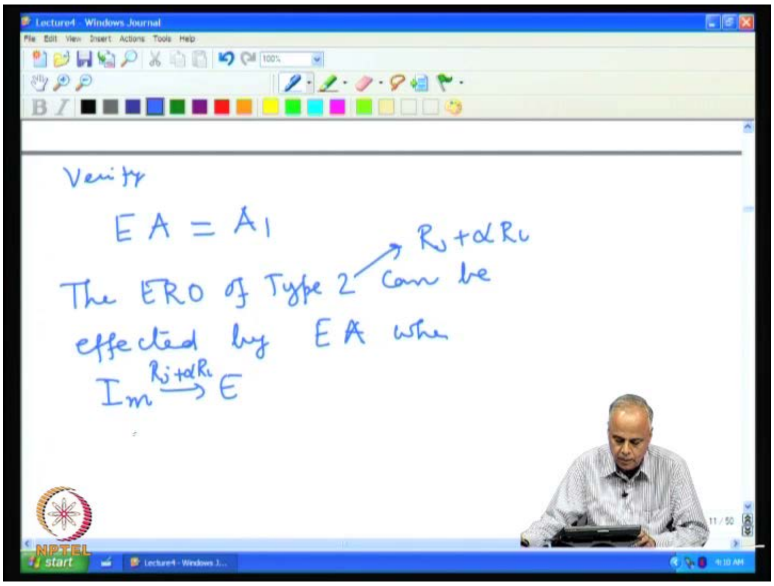

Now, easy to verify when we pre multiply E and A what we get is precisely A 1. So, thus we see that, the ERO of type 2 can be effected by a pre multiplication E A, where if the ERO type two is R j plus alpha R i, E is obtained from I m by applying R j plus alpha R.

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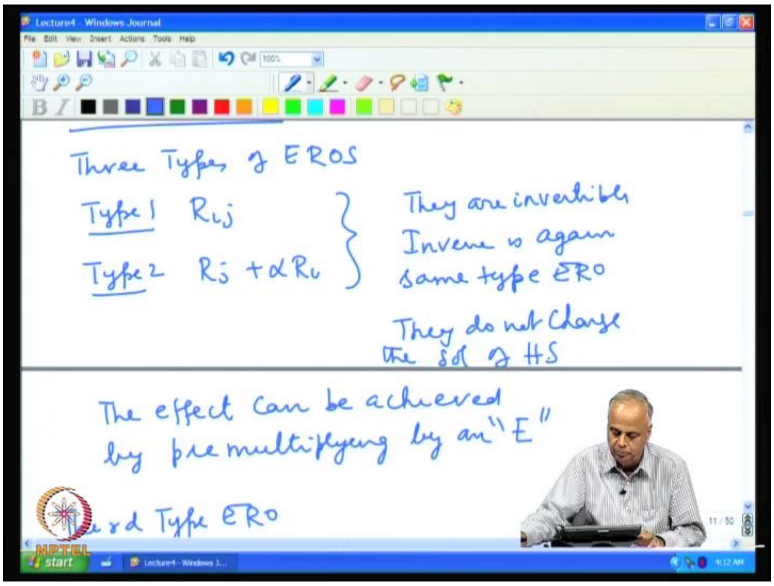

Thus, in all the three cases if we summarize see the following. We had three types of ERO s, type 1 which we simply denoted by R i j interchange of i th row and j th row, type 2 R j plus alpha R i. Now, what is the third type, what are the two these two fellows posses in common. The common thing is they are invertible; inverse is again same type ERO . The effect can be achieved by pre multiplying, you will simply write by any E understand what we mean by any by E means and the identity matrix apply the same type of ERO . The third type, we will introduce again going to posses the same property it will also be invertible. They do not change the solutions of Homogeneous System. The third type also is going to posses the same properties.

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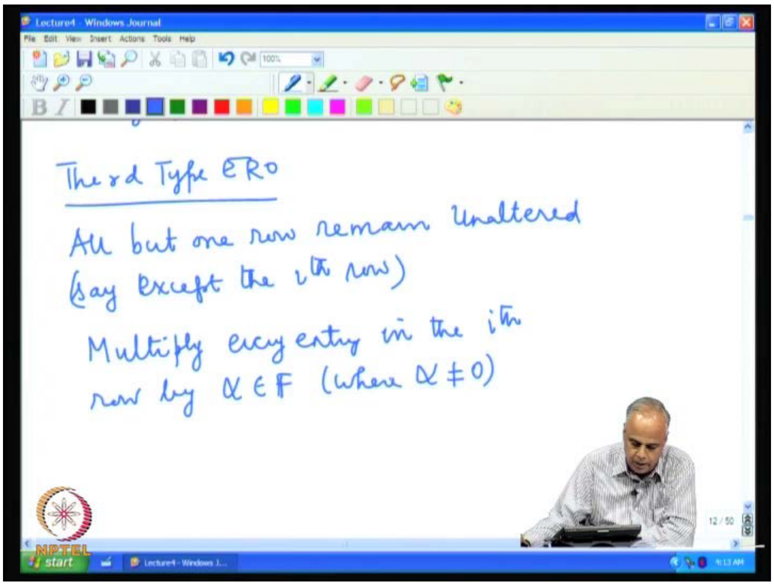

It is again on simple ERO again all, but one row remain unaltered all that row remain unaltered only 1 row undergoes a change say the except the i th row. all remain unaltered except the i th row what do we do on the i th row? It is the i th row that is going to undergo the change, but the type of change that, will make is we are going to multiply every entry in the i th row by a number alpha in F, where alpha is not 0. So, we are going to multiply every entry in the i th row by a non 0 number. This is a very important operation, later we will see brings in some normalization in the reductions that we do.

