Advanced Matrix Theory and Linear Algebra for Engineers

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Module No. # 10

Lecture No. # 37

Singular Value Decomposition (SVD) – Part 2

We have seen how to use the notions of positive (()) definite operators or transformations or matrices to find the right basis for our four sub spaces. Let us recall some of these aspects.

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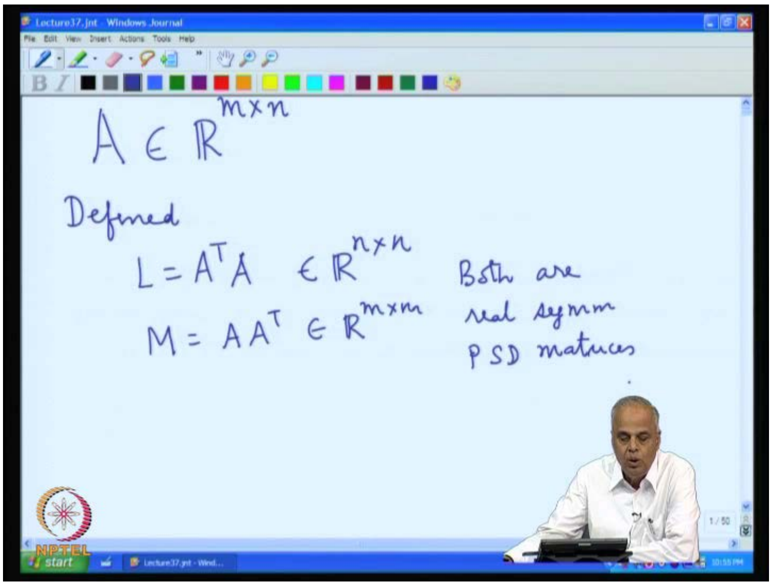

So, we first look at a matrix A which is an m by n matrix, real then, connected with that we defined 2 square matrices 1 L equal to A transpose A which is an n by n square matrix and m which is A A transpose which is an m by m square matrix.

Both are real symmetric positive semi definite matrices. This we saw last time. So, we define connected or associated with the given m by n matrix, m by n real matrix. Two positive semi definite real matrices one was n by n the other one was m by m.

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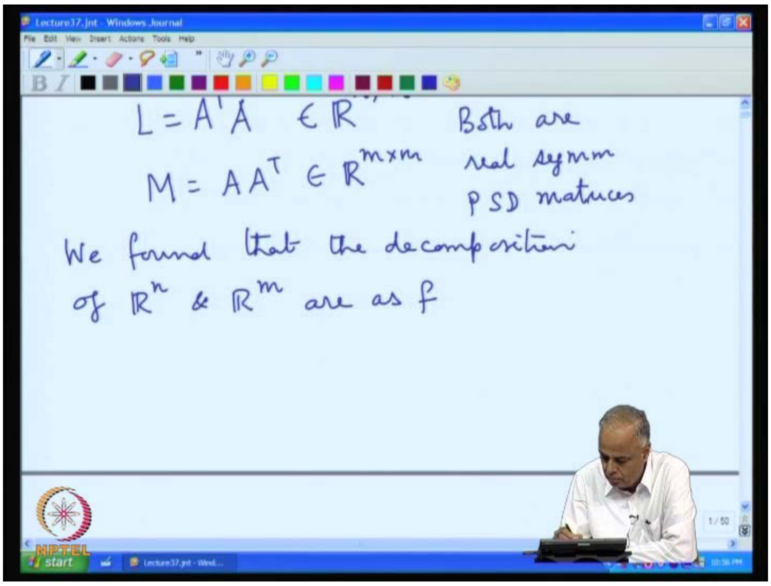

And we found that when we look at the decomposition of R n and R m are as follows.

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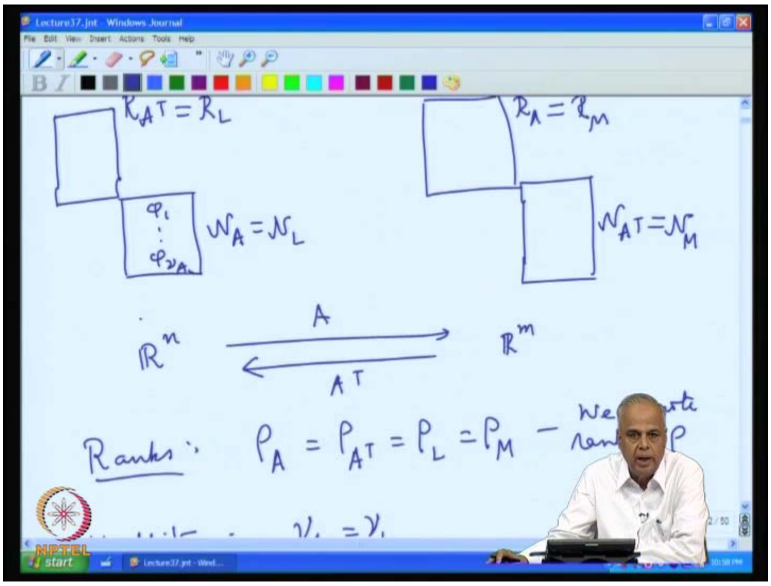

If we look at the R n side and the R m side and we have A going this way and A transpose going this way and R n the fundamental decomposition corresponding to the matrix A for the 2 orthogonal spaces null space of A and the range of A transpose.

We found that the null space of A is the same as null space of the matrix square positive semi definite matrix L and the range of A transpose was the same as range of L. On the other side we found that the fundamental decomposition with respect to the matrix was null space of A transpose and the range of A and we found that the null space of A transpose is the same as null space of m and the range of A was the same as range of m.

Now, therefore, we found that all these four sub spaces can be thought of sub spaces connected not just with the rectangular matrix, but, with positive semi definite matrices. So, using the side here, we found that the rank of all these matrices are connected. The rank of A is the same as rank of A transpose is the same of the rank of L is the same of rank of m.

So, we denote now from now on rank by row. So, see all these are equal, we denote by row. The nullities is the related as the nullity of a is the same as nullity of L because the null space of A the same as null space of L and the nullity of A transpose is the same as nullity of m because the null space of A transpose is the same as null space of m.

Now, using these ideas we found that we can find a basis pi 1 phi 2 phi nu a which is an Ortho normal basis for this null space of A as the Ortho normal Eigen vectors corresponding to the Eigen values 0 of l.

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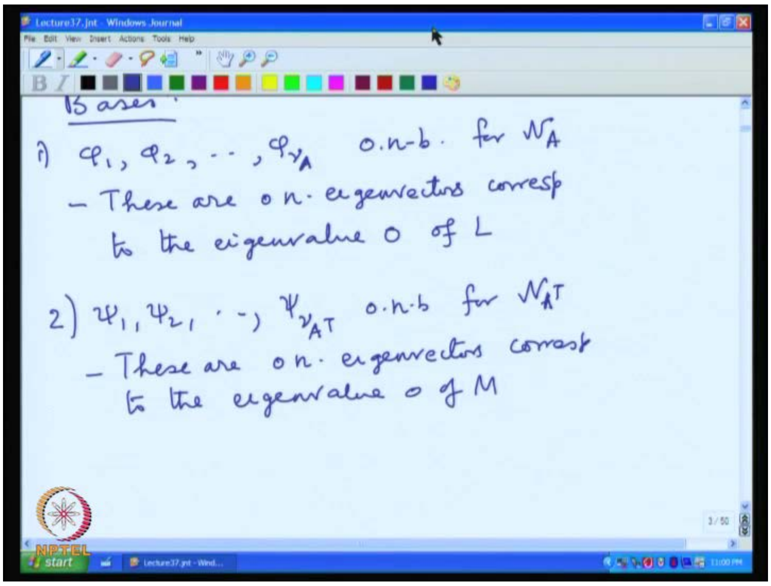

So, the basis that we found that we said we could find where 1 phi 1 phi 2 phi nu a Ortho normal basis for the null space of A. These are ortho normal Eigen vectors corresponding to the Eigen value 0 of the matrix l.

Now, note is that if nu a is 0 then then 0 will not be Eigen value of L and the null space of A will consists only of 0 vector and we have to only focus on range of A transpose. Then we could find psi 1 psi 2 psi nu A transpose an ortho normal basis for the null space of A transpose and these are ortho normal Eigen vectors corresponding to the Eigen value 0 of the matrix m.

So, that gives rise to these two bases, the two null spaces on either side. Then we went about the task of finding the bases for the other two sub spaces.

