Advanced Matrix Theory and Linear Algebra for Engineers

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Lecture No. # 32

Hermitian and Symmetric matrices- Part 1

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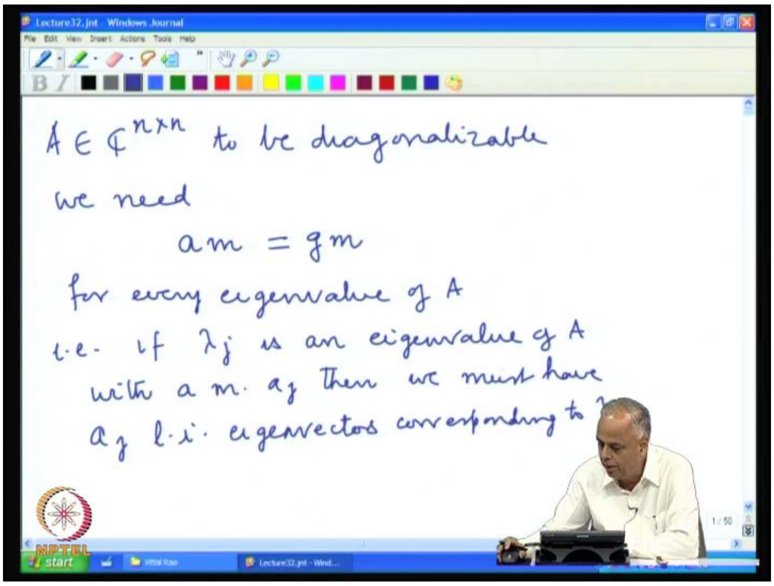

In the last lecture, we saw that if A is in C n n, that is, if A is n by n complex matrix, then for A to be diagonalizable, for A to be diagonalizable, for A to be diagonalizable, we need that the algebraic multiplicity is equal to the geometric multiplicity for every Eigen value of A. What we mean is, that is, if lambda j is an Eigen value of A with algebraic multiplicity a j, then we must have n, we must have a j, linearly independent eigenvectors corresponding to lambda j. So, depending on the multiplicity of the Eigen value we must have that many linearly independent eigenvectors.

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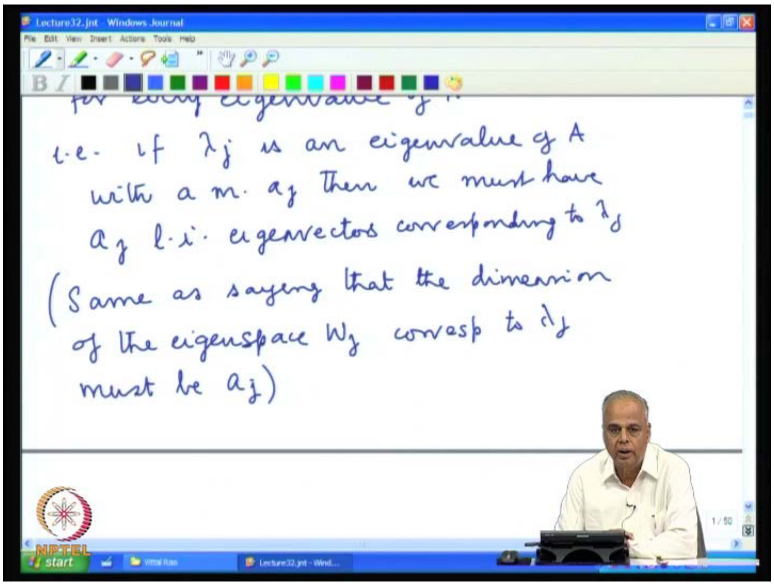

This is the same as saying, that the dimension of the eigenspace W j corresponding to lambda j must be, must be a j, must be a j. So, we need this condition for diagonalizablity.