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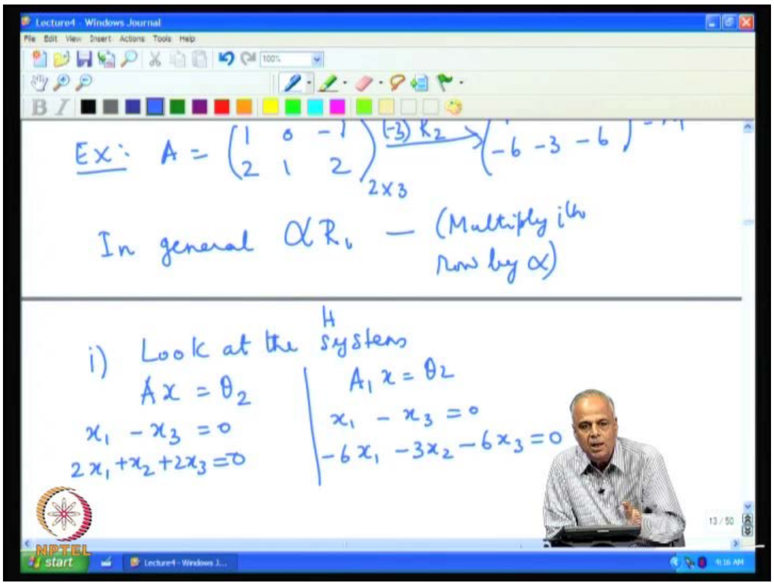

For example, let us take the matrix 1 0 minus 1, 2 1 2. Let us say we are going to change the second row, what is the change we are going to make we are going to multiply every entry by minus 3. So, what do we get 1 0 minus 1, minus 6, minus 3, minus 6 we call this matrix as A 1. So, again we look at from. So, in general let us fix the notation in general alpha R i would mean they are multiplying i th row by alpha. Where, of course alpha will be non 0.

So, again we observe some simple properties of the Elementary Row Operation of type 3. Now, look at the above example look at the systems Homogeneous Systems A x equal to theta. In this case m is 2, m is 3. So, we have to take theta m which is theta 2 and A 1 x equal to theta 2. What are these 2 systems the equations here are x 1 minus x 3 equal to 0 2 x 1 plus x 2 plus 2 x 3 equal to 0. The equations here are x 1 minus x 3 equal to 0 minus 6 x 1 minus 3 x 2 minus 6 x 3 equal to 0. Now, look at these two system, the first equations are the same and therefore, whichever satisfy the first equation on this side will also satisfy the first equation on that side and if anyone x 1 x 2 x 3 satisfy the second equation on this side it will also satisfy the second equation on that side, because this just both sides multiplied by minus 3. Similarly, if anybody satisfies the second equation on the right side, it will also satisfy the second equation on the first side, because it is only division by minus 3. That is why we want it to assume alpha not equal to 0, because we want to divide by alpha.

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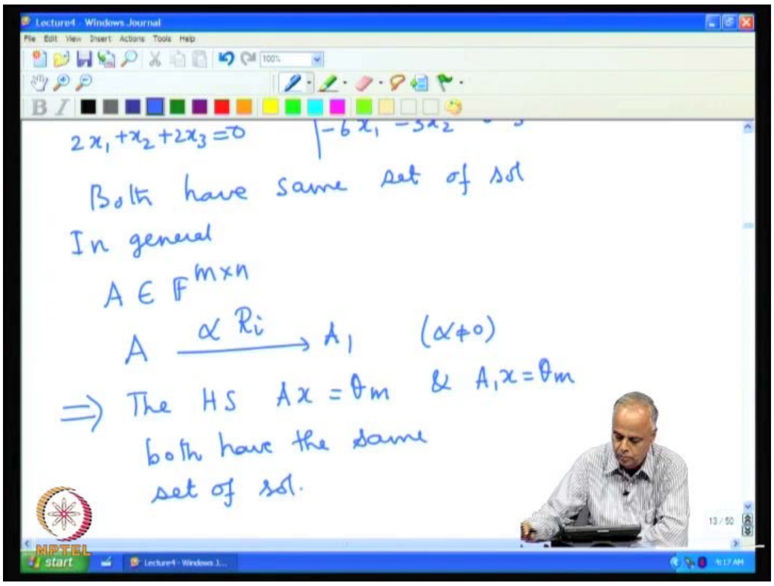

So, both have same set of solutions. Thus, see in general if, I have A matrix which is m by n and I get a from A to A 1 by multiplying the i th row by alpha i, alpha not equal to 0. Then, the Homogeneous Systems A x equal to theta m and A 1 x equal to theta m both have the same set of solutions.