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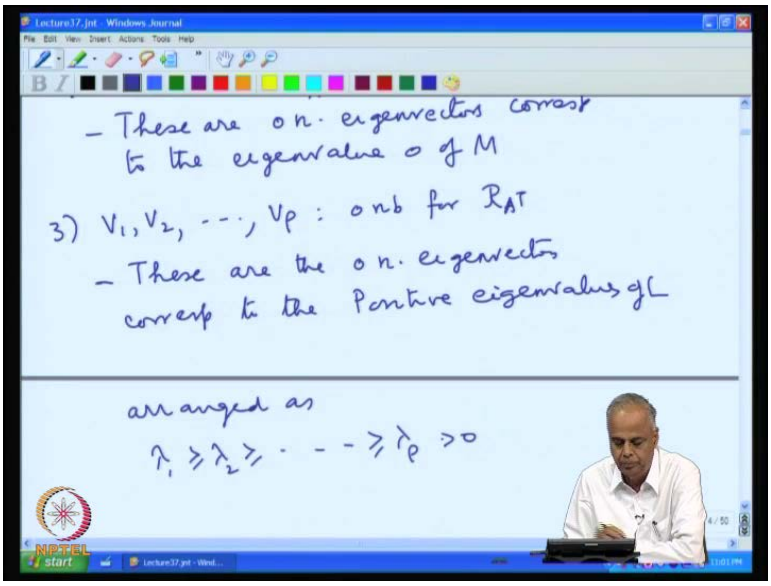

Then we found that V 1 V 2 V rho ortho normal basis for the range of A transpose these are the ortho normal Eigen vectors corresponding to the positive Eigen values which we arrange as arranged as Eigen values of L, arranged as lambda 1 greater than equal to lambda 2 greater than equal to lambda rho greater than 0.

So, we found the basis for the range of A now. Now, using this basis we generated the basis for the range of A. How did we get that basis?

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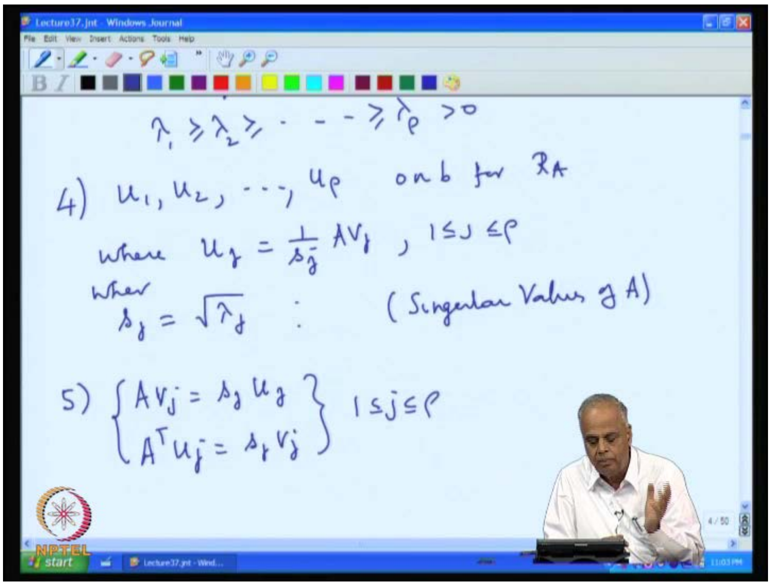

If you recall we define u 1 u 2 u rho ortho normal basis for range of A where u j is defined as 1 by s j A V j for j equal to 1 to rho s j is square root of lambda j and these are called the singular values of A. And we had the fundamental relation that A V j was s j u j and A transpose u j was s j V j 1 less than equal to (( )). So, our choice of basis the way we selected the u's from the v's make it that the A V j and A transpose u j are related this way.

So, our choice of basis is very particular for the range of A, range of A transpose we picked the Eigen vectors the ortho normal Eigen vectors corresponding to the positive Eigen values of L. And from that we generate the basis u 1 u 2 u row for the range of A transpose and this way of choosing the basis makes this relationship between the basis for range of A transpose and the range of A connected such that the V j basis for range of A transpose goes to the direction of the u j basis of range of A with the multiplying or a scaling factor of s j and the u j basis of the range of A goes to the V j direction of the range of A transpose with a scaling same scaling factor as before s j. So, these are facts that we observed in the last lecture.

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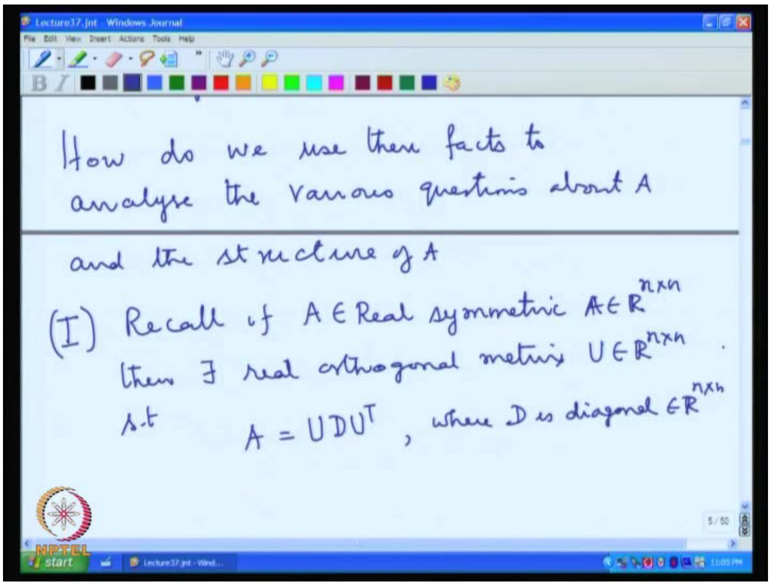

Now, we shall see how we use this how do we use these facts how do we use these facts to analyze the various questions about a and the structure of a. So, we would like to look at the general structure of an m by n matrix and we should want to look at the answers the various questions that we raised about the matrix A.

So, to this end we shall now begin our analysis using these facts of a general matrix. The first you may recall that we showed that if we have a hermitian matrix we could decompose this as the product of three matrices; the two extreme once being unitary matrices. In the case of real they were orthogonal matrices and the middle one was diagonal.

So, recall if A is a hermitian matrix, there exists since we are dealing with real matrices at the moment let us say A is real symmetric real symmetric matrices. So, R n a belongs to R n n. So, if a is a real symmetric matrix then there exists unitary in the case of real it will be orthogonal again. Let us restrict ourselves in the real case real orthogonal matrix u belonging to R n n such that we could write a as the product of u and a diagonal matrix and u transpose where u is d is diagonal.

This was the result we had for a real symmetric matrix. Now, what we will do is we will look at a general version of this result for a general m by n matrix.

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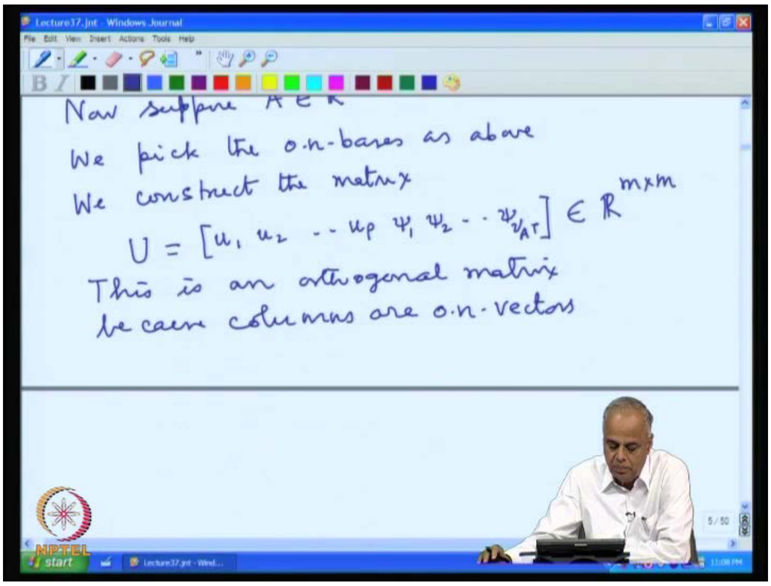

So, now suppose a is a general m by n matrix and then we pick the ortho normal basis as above. That is we construct the matrix L and then we pick up the phi 1 phi 2 phi nu a psi 1 psi 2 psi nu a to transpose V 1 V 2 V rho u 1 u 2 u rho as mentioned starting from the matrix L and m.

So, we pick that ortho normal basis as above. Then we construct the matrix u whose columns are the u vectors u 1 u 2 u rho that is these are the ortho normal basis for the range of A and then the remaining basis coming from the null space of A transpose.

So, u is the matrix whose columns are formed by if you look at this picture whose columns are formed by this u 1 u 2 u rho and psi 1 psi 2. So, this together all these form a basis. Each one of them is an m component vector. We have m of them and what we do is we put them all together and we have this matrix u whose columns are these u 1 u 2 u rho.

So, this is a matrix which is an m by m matrix and now this is an orthogonal matrix because the columns because columns are ortho normal vectors. Whenever you have a real matrix with columns ortho normal becomes an orthogonal matrix. Whenever you have may complex matrix these columns are complex ortho normal then it becomes a unitary matrix.

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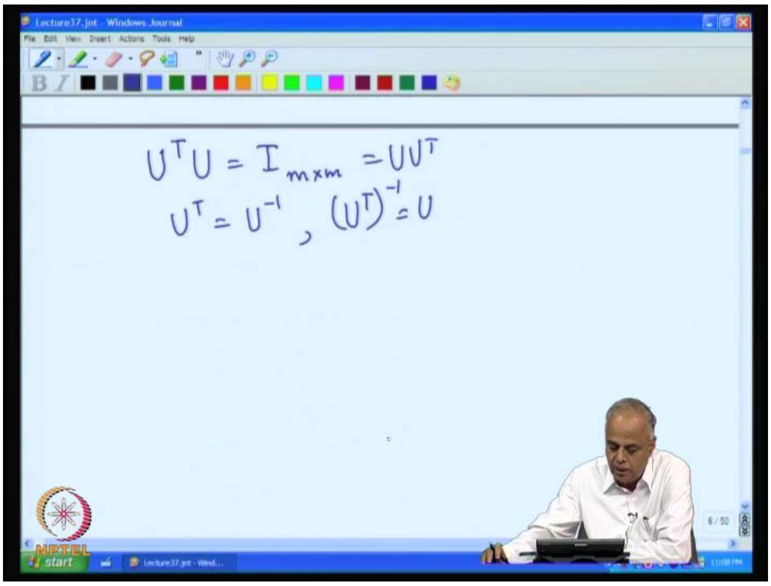

What does it mean to say it is an orthogonal matrix? U transpose U is the identity matrix which is the same as U U transpose which means U transpose is U inverse and u transpose inverse is U.

