**Advanced Matrix Theory and Linear Algebra for Engineers**

**Prof. R. Vittal Rao**

**Centre for Electronics Design and Technology**

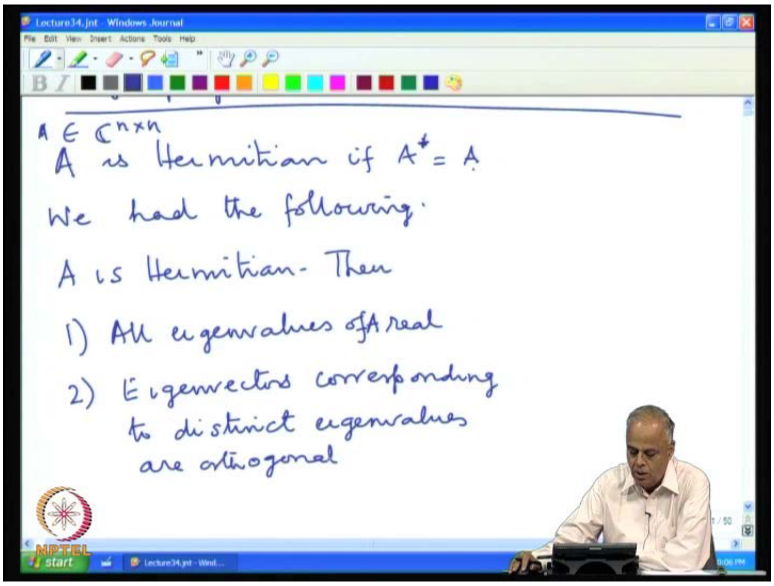
**Indian Institute of Science, Bangalore**

**Lecture No. # 34**

**Hermitian and Symmetric Matrices – Part 3**

We have studying the Eigen properties of particularly Hermitian matrices.

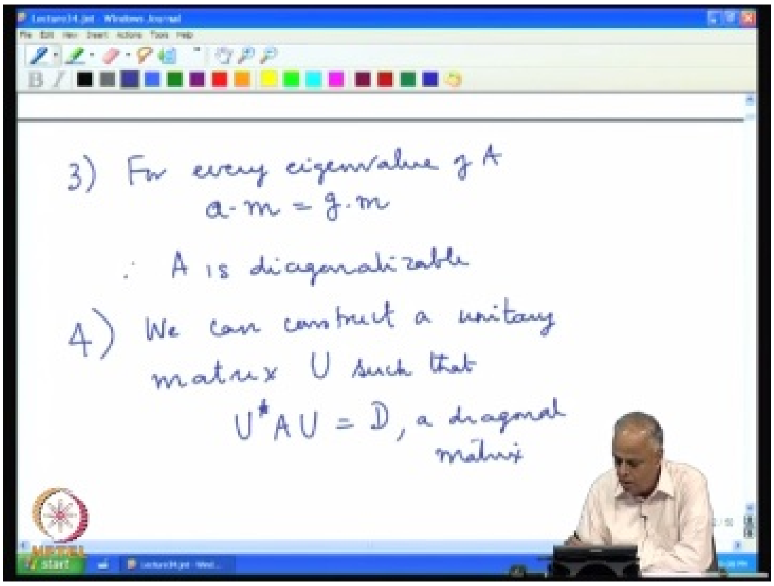
(Refer Slide Time: 00:19)




Recall that we say A is Hermitian if A star equal to A; that is, the matrix is self conjugate (( )) A star is called Hermitian conjugate of A. And, if A is its own conjugate, then we call it a Hermitian matrix. So, A belonging to C n cross n is an n by n complex matrix, is called Hermitian if it is its own conjugate. And, we were looking at some very specific Eigen properties of such a matrix.

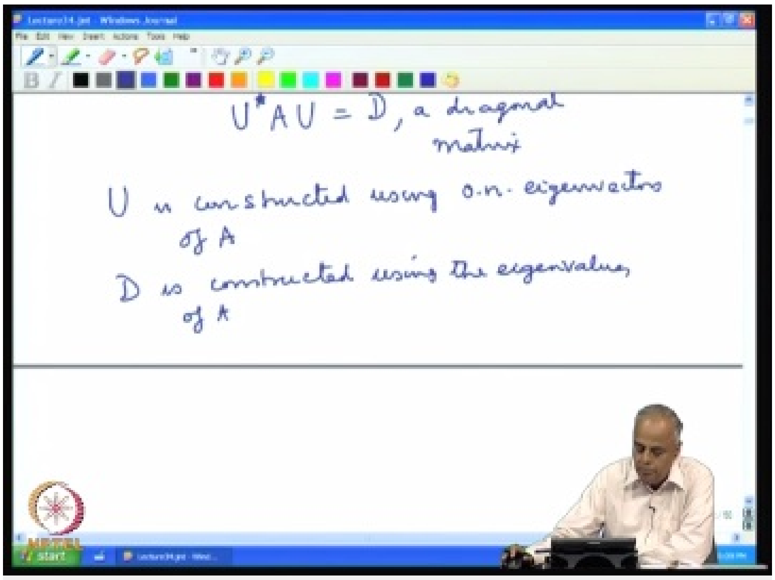
We had the following important properties of such a matrix. A is Hermitian; then, we have one – All eigenvalues of A are real. This is a very important property of Hermitian matrices. All the eigenvalues of A must be real. The second important property is that the eigenvectors corresponding to distinct eigenvalues are orthogonal. So, these are the two important properties. The first one says about the structure of the eigenvalues; says that they are all real; they will all be… Even though the matrix may be complex, as long as the matrix is Hermitian, the eigenvalues are bound to be real. So, that is the first property, which is the structure of the eigenvalues. Secondly, the second property is about eigenvectors. The eigenvectors are always orthogonally oriented. If you take two different eigenvalues, then take the eigenvectors corresponding to each one of them, they are bound to be oriented orthogonally to each other. These are the two important properties of Hermitian matrices.

(Refer Slide Time: 02:57)



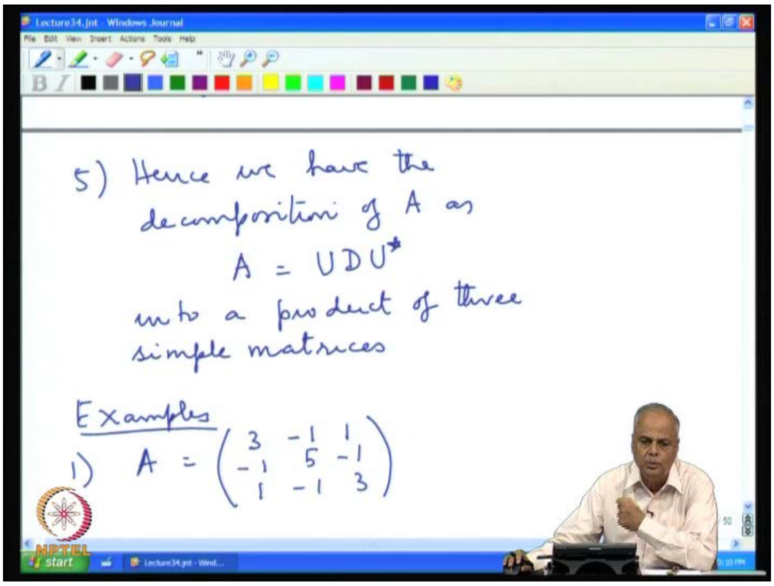

We also found that… Without proving, we said that for every eigenvalue of A, the algebra multiplicity is equal to geometrical multiplicity. And therefore, A is diagonalisable. Using these properties, we showed that we can construct a unitary matrix; that is, we want eventually assert that A is not only diagonalisable, but A is unitarily diagonalisable. So, we can construct a unitary matrix U such that U star A U is a diagonal matrix.

(Refer Slide Time: 04:00)




And, we observed that U is constructed using orthonormal eigenvectors of A. And, D is constructed using the eigenvalues of A. These were the two important constructions. The orthonormal eigenvectors give rise to the unitary matrix U and the eigenvalues give rise to the diagonal matrix D. Since U is unitary, U star is U inverse and U star inverse is U. So, this U star A equal to D gives us the decomposition.

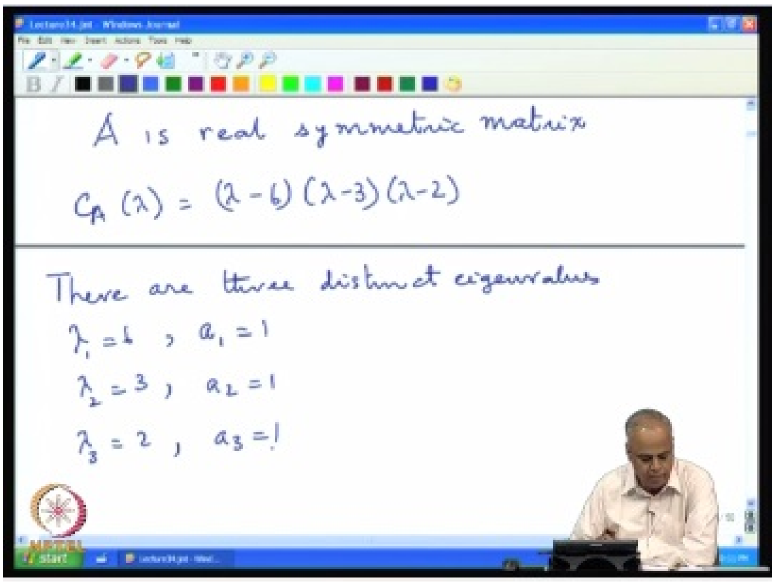
(Refer Slide Time: 04:56)




Hence, we have the decomposition of A as A equal to U D U star into three simple matrices. The two extreme matrices are unitary and therefore, easily invertible; and, the medium one is diagonal and therefore, easily handleable – into a product of three simple matrices.

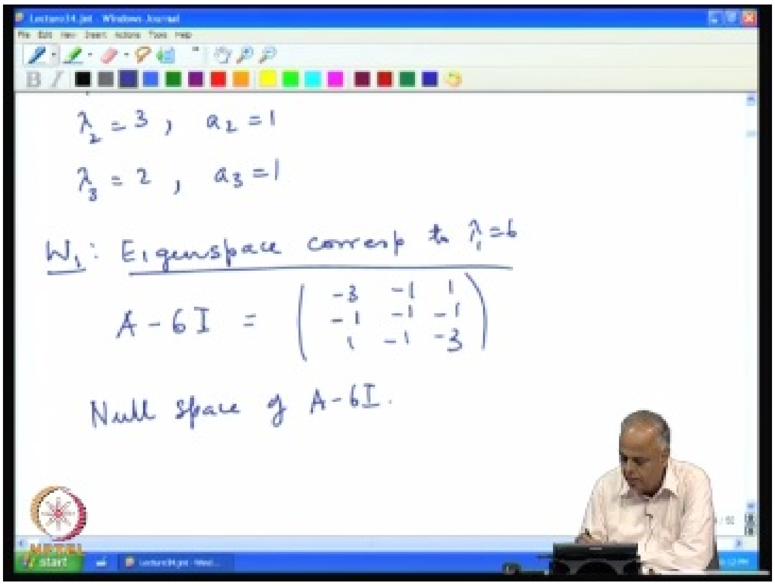
We shall now look at some simple examples of these calculations. Consider the matrix A equal to 3, minus 1, 1, minus 1, 5, -1, 1, minus 1, 3. Notice that this is a real symmetric matrix. And, as we observed yesterday in the last lecture, the real symmetric matrix case, we get everything; in place of unitary, we get orthogonal matrices; that is, U star is… U is real and U star becomes transpose.