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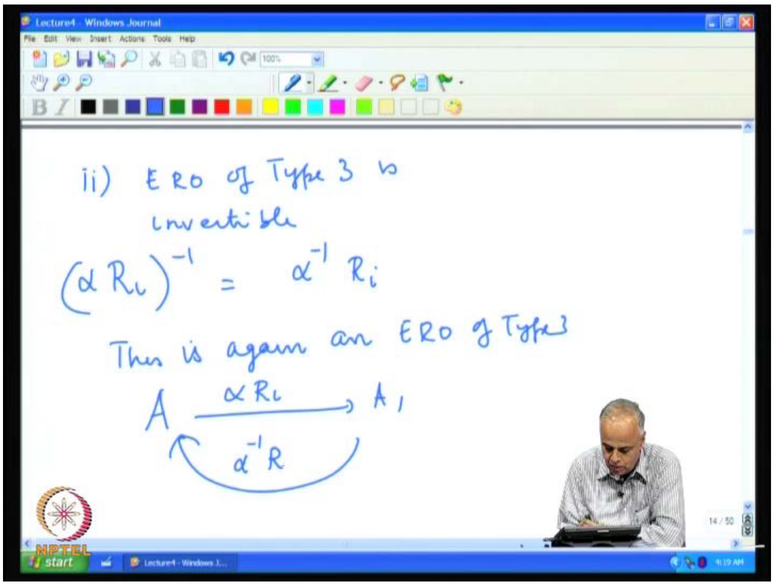

The second property, that we observe about the ERO of type 3 is the just like the other 2 ERO s, this is also invertible. ERO of type 3 is invertible, because if you obtain A 1 from A by multiplying all the entries in a particular row by a non 0 constant. We can anal this effect by multiplying by the inverse of that non 0 constant the same row. So therefore, we have if alpha R i is the row operation, type 3 then, it is inverse will be multiply the i th row by alpha and now this alpha inverse make sense, because we have assume alpha not equal to 0. That is why it is important to state in the Elementary Row Operation type 3, that it is multiplication of a row by an non 0 alpha in F. So, the ERO of type 3 again an ERO of type 3 because is again multiplying the i th row by some number. So, this is again an ERO of type 3. So, therefore, if you have A and we get A 1 by alpha R i then, will get back to A from this by alpha minus 1 R i alpha to the power of minus 1 R i.

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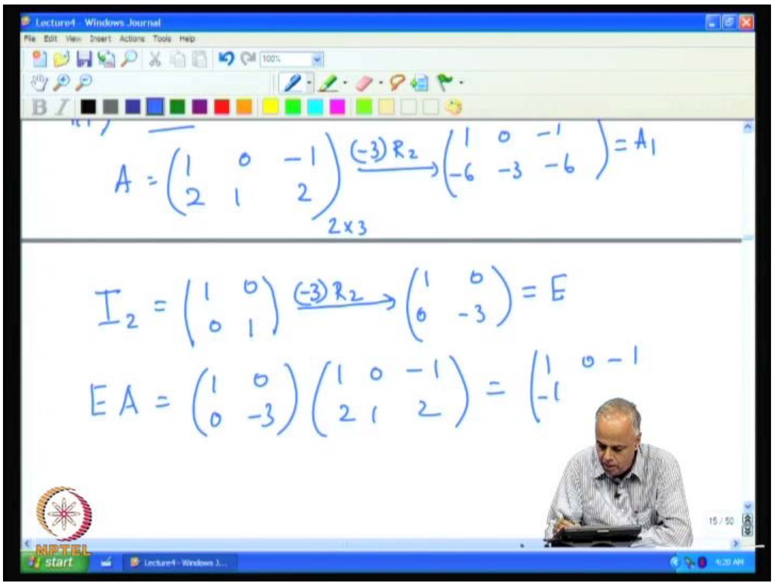

The third important property, let us look at our example A was 1 0 minus 1 2 1 2 and we applied minus 3 times R 2 to get minus 6 minus 3 minus 6. Which we called as A 1. Again as before observe this is a 2 by 3 matrix, therefore n is 2 in this case. So, we consider the identity matrix 2 by 2, which is 1 0 0 1. Suppose, we apply the same transformation on this, we get 1 0 0 minus 3 which will call as E. If, you now multiply E and A, what we are going to get is that 1 0 0 minus 3 into 1 0 minus 1 2 1 2 which is just 1 0 minus 1 minus 6 minus 3 minus 6 which is A 1.

