Advanced Matrix Theory and Linear Algebra for Engineers
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Lecture No. # 35

Hermitian and Symmetric Matrices – Part 4

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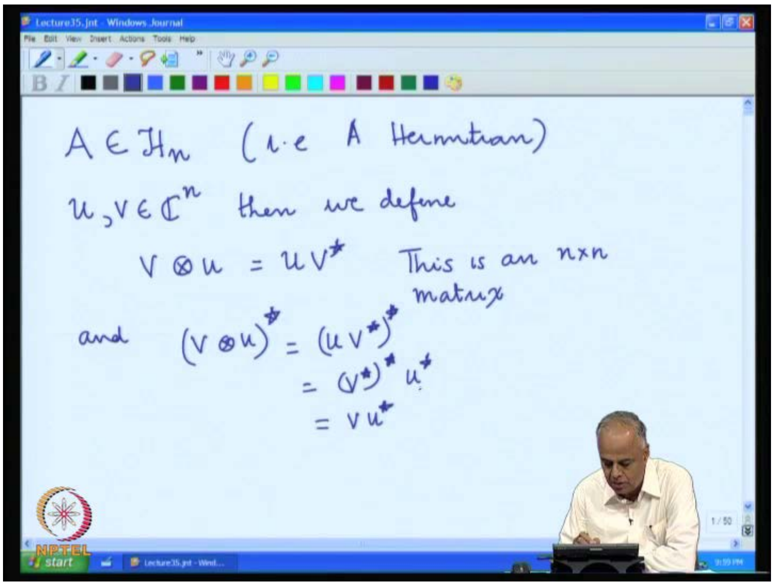

We have been looking at the decomposition of a Hermitian matrix or the sum of rank one matrix; let us recall some of the things that we obtained in this context. So, we shall consider a matrix A, which is H n, that is, A is Hermitian, we want to express this or the sum of rank one matrices.

Now, we will follow the notation. Let us recall, if u and v are in C n, and then we define v tenser u to be matrix uv star, notice that this is an n by n matrix; this is an n by n matrix. And if we now look at this matrix and look at its Hermitian conjugate, it is uv star star, which is v star star into u star. So, when you take transpose of the product, the product of the transpose in the reverse order. But when you take the Hermitian conjugate of a product, the product comes in reverse order, but v star star is v, so it is vu star. So, we have this notation, that v cross u is equal to uv star star, which is v star star into u star is equal to vu star.

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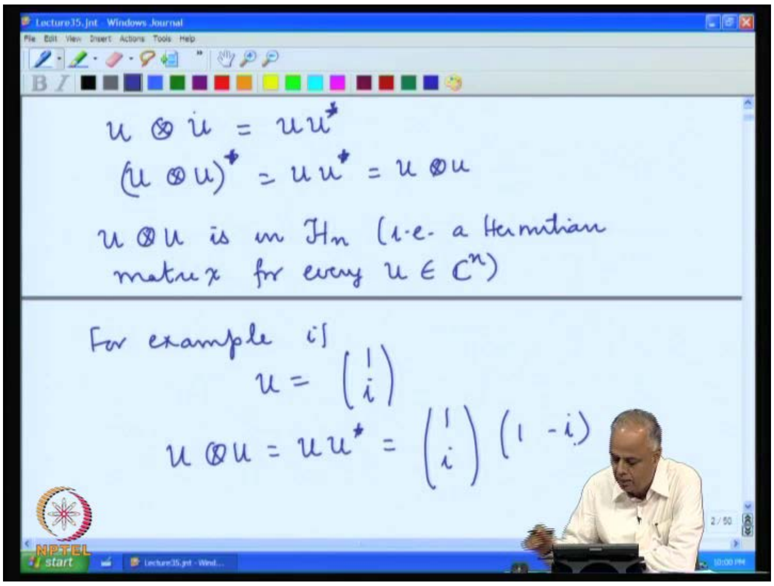

Now, we have this tenser notation. With this notation we shall particularly look at the idea of taking the tenser product of a vector with itself. Then, we get u cross u as uu star, and then uu star. Since u is equal to v above is equal to uu star, which is u tenser u and therefore, u tenser u is a Hermitian matrix. u tenser u is in H n, it is a Hermitian matrix; it is a Hermitian matrix for every u in C n.

For example, if u equal to say, 1, i, then u tenser u is equal to uu star; u is 1, i; u star is the transpose conjugate, so it is 1 minus i. So, when you take the product to get 1 minus i, i and minus i square, which is 1 and which is Hermitian matrix, which is a Hermitian matrix. So, if you take a vector in C n and take the tenser product of the vector with itself, we get a Hermitian matrix. Now, we use this notation in the decomposition of the matrix.

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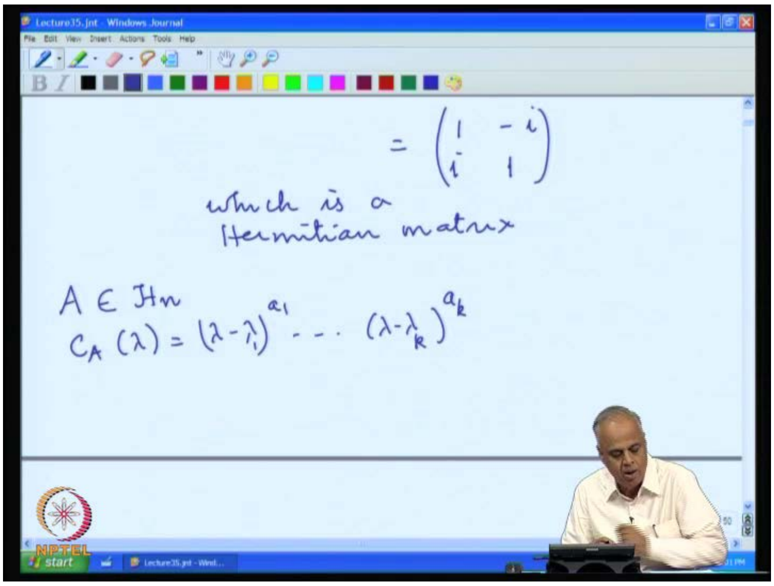

Recall, if A is a Hermitian matrix and its characteristic polynomial is lambda minus lambda 1 power a 1, etcetera, lambda minus lambda k power a k. Our usual notations were lambda 1, lambda 2, lambda k are the distinct Eigen values of a and a 1, a 2, a k, are the algebraic multiplicities.

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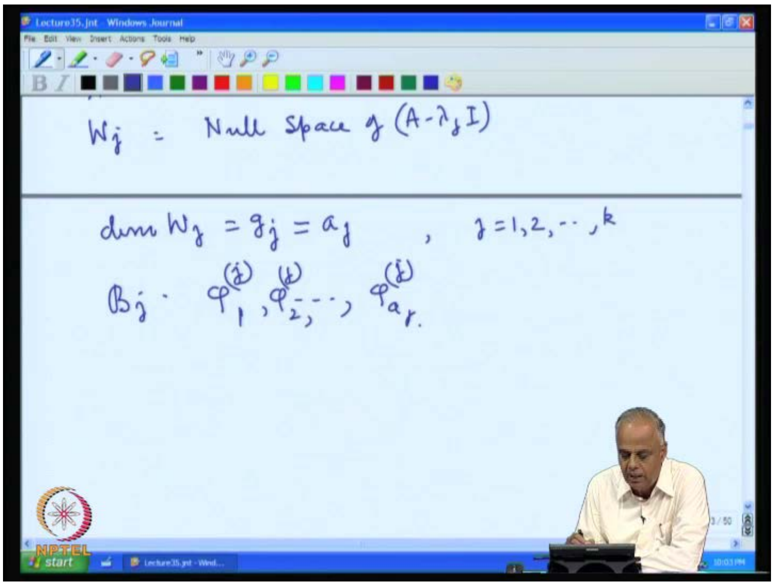

Then, we have W j, the null space of A minus lambda j I, which is the eigenspace corresponding to the Eigen value lambda j and the dimension of W j, which is the geometric multiplicity, in the case of Hermitian matrix will always be equal to the algebraic multiplicity. This is true for each j equal 1, 2, k.

So, for each one of the Eigen values, the corresponding Eigen space has the same dimension as the algebraic multiplicity. Then, we denote it by B j and ortho-normal basis for the eigenspace W j. The superscript j says that it is a basis for the eigenspace W j and the subscript is the index of the basis vector. This is first basis vector for phi j 1, phi j 2 the second basis vector, phi j a j if the a j, this is vector for W j.

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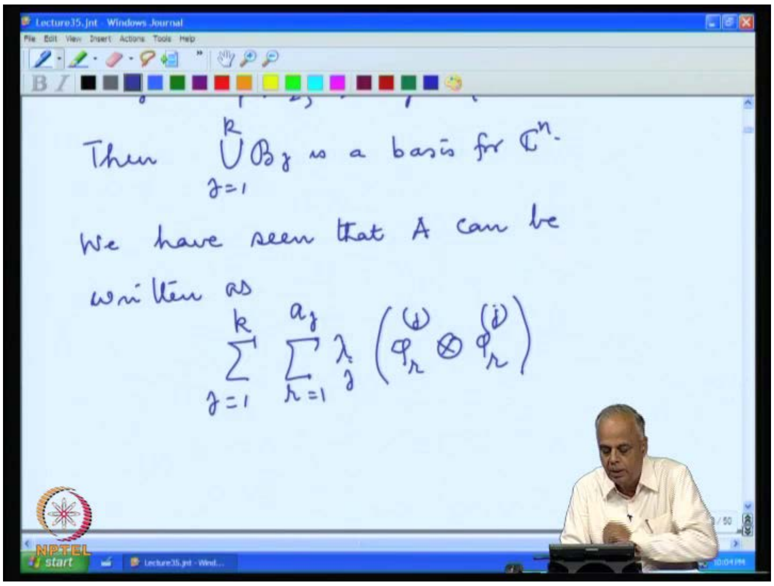

Then, the union of all this j equal to 1 to k is a basis for the whole space. And we saw, that this matrix A, we have seen, that A can be written as the following sum. What is that sum? For each one of this Eigen values and eigenspace we look at first the B j basis, there are a j of them, for each one of these vectors we construct the tenser product.

So, we construct phi j r tensered with phi j r and this is going to be a matrix. As observed above, this is going to be a n by n matrix and since it is the tenser product of a tenser with itself, is going to be a Hermitian matrix. And so, that is the Hermitian matrix of order n by n, and it is multiplied by the corresponding Eigen values j, and we look at the sum from r equal to 1, 2, a j, that is, for each one of these basis vectors in the B j basis or the basis for W j, we look at the tenser product. Note, that this is an orthonormal basis, in addition. Now, we do this for every one of the Eigen values, so j equal to 1 to k. So, totally, we get a 1 plus a 2 plus a k n term.

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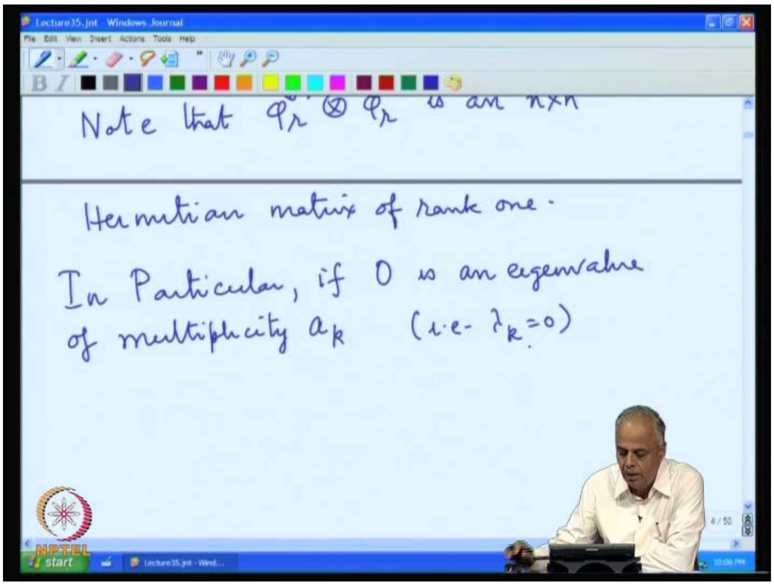

So, notice, that phi j r, phi j r is a n by n Hermitian matrix of rank one. Thus, we have expressed the matrix A and the sum of matrices of rank one. In particular, if 0 is an Eigen value of multiplicity a k, recall kth Eigen value is leave the 0, then that is, we say lambda k equal to 0.