(Refer Slide Time: 06:24)



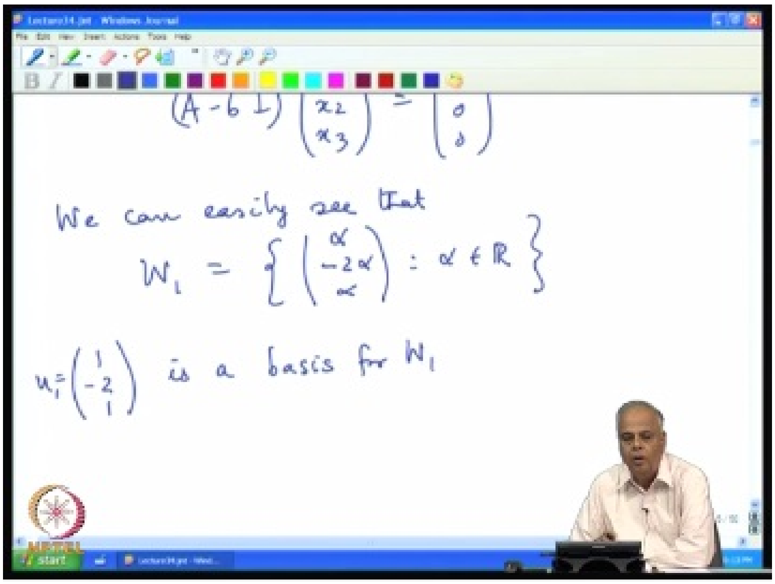

Now, look at this matrix. A is a real symmetric matrix. Then, let us find… The characteristic polynomial of this can be shown to be lambda minus 6 into lambda minus 3 into lambda minus 2. We have to just write the determinant of the lambda m minus A and expand it. Then, we find that it can be factored as lambda minus 6 into lambda minus 3 into lambda minus 2. Therefore, there are 3 distinct Eigen values. They are lambda 1 equal to 6, lambda 2 equal to 3, and lambda 3 equal to 2. And obviously, their algebraic multiplicities are all 1. So, we have a real symmetric matrix now, whose eigenvalues are known; we know the algebraic multiplicities.

(Refer Slide Time: 07:33)



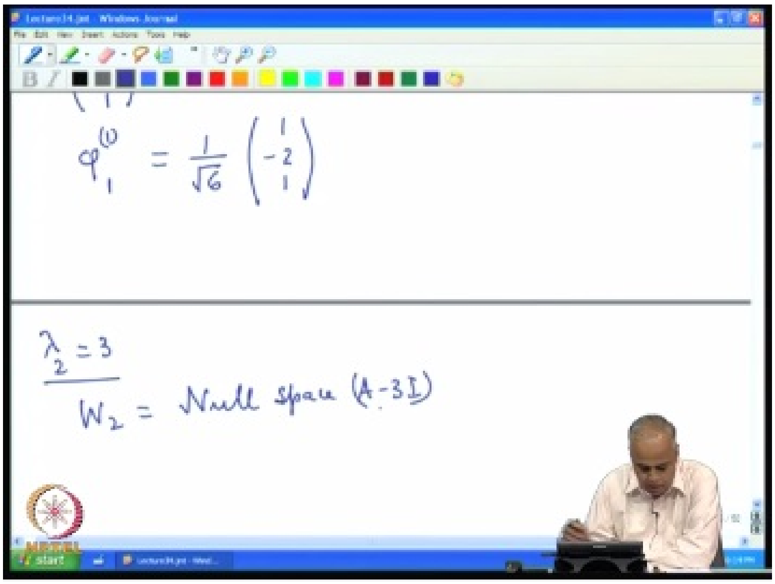

Let us find the eigenvectors. The first, the eigenspace corresponding to lambda 1 equal to 6. For this, we must look at the matrix A minus 6 I, because lambda 1 is 6. A minus 6 I is this matrix minus 3, minus 1, 1, minus 1, minus 1, minus 1, 1, minus 1, minus 3. This matrix taking obtained by taking the matrix A (Refer Slide Time: 08:10) and subtracting 6 from the diagonal. So, if you subtract 6 from the diagonal, you get this (Refer Slide Time: 08:18) and all the other items remain unchanged. Now, to find the eigenspace W 1, we must find the null space of A minus 6 I.

(Refer Slide Time: 08:36)



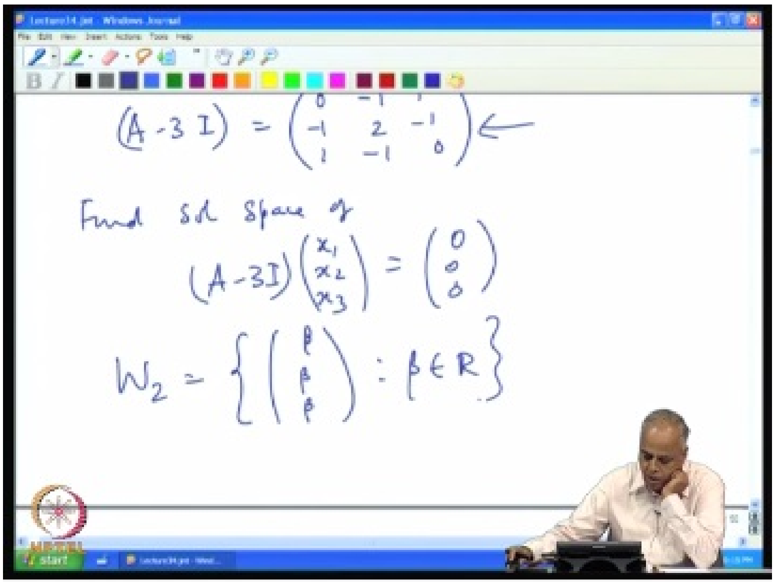
That is, to find the solution space of system A minus 6 I into x 1, x 2, x 3 equal to 0, 0, 0. We have to solve this system. When we solve this system, we can easily see that the set of all solutions we get is of the form alpha, minus 2 alpha and alpha; where, alpha is any real number. And therefore, we see that 1, minus 2, 1, is a basis for W 1. Recall the way we construct the matrix U and the way we treat the eigenvectors for a Hermitian or a real symmetric matrix. It is always the orthonormal eigenvectors. So, this is an eigenvector. Any nonzero vector in W 1 is an eigenvector; U 1 is an eigenvector, but it not normalized to one. So, we normalize it.

(Refer Slide Time: 10:00)



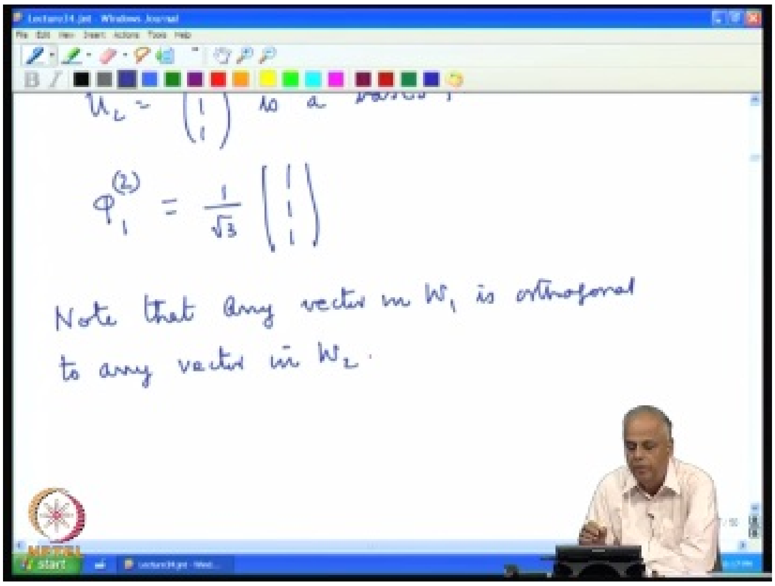

That is, the eigenvector corresponding to the eigenvalue 1. And, there is only one eigenvector, because algebraic multiplicity. So, we divide by its length, which is square root of 6 and we get the eigenvector corresponding to the value lambda 1 equal to 6. Similarly, we have the eigenvalue lambda 2 equal to 3. So, we look at W 2, the eigenspace, which is a null space of A minus 3 I.

(Refer Slide Time: 10:40)




Now, if you write A minus 3 I… A minus 3 I is obtained from subtracting 3 along the diagonal of A. When we do that, we get this matrix. And then, we have to find the solution space of A minus 3 I into x 1, x 2, x 3 equal to 0, 0, 0. And, A minus 3 I matrix is given here. And so, when we solve the system, we get the solution as W 2 equal to the set of all vectors of the form beta, beta, beta, where beta belongs to R.

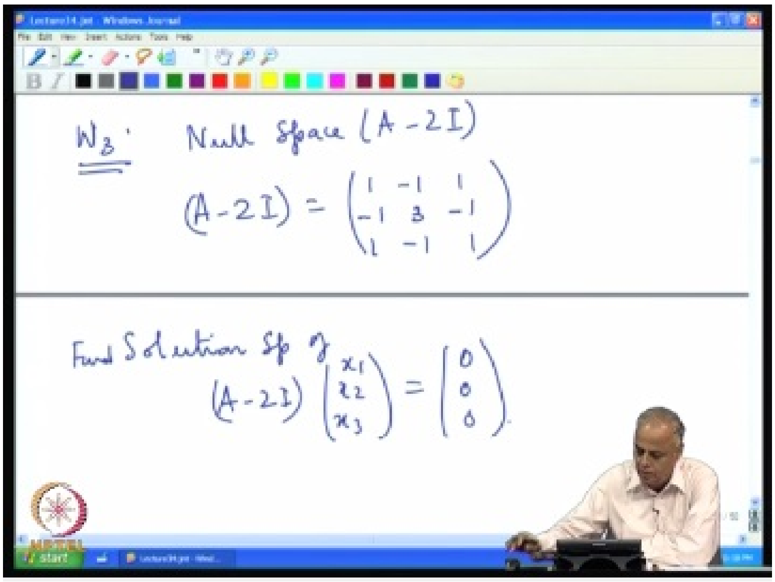
(Refer Slide Time: 11:28)




Now, we see that u 2 equal to 1, 1, 1 is a basis for W 2 and this is not normalized. So, we construct the orthonormal vector corresponding to the eigenvector 2. And there is only one of them, because of the algebraic multiplicity 1. If we divide by the length, we get 1, 1, 1. So, that is the orthonormalized eigenvector corresponding to lambda 2.