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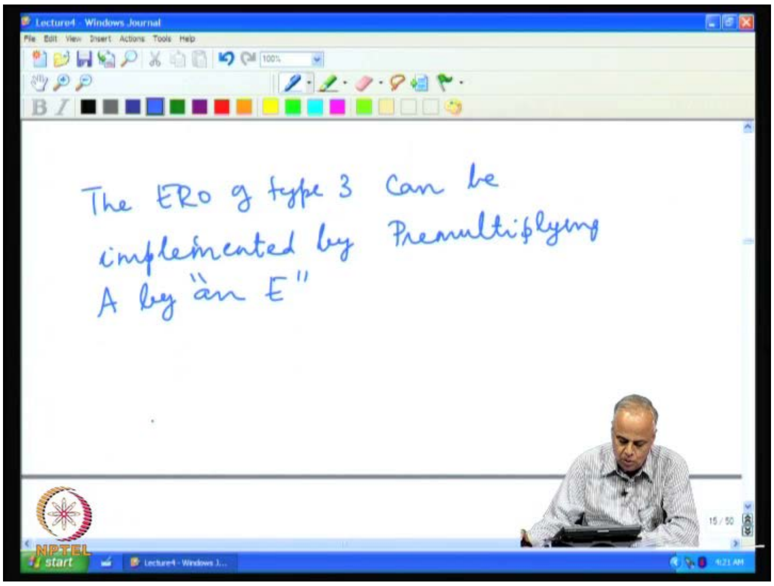

So, once again the ERO of type 3 can be implemented or effected by pre multiplying A by an E. we know what is mean by an E, take the identity matrix and apply the same transformation to the identity matrix. So, that is in all the three cases of this Elementary Row Operations the three fundamental properties that we observe are.

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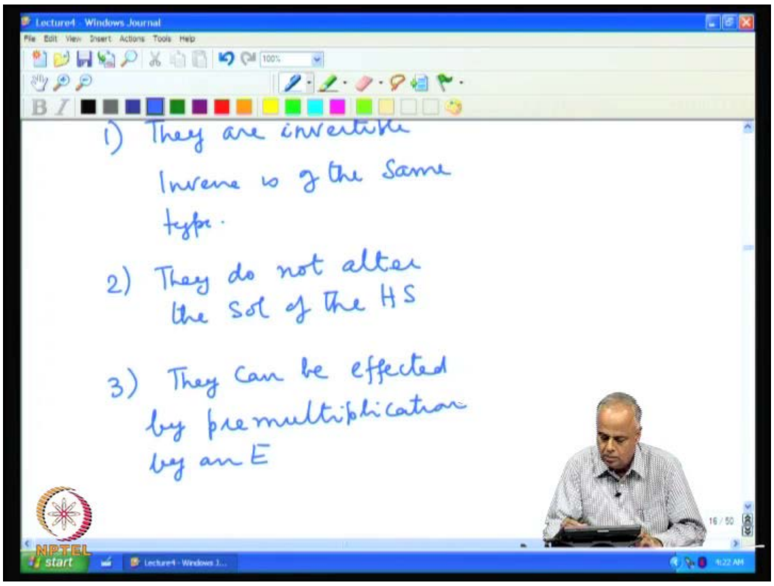

So, all ERO 's have the following properties. One, they are invertible, inverse is of the same type. Two, they do not alter the solutions of the Homogeneous System and three, they can be effected by pre multiplication by an E. pre multiplication by A an E again we understand what we mean by E.

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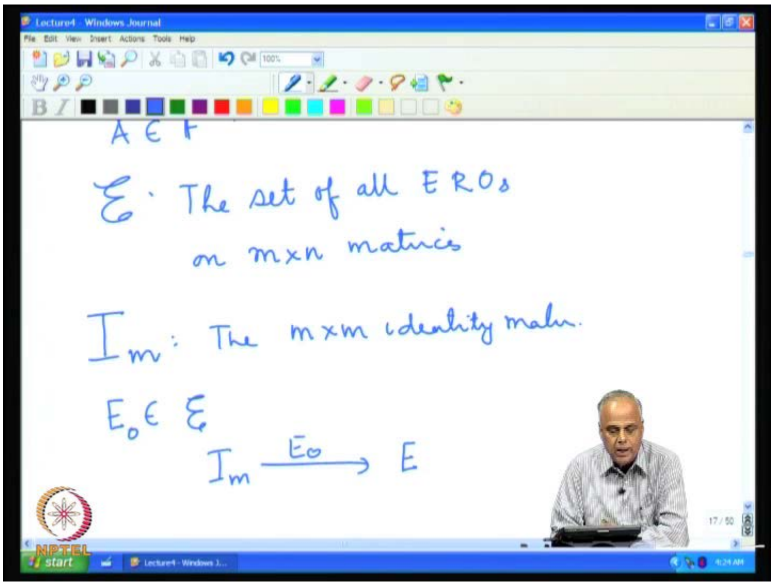

So now, what we will do is let us taking F m n and let us denote by E to be the set of all Elementary Row Operations on m by n matrix. All possible Elementary Row Operations on m by n matrices. They may be of type 1, they may exchange choose any arbitrarily two rows and exchange them or add a multiple of some row to another row or multiply some row by a non 0 constant. So, take all possible elementary row operations on m by n matrices and call them as E. This is the collection of all elementary operations on m by n matrices. Now, suppose I take I m the m by m identity matrix and take E and e; that means, E is an Elementary Row Operation and I apply, E to this I am going to get let us call it as E 0 and then we get a matrix E. That is by applying an Elementary Operation on identity matrix, we get a matrix E such matrices are called Elementary Matrices.