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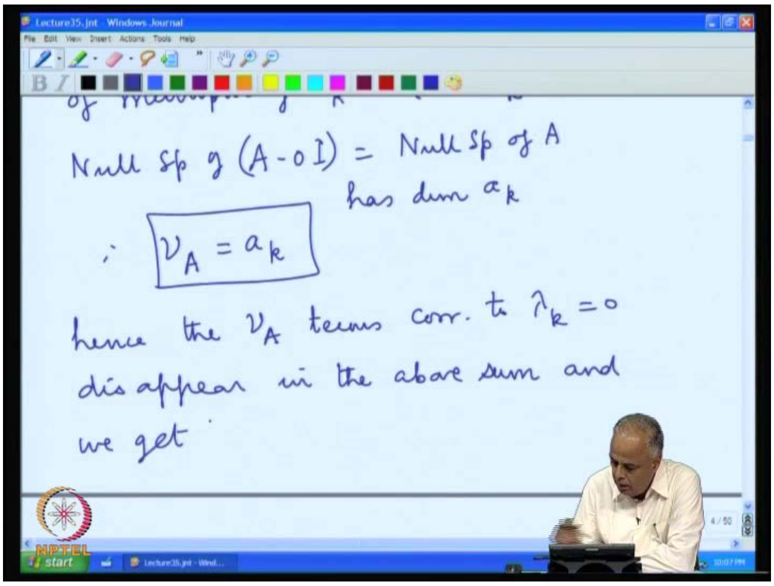

Then we know, that the null space of A minus the corresponding Eigen value is 0, so A minus 0 I, which is the same as the null space of A, must have dimension same as a k, but we know, that the dimension of the null space of A is nu A. Therefore, the nullity must be equal to a k.

So, suppose we have 0 as an Eigen value, then nu a must be equal to the algebraic and geometric multiplicity of this Eigen value. And therefore, if you look at the sum, in the sum corresponding to the term lambda k we will be multiplying by lambda k every one of these a k terms, but lambda k being 0, these terms will disappear and hence, the nu A terms corresponding to lambda k equal to 0 disappear in the above sum.

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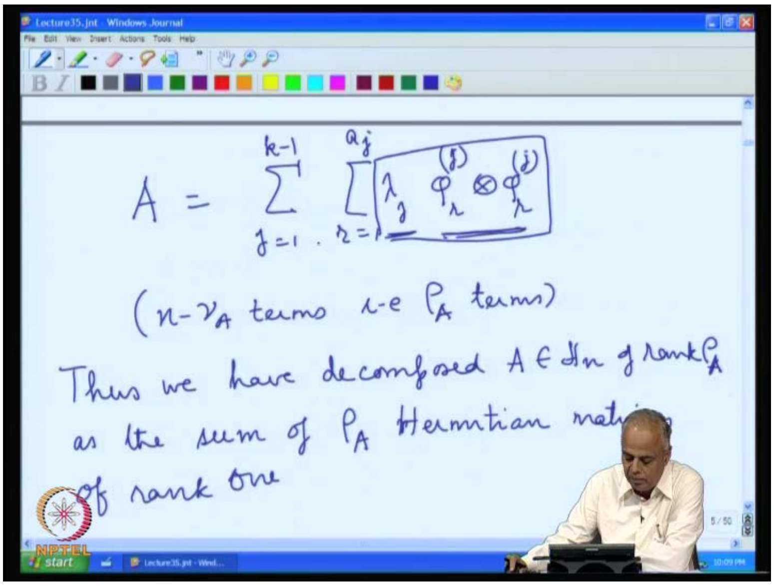

And we get A equal to summation j equal to 1 to k minus 1, summation r equal to 1 to a j lambda j phi j r tensered with phi j r. Now, obviously there are a 1 plus a 2 plus a k minus 1, which is n minus nu A, so n minus nu A terms, but n minus nu A is the rank of the metrics, so rho A terms. So, we have, now each one of them is non-zero, because these are orthonormal vectors, they are non-zero vectors and lambda js are non-zero.

So, thus we have A, we have decomposed A, which is a Hermitian matrices, as and its rank is rho A, so its rank is rho A as the sum of rho A matrix. In fact, rho A Hermitian matrices of rank one, each one of these terms is Hermitian. This is real because Hermitian matrices, the Eigen values are real. So, when you multiply a Hermitian matrix by real number you get a Hermitian matrix, so this all quantity is a Hermitian matrix. So, therefore, the whole sum, if the sum of Hermitian matrices and each as rank 1.

So, thus we have seen, that if I have a Hermitian matrix of rank rho, it can there be split into the sum of rho 1 rank matrices. We will always reduce everything to one rank level. Let us look at some examples of this decomposition.

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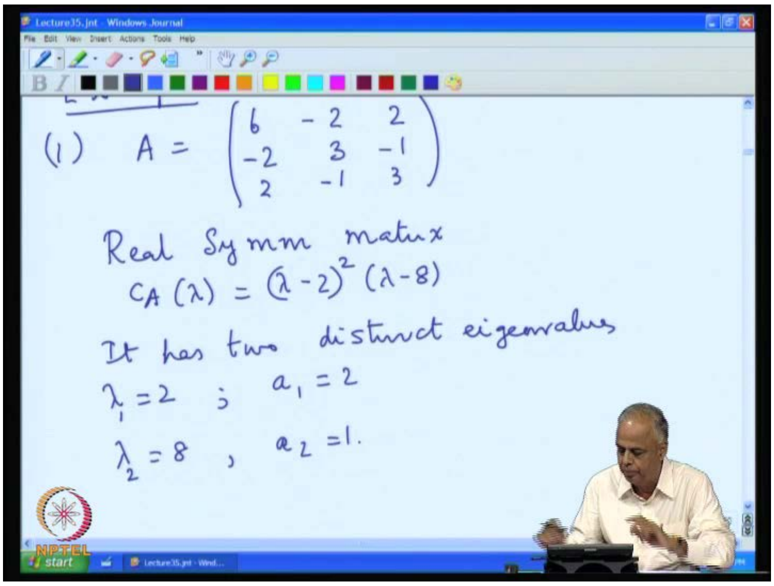

So, let us look at the first example, the matrix, which you have seen in the last lecture. You see, that in all the above decompositions, in particular, if A is real symmetric, we replace star by transpose everywhere because conjugation does not give anything new in the real situation. So, this a real symmetric matrix and we have seen, that in the previous lectures, that its characteristic polynomial is lambda minus 2 square into lambda minus 8 and therefore, it has, it has two Eigen values. It has two distinct Eigen values, one of them is 2 and its multiplicity is 2 because they have lambda minus 2 square. The second Eigen value is 8 and its multiplicity is 1.

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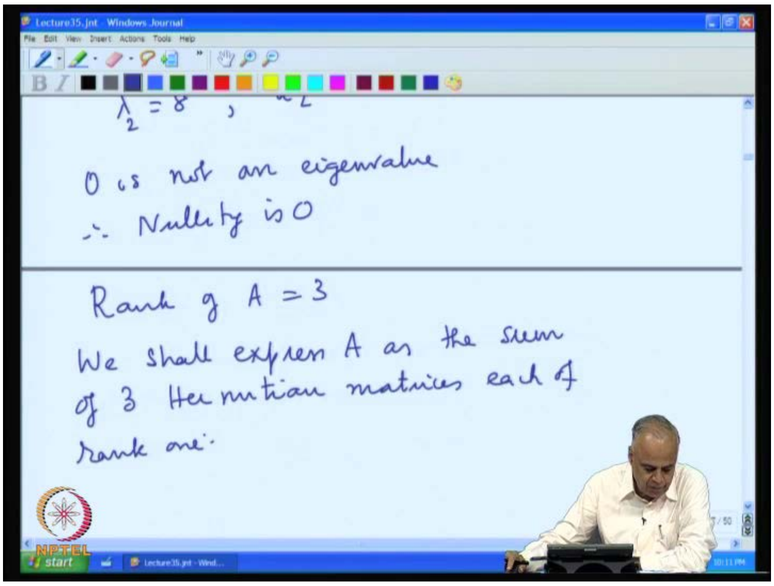

We see therefore, that 0 is not an Eigen value, therefore nullity is 0 and hence, rank is three. Since the rank is 3, we shall express A as the sum of thee Hermitian matrices, each of rank one. How do we do this? For this, define the eigen spaces corresponding to these eigen values.

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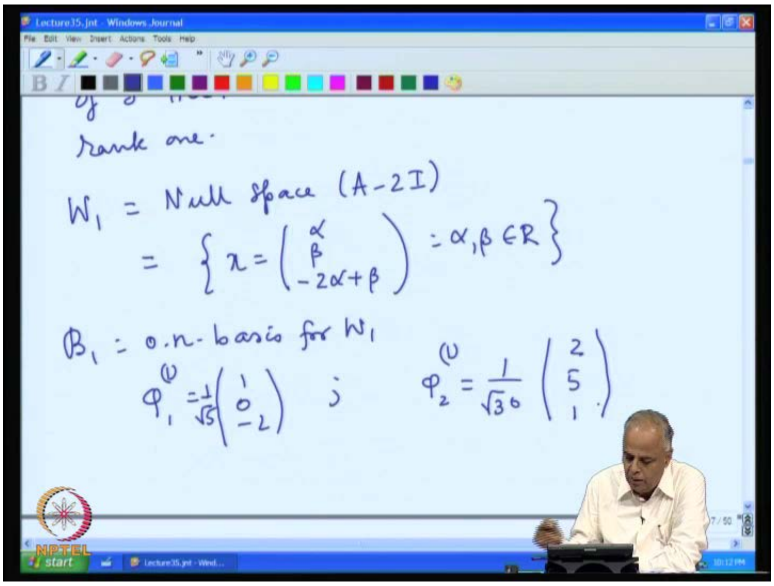

The W 1, in the null space corresponding to the Eigen value 2, it is A minus 2I and we have found in the last lecture, that this consist of all vectors of the form alpha, beta, minus 2 alpha plus beta, where alpha and beta real. And we found on orthonormal basis B 1, orthonormal basis for W 1. We found there will be two vectors, phi 1 1 and phi 1 2. Because the multiplicity is 2 corresponding to the Eigen value lambda 1, there will be phi 1 1 and phi 1 2. The phi 1 1 we found as 1 by root 5 into 1, 0, minus 2 and the other we, we found as 1 by square root of 30, 2, 5, 1. These are the two, two orthonormal vectors, which form a basis for the eigen space corresponding to the eigen value lambda 1, which we have found in the previous lectures.