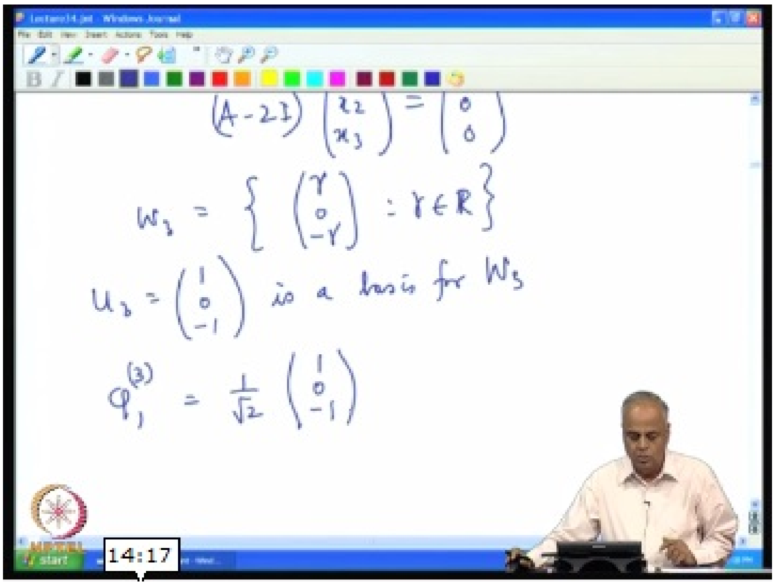
Now, observe that any vector in W 1 is of the form (Refer Slide Time: 12:12) alpha, minus 2 alpha, alpha. And, any vector in W 2 is of the form (Refer Slide Time: 12:17) beta, beta, beta. And, if you take the dot product, you find that alpha beta, minus 2 alpha beta; they get cancelled and therefore, they are orthogonal. Note that any vector in W 1 is orthogonal to any vector in W 2. And, this is what we have had that when we have a real symmetric matrix or a complex Hermitian matrix. The eigenvectors corresponding to distinct eigenvalues will be orthogonal.

(Refer Slide Time: 13:00)



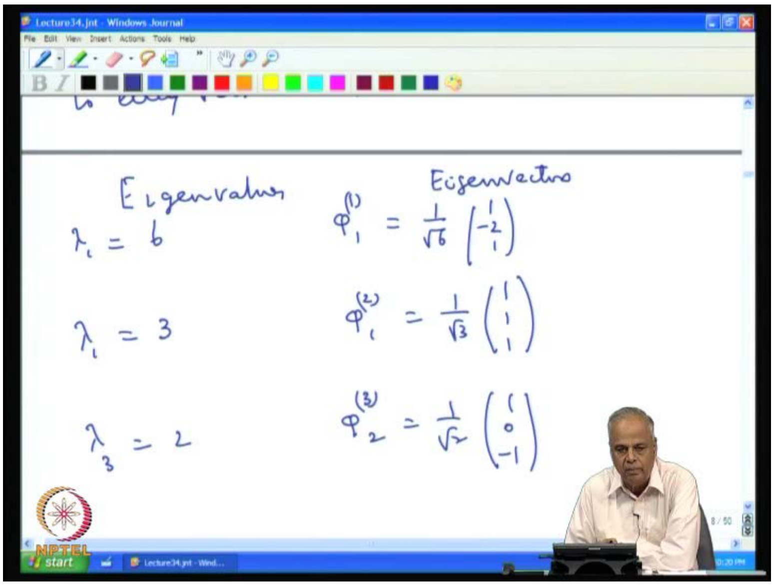

Now, let us look at the corresponding eigenspace for W 3. So, we must look at null space of A minus 2I, because lambda 3 is 2. Now, A minus 2I is the matrix obtained by subtracting 2 from the diagonals and we get this matrix. And therefore, to find a null space, we must find the solution space of A minus 2 I x 1, x 2, x 3 equal to 0, 0, 0.

(Refer Slide Time: 13:50)




Now, when we have the A minus 2I here, when we solve the system, we get the solution space W 3 to be the set of all vectors of the form gamma, 0, minus gamma; where, gamma belongs to R. And, we see that 1, 0, minus 1 is a basis for W 3. And, when you normalize it, you get the eigenvector corresponding to the third eigenvalue. And, there is only one – 1 by root 2 into 1, 0, minus 1.

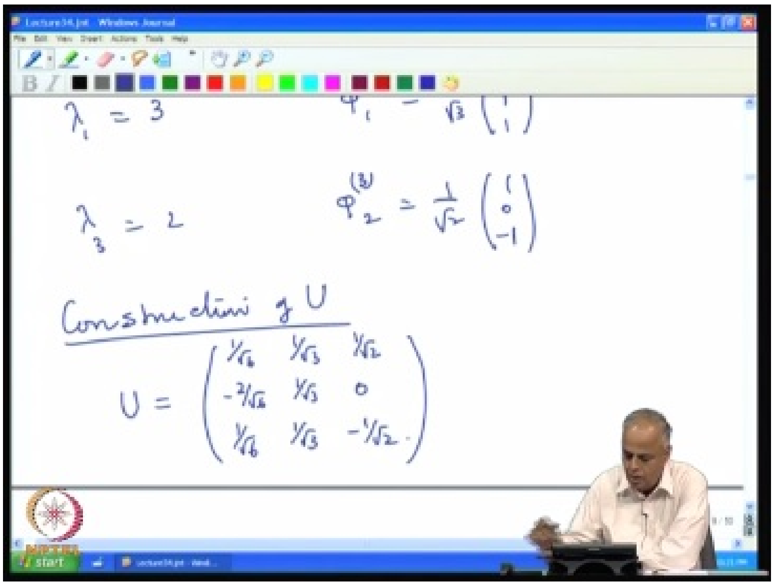
(Refer Slide Time: 14:29)




Therefore, if we summarize, we have the eigenvalues on one side, eigenvectors normalized on other side; we have lambda 1 equal to 6 and the eigenvector was 1 by root

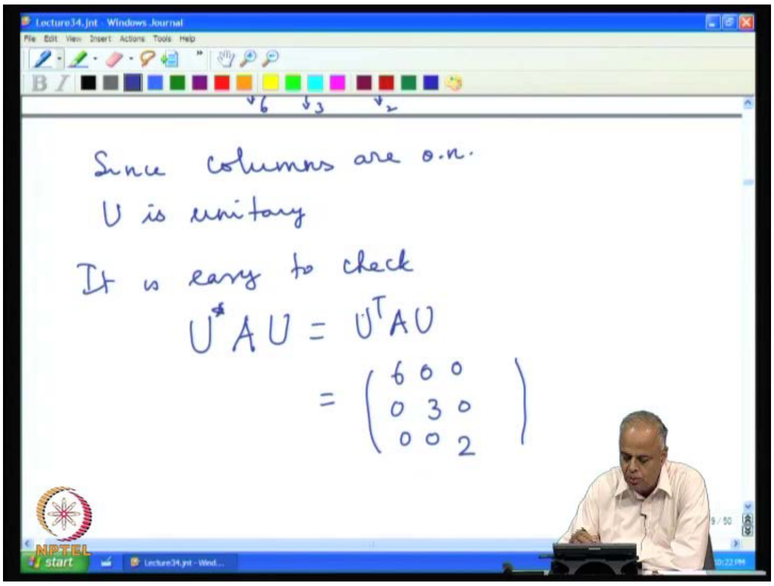
6 into 1, minus 2, 1. This is what we called as phi 1 1. And similarly, for lambda 2 equal to 3, we had the eigenvector 1 by root 3 into 1, 1, 1. And, lambda 3 equal to 2; we get phi 3 2 equal to 1 by root 2 1, 0, minus 1. Now, observe that all these vectors are length 1 and they are orthogonal to each other. Also, observe that (Refer Slide Time: 15:19) every vector in W 3 is orthogonal to every vector in W 1 and W 2. That is the statement that eigenvectors corresponding to different eigenvectors are orthogonal to each other. So, for the three eigenvalues, now, (Refer Slide Time: 15:45) we have got the three eigenvectors. They are orthonormalized. Notice that the eigenvalues are all real, because the matrix is real symmetric. Whenever you have a real symmetric matrix or a complex Hermitian matrix, all the eigenvalues will be real.

(Refer Slide Time: 16:03)




Now, we look at the construction of U, the unitary matrix. How do we construct? We put these eigenvectors along the diagonal. If we put the eigenvectors, we have 1 by root 6, minus 2 by root 6, 1 by root 6. That is the first eigenvector. Then, the second eigenvector comes along the second column. So, it is 1 by root 3, 1 by root 3, 1 by root 3. And, the third eigenvector comes along the third column – 1 by root 2, 0, minus 1 by root 2.

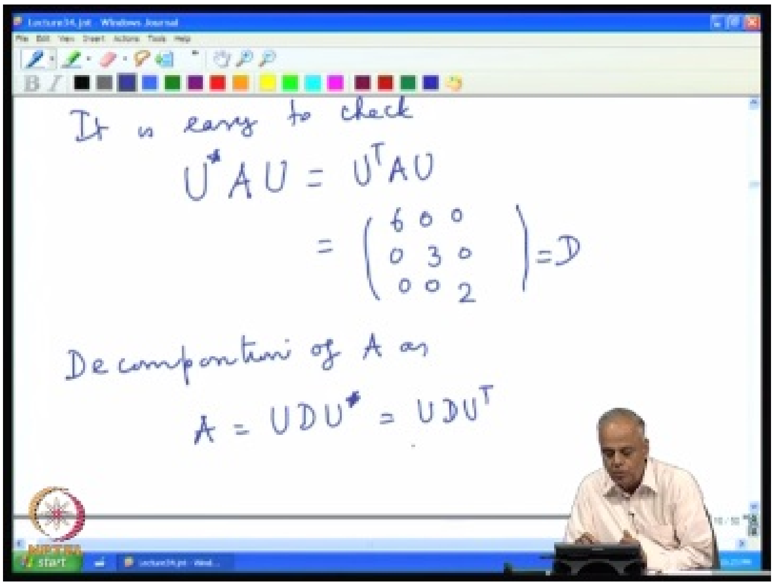
(Refer Slide Time: 16:45)




Now, since the columns are orthonormal, U is unitary. And now, it is easy to check that if we now take the matrix A and multiply U star A U, we would get… What is the same in this case? It is the same as U transpose A U, because U star is U transpose, because we are dealing with real matrices. And, this will be the diagonal matrix. And, what will be the diagonal entries? The eigenvalues corresponding to the columns eigenvectors – the first column corresponds to this eigenvalue 6; the second column corresponds to the eigenvalue 3; and, the third column corresponds to the eigenvalue 2. These three eigenvalues will come along the diagonal. The rest of the entries will be 0 and we have this diagonalisation process.

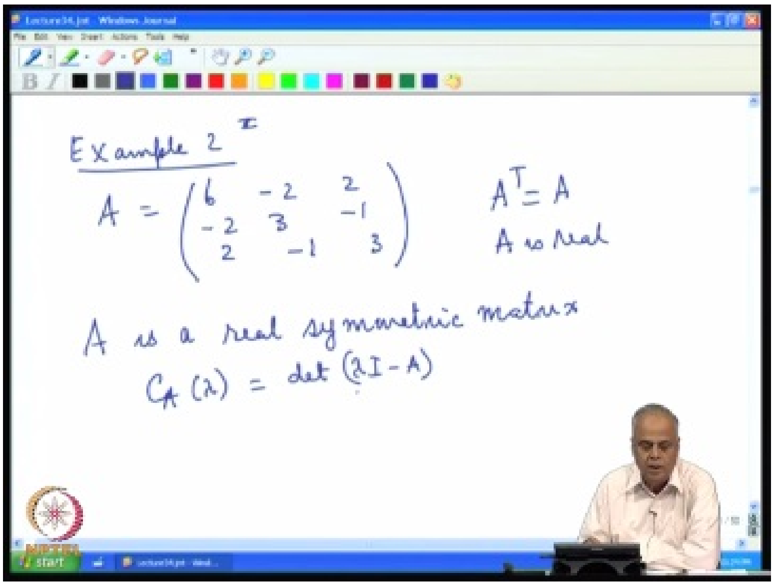
Now, we can actually substitute (Refer Slide Time: 17:56) U and therefore, U transpose; and, we know the matrix A. We can actually carry out the multiplication and verify that it is indeed the diagonal matrix 6, 0, 0; 0, 3, 0; 0, 0, 2.