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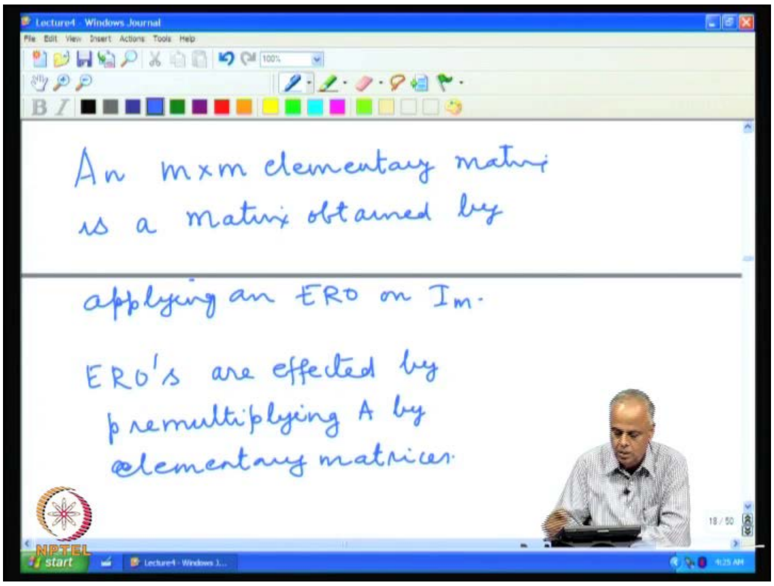

So therefore, An m by m elementary matrix is a matrix obtained by applying an Elementary Row Operation any one of the Elementary Row Operations from this collections on I m. So therefore, what we had observe was that ERO s are effected by pre multiplying A by an elementary matrices, that is the important observation made all A any type of ERO can be effected by pre multiplying the given matrix by a suitable elementary matrix.

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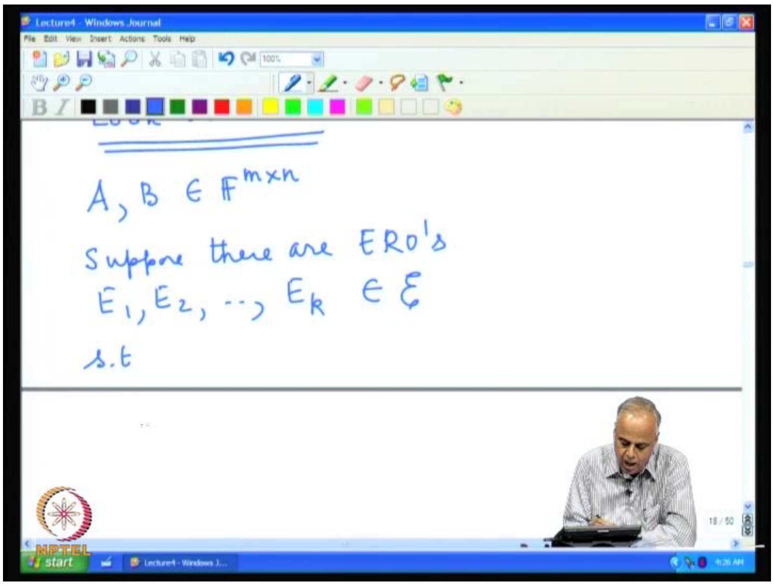

Now, let us look at F m n. Take two matrices in F m n. So, I am looking at two matrices of the same size and suppose, there are Elementary Row Operations let us call them as E 1 E 2 some E k that is these are all in the collection of Elementary row Operations each one is an Elementary Row Operation.

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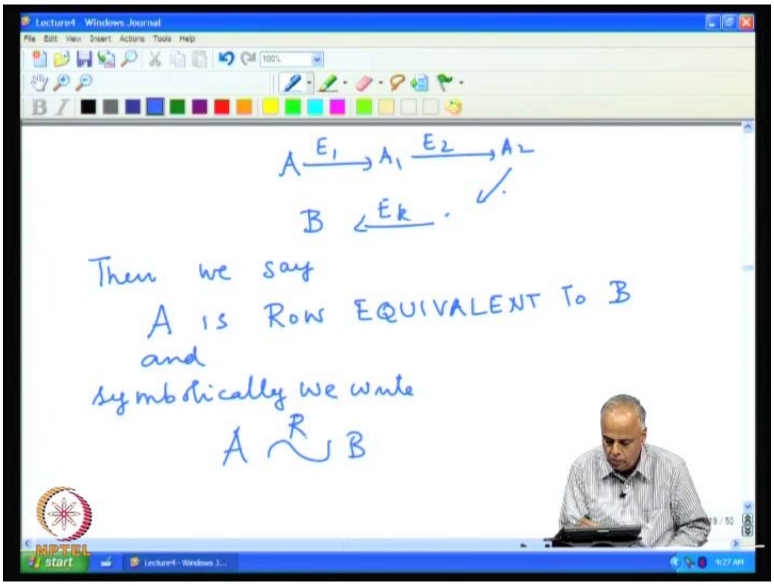

Suppose, they are such that when I apply on E successively the Elementary Row Operations, I get A 1 then I apply the Elementary Row Operation, I get a two and so on and so forth and finally, I apply the Elementary Row Operation and I get B. Suppose, I have two matrices A and B and the succession of Elementary Row Operations, such that when I apply the successive of elementary row operations on A, I get the matrix B then, we say A is row equivalent to B and symbolically, we write this as and symbolically we write A row equivalent to B.