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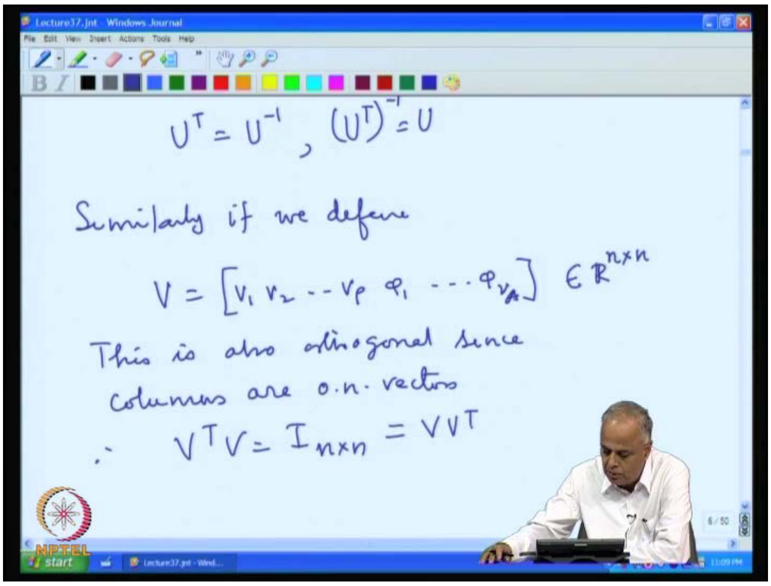

Analogously, similarly if we define we now we constructed the u matrix taking care of the basis on the R m side. Now, we construct the matrix V taking care of the basis on the R m side that is V 1 V 2 V row pi 1 pi 2 pi nu a.

Now, all these vectors in R n (( )) of them. So, this becomes an n by n matrix and this is also orthogonal matrix. Now, it is an orthogonal n by n matrix since columns are ortho normal columns are ortho normal vectors and therefore, V transpose V equal to identity. Now, n by n matrix all are n by n matrices V v transpose.

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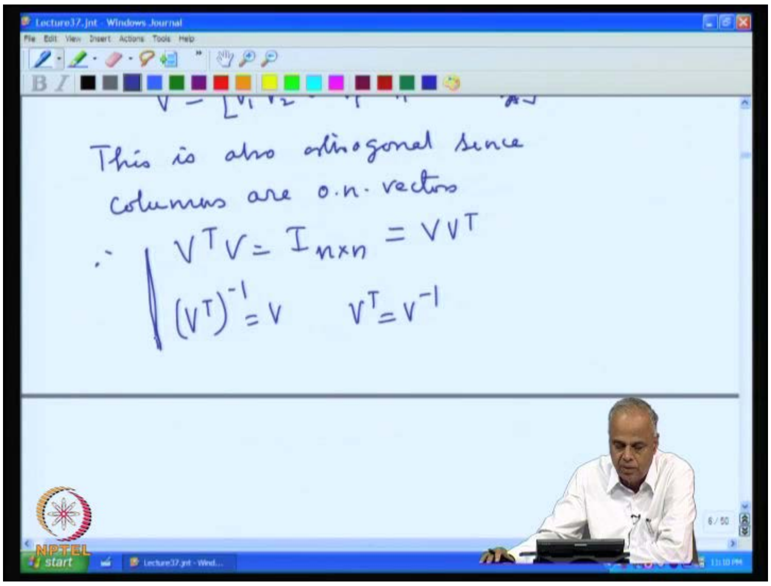

And V transpose inverse is V and V transpose is V inverse. So, these are two important matrices that we have constructed.

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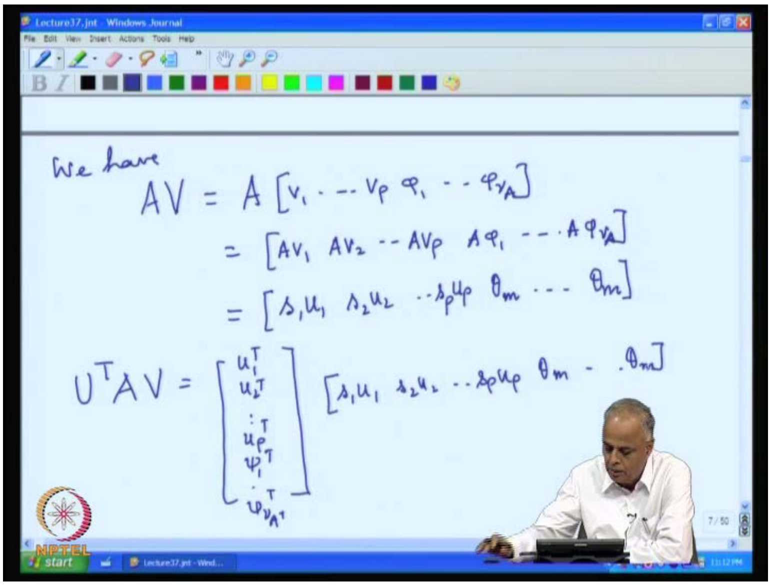

Now, we look at A V. This is the same as A V is this matrix phi 1 phi 2 phi nu a. Now if we multiply you get A V 1 the first column is A V 1 second column is A V 2 row column is A V row then we get a phi 1 a phi nu a.

This is the same as A V 1 we knew from our choice of basis A V 1 is s 1 u 1 A V 2 if you look here we have A V j is s j u j. So, A V 1 will be s 1 u 1 A V 2 will be s 2 u 2 and so on. We use that fact and therefore, we can write this as s 1 u 1 then s 2 u 2 s rho u rho then phi 1 is in the null space of a. So, a phi 1 will be theta m a phi 2 will be theta m. All these will be theta m 0 columns because phi 1 phi 2 phi nu a are all in the null space of a.

So, A V is simply s 1 u 1 s 2 u 2 s row u row and the last n minus row columns are all 0 now let us multiply this matrix by u transpose u transpose is u 1 transpose u 2 transpose u rho transpose psi 1 transpose and psi new A transpose transpose that is u transpose into s 1 u 1 s 2 u 2 s rho u rho theta m etc theta m.

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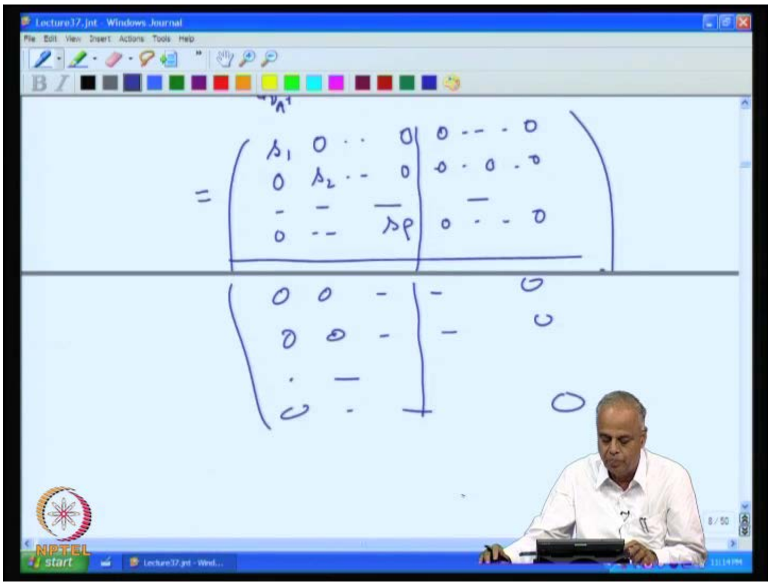

Now, if you multiply what do we get first is u 1 transpose we get u 1 transpose into u 1 u 1 transpose u 1 is 1 because these vectors are of length 1 are all ortho normal vectors. So, they are all of length 1. So, u 1 transpose u 1 is 1. So, I will just get s 1. So, the first entry is s 1. Next we have u 1 transpose u 2 since u 1 and u 2 are orthogonal to each other u 1 transpose u 2 will be 0.

So, next one will be zero and similarly, u 1 transpose u row will be zero. So, all the remaining are zero then we will get u 1 transpose the 0 vector u 1 transpose the 0 vector and so on. So, all the remaining things will be zero.

So, the first row will have only the leading entry as s1 and all others are zero similarly, the second row as 0 s 2 0 0. It goes on up to the say rho rho th. Then beyond the rho th row. We have psi 1 transpose u 1 which is 0 psi 1 transpose u 2. That is zero because they are orthogonal psi 1 transpose u row that is zero and psi 1 transpose the 0 vector it is zero. So, all others are zero value all the other entries are 0 0 0 0 0 0 and so on.