(Refer Slide Time: 18:11)



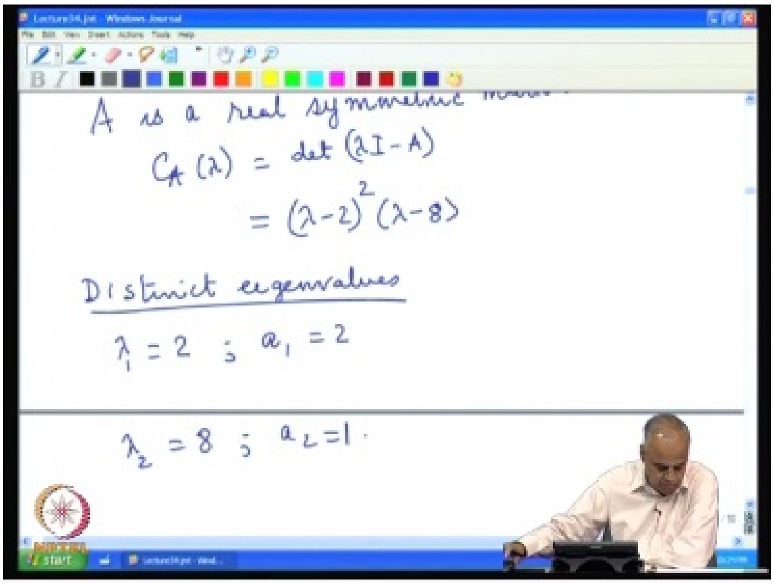

And, we get the decomposition of A as A equal to U D U star or U D U transpose, because now, U star is U transpose. Now, what is the U? We have the U here. We can substitute that and we have the diagonal matrix here; we can substitute that. And, since we know (( )) U transpose, so that we see that now, the decomposition of A as the product of three simple matrices. This is the demonstration of all the results that we obtained for real symmetric matrix or complex Hermitian matrix.

(Refer Slide Time: 19:04)



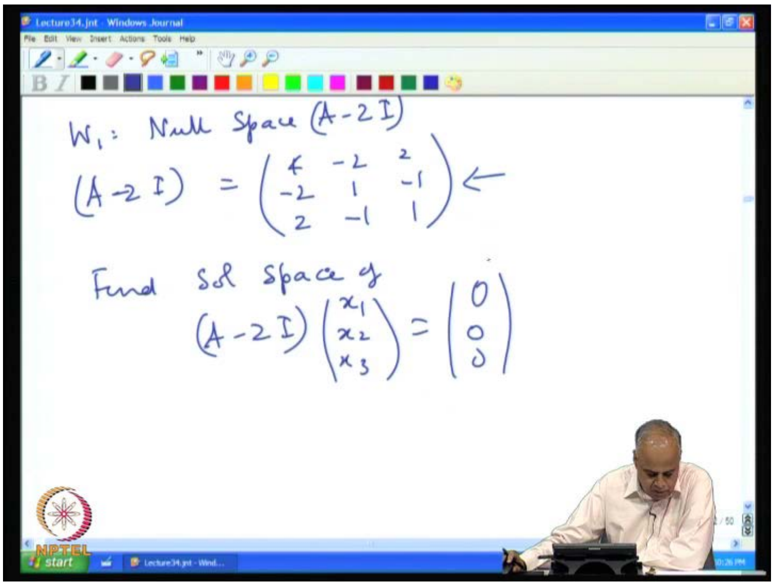
Now, let us look at another example without too many details. Look at the matrix A – 6, minus 2. 2; minus 2, 3, minus 1; 2, minus 1, 3. Again, notice that A transpose is equal to A and A is real. And, therefore, A is a real symmetric matrix.

(Refer Slide Time: 19:45)



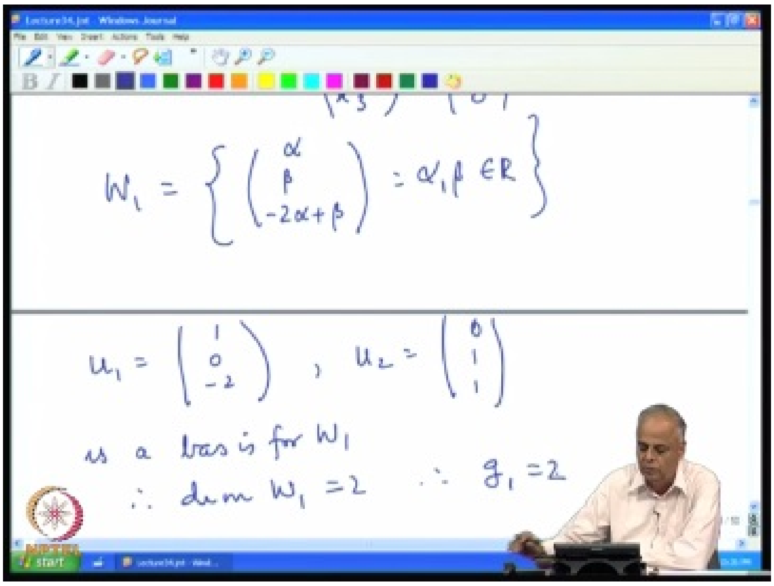

Once again, if you calculate the characteristic polynomial of lambda, which is the determinant of lambda I minus A, we have the A, we have the I. If we substitute, we get the determinant. When we expand the determinant, we get lambda minus 2 square into lambda minus 8. Now, in this case, we have a multiple eigenvalue. So, the distinct eigenvalues are lambda 1 equal to 2 with algebraic multiplicity is 2. And, the eigenvalue lambda 2 is 8 with algebraic multiplicity is 1. Notice that all the eigenvalues are real.

(Refer Slide Time: 20:51)



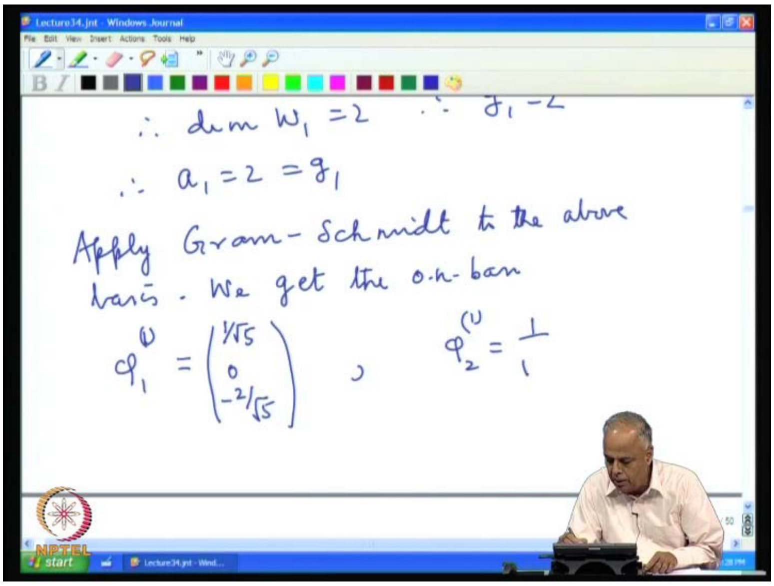

Now, let us find the eigenspaces. W 1 is the null space of A minus 2 I, because lambda 1 is 2. So, A minus lambda 1 I is A minus 2I. So, A minus 2I with is given matrix is just 4, minus 2, 2; minus 2, 1, minus 1; 2, minus 1, 1. We get this matrix by subtracting 2 from the diagonal. Now, you find the solution space of the system A minus 2I x 1, x 2, x 3 equal to 0, 0, 0. We have the matrix A minus 2I; we have homogeneous system; we know how to solve it.

(Refer Slide Time: 21:45)



And, when we solve this, we get W 1; the set of all solutions is of the form alpha, beta, minus 2 alpha plus beta; where alpha and beta are real. So, this is the set of all solutions of this homogenous system. And, we see easily therefore, U 1 equal to 1, 0, minus 2; and, U 2 equal to 0, 1. 1 is a basis for W 1. And therefore, dimension of W 1 is 2. And therefore, g 1, the algebraic multiplicity is 2.

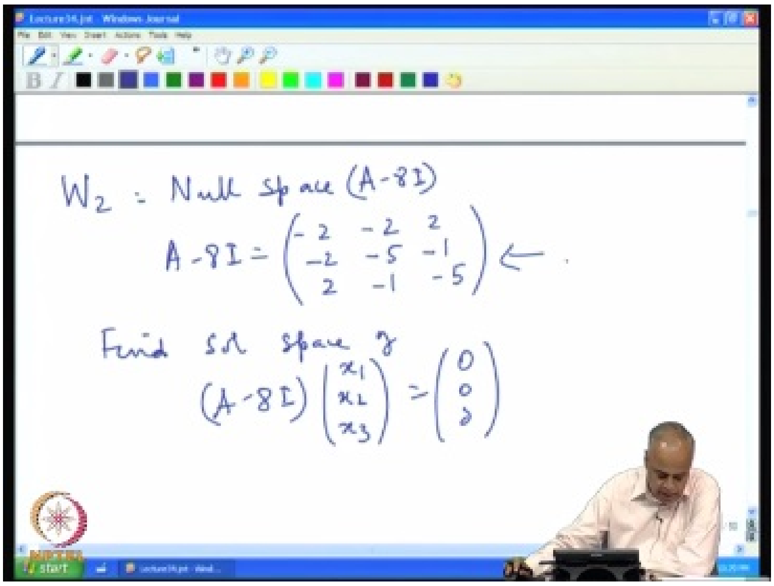
(Refer Slide Time: 22:27)




And therefore, see the algebraic multiplicity was 2; and, this is also the geometric multiplicity. Thus, the geometric multiplicity and algebraic multiplicity are the same. The basis we have found for W 1 is not orthonormal. So, we apply Gram-Schmidt to the above basis. Then, we get an orthonormal basis. We will leave it as an exercise to find this orthonormal basis using Gram-Schmidt. We get the orthonormal basis.

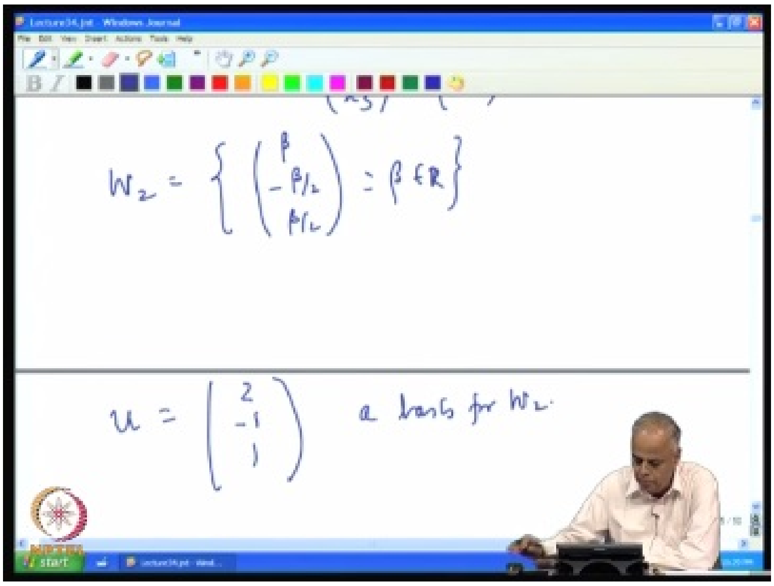
Now, these are orthonormal eigenvectors corresponding to the eigenvalue 1. But, there are two eigenvectors and they turn out to be 1 by root 5, 0, minus 2 by root 5. And, phi 2 1 turns out to be 1 by root 30 into 2, 5, 1. So, we get the two eigenvectors corresponding to the eigenvalue lambda 1 equal to 2. And there are two eigenvectors: algebraic multiplicity is 2; geometric multiplicity is 2. We get two orthonormal eigenvectors.