Now, observe the following suppose A is row equivalent to B, what do we mean we can step by step from A applying one Elementary Row Operation at each step to finally, reach the goal B. Now, each step is an Elementary Row Operation and we have seen that each Elementary Row Operations invertible and therefore, we can reverse each step and then, we can go back from B to A poorly by A sequence of Elementary Row Operations.

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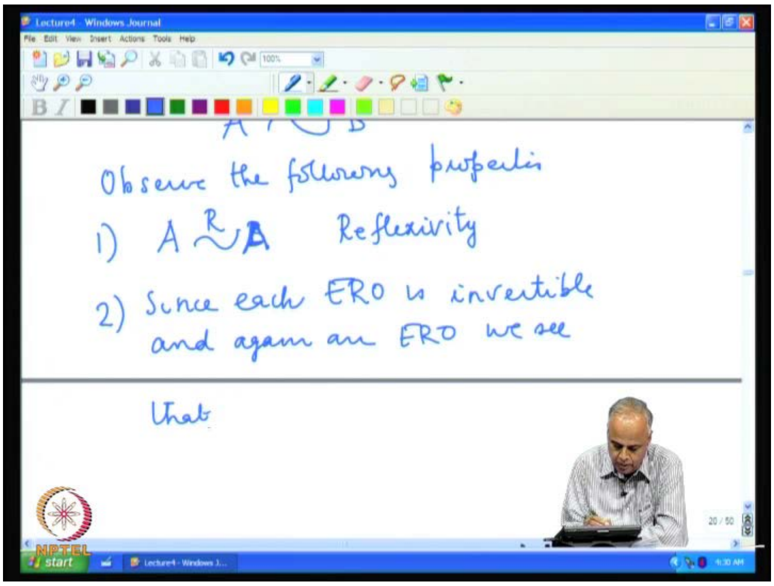

So, we will now apply all these together and observe the following properties. 1, we can go from A to A by one step namely simply multiply the first row by 1. So, we can go from A to B by one Elementary Row Operation. So, A is row equivalent to A itself a is row equivalent to itself, because we can go from A to A by A finite number of successive Elementary Row Operations. We call this property, the reflexivity property of row equivalent. Secondly, as we have observed if I can go from A to B by Elementary Row Operations and each Elementary Row Operation is reversible we can also go from B to A by Elementary Row Operations.

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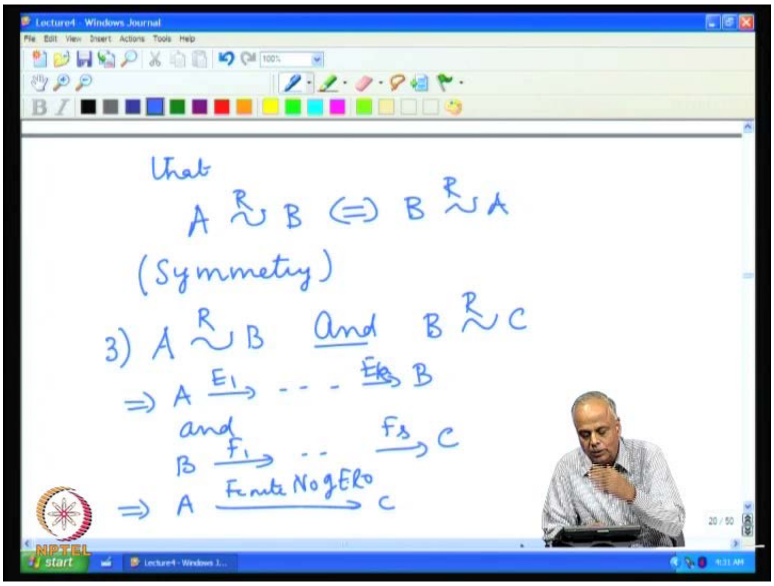

So, since each ERO is invertible and again an ERO the inverse is again an ERO, we see that A will be row equivalent to B, if and only if B is row equivalent to A. If, I can go from A to B by retracing A pass I can go from B to A and vice versa. So, A is row equivalent to B if and only if B is row equivalent. So, this is called the symmetry property of row equivalence. The third property is suppose, A is row equivalent to B and B is row equivalent to C, what is that mean; that means, I can apply a number of steps to get to B and this is a finite number of Elementary Row Operations and maybe I can apply let us call it as k F 1 F 2 F s to go from B to C. The A to B there are a finite number of Elementary Row Operations; B to C there are finite number of Elementary Row Operations; that means, I can go from A to C by A finite number of ERO's. While, first apply all the E 1 E 2 E k then, apply F 1 F 2 F s you will go from A to B and then to C. So, you will go all the way from A to C.