So, we get a matrix like this and let us see what it means it means. It has a nice diagonal block which has s 1 s 2 s row on it and all the others are zero.

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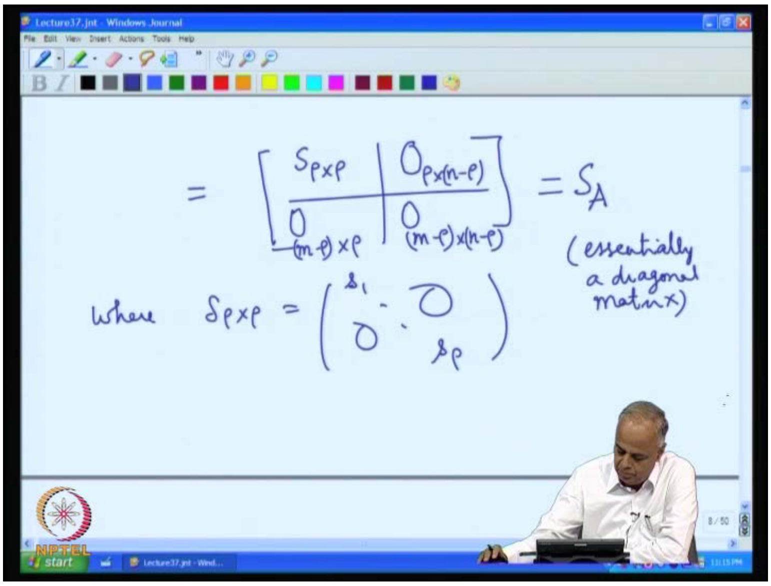

So, we will write it in this form it in block form a matrix s rho by rho. All the others are zero. So, this will be the zero matrix with row columns and n minus row th row rows and n minus row columns. This will be the zero matrix. We have already taken row rows here. So, the remaining m minus rhos into row and this will be zero m minus rho into n minus rho.

So, basically where s rho is the diagonal matrix s 1 the the diagonal entries are the singular values we will call this matrix as s a. So, S A is essentially a diagonal matrix essentially a diagonal matrix. Only significant part, the non zero part is the diagonal block its are the leading diagonal block and the diagonal entries are all the singular values of a.

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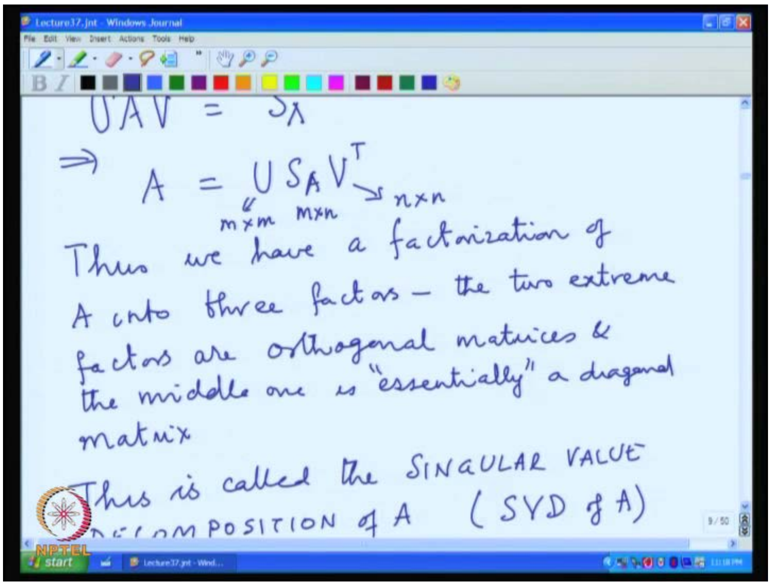

So, therefore, we have u transpose A V is equal to S A or a equal to e transpose inverse is u and V if we take the other side V inverse is V transpose because these u and V are orthogonal matrices. The inverse of u is u transpose, the inverse of u transpose u inverse, the inverse of V in is V transpose.

So, thus we see that A has been factorized into three factors. The two extreme factors are orthogonal matrices and the middle one is essentially a diagonal matrix. Thus, we have a factorization of A into three factors. The two extreme factors are orthogonal matrices and the middle one is essentially a diagonal matrix.

When we say essentially a diagonal matrix what we mean is that the only significant part of the matrix is a leading diagonal block consisting of the, I singular values along the diagonal and all other entries being 0.

Note that, in the two extreme factors the u is an m by m factor matrix and this V transpose is an n by n and this the middle one SA is an m by n matrix. So, that the product is again m by n. This is called the singular value decomposition of A. We will generally denote it by SVD of a. SVD means singular value decomposition and in the case of hermitical matrices since everything in the square the extreme matrices where I can be chosen to be the same. And that is what we got in the you got u diagonal u transpose.

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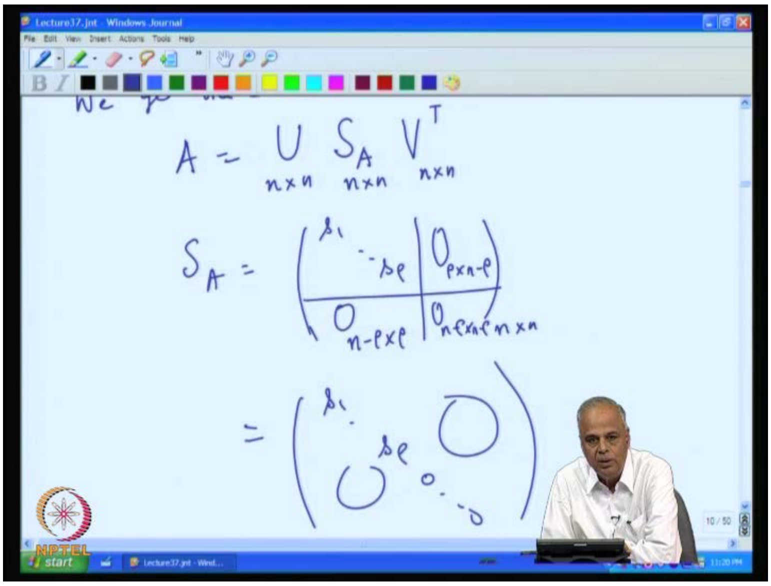

In the case when m equal to n in case we have m equal to n. So, let us say we have a square matrix. We get a equal to u which is now n by n orthogonal matrix, S A which is now an n by n matrix because m equal to n V transpose which is again an n by n orthogonal matrix.

Now, how does SA look like? We get the SVD as the singular value decomposition. SA is now an n by n matrix it has s 1 s 2 s rho along the diagonals and all the others are zero. Zero mn minus rho I am sorry this is the rho remaining number of rho's is n minus rho columns is rho here is remaining columns is n minus rho here is n minus rho n minus rho which simply means it is the diagonal matrix s 1 s 2 s rho and all the other entries are zero.

So, therefore, we see a general version of the diagonalization theorem. If we have any matrix a any real matrix A which is square. So, if we have any real square matrix a it can be always written as the product of three matrices; the extreme two being orthogonal n by n matrices and the middle one being a diagonal matrix.

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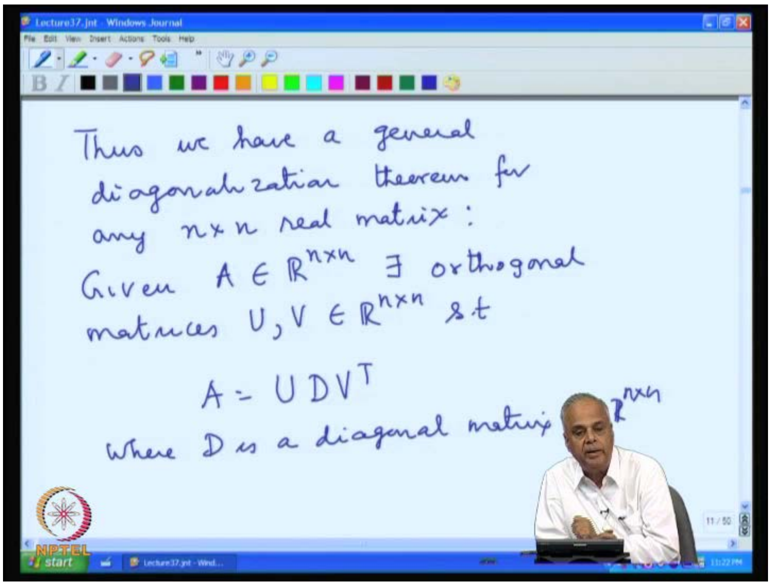

So, thus we have a general diagonalization theorem for any n by n real matrix. Recall that whenever we when we talk of diagonalization of the form p inverse a p should be diagonal, we always required that algebraic multiplicity of the Eigen value to be equal to the geometric multiplicity of the Eigen value and we found examples where this was not valid and therefore, there were matrices which were not diagonalizable in that format.

But now, we see that if you allow the two extreme factors to be two different orthogonal matrices then we are always able to diagonalizable a matrix. So, we have this general diagonalization theorem for any n by n matrix. So, given a belonging to R n cross n there exists unitary the then now since you are in the real case I should write orthogonal matrices. There exists orthogonal matrices u V in n by n such that u a equal to U D V transpose where d is a diagonal matrix.

So, this is the the diagonal entries are now not connected with the Eigen values, but, they are connected with the singular values.