(Refer Slide Time: 24:08)



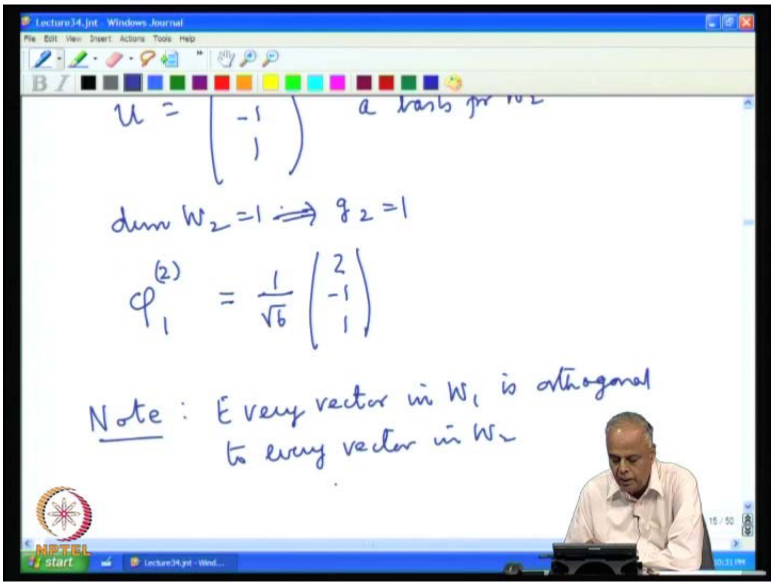

Let us next find W 2, which is a null space of A minus lambda 2I; lambda 2 is 8. So, A minus 8I. Now, A minus 8I is obtained from the given matrix A by subtracting 8 from the diagonal. When we do that, we get this matrix. Now, we have to find the solution space of the homogenous system A minus 8 I x 1, x 2, x 3 is equal to 0, 0, 0.

(Refer Slide Time: 25:01)



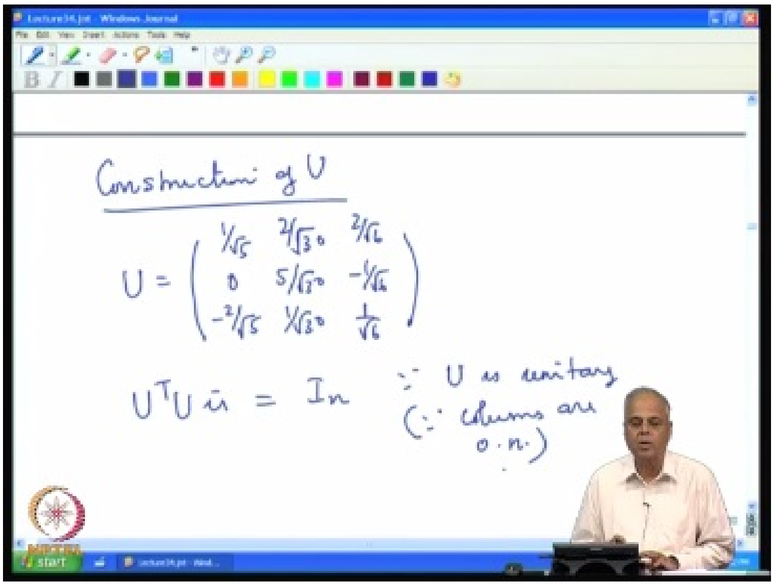

Now, we have the matrix A minus 8 I. We can apply the rho operations or any other trick to solve the system. We get W 2 to be the set of all vectors of the form beta, minus beta by 2, beta by 2; where, beta is any real. And therefore, we take beta equal to 2; we get U equal to 2, minus 1, 1 – a basis for W 2.

(Refer Slide Time: 25:32)



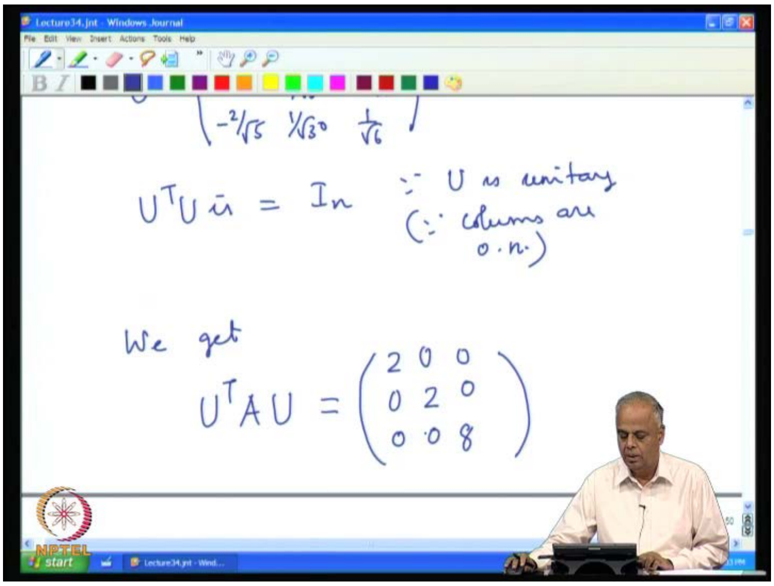

And therefore, dimension of W 2 equal to 1, which implies g 2 equal 1. Again, we find the algebraic multiplicity of the second eigenvalue is 1, the geometric multiplicity is also equal to 1. Now, this eigenvector is not normalized; it is not (( )) 1. So, by normalizing it… And, there will be only one eigenvector corresponding to the eigenvalue 2; and, that is given by 1 by root 6 into 2, minus 1, 1, because the length is root 6. Now, notice that every vector in W 1 is orthogonal to every vector in W 2. It is easy to check from the structure of these two spaces that every vector in W 1 is orthogonal to every vector in W 2. This is the corroboration of the statement that the eigenvectors corresponding to distinct eigenvalues must be orthogonal to each other.

(Refer Slide Time: 26:53)



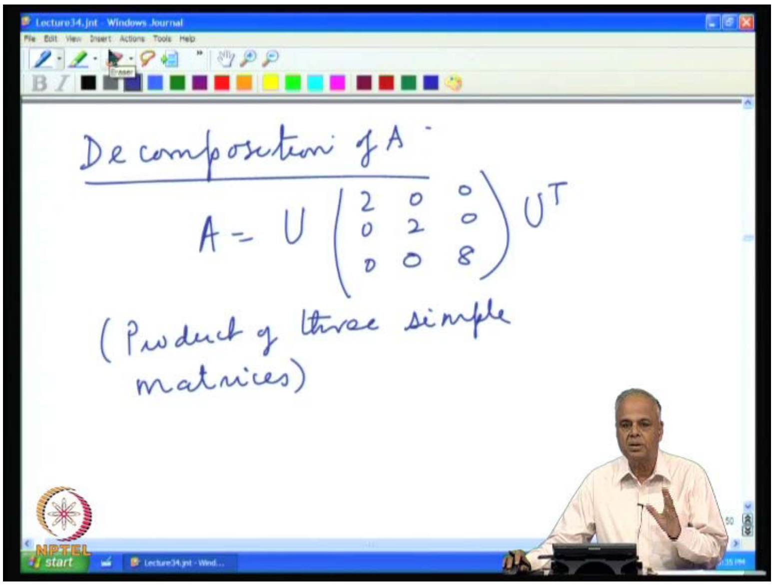

Now, the construction of U – the unitary matrix U. Now, to get U, we must put the eigenvectors along the orthonormal eigenvectors that we found. We have 1 by root 5, 0, minus 2 by root 5; then, 2 by root 30, 5 by root 30, and 1 by root 30; and then, we have 2 by root 6, minus 1 by root 6, 1 by root 6. So, this is the matrix obtained by putting the three eigenvectors along the three columns. Then, U transpose U is equal to identity, because U is unitary. Why is U unitary? Because columns are orthonormal. The moment the columns are orthonormal, the matrix becomes unitary. So, that is the matrix U.

(Refer Slide Time: 28:06)



And then, we get – if we know put U transpose A U – U star is U transpose in this case, we can easily verify. We can substitute for U; we can substitute for U transpose; the matrix U is given. And, verify that the product is going to give a diagonal. Which diagonal? The first two columns corresponds to the eigenvalue 2. So, the first two diagonal entries will be 2. Then, the third column corresponds to the eigenvalue 8. And therefore, the third diagonal entry will be 8 and therefore, this is the diagonalisation of U.

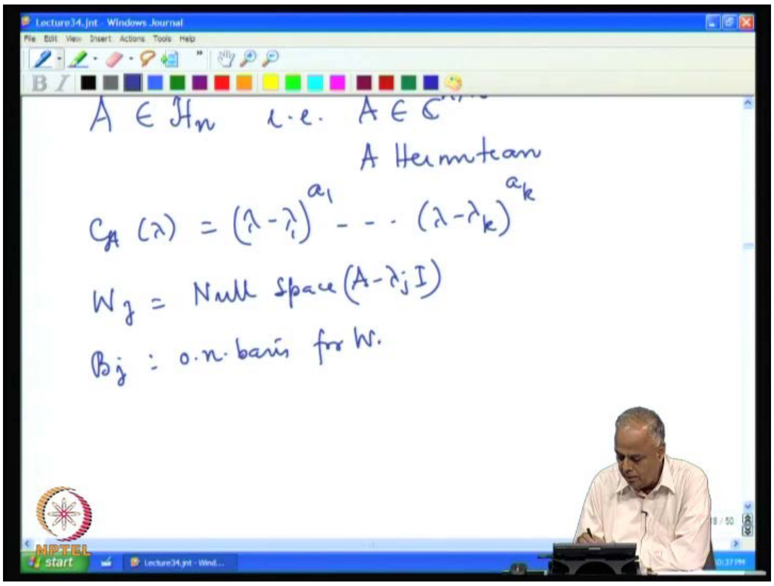
(Refer Slide Time: 28:48)




And, the decomposition of A is given by A equal to U – this diagonal matrix into U transpose; where, U is as obtained here (Refer Slide Time: 29:09). And therefore, we see that A has been decomposed into the product of three simple matrices; the two extreme factors being unitary and the middle one being diagonal. We seen therefore the structure of real symmetric and complex Hermitian matrices through their eigenvalues. We observed that the eigenvalues are real; eigenvectors are orthogonal to each other when the eigenvalues are distinct. And, we can find eigenvector basis for the whole space.