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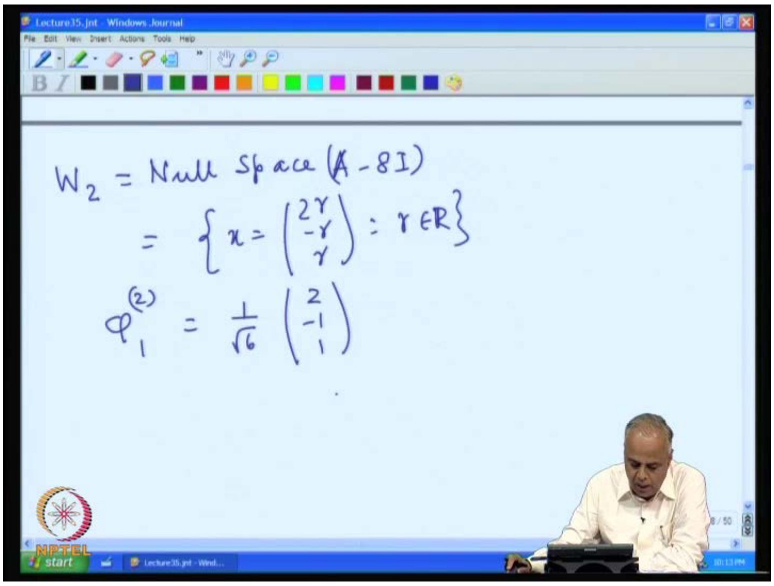

Similarly, w 2 is the eigen space corresponding to the eigen value 8, so it is the null space of A minus 8I, and we found this to be consisting of all vectors of the form x is equal to 2 gamma, minus gamma, gamma; the gamma is real. And there is going to be only one, the dimension being 1, there is going to be only one orthonormal basis for that, and this we found to be 2, minus 1, 1, square root of 6. So, we have the three Eigen values. Now, we and one of them is repeated twice, two, two are the Eigen values and then, the other Eigen values (( )) corresponding to them.

We have the three Eigen vectors, we call the, this corresponds to the lambda 1 equal to 2. This also corresponds to lambda 1 equal to 2, and this corresponds to lambda 1, 2, equal to 8.

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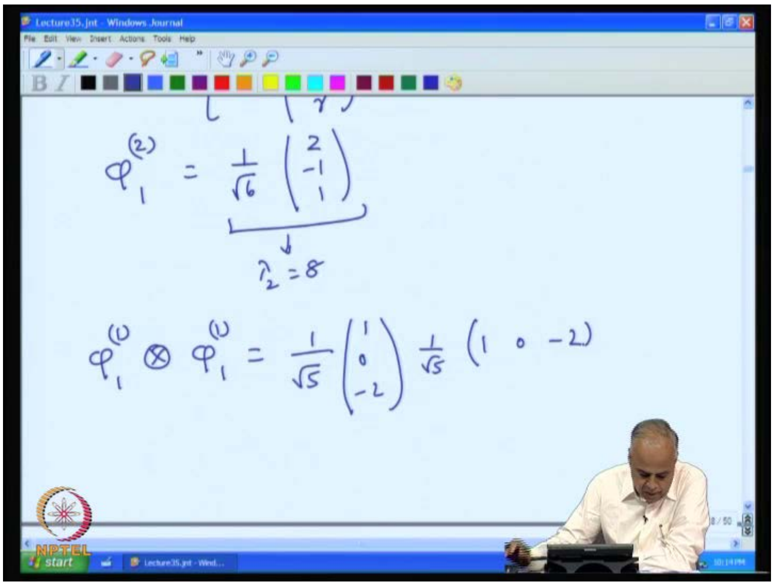

Now, we form the (( )) the tenser products corresponding to each one of these Eigen vectors. So, we first calculate phi 1 1, phi 1 1, and it what is phi 1? It is just the 1 by root phi of 1, 0, minus 2 times phi 1 transpose, which is 1, 0, minus 2. Remember, everything is real here, so we have to look at the transpose.

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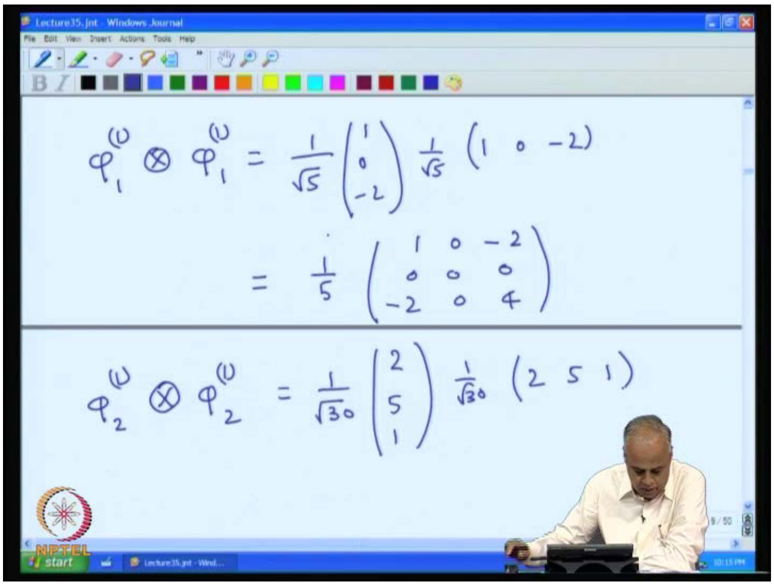

And the matrix is 1 by 5. If we carry out the product we get this matrix, we have done this in previous lecture also, so it is a simple matrix multiplication, we get this. Similarly, we look at the, again the first Eigen value itself. But look at its second Eigen vector and take the cartesian or the tenser product, it is 1 by root 30 into 2, 5, 1 into the transpose of that.

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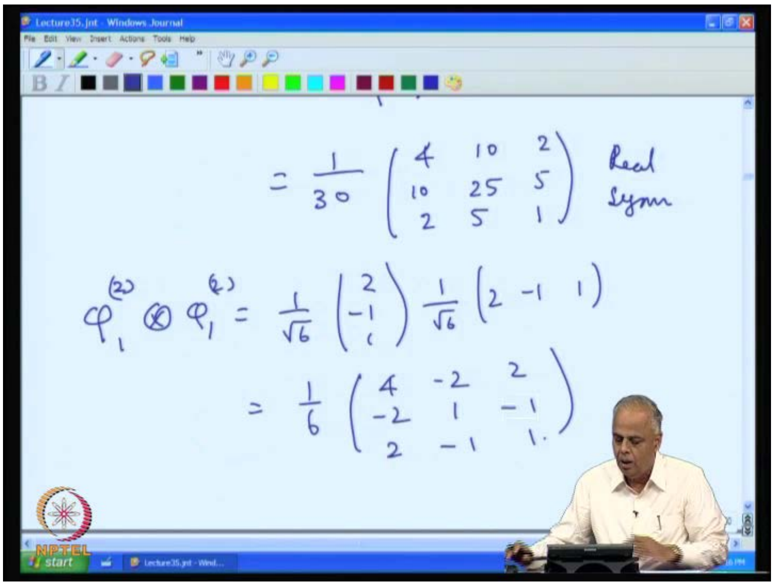

And when we carry out the multiplication we get, 1 by 30 into 4, 10, 2, 10, 25, 5, 2, 5, 1. Notice, that this is a Hermitian or as real symmetric matrix. This is what we observed, that when you take Hermitian vector in c n and tenser with itself we get a Hermitian matrix. And the real case, if we take a real vector and tenser it with itself we get a real symmetric matrix; again, this is a real symmetric matrix.

And then, finally we look at the Eigen vector corresponding to the second Eigen value, and do the tenser calculation with respect to the, it is 2, minus 1, 1, 1 by root 6 into 1 by root 6, the rho vector, 2, minus 1, 1.Then, we carry out this product we get 1 by 6, 4, minus 2, 2, minus 2, 1, minus 1, 2, minus 1, 1.

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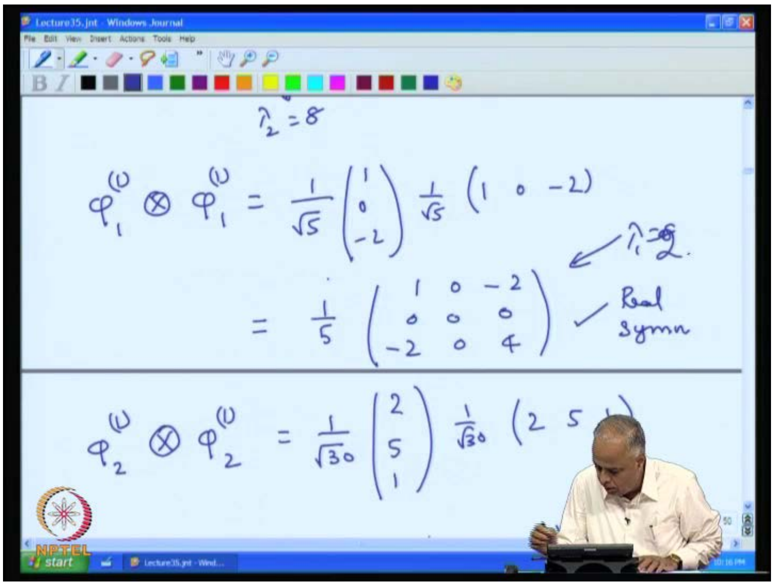

Now, this corresponds to still the Eigen value lambda 1 equal to 2 and this corresponds to the Eigen value lambda 1 equal to 2 and the last one corresponds to lambda 2 equal to 8. Now, having constructed the tensor products of each one of these Eigen vectors, we multiplied them by the corresponding Eigen values.

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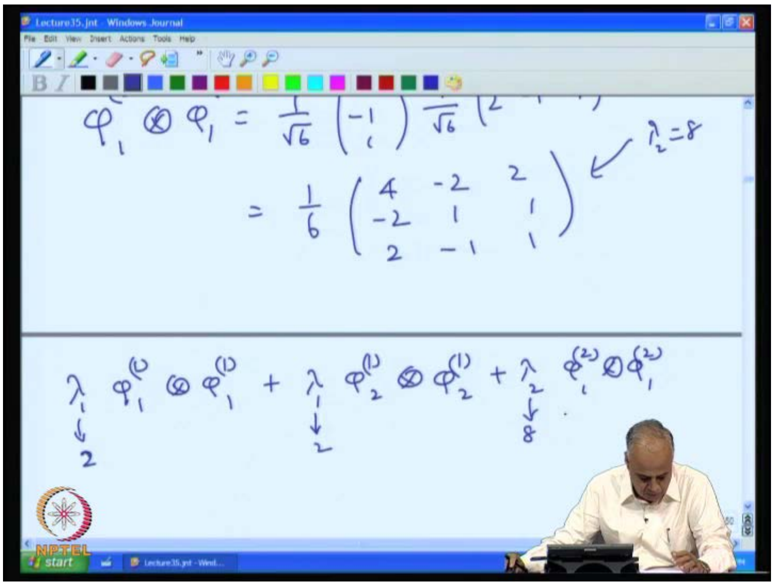

So, we look at lambda 1. This phi 1 1 corresponds to Eigen value 1 lambda 1. The phi 1 2 also corresponds to the Eigen value lambda 1 and the phi 2 1 corresponds to the Eigen value lambda 2. Lambda 1 is 2, lambda 1 is 2, lambda 2 is 8 and we have calculated the tensor products above.