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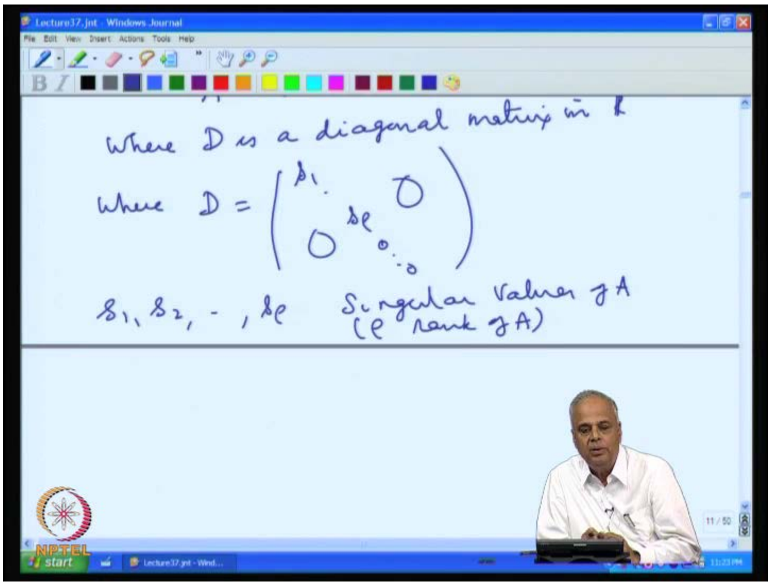

So, where d equal to the diagonal matrix s 1 s 2 s rho 0 0 0 0 s 1 s 2 s rho the singular values of a and of course, rho is the rank of a. If rho is n, it is s 1 s 2 s n full diagonal matrix. So, thus we have this generalization of the idea of the factorization we had for hermitical matrices. We have the generalization of the idea of diagonalization we had for matrices with algebraic multiplicity equal to geometric multiplicity for all Eigen values.

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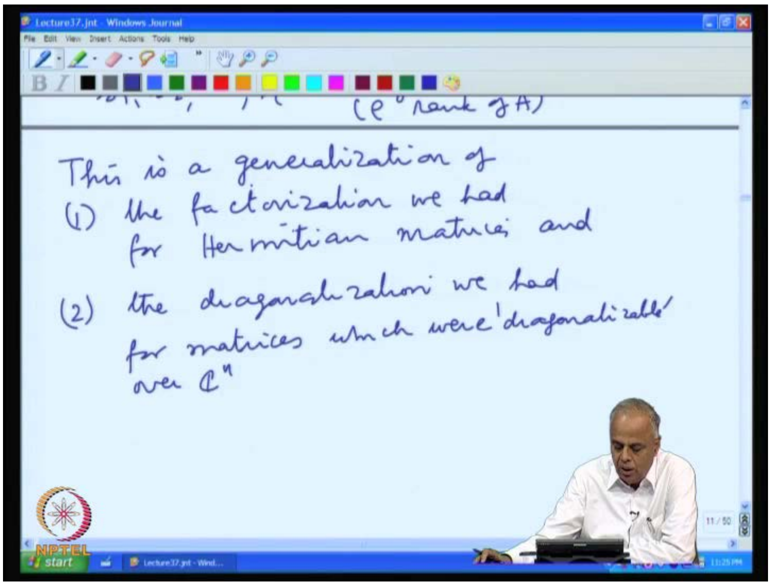

So, this is a generalization of one; the factorization we had for Hermitian matrices and two, the diagonalization we had for matrices which were diagonalizable over c n. So, these are the to here in the case of this sort of general diagonalization we do not have to resort to complex polynomial etc. We are always guaranteed that all these can be done over the (( )). So, if you want to restrict over (( )) then any matrix a can be diagonalized in this format namely U D V transpose where d is diagonal and u and V are orthogonal matrix and also if you take any general matrix m by n matrix non square. Even rectangular matrix then a can be factorized and to this form this. So, called SVD in the case in the in the general case the SVD involves m by n matrices the general a belonging to R m m can be factored as U S A V transpose where u and V are orthogonal and S A is essentially diagonal.

So, thus we have the factorization theorems and the diagonalization theorem all of them coming out as a consequence of our basis that we chose. This worked out well because our u and V that we chose were well behaved. However, the U S and V S coming from U S were coming from the basis for R m that we chose and the matrix V was coming from the basis we chose for R n.

So, it is this right choice of basis that we made that made things work and gave us the factorization of the matrix a. So, that is the first consequence. This generalization of this factorization ideas and the generalization of this diagonalization ideas are the first consequences of the type of basis that we have chosen in studying the structure of the matrix a.

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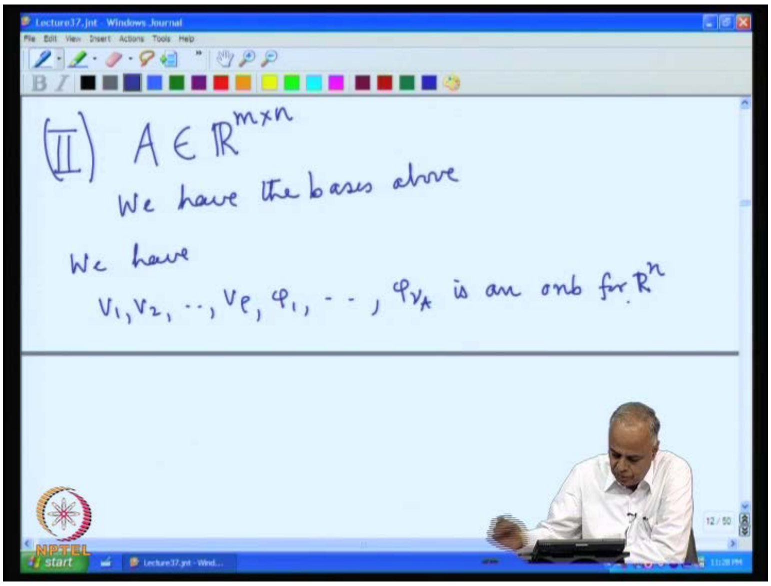

Now, let us look at the same thing in a different context, in a different view point. Remember we also studied that when we had hermitian matrices on the one hand we had the factorization of the matrix into three parts; unitary, diagonal, unitary. Also on the other hand we said that a Hermitian matrix of rank rho can be decomposed at the sum of rho 1 rank matrices.

Analogously, we shall now see that if you take any general rectangular matrix also or any square matrix or any matrix real matrix for that matter and this could be extended to complex matrices. So, if you take any matrix a belonging to R m cross n; m may be equal to n m may not be equal to n m may be greater than n m may be less than n it is irrelevant.

If you give any matrix a belonging to Rmn; if its rank is rho we shall see that it can be written as the sum of rho 1 rank matrices. This is the sum decomposition and this will be the some version of the singular value decomposition. We have obtained above the singular value decomposition in the product form now we will get the singular value decomposition in the sum form. So, let us now start with any matrix or m cross n and again we have the basis as above.

The moment matrix is given will construct the L and m and we will choose the basis for R n and R m as explained above, the u 1 u 2 u rho phi 1 phi 2 phi new a for the R n side and the u 1 u V 1 V 2 V rho and the phi 1 phi 2 phi nu a for the R m side u 1 u 2 u rho and the psi 1s on the other side.

So, we have the above basis what is the consequence of the basis in terms of the sum now? We have V 1 V 2 V rho psi 1 psi 2 psi nu a is an ortho normal basis for R n. The moment we have an ortho normal basis for a vector space we can always expand any vector in terms of this ortho normal basis.

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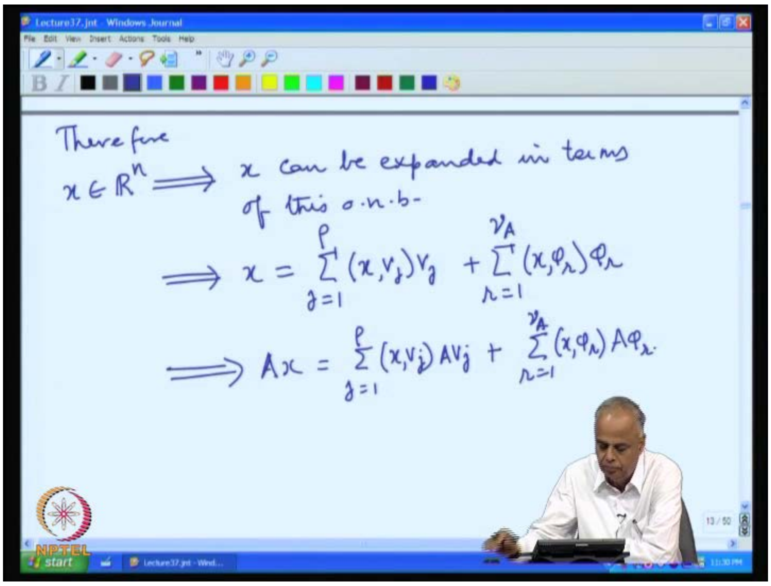

Therefore, x belongs to R n implies x can be expanded in terms of this ortho normal basis. What is this expansion? That is x is equal to we have to take the components along the V directions and the components along the phi directions. So, we will write it as j equal to 1 2 rho, x comma V j V j. These are the components along the V j direction x V j times the vector the base vector and then will write R equal to 1 to nu a. The components along the phi directions this is the so called Fourier expansion in terms of the ortho normal basis.