And, corresponding to each eigenvalue, if the algebraic multiplicity is A j, then exactly A j, orthonormal eigenvector is corresponding to them. And, when you put all these orthonormal vectors along the columns of the matrix, it become the unitary matrix and we get the unitary diagonalisation in the case of complex; and orthogonal diagonalisation in place of real. And, we get the decomposition of a given Hermitian matrix or a real symmetric matrix into a product of three simple matrices – two of them being unitary and the middle one being a diagonal matrix. And, the unitary matrix U is constructed using the eigenvectors of… See when we construct (Refer Slide Time: 30:44) this U, it is constructed using the three eigenvectors of U. And, the diagonal matrix is constructed using the eigenvalues. And, each eigenvalue appears as (Refer Slide Time: 30:57) many times as this algebraic multiplicity. So, the construction of this decomposition involves both the eigenvalues and the eigenvectors. Thus, the eigenvalues and the eigenvectors of a Hermitian matrix play a very crucial in looking at the structure of the matrix in a very simple form.

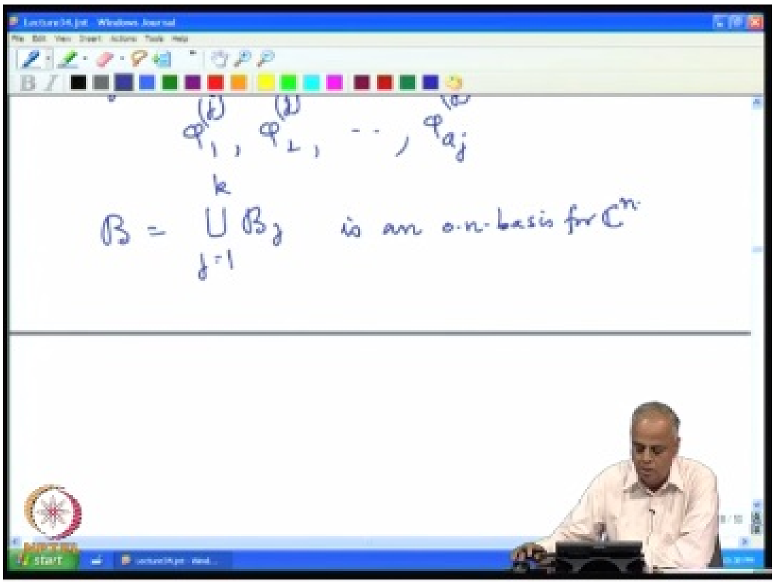
(Refer Slide Time: 31:46)




We have see the decomposition of a Hermitian matrix into a product of three simple matrices. Now, we shall look at the same in a different angle and look at the decomposition of matrix, not as a product of simple matrices, but as the sum of simple matrices. So, decomposition of A as the sum of simple matrices – this is our goal. We say what is meant by simple matrices as the goal (( )) This is connected with one of the question that we raised at the beginning of the course. So, what we do is the following. We start with the matrix A; H n is the collection of all Hermitian matrices; that is, A belongs to C n by n and A is Hermitian.

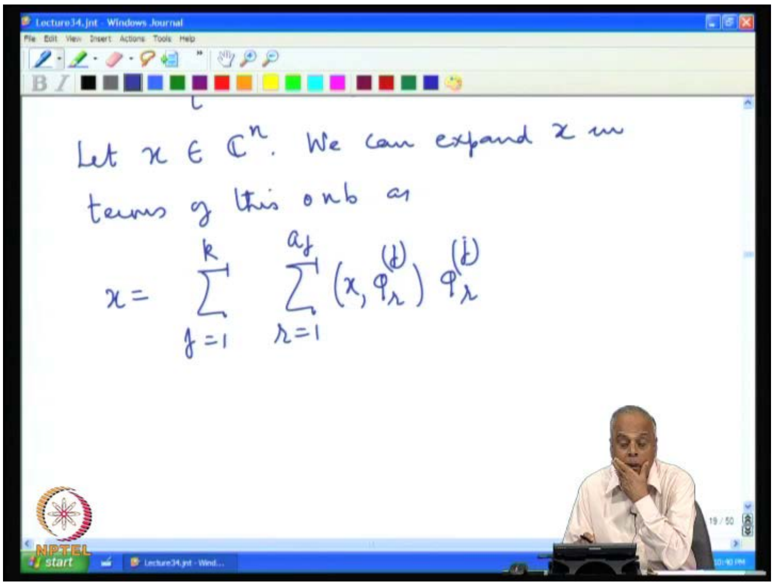
Now, we start with a Hermitian matrix. Then, we have its characteristic polynomial with the usual (Refer Slide Time: 32:36) notations; lambda 1, lambda k are the distinct eigenvalues. And, a 1, a 2, a k are the algebraic multiplicities of these eigenvalues. And then, we have the W j to be the null space of A minus lambda j I, which is the eigenspace corresponding to lambda j I.

(Refer Slide Time: 33:18)



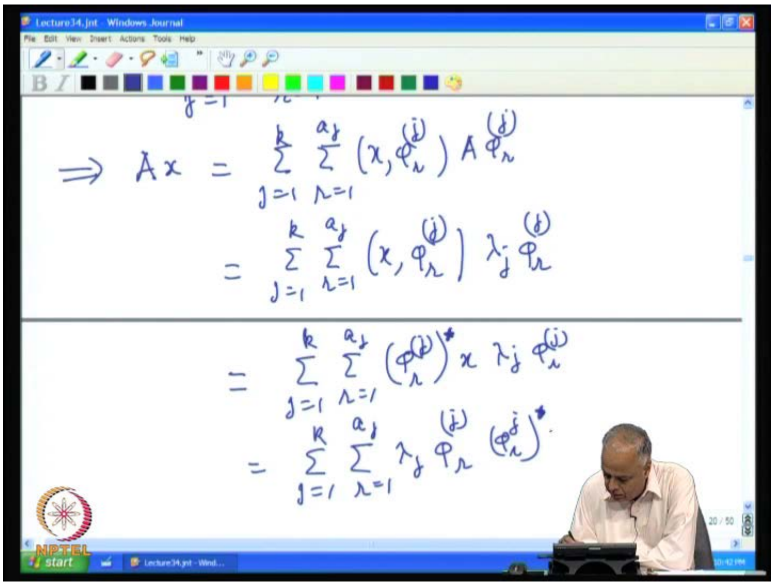

And then, we have the orthonormal basis for W j, which we denote by phi j 1, phi j 2, phi j a j. This is the notation we have used in the last lecture corresponding to the j th eigenvalue. So, the superscript j says that we are talking about the eigenvalue lambda j; and, the subscript says the ordering of the eigenvector. So, there are a j orthonormal eigenvectors corresponding to the eigenvalue lambda j. And then, we said that… We observed last time that if we put all these things together j equal to 1 to k – these basis, the individual orthonormal basis is an orthonormal basis for C n. So, how does this B looks like? It starts with phi 1 1; then, goes on up to phi 1 a 1; then, it goes to phi 2 1; then, phi 2 a 2 (Refer Slide Time: 34:24). So, at least first, the a 1 eigenvectors, orthonormal eigenvectors corresponding to lambda 1; then, the a 2 eigenvectors corresponding to lambda 2. And, in the end, it (( )) the eigenvectors corresponding to lambda k and there are a k of them. So, this is how the basis looks like. Now, we have an orthonormal basis.

(Refer Slide Time: 34:53)



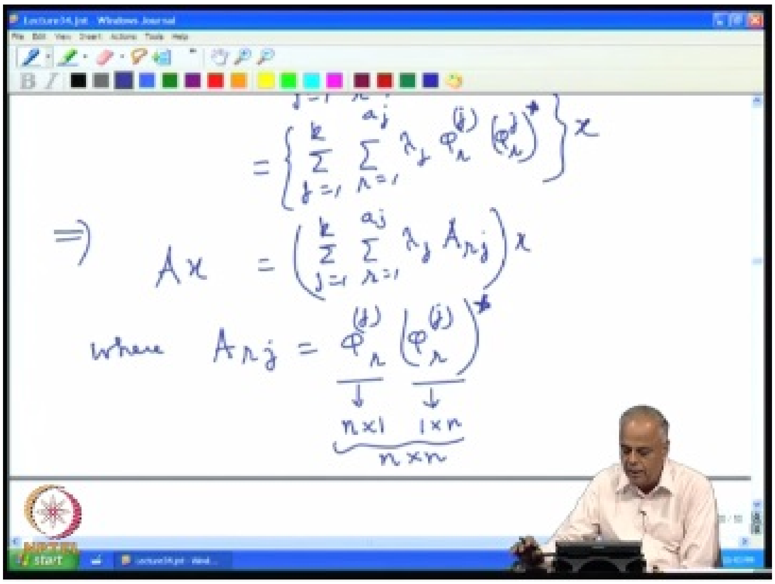
Now, any vector can be expanded in terms of the orthonormal basis. Let x be in C n. We can expand x in terms of this orthonormal basis as… Now, see first, we will write all the terms corresponding to the first a 1 base vectors; then, a 2 base vectors. So, we can use this notation x is equal to… First, expand with respect to the jth basis vectors and do this for every one of the j's. So, we can write the expansion of x in terms of this orthonormal basis in this form – x phi r j – the orthogonal projection of x along the direction times the projection. This is the Fourier expansion, which we have seen before. For any orthonormal basis, we can expand a vector in terms of the orthonormal basis.

(Refer Slide Time: 35:58)



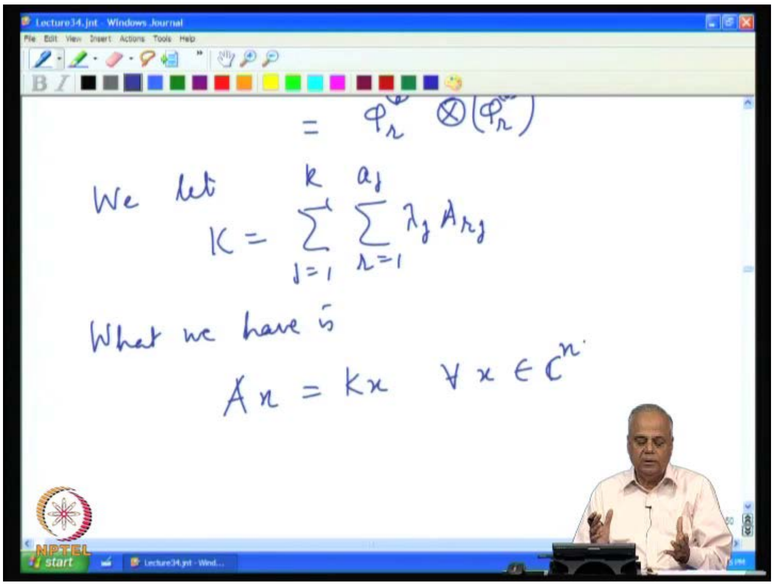
Now, this implies if we now look at A x, I have to multiply by A; it is a sum. So, I can distribute the multiplication to each one of these terms. This is a constant; this is a number; (( )) product is a number. So, I have look at A phi r j. But, I know the moment as this superscript is there, that means this is an eigenvector corresponding to the eigenvalue lambda j. So, that becomes the two summations and this coefficient. And then, this is the eigenvector corresponding to eigenvalue lambda j. So, it can be written as lambda j phi j r. If we recollect, we can write the inner product of two vectors: x comma y as y star x. So, this is the same as phi r j star x. Inner product is written in terms of the matrix dot product, but now, it is a conjugate. So, phi r j star x; and then, lambda j and then phi r j.