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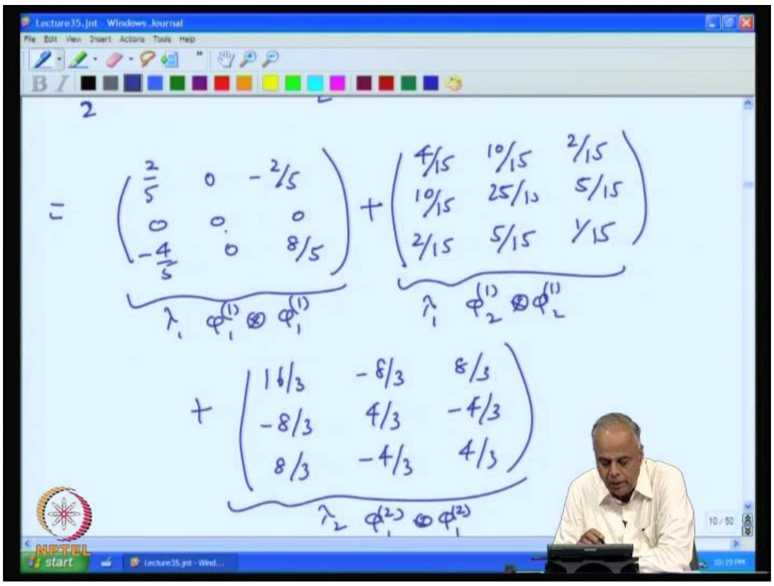

When you substitute all that, we get the summation as 2 by 5, 0, minus 2 by 5, 0, 0, minus 4 by 5, 0, 8 by 5 plus 4 by 15, 10 by 15, 2 by 15, 10 by 15, 25 by 15, 5 by 15, 2 by 15, 5 by 15, 1 by 15. These are the two terms corresponding to the first Eigen value. This is lambda 1; this is the term, which is lambda 1, phi 1, 1 phi 1, 1 with the value lambda 1 equal to 2 and phi 1 1 tenser phi 1 1, which we have found above. This is lambda 1 again, with lambda 1 equal to 2, but phi 1 2 tenser phi 1 2 plus the next one is the other Eigen value, which is 16 by 3 minus 8 by 3 8 by 3 minus 8 by 3 4 by 3 minus 4 by 3 8 by 3 minus 4 by 3 4 by 3. Now, this is the term, which is lambda 2 phi 2 1 tenser phi 2 1.

And when we add all this we get check, this exactly adds up to the given matrix A. Now, notice, that the first matrix we have here is of rank 1, because every rho is a multiple of the first rho. Similarly, the second matrix is a matrix of rank 1, because every rho is a multiple of the third rho and the third matrix is a multiple of either matrix of rank 1 because every rho is a multiple of the third rho. Observe also, that each one of this is a Hermitian matrix.

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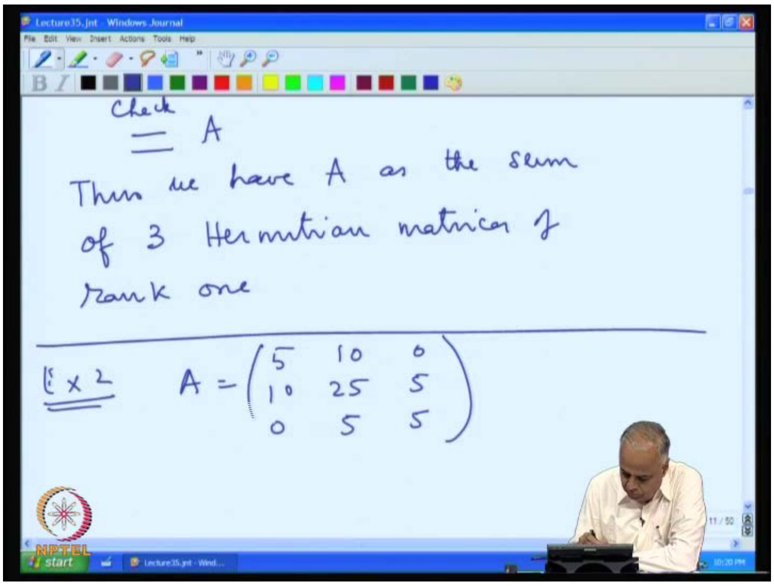

Thus, we have A as the sum of three Hermitian matrices of rank one. Now, we have three of them because the rank of the matrix was three. So, whenever we have rank rho, you will have rho matrix for which we will take of each of rank one.

Let us look at another example, A equal to 5, 10, 0, 10, 25, 5, 0, 5, 5.

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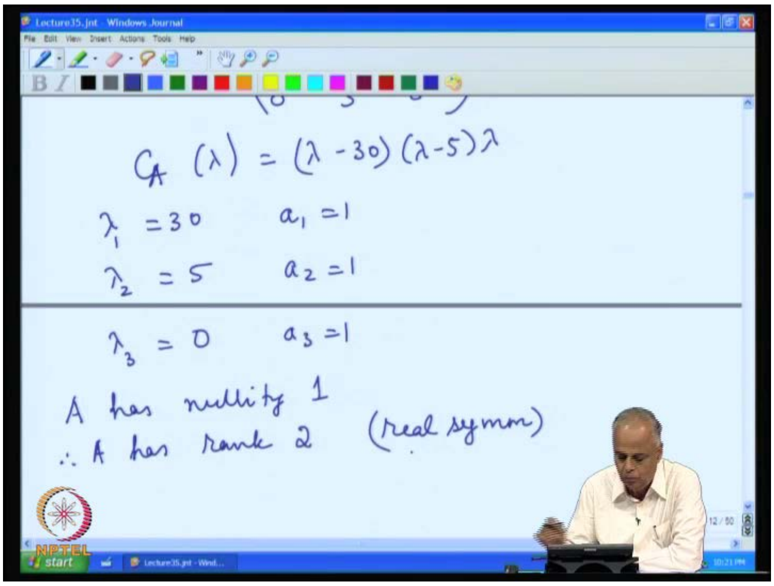

Now, if we calculate the characteristic polynomial we get lambda minus 30 into lambda minus 5 into lambda. If we write down the determinant lambda minus A and expand it, you will see, that it can be factored as, (( )) lambda equal to lambda minus 30 into lambda minus 5.

What are the Eigen values? Lambda 1 equal to 30, algebraic multiplicity 1; lambda 2 equal to 5, algebraic multiplicity is 1. Notice, that lambda 3 is 0, which means, that A has nullity 1, because lambda equal to 0 is an Eigen value of multiplicity 1 and therefore, A has rank, the matrix is 3 by 3, nullity is 1. So, the rank is 3 minus 1, which is 2, therefore A is real symmetric and has rank 2.

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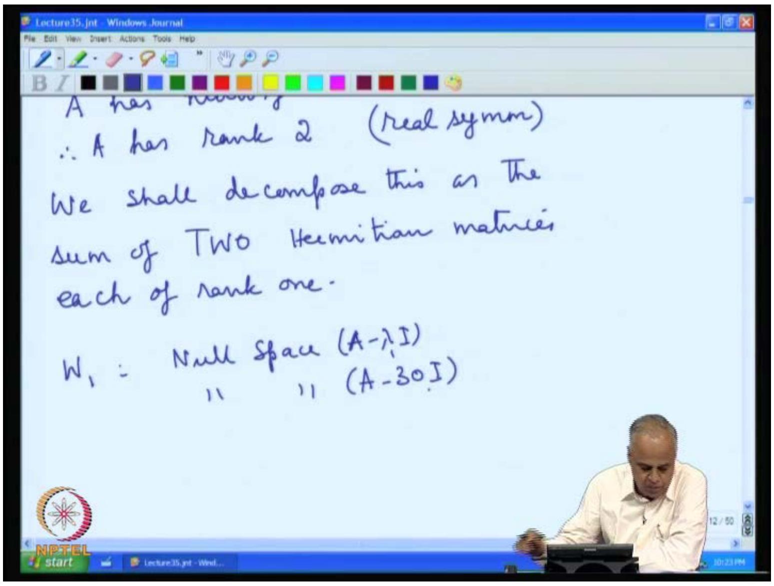

And therefore, we have, we shall decompose this as the sum of, now since the rank is two even though the matrix is 3 by 3. Since the rank is two, we decompose there the sum of two Hermitian matrices, each of rank one.

Now, how do we do the decomposition? Once again we require, notice, that as we observed in the beginning, when you do this decomposition, the terms in the decomposition corresponding to the Eigen value zero disappear because we multiply by the Eigen value. So, we have to only concentrate on the non-zero Eigen values. Therefore, we must look at only the null space of the first two Eigen values. The first Eigen value is, thus, the null space of A minus 30I because the first Eigen value was 30.

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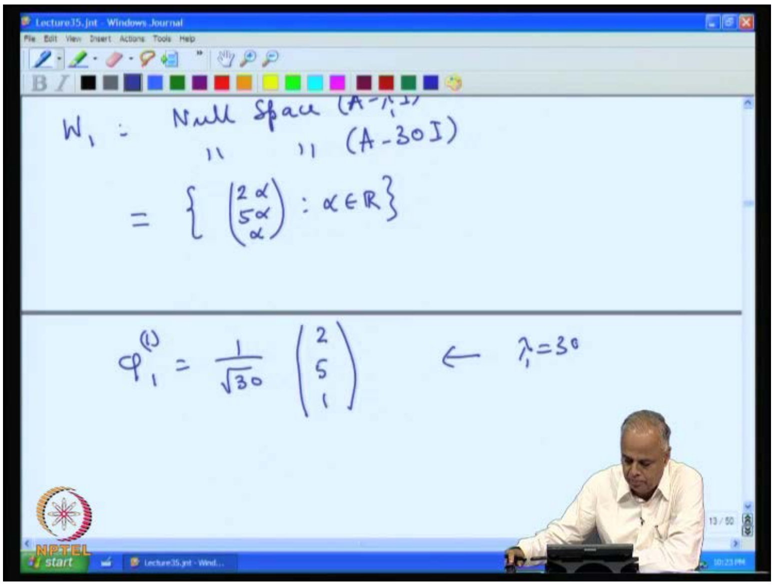

And it can be shown, you can easily compute the solutions to be of the form 2 alpha, 5 alpha, 1 as alpha belongs to R and therefore, an orthonormal basis for that is 1 by root 30 into 2, 5, 1. So, that is corresponding to lambda 1 equal to 30.

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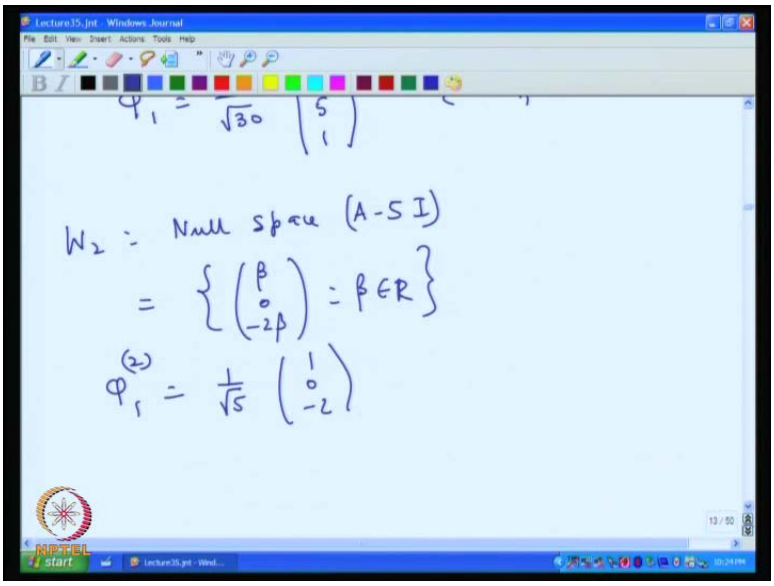

Then, we look at W 2, which is the null space of A minus the second Eigen value, which is 5I. Now, we tenser, we can verify, that again this is consisting of all the vectors of this form and the corresponding orthonormal Eigen vector is 1, 0, minus 2.

Now, in the decomposition, only the Eigen vectors corresponding to the non-zero Eigen values appear. So, we do not have to worry about W 3, because W 3 is the eigen space corresponding to the Eigen value 0.