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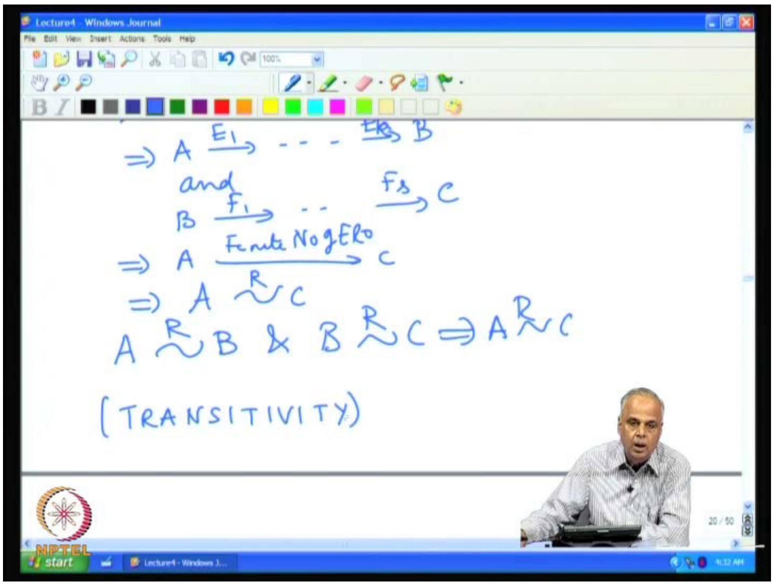

So, that means, since I can go from A to C by A finite number of ERO's you call as that a will be row equivalent to C. So, the conclusion is if A is row equivalent to B and B is row equivalent to C then, A is row equivalent to C. This is called the Transitivity property of the row equivalence. If, any relation has these three properties of ref[lexivity]- reflexivity, symmetry and transitivity it is called an equivalence relation.

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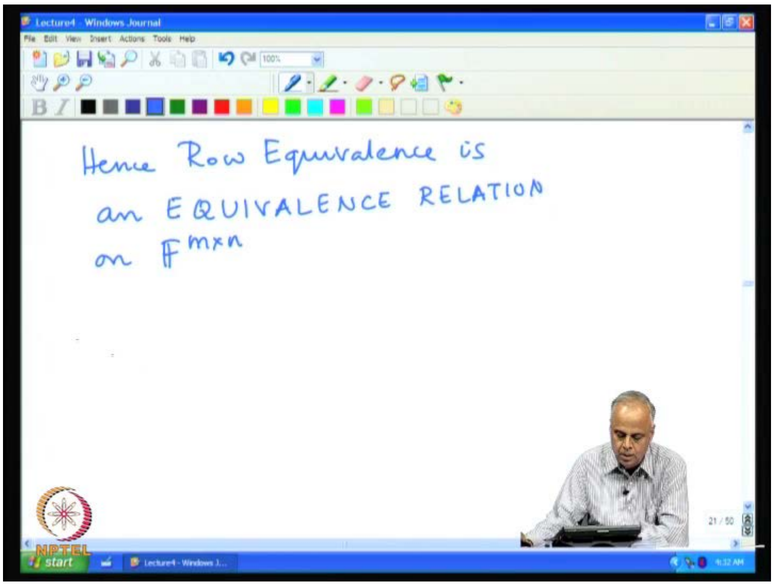

And hence, row equivalence is an Equivalence Relation on the collection of all m by n matrices.

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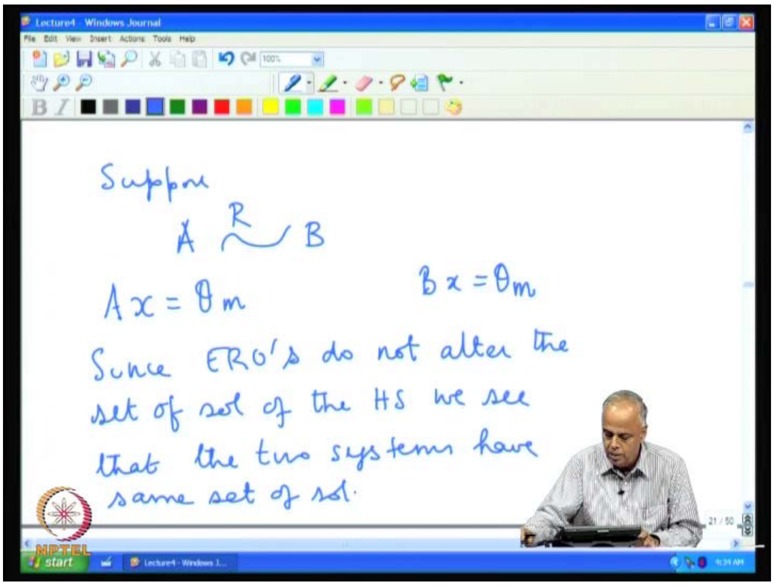

Now, suppose A is row equivalent to B. Now, we get back over basic problem of the Homogeneous System of equations. Look at the system corresponding to A x and look at the system corresponding to B; the matrix B x equal to theta m. Now, how do I go from A to B; I go from A to B step by step by Elementary Row Operations and we have observed, that the Elementary Row Operations do not alter the set of solutions of the

Homogeneous System and therefore, at each step, the system posses the same solutions and therefore, at the end A and B the corresponding Homogeneous System will possess the same solutions. Since, ERO's do not alter the set of solutions of the Homogeneous System. We see that the two systems have same set of solutions.