We have seen that the moment we have an ortho normal basis we always have a Fourier expansion of this form. Now, once we have this what does that say about A x? Now, A x can be matrix multiplication is distributive over the sum. So, we can take the product sum term by term x V j is a number because it is the component times A V j plus summation R equal to (( )) new A x phi R is a number and a phi r.

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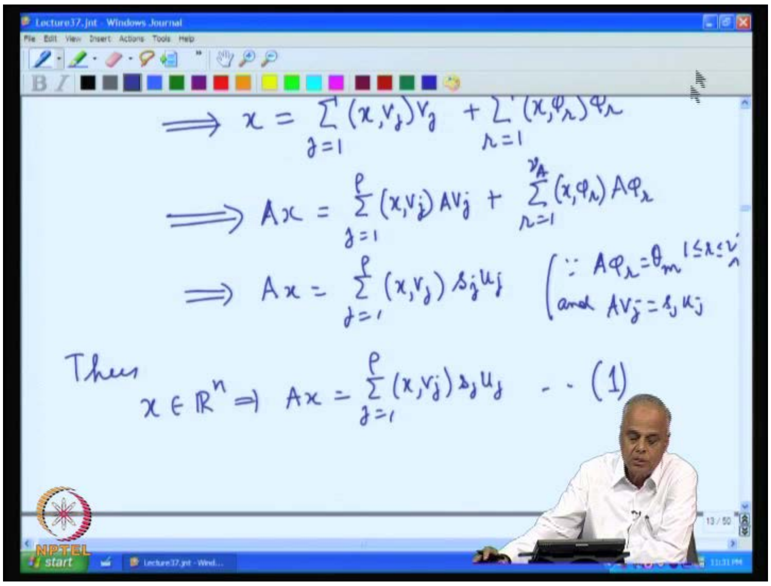

That says j equal to 1 to rho remember that according to our choice of basis A V j was s j u j. That is how we chose the basis the way we have chosen the basis forced a b j is to be s j u j the V j direction went to the u j direction with the scaling factor s j plus R equal to 1 to nu a. What is a phi R? phi rs are all in the null space of a. So, a phi R is the 0 vector. So, all these terms just give the 0 vector.

So, we can ignore them because a phi R is equal to theta m for 1 less than R less than equal to nu a and A V j is equal to s j u j. These two facts we know by the choice of our basis. So, we have Ax equal to s. So, thus x belongs to R n implies A x equal to summation j equal to 1 to rho x V j s j u j. This lets call this a equation 1.

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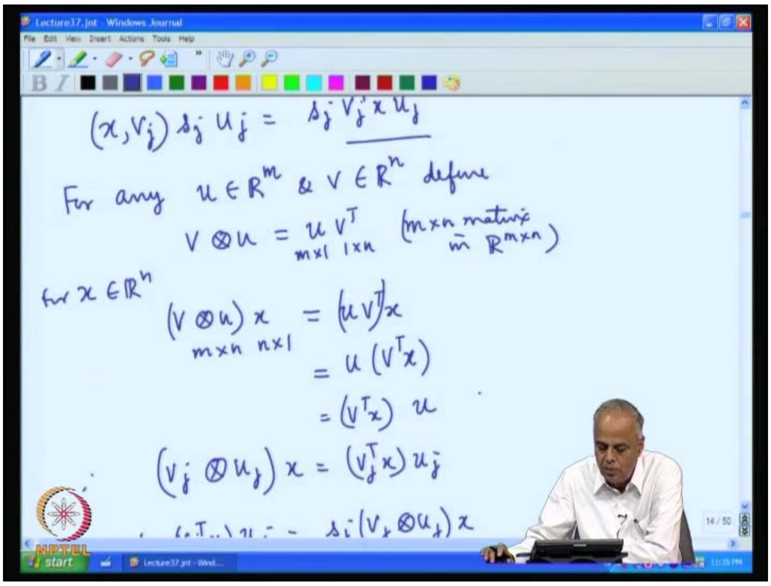

Now, lets put this equation 1 in a different form. First of all we observe that x V j we are all in the real case. So, the x V j the inner product between x and V j is just V j transpose x. It is the inner product in the real case V j transpose x. In the complex case it is V j star x.

So, since we are dealing with real vectors x comma V j is V j transpose and therefore, x comma V j s j u j is the same as s j V j transpose x into u j. Now, for any u in R m and V in R n, we define the tenser product V tenser or Vcross u has u V transpose. Now, since V is in R m this is m by 1 V v is in n. So, this will be 1 by n. So, this will be n m by n matrix, n R m n so on its real.

So, if u is in R m, V is in R n, V tenser u defined as V tenser u is equal to u V transpose is an m by n matrix and now if we take x in R n, V tenser u this is the matrix multiplying x. So, this is an m by n matrix, this is an n by n vector, I should get an m by 1 vector by V tenser u is u V transpose x.

So, this is nothing, but, V transpose is now matrix multiplication is associative. I can write it as u V transpose x. V transpose x V is in R n x is an R n V transpose x is a number. So, we can write this as V transpose x times the vector u.

So, for any x in R n, V tenser u x is equal to V transpose x into u. If you use that notation we will see that V transpose x into u j can be written as V j transpose V j tenser u j acting on x. Therefore, V j tenser u j acting on x is V j transpose x u j. So, therefore, we can now write s j times V j transpose x times u j as s j into V j tenser u j acting on x. So, let us now substitute this in one.

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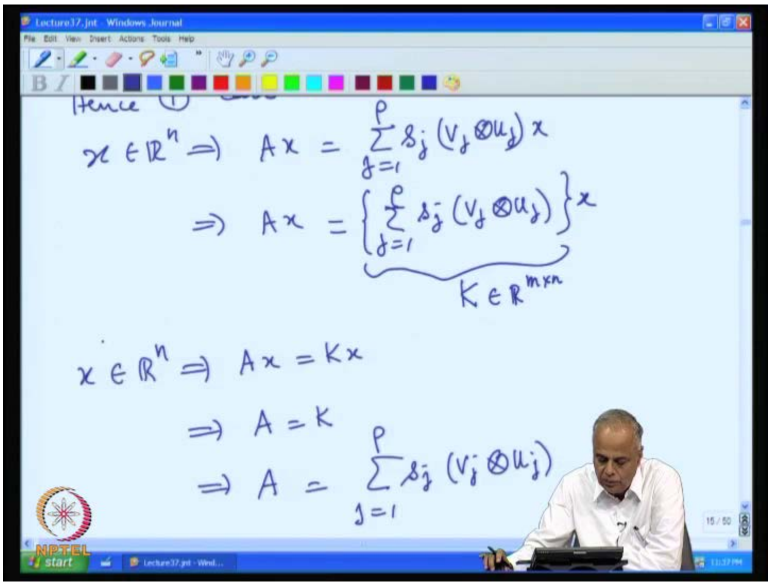

So, therefore, one can be written as x belongs to R n implies A x equal to summation j equal to 1 to rho we have 1 here, s j s j and x V j as we observed V j transpose x and combining that we get V j tenser u j x.

So, in other words it is the action of x on a number of matrices. We can write this as summation j equal to 1 to rho s j V j tenser u j. This whole thing is acting on x. Each term in the sum is an m by n matrix because the tenser product of V j and u j. V j is in R n and u j is in R m by our definition this is an m by n matrix.

So, we have a sum of m by n matrices. So, this is an m by n matrix x. So, lets call it as k. This k belongs to m by n. So, therefore, x belongs to R n implies A x equal to k x. Now, if two matrices coincide at all the vectors if A x equal to k x for all the x; then a must be equal to k. So, that says a equal to summation j equal to 1 to rho s j V j u j and each term is a matrix of rank 1 because as you see here in the tenser product, if you when you take the tenser product the range is consists of any vector of this form and it is always a multiple of the vector u, constant times u.

So, u spans the range. So, therefore, if you take this V j tenser u j, u j will span the range therefore, the rank is 1. So, each one of the matrices in the sum is rank 1 and therefore, we have a sum of rho 1 rank matrices.

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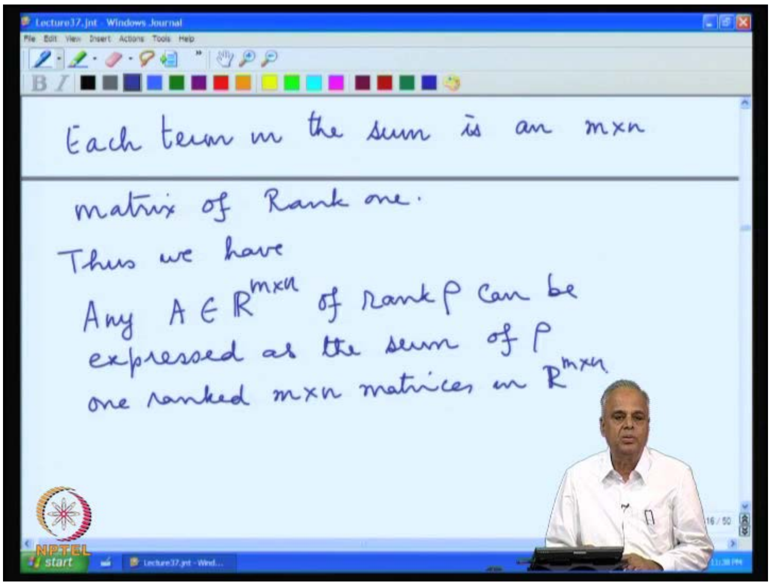

So, each term in the sum is an m by n matrix of rank 1. So, thus we have a is any any a belonging to R m cross n of rank rho can be expressed as the sum of rho 1 ranked m by n matrices.