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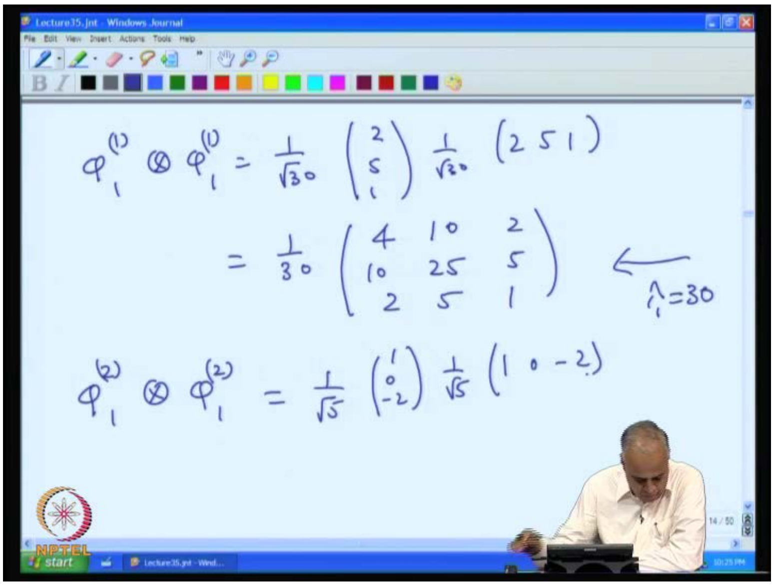

Now, we compute phi 1 tenser phi 1, which is 1 by root 30 into 2, 5, 1 into 1 by root 3 into 2, 5, 1, which we have calculated earlier in the previous example, and it turns out to be 1 by 30, 4, 10, 2, 10, 25, 5, 2, 5, 1. This is the same as what was obtained in the previous example and this corresponds to, we keep reminding, that this corresponds to the Eigen value 30.

Similarly, the 2nd Eigen value we calculate the tenser phi 2 1, phi 2 1, which is 1 by root 5 into 1, 0, minus 2 into 1 by root 5 into 1, 0, minus 2.

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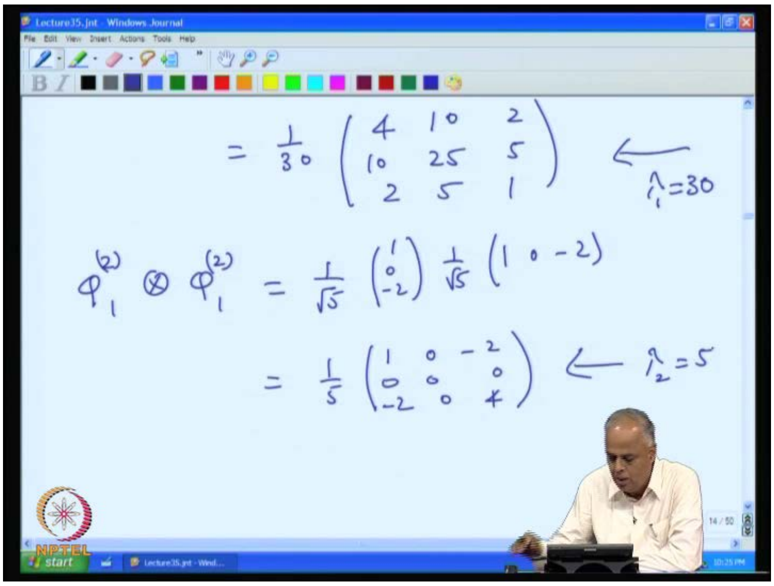

Again, we have calculated this in the earlier example, it turns out to be 1, 0, minus 2, 0, 0, 0, minus 2, 0, 4 and this corresponds to the eigenvalue lambda.

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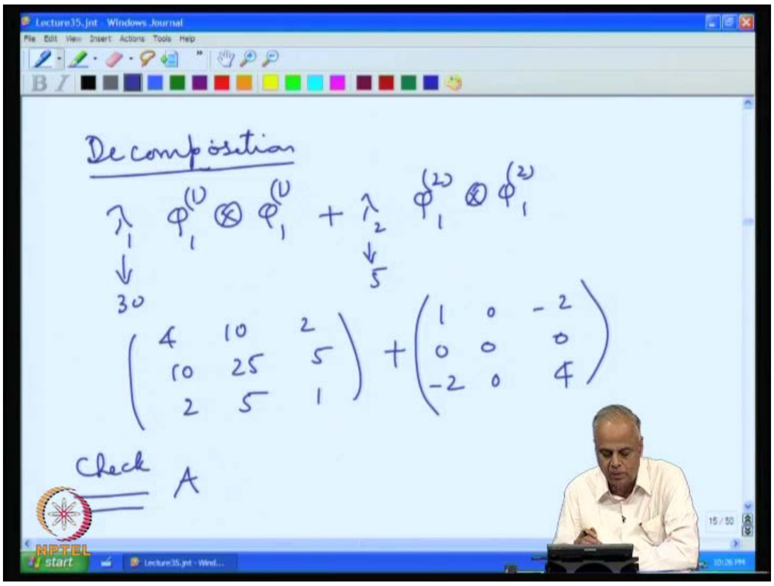

Now, for the decomposition we have, therefore how does the decomposition look like? We have to look at the lambda 1, the tensor product of the Eigen vectors must be multiplied by the corresponding eigenvalue and they have only two non-zero1s here. Now, we have lambda 1 is 30, lambda 2 is 5. So, if we now multiply this by 30, this has to be multiplied by 30, so the 1 by 30 and 30 get cancelled, we get 4, 10, 2, 10, 25, 5 and 2, 5, 1. And then, when we multiply this matrix by 5, the 1 by 5 gets cancelled we get 1, 0, minus 2, 0, 0, 0, minus 2, 0, 4. And when we add, check this is exactly equal to the given matrix here.

Notice, that each one of the matrices in above sum is a Hermitian matrix, real symmetric in this case. And the first matrix is of rank one because every row is a multiple of the 3rd row and a 2nd matrix is of rank one because every row is a multiple of the 1st row.

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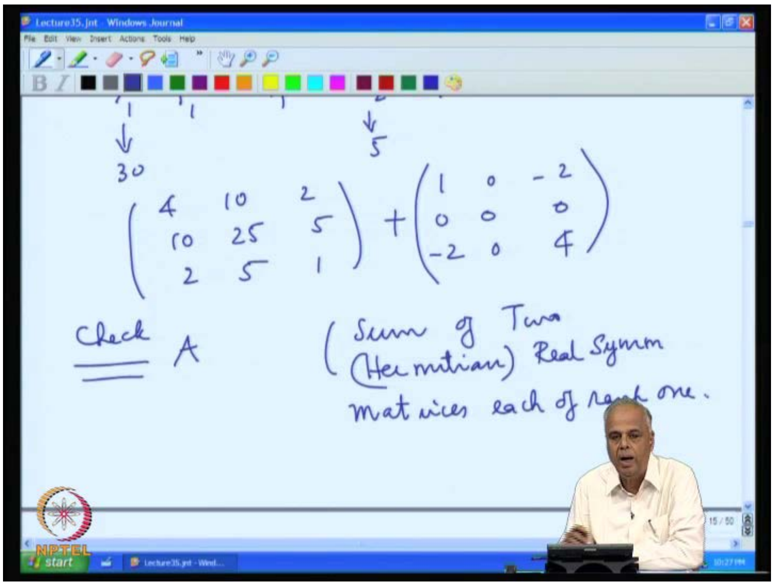

So, therefore, we have expressed A as the sum of two Hermitian matrices, in this case real symmetric matrices because you are with real symmetric, real symmetric matrices, each of rank one.

And why do we have only two matrices in the sum or two terms in the sum? It is because of rank two. So, thus, if we have a Hermitian matrix of rank rho, it can be always split into your sum of rho terms of rho matrix. Each matrix is Hermitian, each matrix is of rank 1, so rank rho matrix is the sum of row one rank matrices, so the ranks can be split. This is the decomposition of a Hermitian matrix into rank 1 matrices, which we saw last time. These are the two examples.

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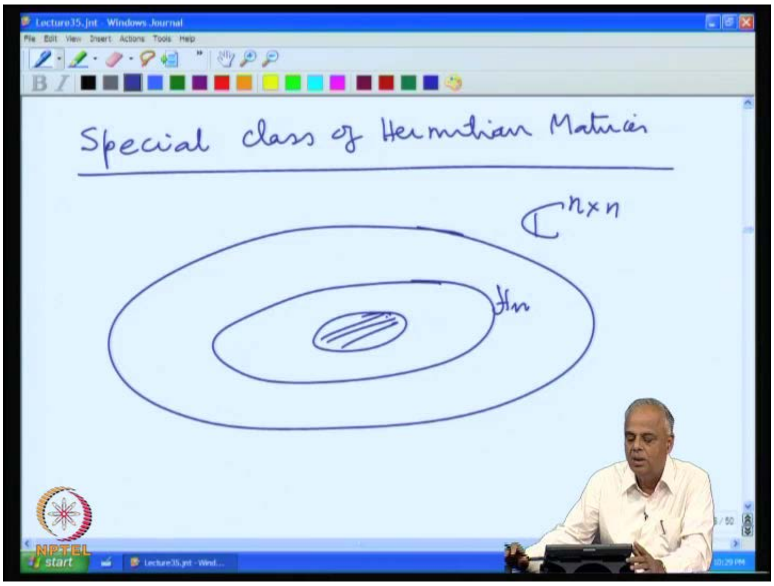

Now, we look at a special class of Hermitian matrices; special class of Hermitian matrices. Now, first we looked at all n by n matrices, then we saw the various diagonalizability criteria, namely a m equal to g m and then we found, that there are matrix for which a m may not be equal to g m. Therefore, all n by n matrices are not diagonalizable. Then, we looked at a special class of n by n matrices, namely Hermitian matrix where diagonalizability was guaranteed.

Now, inside this Hermitian matrix class we are going to look at a special class. So, we had first the collection of all n by n matrices, in that we had the H n, the Hermitian matrices. Now, we are going to look at a sub-class of the Hermitian matrices.

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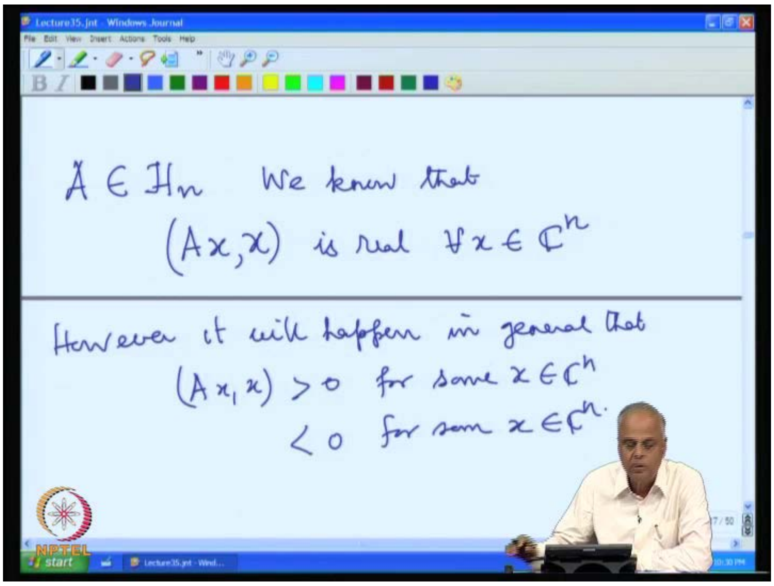

Now, suppose I take any Hermitian matrix. We know, that one of the fundamental properties of Hermitian matrices is that (A x, x) is real for every x in C n. If you take even a complex matrix n by n and even a complex vector x, as long as the matrix is Hermitian (A x, x) always turns out to be real. This is a typical property of Hermitian matrix. So, for Hermitian matrices, (A x, x) is always real. However, it may, it will happen in general, that (A x, x) is positive for some x, negative for some x and of course, is obviously 0 for x equal to 0, x equal to 0 for x equal to theta n.