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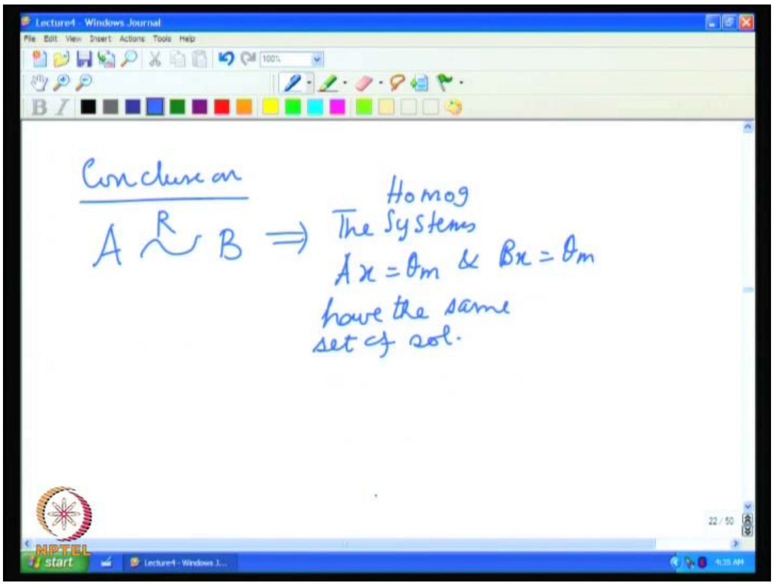

Therefore, conclusion A is row equivalent to B implies, the systems say systems. We need a Homogeneous Systems A x equal to theta m and B x equal to theta m, have the same set of solutions.

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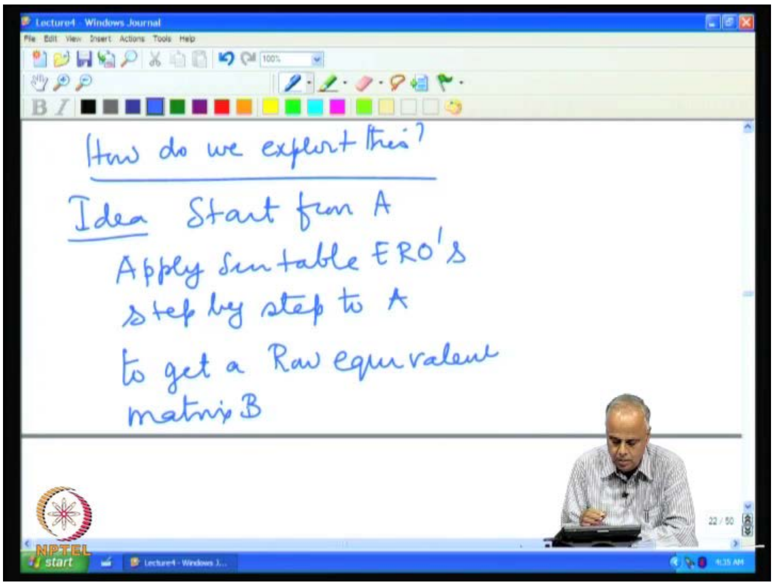

How do we exploit this? The idea is, start from A, apply suitable ERO’s, step by step to A, to get a row equivalent matrix B.

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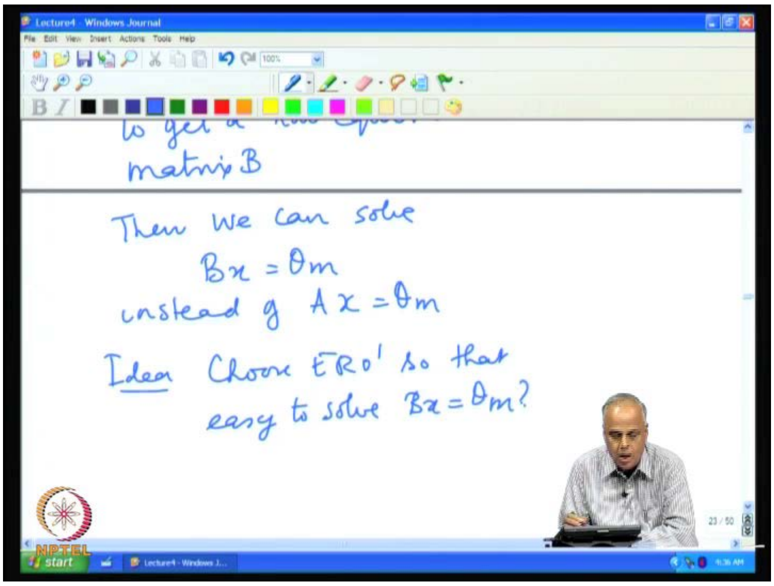

Then, we can solve B x equal to theta m instead of A x equal to theta m because both are same set of solutions. The idea is choose ERO’s, so that easy to solve B x equal to theta m, how do we do this? We shall see this in the next lecture.