(Refer Slide Time: 37:50)



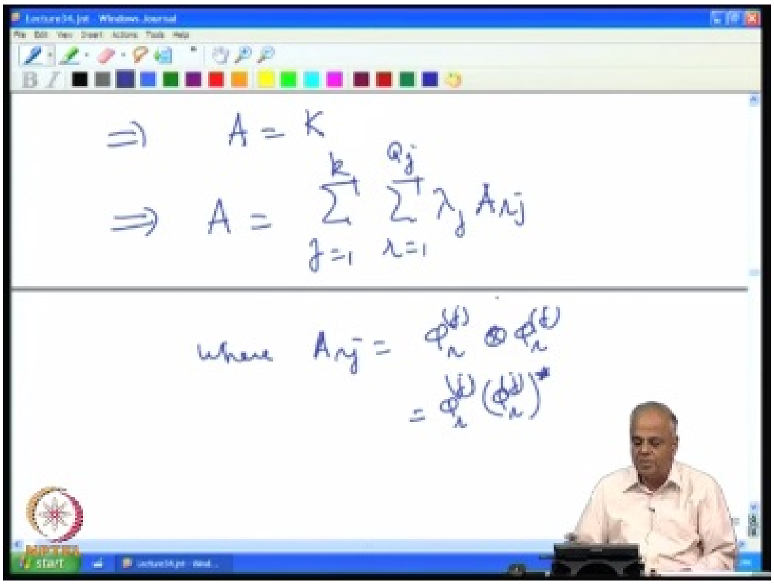
We can rewrite this as summation j equal to 1 to k summation r equal to 1 to a j lambda j phi j r phi r j star x; phi r j star x is a number. So, this is the complete product. And, we can rewrite as lambda j phi r j phi r j star – this whole thing multiplying x, because matrix multiplication is distributive. So, we will write this now as j equal to 1 to k; r equal to 1 to a j lambda j – we will write this as A r j x; where, A r j is just phi j r pie j r star. So, this is what A x is. Now what is the A r j? If U look at phi r j, it is an eigenvector; it belongs to C n. So, that is an n by 1 vector. Therefore, this is obtained by transposing and conjugating. So, that is the 1 by n vector. So, the product is going to be an n by n matrix. So, A r j is an n by n matrix.

(Refer Slide Time: 39:08)



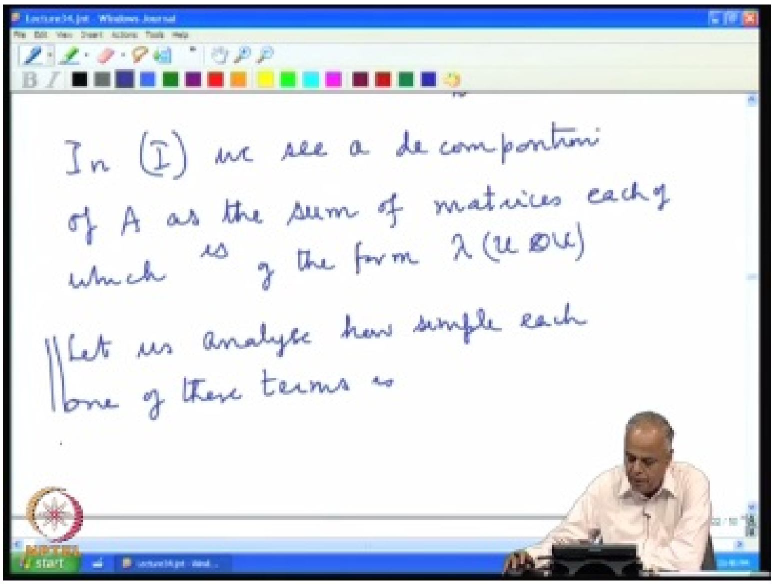

In short notation, we will write it as phi r j. This is what in the beginning of the course we called as the outer product or the tensor product. A r j is phi r j tensor phi r j. It is the tensor of phi r j, which is (( )). So, phi r j tensor phi r j means phi r j phi r j star. That is the matrix phi r j phi r j star. So, this phi r j phi r j star is written in this form. Therefore, A r j is a matrix. Now, (Refer Slide Time: 39:54) A r j is a matrix; lambda j is a number. So, number times matrix is a matrix. We are adding a number of n by n matrices; that is going to be another n by n matrix. So, we let K to be all that sum – j equal to 1 to k; r equal to 1 to a j lambda j phi A r j. Now, this is an n by n matrix. What we have is A x. This A x is equal to k x and this is true for every x; A x is equal to k x for every x in C n. Now, if there are two matrices A and k such that A x equal to kx for every x in C n, then A must be equal to k.

(Refer Slide Time: 40:58)




This implies A equal to K. Therefore, we have A equal to summation j equal to 1 to k; r equal to 1 to a j lambda j A r j. Where, A r j again I repeat is phi r j tensor phi r j, which is the same as phi r j phi r j star. Now, we see already we have some decomposition. We have decomposed the matrix A as the sum of a number of matrices lambda j A r j; A r j is a matrix; so, we are looking lambda j A r j. Thus, we have decomposition.

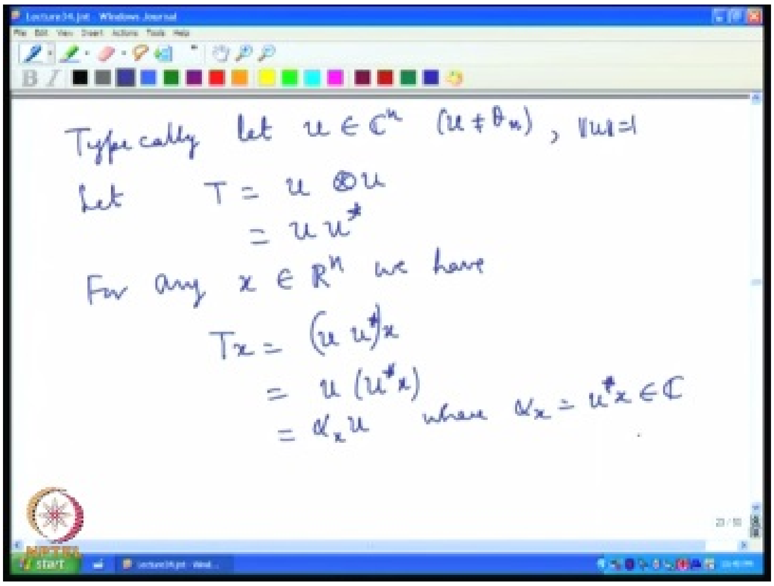
(Refer Slide Time: 42:04)




Let us analyze this decomposition little bit carefully by looking at each term. Let us call this as 1. In 1, we see a decomposition of A as the sum of matrices is of the form lambda

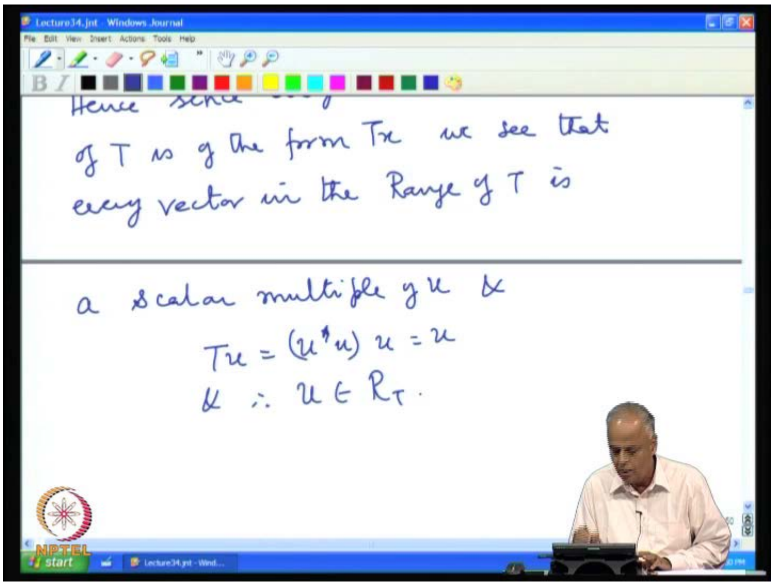
U cross U by suitable values of lambda and U. So, let us analyze each one of these. So, let us analyze how simple each one of these terms is.

(Refer Slide Time: 43:40)



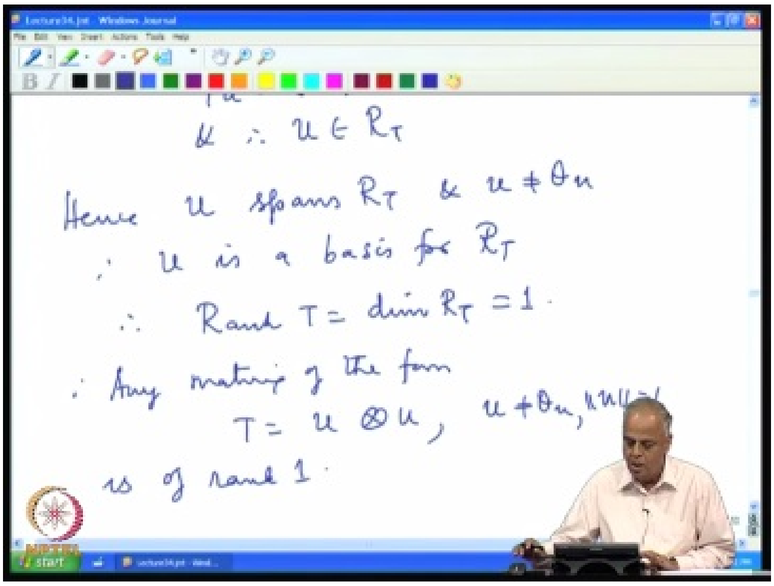

Typically, let U belong to C n and U not equal to theta n. And, let say T be a matrix, which is U tensor U, which is equal to U U star. Now, for any x in R n, we have… – and, let us without the loss of generality even take length of U to be 1, because in our construction, all the vectors involved are of length 1. So, we can as well look at these things with the length 1. For any x in R n, we have T x is equal to U U star x. Now, matrix multiplication is associative. So, it is U star x. Now, U star x is a number. So, it is of the form some alpha x U, where alpha x is U star x, which belongs to C. And therefore, we find that any vector in the range of T is of the form T x. And therefore, every vector is a multiple of U.

(Refer Slide Time: 45:10)



Hence, since every vector in the range of T is of the form T x, we see that every vector in the range of T is a scalar multiple of U. And, T of U is U star U into U, which is equal to U, because U is unitary; U star U is the length U square, which is 1. And therefore, U belongs to range of T, R T. Therefore, U is a vector in the range of T; and, every other vector in the range of T is a multiple of U.