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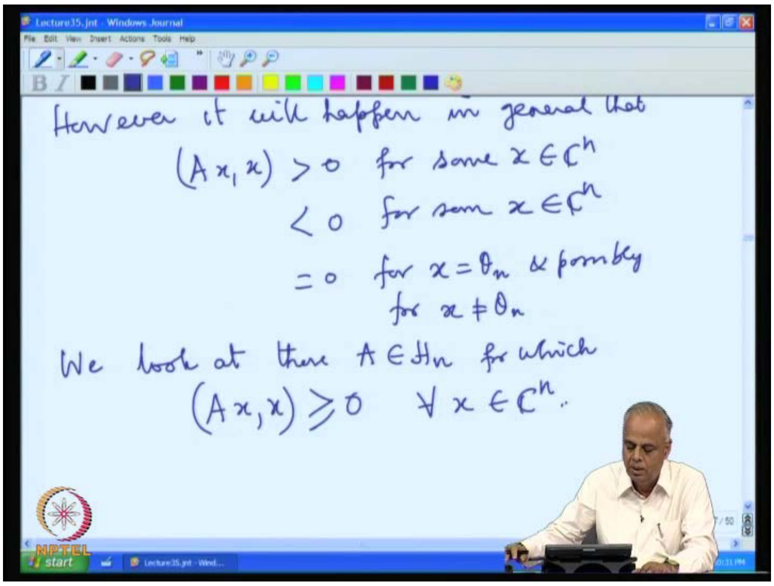

And possibly, for x not equal to theta n. So, in general even though we know, that (A x, x) is real, we cannot say precisely, whether it is going to be positive or negative. For a general Hermitian matrix it could turn out to be positive for some x, negative for some x and zero for some.

Now, we are going to look at a special subclass for which it always maintains the same sign. So, we look at those A, which are Hermitian, a number we write here belongs to H m, what we mean is that A is a Hermitian matrix. So, we look at those A in H m for which (A x, x) is always greater than or equal to 0 for every x in 0. Such matrice are called positive.

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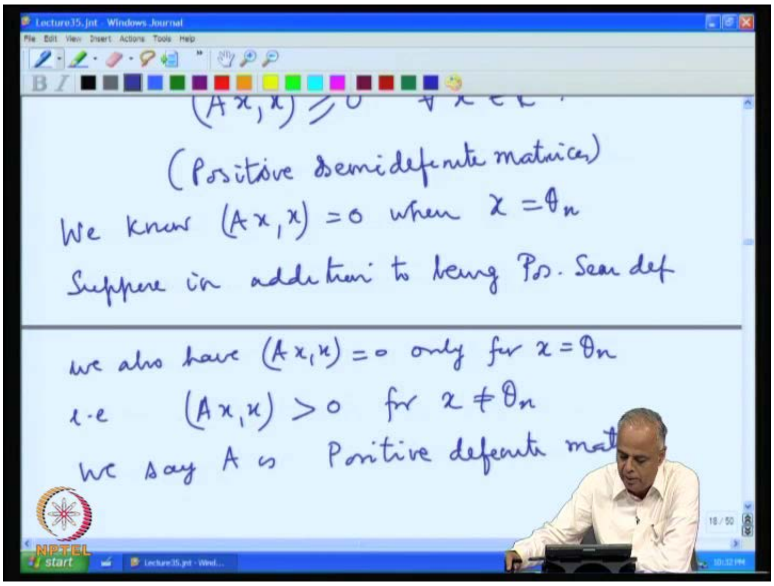

We will write the formal definition, positive semi-definite matrix.

Now, we know, that (A x, x) is 0 when x is equal to theta n. We know, (A x, x) is equal to 0 where x is the 0 vector. Suppose, in addition to being positive semi-definite, we also have 0 is the only vector for only, for x equal to theta n, which means what? It is always greater than or equal to 0 and equal to 0 only for x equal to theta n, that is, (A x, x) will be strictly positive for x not in 0. We, then we say A is positive definite.

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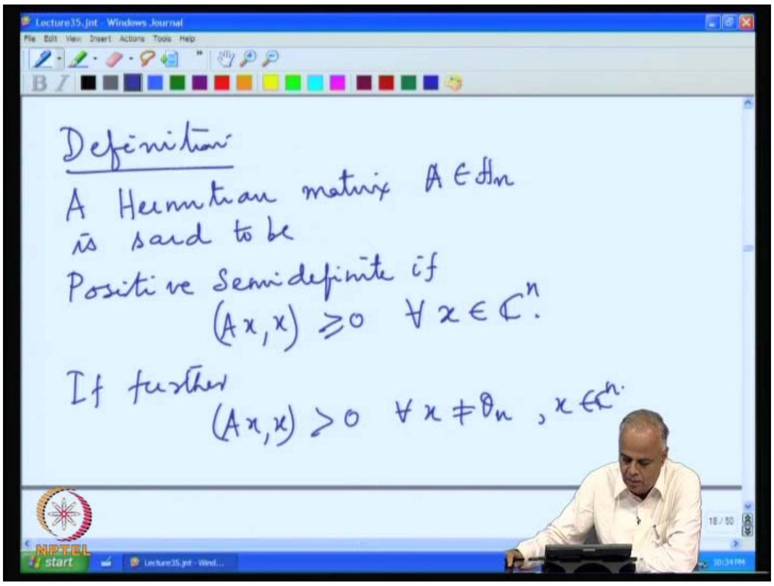

So, we will right the formal definition. So, definition, first of all, all these notions are positive semi-definite, positive definite. We are introducing only for Hermitian matrix.

So, a Hermitian matrix, that is, n by n Hermitian matrix is said to be positive semi-definite if (A x, x) is greater than or equal to 0 for every x belonging to, for every x belonging to C n.

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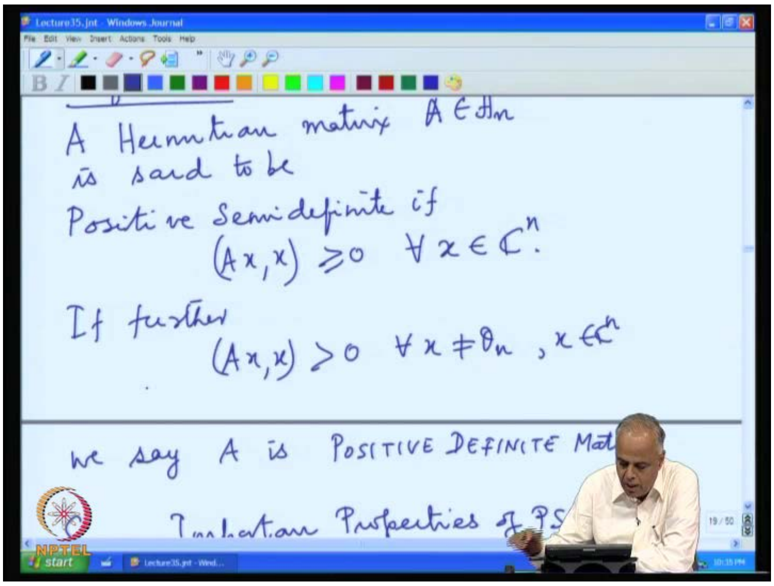

If further, (A x, x) is strictly positive for every x not equal to theta n, x belongs to C n, we say, A is positive definite matrix.

Analogously, we can define negative semi-definite by replacing greater than or equal to 0 by less than or equal to 0 and negative definite by replacing greater than 0 by less than 0 above. So, we have the notions of positive semi-definite matrices and positive definite matrices. What are the some important properties of such matrix? We are going to use all these properties eventually to analyze a general (( )) of, I let PSD for positive semi-definite matrix.

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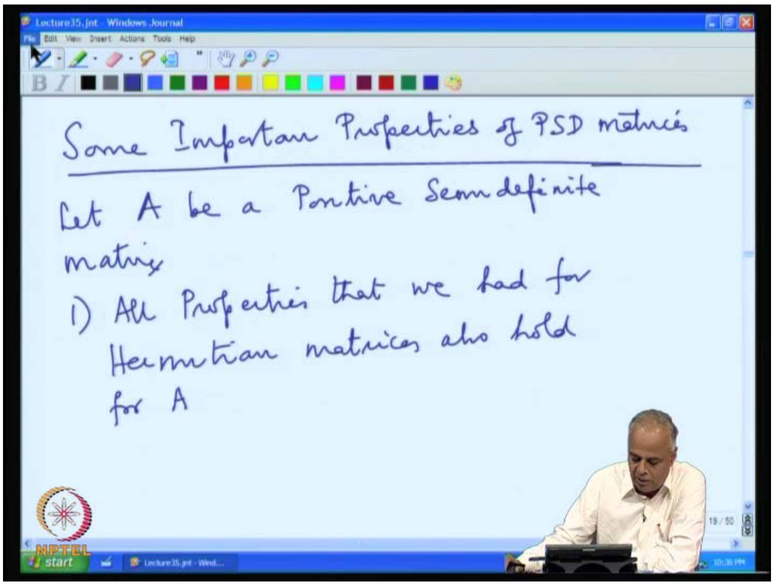

Let A be a positive semi-definite matrix. Now, we have seen, that the notion of positive definiteness, a prior we assume, that it is a Hermitian matrix and therefore, all properties, that we had for Hermitian matrices also hold for A because A is positive semi-definite.

Recall our picture, the positive definite matrices are, what are these? They we are now looking at the positive semi-definite matrices, they are sitting inside the H n. So, whatever, properties hold for H n, hereditarily they hold for positive semi-definite matrix, all the properties that we had for Hermitian matrix. What are some of the properties? Eigen values will be real; Eigen vectors corresponding to distinct Eigen values will be orthogonal; algebraic multiplicity will be equal to geometric multiplicity for all Eigen values, the matrix, if it has rank rho can be decomposed in terms of rho 1 rank matrix.

All these properties will now sweepingly, can be applied for positive semi-definite, but that is only hereditary. They have acquired this property by being Hermitian matrices. What are the properties, that they have going to get in addition to all these properties as extra properties because they are positive semi-definite. These properties are specific to positive semi-definite matrices.

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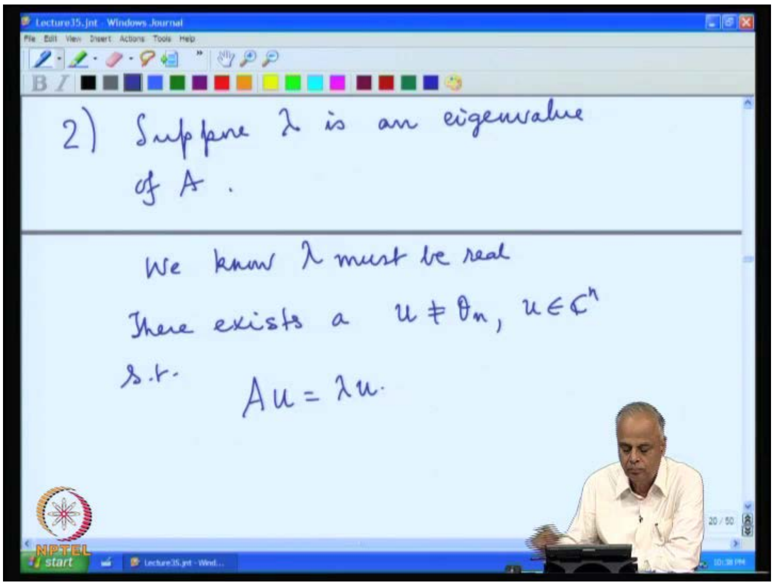

Now, suppose lambda is an Eigen value of A, so I have a positive semi-definite matrix and I consider an Eigen value of A. I, we know lambda must be real, why lambda should be real? Because A is positive definite and therefore, it is Hermitian and we know, that the Eigen values of Hermitian matrices are real and therefore, lambda must be real.

So, once it is real, either it is positive or it is negative or it is zero, you would like to make some statement about the sign of the Eigen values. Now, because lambda is an Eigen value, there exists an Eigen vector such that A u equal to lambda u.