So, thus we have the sum decomposition of a rho rank matrix. This decomposition that we have called this will be called the SVD, the sum version. We solve the product version of the decomposition into three matrices; product of three matrices. Now, we have the sum decomposition of any m by n real matrix. We have the sum decomposition of the rank of the matrix is rho then, we can decompose it into rho matrices each of rank 1. So, this is the generalization of such a decomposition we had for a hermitian matrix.

So, for hermitian matrix we had product decomposition, analogous to that for a general m by n matrix we have product decomposition. For hermitian matrix we had a sum decomposition. Analogous to that we have for any general m by n matrix we have sum decomposition. For a hermitian matrix we had a diagonal decomposition corresponding to that for a general m by n matrix we have essentially diagonal decomposition. All these are related to the SVD of the matrices.

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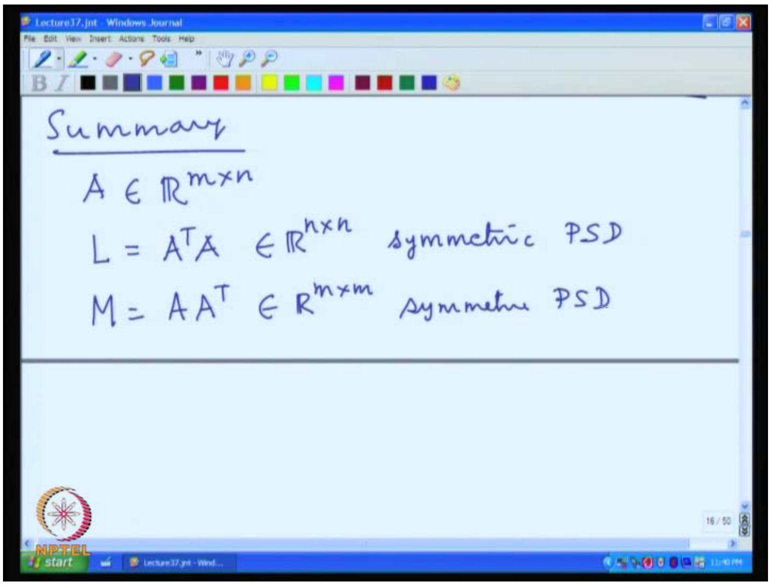

So, let us like write down this summary of all our discussions so far. So, the summary is a belongs to m by n matrix. We will repeat the whole structure. We define L to be A transpose a. This belongs to R n n symmetric. So, it is a real symmetric n by n matrix, m to be A A transpose. It is an m by m matrix and symmetric and both are positive semi definite.

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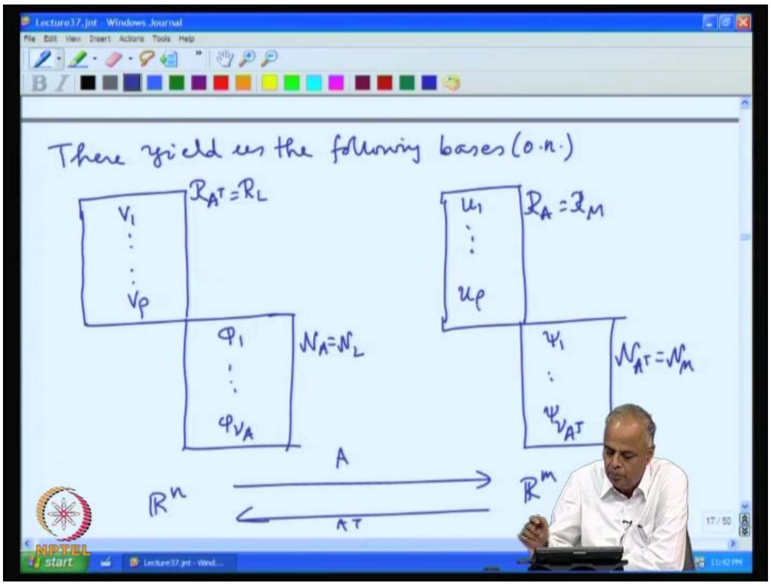

These yield us the following bases which are ortho normal. Again we will write the picture that we had. This is R n and this is R m a is the mapping from here to there and A transpose on this side and this is the range of A transpose which was the range of L. This is the null space of A which was equal to null space of L. This is the range of A which was equal to range of m and this was the null space of A transpose which was equal to null space of m.

We found ortho normal basis for this which were coming from the Eigen vectors corresponding to the 0 Eigen value of the matrix L and we found the beds ortho normal basis V 1 V 2 0. These were the ortho normal Eigen vectors coming from the positive Eigen values of L arranged in the form lambda 1 greater than equal to lambda 2 greater than equal to lambda rho greater than 0.

Then we got the basis here and these came from the ortho normal Eigen vectors corresponding to the Eigen value 0 of the matrix m.

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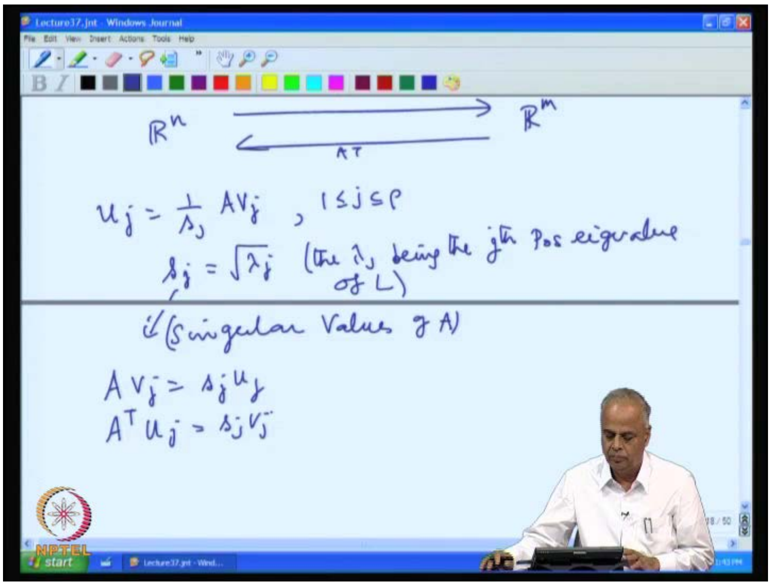

Then we got the basis here. U 1 u 2 u rho with u j depend as 1 by s j A V j 1 less than equal to j less than equal to rho where s j is the square root of lambda j, the lambda j being the jth positive square root of positive Eigen value of l.

Remember, we arranged them as lambda 1 greater than equal to lambda 2 greater than equal to lambda rho and the jth 1 and these are called the singular values of the matrix. So, these are called singular values of A. And then we had the relation that A V j is s j u j this is from the definition itself and similarly, A transpose u j is s j V j.

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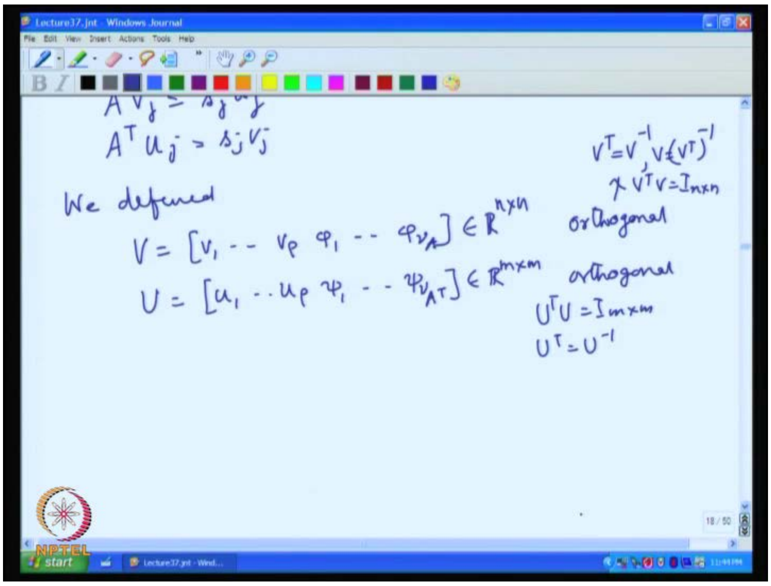

Then, we defined V to be the matrix whose columns are V 1 V 2 V rho phi 1 phi 2 phi nu a u to be the matrix u 1 u 2 u rho psi 1 psi 2 psi nu A transpose. This was an n by n matrix, this was an m by m matrix and this was orthogonal m by n matrix. That is V transpose V is identity. This was orthogonal m by n matrix. So, orthogonal here means V transpose is V inverse, V equal to V transpose inverse and V transpose V is identity. Similarly, u transpose u is identity and each is the inverse of the other. We had these two orthogonal matrices.