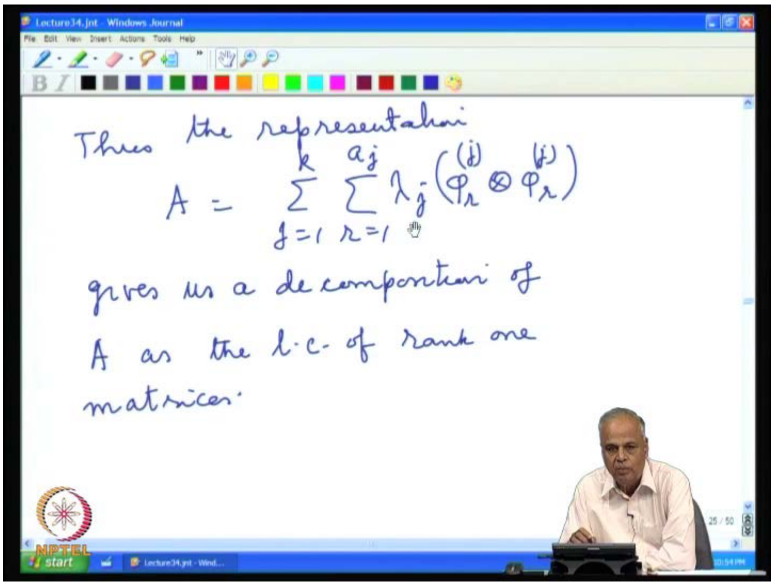
(Refer Slide Time: 46:15)



And hence, U spans the range of T and U not equal theta n. Therefore, U is a basis for range of T. And therefore, rank of T, which is equal to the dimension of range of T…

Now, since there is only one vector in the basis for the range of T, the dimension is 1. Therefore, any matrix of the form T equal to U tensor U; U not equal to theta U normally equal to 1 is of rank one. Now, since all the matrices A r j are all of this form (Refer Slide Time: 47:12) – they are all of the form U tensor U; U is phi r j; and they are all of length 1. Then therefore, by the structure of A r j we get that. Hence, (Refer Slide Time: 47:30) each A r j is of rank one. And therefore, we have seen that we are able to express the matrix A.

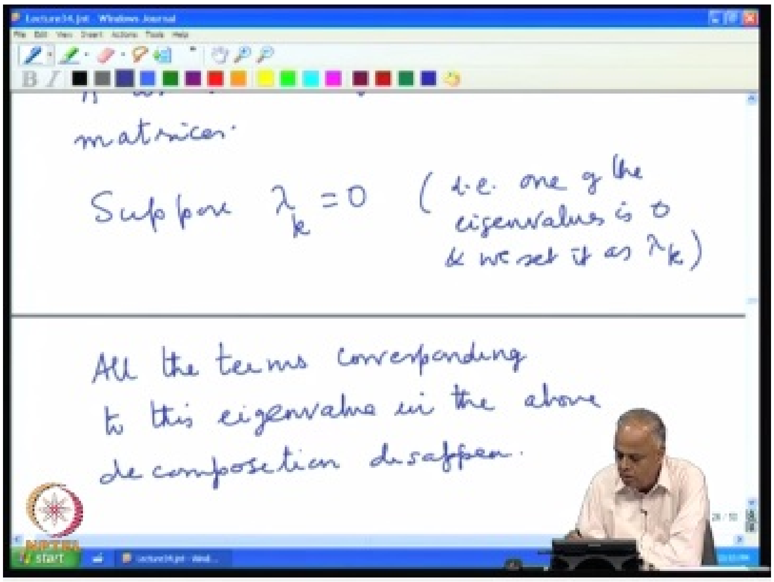
(Refer Slide Time: 48:04)




If we look at this (Refer Slide Time: 47:48) representation, these are all of rank one. So, this is only a linear combination of rank one matrices. So, we have expressed… The first thing, we observe the following. Thus, the representation A equal to summation j equal to 1 to k summation r equal to 1 to a j lambda j phi j r tensor phi j r gives us a decomposition of A as the linear combination of rank one matrices. This is the first observation that we have about the decomposition of the matrix as the sum of matrices. Now, they are sum of simple matrices, because now, they are all linear combinations of rank one matrices.

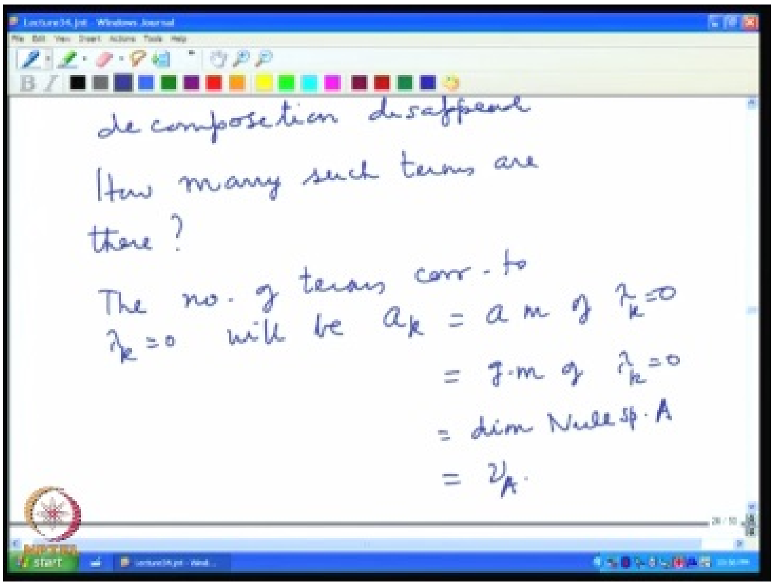
Now, let us analyze this a little more. What we observe is that when we have an eigenvalue 0, the terms corresponding to that eigenvalue will all disappear, because we will be multiplying by that eigenvalue lambda j. For example, we are having this multiplier (Refer Slide Time: 49:31) lambda j. So, whenever we have the terms corresponding to an eigenvalue 0, these terms will completely disappear from this summation. So, let us make that point.

(Refer Slide Time: 49:54)



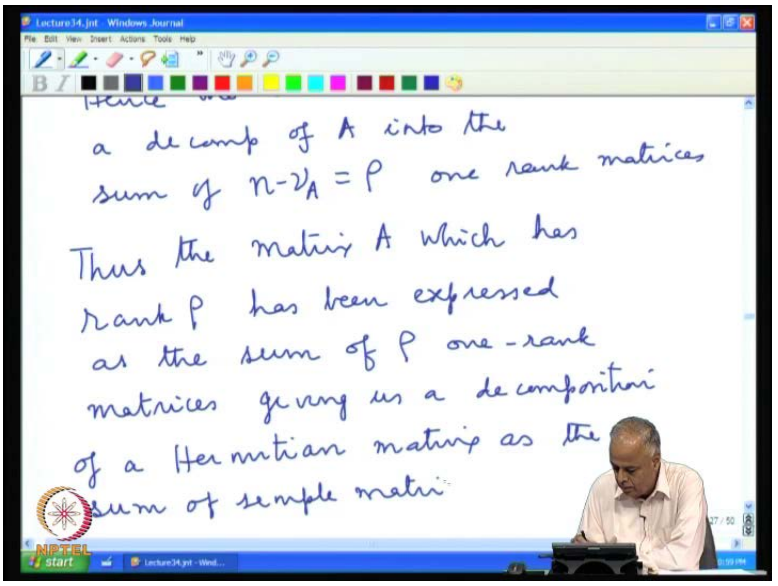

Suppose let us say for convenience last eigenvalue is equal to 0; that is, one of the eigenvalues is 0 and we set it as lambda k. So, one of the eigenvalues is 0 say. What does that mean? This means all the terms corresponding to this eigenvalue in the above decomposition disappear, because we will be multiplying by that particular eigenvalue. We are multiply (Refer Slide Time: 50:59) lambda j times this decomposition; you will see that there is a multiplier lambda j, so that the eigenvalue corresponding to k is lambda; k term we take. Then, all those terms will disappear. Now, how many terms are there? What is the effect of the disappearance? All these if you look at, then we get a decent decomposition (( ))

(Refer Slide Time: 51:25)



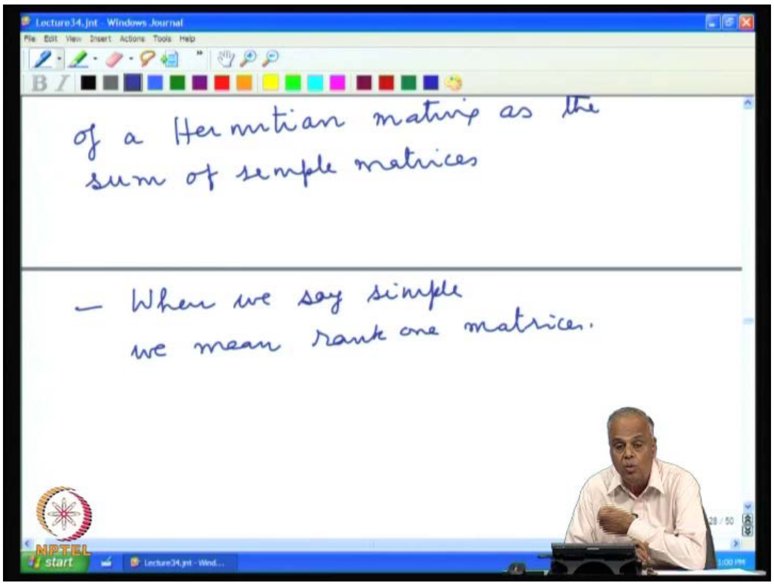

How many such terms are there? The number of terms corresponding to lambda k equal to 0 will be a k. What is a k? That is equal to the algebraic multiplicity of lambda k equal to 0, the eigenvalue lambda k, which means this is the same as the geometric multiplicity of lambda k equal to 0, which is the dimension of null space of A minus lambda k, but lambda k is 0 as the dimension of null space of A, which is just the nullity of A. In other words, nu A terms will totally disappear from the representation of A. So, we have the decomposition of A as the sum as the linear combination of matrices, linear combination of rank one matrices. A prior, we had n terms and because of the multiplier factor of the eigenvalue, whenever we have an eigenvalue 0, nu A, a nullity times, that many terms will disappear from the eigenvalue.

(Refer Slide Time: 52:54)




Hence, the above sum becomes a decomposition of A into the sum of – now, totally we had n terms – we had a 1 terms corresponding to lambda 1, a 2 terms corresponding to lambda 2, etcetera. Out of which now nu A disappeared; n minus nu A terms will be there, but n minus nu A is the rank of the matrix of rho one-rank matrices. Now, each one is one rank, because it is one rank multiplied by a constant – nonzero constant. Remaining eigenvalues are nonzero. So, all other terms are of rank one matrices. Thus, we see that the matrix, which has rank rho has been expressed as the sum of rho one-rank matrix. Thus, the matrix, A which has rank rho has been expressed as the sum of rho one-rank matrices giving us a decomposition of – remember, we are dealing with the Hermitian matrix – decomposition of a Hermitian matrix as the sum of simple matrices.

(Refer Slide Time: 55:03)




And, when we say simple, we mean rank one matrices. Now, we see therefore, in the case of Hermitian matrices, we have very nice decompositions; we have a product decomposition; we can decompose the matrix as the product of three simple matrices: two of them being unitary, the middle one being diagonal. We can diagonalise the matrix A; we have all eigenvalues real. And, we can also express the matrix as the sum of simple matrices namely sum of rank one matrices. And, if the rank of the matrix is rho, we get exactly A as the sum of rho one-rank matrices. And, these one-rank matrices are easily constructed as the tensor product of an eigenvector with itself. And, these are the eigenvectors corresponding to the nonzero eigenvalue of the matrix. So, in the case of Hermitian matrices, we have very nice structure, we have very nice decompositions, which should automatically help in the analysis of the matrix as well as the systems of equations connected with such Hermitian matrices. But, all these we have seen only for Hermitian matrices.

What does one do when the matrix is not Hermitian or if you have a real matrix, which is not real symmetric, what do we do in such cases? Or, if you do not even have a square matrix and you are dealing with a rectangular matrix, what do we do in such cases? So, what we shall see is we can reduce all these problems to the analysis of some suitable Hermitian or real symmetric matrix. Therefore, we will refer to all the results we have. And, through this, we can conclude about the decompositions as product, decompositions as sum and so on of any general matrix. To this end, we shall start looking at a very special class of Hermitian matrices, which we shall be using in our analysis of a general matrix.