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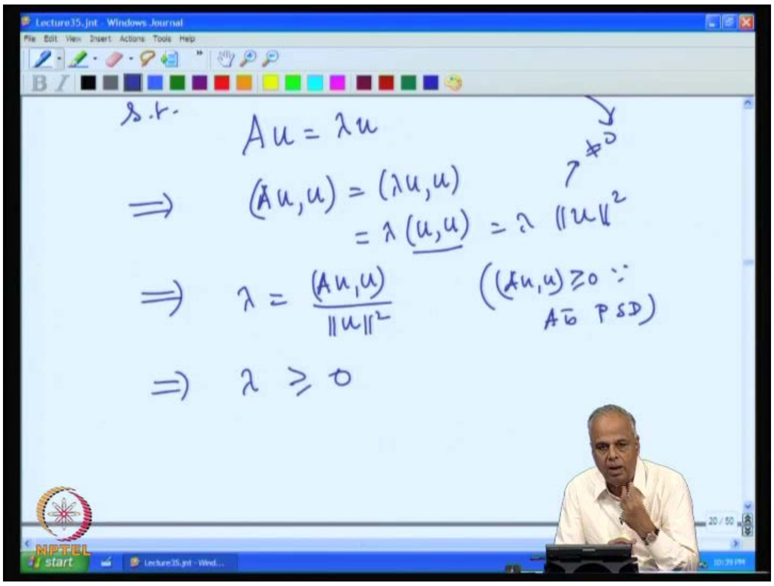

Now, that says, if I now take the inner product of u, I get (lambda u, u). Now, lambda is a constant, it can be pulled out. Now, (u, u) cannot be 0 because u is a non-zero vector and therefore, a non-zero vector inner product with itself will give a length of u square and therefore, the length u will not be zero. If u is not 0 and so, (u, u) is not zero. We will write it as this non u square cannot be 0 because of u being not theta m. Therefore, we can divide by non u square, we get lambda equal to (A u, u) by non u square.

So, the lambda is the ratio of these two quantities, but now since A is positive semi-definite, this is the time we are using the property, that A is positive semi-definite. This is precisely the place there we use the fact, that A is positive semi-definite. Since, A is positive semi-definite, the numerator is greater than or equal to zero because A is positive semi-definite and the denominator is automatically greater than or equal to zero because length square. So, it is a ratio of two non-negative quantities that says, that should also be greater than or equal to zero.

So, therefore, we have in addition to being the Eigen value being real, we have the fact, that Eigen value cannot be negative, it has to be non-negative, it is either zero or positive.

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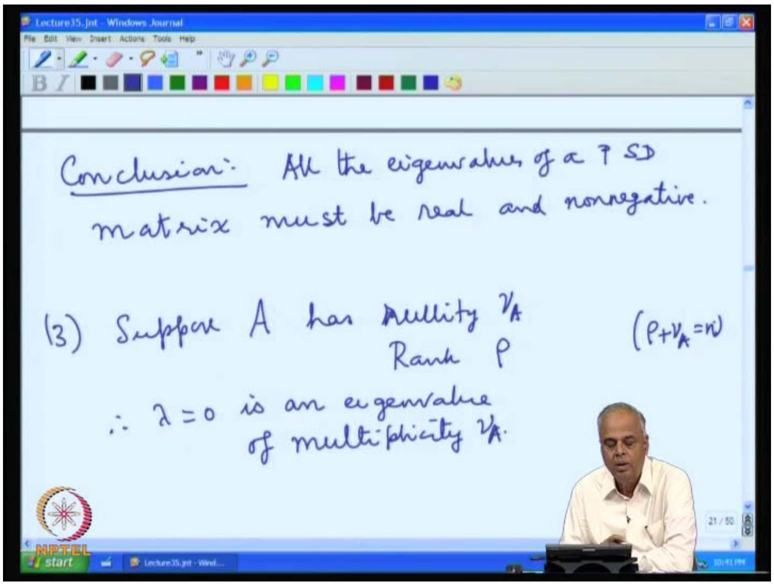

So, the conclusion is, all the Eigen values of a positive semi-definite matrix must be real and non-negative. Now, zero can become an Eigen value only if the null space has non-zero vectors only if nullity is greater than or equal to 1. Therefore, we have all the Eigen values of the positive semi-definite matrix or real and non-negative.

Now, what are the consequences of this? Suppose, again we are all looking at positive, A is always a positive semi-definite matrix, so A has nullity nu A, that is assumed it is, there is some nullity, nu A greater than or equal to. Then, what is rank? Rank is rho and we are rho plus nu A is equal to n.

Now, what does it mean to say, that nullity is nu A? It means lambda equal to 0 is an Eigen value; lambda equal to 0 is an Eigen value of multiplicity nu A. A is n by n matrix, there must be n Eigen values, nu A of these Eigen values are 0; 0 appears as an Eigen value nu A times.

Now, how many more Eigen values we require? We require n minus nu A Eigen, which is equal to rho Eigen values and since all the Eigen values are greater than or equal to 0 and 0 has been taken care of, here all the remaining Eigen values must be positive.

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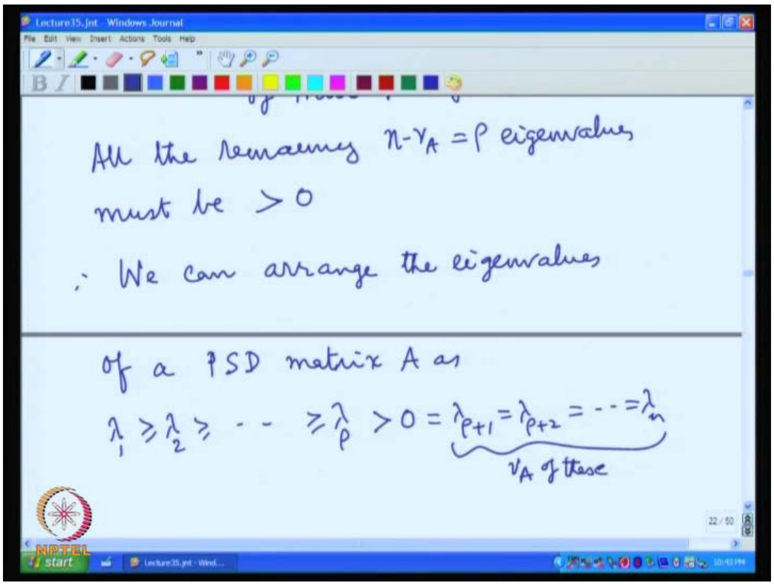

All the remaining n minus nu A equal to rho eigenvalues must be strictly positive. Therefore, we can arrange the eigenvalues of a positive semi-definite matrix A as an eigenvalue, may be the next eigenvalue is equal to… and so on. So, there are rho eigenvalues, which are all strictly positive and the remaining eigenvalues are all equal to 0. So, there are nu A of them. So, nu of these eigenvalues is 0 and rho of these eigenvalues are positive.

So, therefore, if we have a positives semi-definite matrix of rank rho, we can always split the eigenvalues into two groups, one group of eigenvalues, which are all strictly positive than the eigenvalue zero, the eigenvalue zero because nullity nu A will appear nu A times and all the other eigenvalues put together will give us rho eigenvalue.

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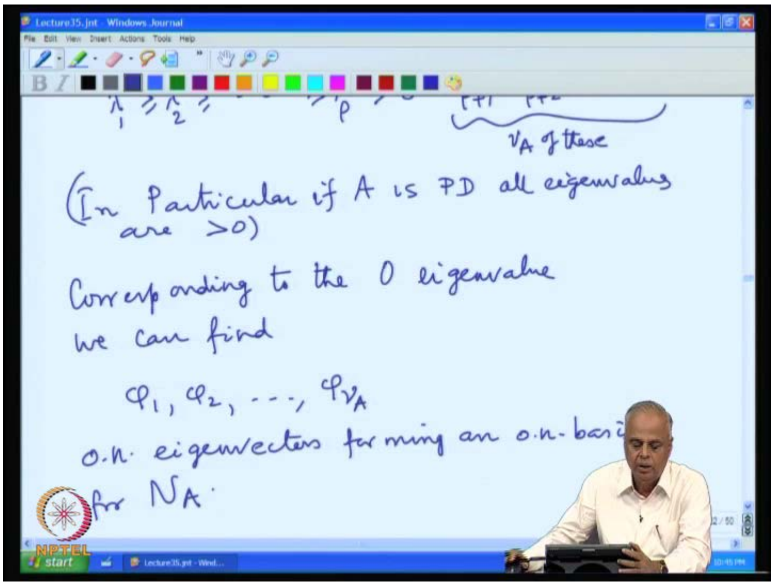

In particular, if A is positive definite, then zero is not an eigenvalue at all, all eigenvalues are strictly positive. That means, we will have lambda 1 greater than or equal to lambda 2 greater than or equal to up to lambda n, all of them greater than 0. So, the zero eigenvalue will not appear at all.

Now, once we have these eigenvalues, because it is Hermitian, we will be able to find corresponding orthonormal Eigen vectors. So, now corresponding to the zero eigenvalues we can find, since multiplicity is nu A we will find nu A will, I will write it as phi 1, phi 2, phi nu A orthonormal Eigen vectors. And since these are eigenvalues corresponding to the eigenvalue zero forming an orthonormal basis for null space of A, the Eigen vectors corresponding to the eigenvalue zero will give us a basis, an orthonormal basis for the null space of A in the case of positive semi-definite matrix.

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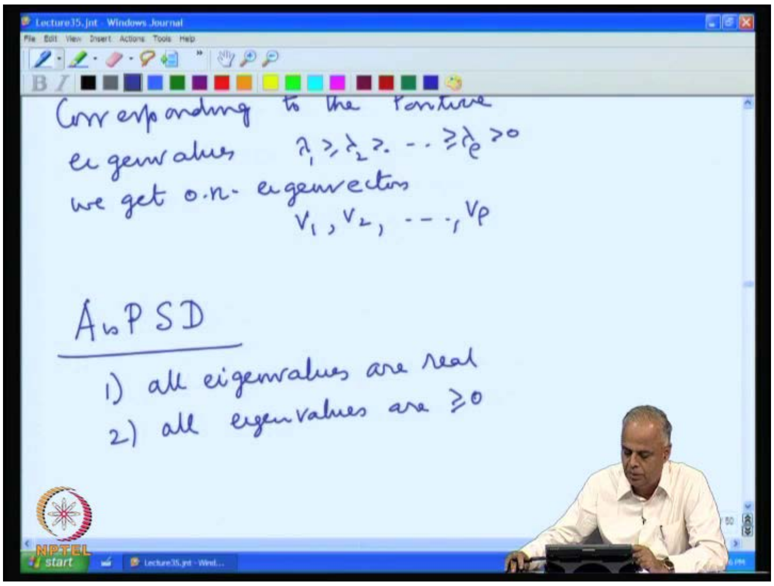

Next, corresponding to these positive Eigen values we will get Eigen vectors, say v 1, v 2, v 3, v rho and there will be orthonormal because we know, that for Hermitian matrices we can always get the orthonormal Eigen vector. So, corresponding to the positive eigenvalues lambda 1 greater than or equal to lambda 2 greater than or equal to lambda rho greater than 0, we get orthonormal Eigen vectors v1, v2, v rho. So, therefore, if A is positive semi-definite, so let us summarize this, A is positive semi-definite, all eigenvalues are real, all eigenvalues are greater than or equal to 0.