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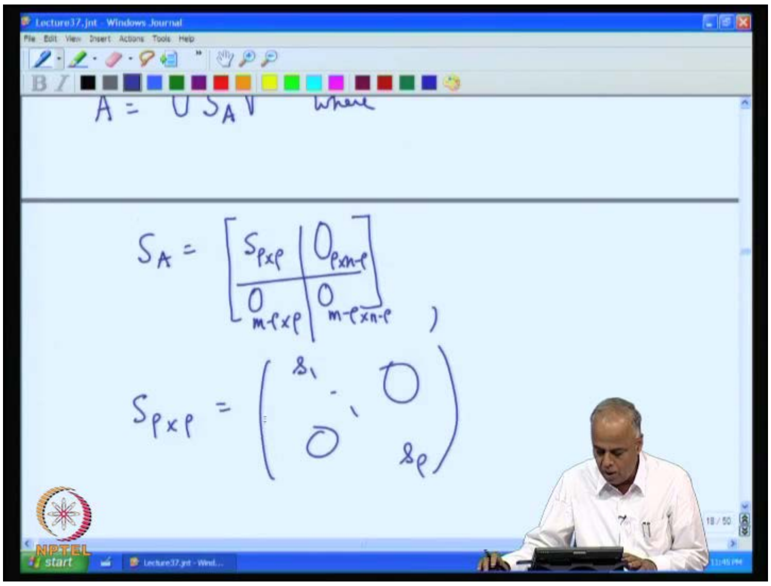

Then, we had this product decomposition. The product decomposition is of A is a equal to u S A V transpose where S A was block form s rho by rho matrix a diagonal block. The rest of them are 0. If this is 0 m minus rho n minus rho this is 0 this is 0 m minus rho rho and this is 0 m minus rho n minus rho.

So, this was in diagonal block and s rho rho was the diagonal matrix whose diagonal entries were s s 1 s 2 s rho, the singular values of the matrix a.

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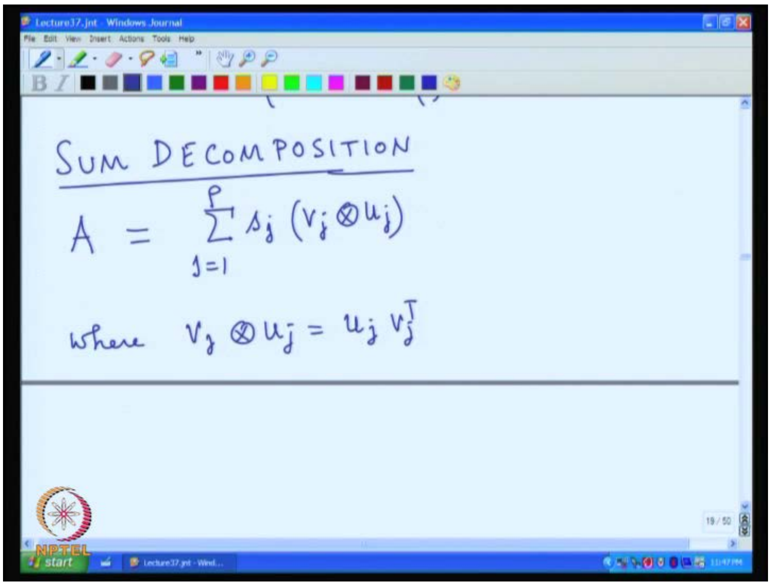

So, the first thing that we had got the product decomposition. Then we had the sum decomposition these are the s V ds the decomposition, when we say decomposition we mean the singular value.

Since the diagonal entries are in singular values here we call it the singular, so, this is the product decomposition the product version of the SVD and then we had the sum decomposition a is equal to summation j equal to 1 rho s j V j tenser u j where V j tenser u j our notation is u j V j transpose.

So, these are the two main decompositions that we get. So, from the bases, from the choice of our bases, we are able to get the generalization of the product decomposition we had for the symmetric or the hermitian matrices. We are also able to get the sum decomposition that we had for hermitian matrices.

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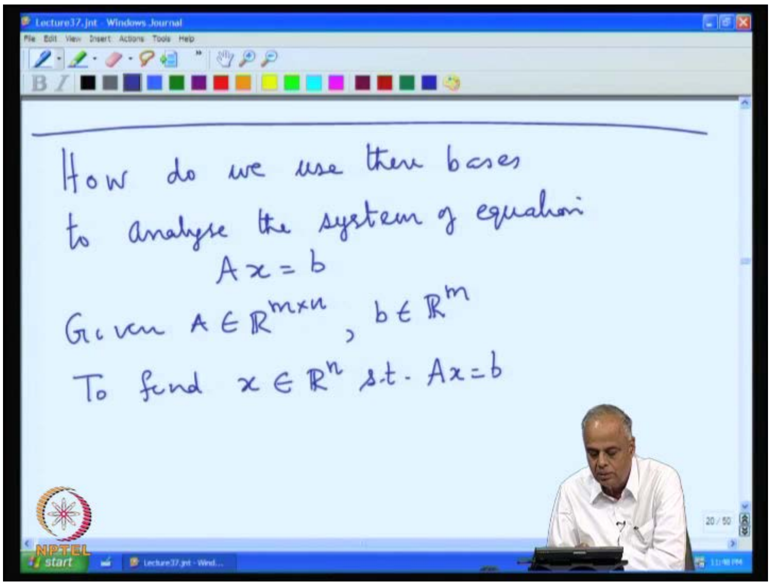

Now, how do we use these basis we have chosen to analyze the system of equations A x equal to b. Remember, we are given a is in R m cross n and b in R m we want to find x in R n such that A x equal to b. This is our problem of the system of equation. Now, we are going to use the basis that we have chosen in order to analyze the system of equation.

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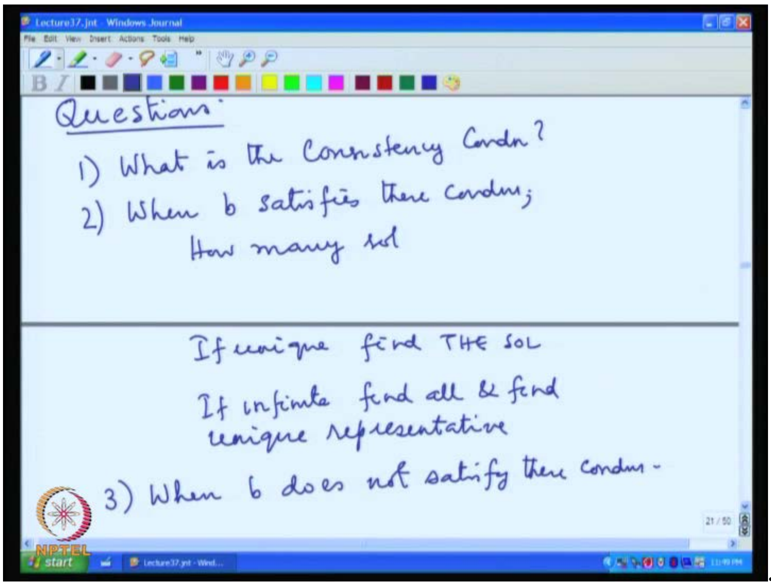

So, remember what are the questions that we asked? The questions that we have to answer is what is the consistency condition the consistency condition that is what condition should be satisfied in order that the system as a solution. Then when b satisfies these conditions; we want to know how many solutions, then if it is unique we want to find the solution. If infinite; the number of solutions is infinite; find all and find unique representative solution.

Among the many solutions that we have, you must say what is some criterion that we can use in order to pick a representative solution?

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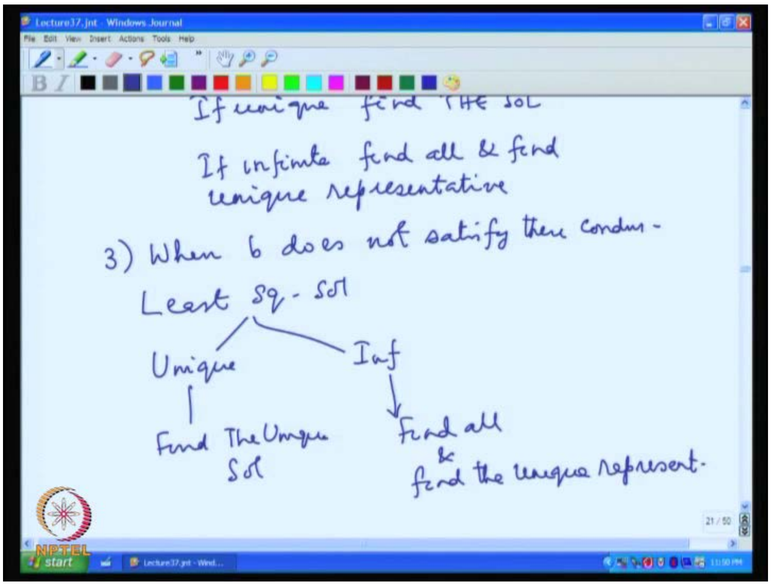

And then we have to worry about when V does not satisfy these conditions. Then we have to worry about what is known as the least square solutions. Then we have to worry about uniqueness or infinite and if it is unique, we want to find the unique solution and if it is infinite find all and find the unique representative unique representative.

Now, these are the fundamental questions that we raised about a system of equations. Consistency condition, if it is satisfied solution exists; unique find it, infinite find all and find a representative. If it does not satisfy then you can do what is meant by least square solutions. Then is it unique. Is it infinite? If it is unique; find it. If it is infinite then find all of them and among them how do we choose a core solution or a representative solution? These are all the fundamental questions.

Now, we shall see how to use the basis that we have chosen in order to analyze all these questions and get satisfactory answers. This will be our next topic.