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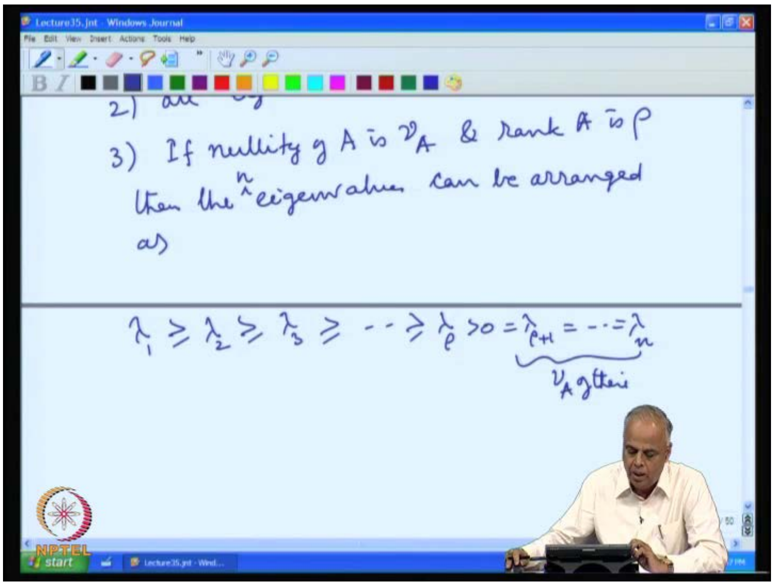

If, nullity of A is nu A and rank A is rho A, then I will just write rho here instead of writing subscript, A is rho, then the eigenvalues can be the n eigenvalues. When I say ((

)) we are looking at multiplicities included, can be arranged as lambda 1 greater than or equal to lambda 2 greater than or equal to lambda 3 greater than or equal to lambda rho greater than 0 and then lambda rho plus 1. All these eigenvalues are 0 eigenvalues and there are nu A of this.

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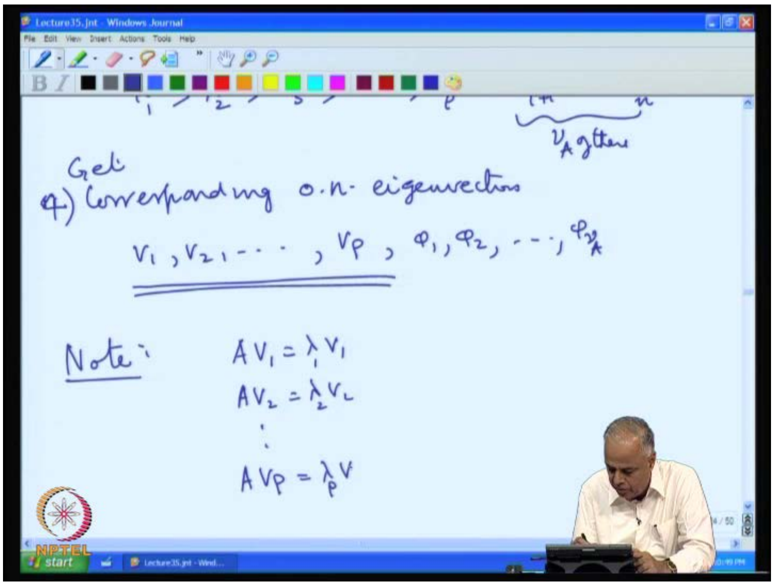

And corresponding, get corresponding Eigen vectors, orthonormal Eigen vectors v1, v2, v rho, phi1, phi2 phi nu A. Now, these are the basic ingredients that we require to analyze a given matrix. We will see how we can convert all the questions, all the competitions, that we require regarding answering the questions for a general matrix to those of some simple Hermitian positive semi-definite matrix.

Now, let us look at this v1, v2, v r, v rho. So, note one important property, we are going to observe these. V 1 is an Eigen vector corresponding to lambda 1, so we have AV 1 is lambda 1 V 1, AV 2 is lambda 2 V 2 and there AV rho is lambda rho V rho.

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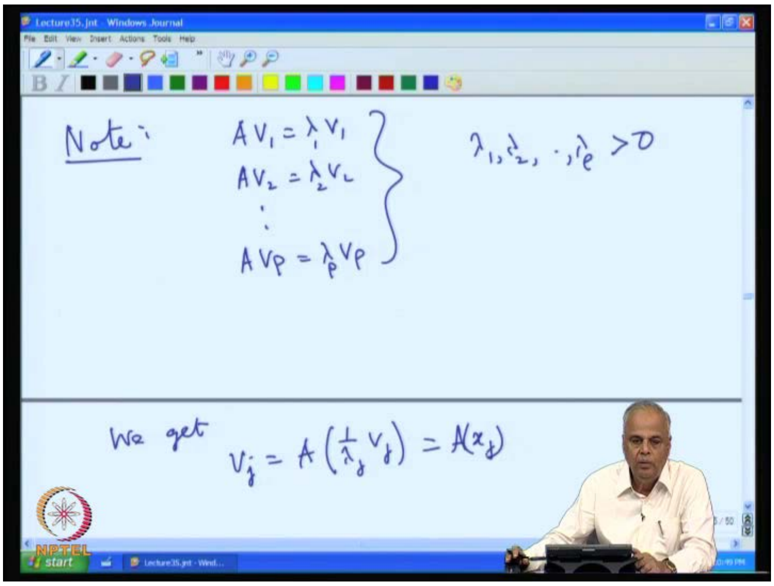

Now, note that the lambda 1, lambda 2, lambda rho is all the positive eigenvalues and therefore, we can divide. So, we get, for each V j can be written as A of 1 by lambda j V j or we can write it as A of x j, which means, V j is a vector of the form A of something.

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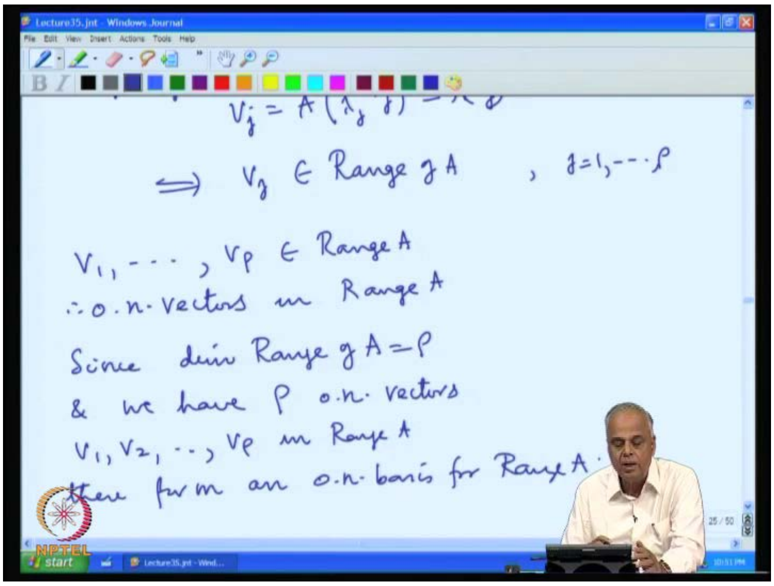

So, therefore, V j belongs to the range of A. This is true for j equal to 1 to rho and therefore, V 1, V 2, V rho belong to range of A, there orthonormal vectors as we have found in range of A. Therefore, orthonormal vectors in range of A, but what is the range of A? The range of A is the rank, which is rho and therefore, the dimension of the range of A is rho. And we have found rho orthonormal Eigen vectors in that space of dimension rho and therefore, they form an orthonormal basis for range of rho.

So, since dimension of range of A equal to rho and we have rho orthonormal vectors V 1, V 2, V rho in range of A, these form an orthonormal basis for range of A. So, we get an orthonormal basis for the range of A from the Eigen vectors corresponding to the non-zero eigenvalues.

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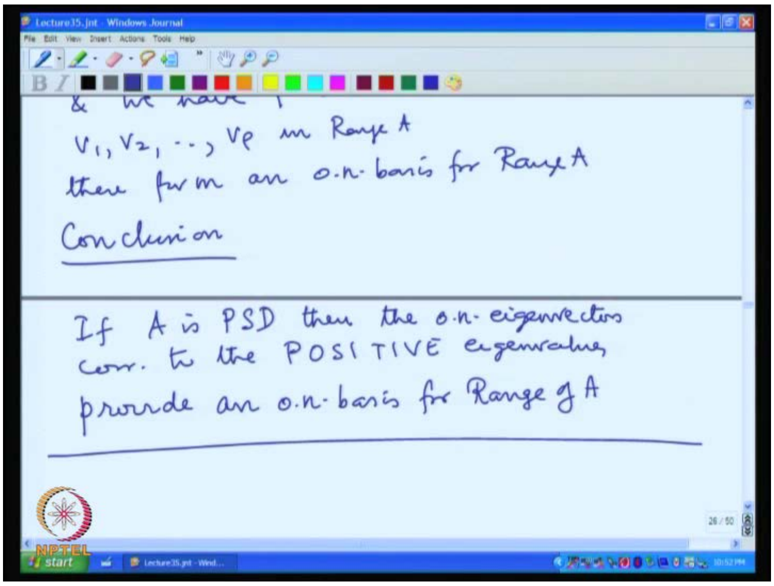

So, that is the important conclusion. If A is positive semi-definite, then the orthonormal Eigen vectors corresponding to the positive eigenvalue, strictly positive eigenvalues. The positive eigenvalues provide an orthonormal basis for range of A.

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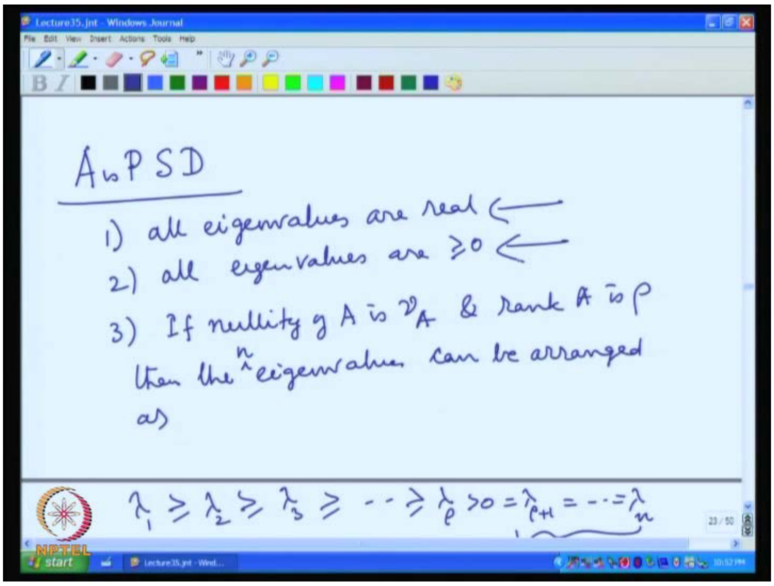

Now, we have studied Hermitian matrices, we have studied positive semi-definite matrices, which is a special class of Hermitian matrices. We have found some special properties of the eigenvalues, and the Eigen vectors of positive semi-definite matrices, which we shall again recall.

The first property is that all the eigenvalues are real, all the eigenvalues are greater than or equal to 0, nullity is nu A, the Eigen values can be arranged in this form, and corresponding to this we will get the Eigen vectors corresponding to these eigenvalues. These are orthonormal Eigen vectors and the Eigen vectors corresponding to the positive eigenvalues provide us a basis with, for the range of A. So, now, we have studied this special class of positive semi-definite matrices.

We have seen the notions of vector spaces; we have seen the notion of subspaces. We have seen that any matrix has four subspaces associated with it, two of them in r n, two of them in r m. We have seen that these pairs are orthogonally oriented, so we introduced the notion of orthogonal complements. Then, we introduced the notion of orthonormal basis and then, we had the notion of Hermitian matrices and then, finally, the special class of positive semi definite matrices.

Now, we are in ready in a position, we have got all the ingredients, all the previous and all the material, that we require to analyze a given general m by n matrix complex or real. Now, we shall put all these ideas together and see how we get all the answers to the fundamental questions that we raised at the beginning of the course. We shall begin this analysis in the next lecture.