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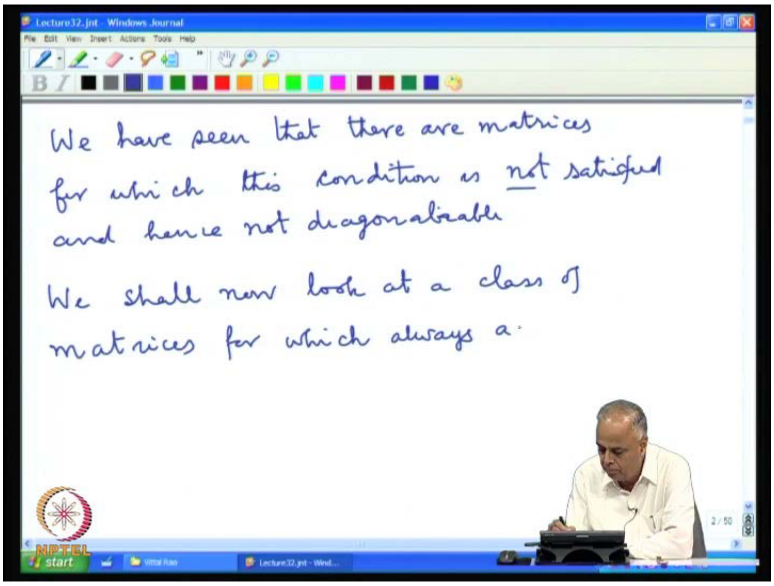

And we have seen that there are matrices, there are matrices for which this condition is not satisfied, this condition is not satisfied, and hence not diagonalizable. So, there are matrices for which the condition is not satisfied, that is, the geometric multiplicity will become less than the algebraic multiplicity for some Eigen values and hence, the matrix fails to be diagonalizable. Therefore, we have this problem, that given a matrix A, you, priory we do not know whether it is going to be diagonalizable or not. We have to look at the geometric multiplicity and the algebraic multiplicity, that is, we look whether we get enough numbers of eigenvectors to form a basis for the whole space.

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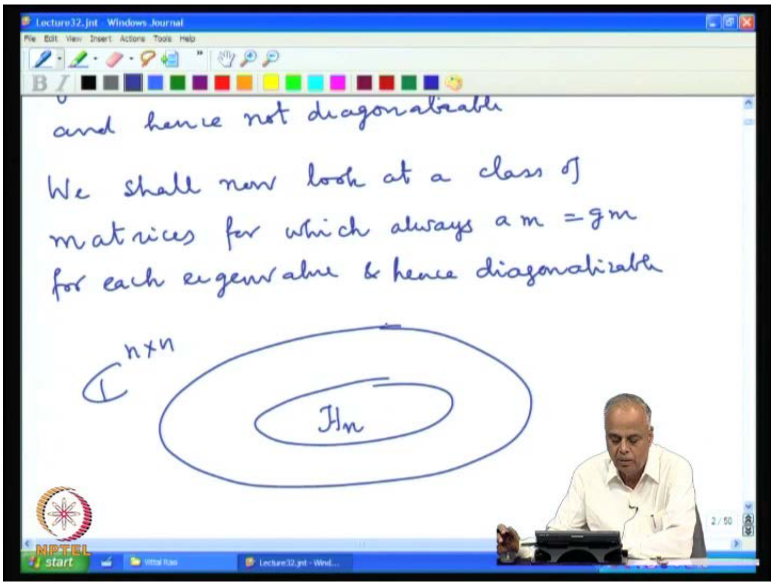

Now, we are going to look at the class of matrices. We shall now look at the class of matrices, which is the sub-class of the entire set of matrices for which always a m equal to g m for each Eigen values. So, we are going to look at matrices, a class of matrices.

Among this whole world of matrices, there is a class of matrices for which this condition is always satisfied and hence, diagonalizable. This is the first sub-class of matrices we look at. What do you mean by following? We have this whole collection of n by n matrices, inside that we are going to look at the sub-class H n. We will define what H n is and this sub-class H n.

You take any matrix A in the sub-class in H n, it will be diagonalizable or for which algebraic multiplicity, it will be equal to geometric multiplicity. So, we will call this sub-class H n, we will explain, what H n is, in short.

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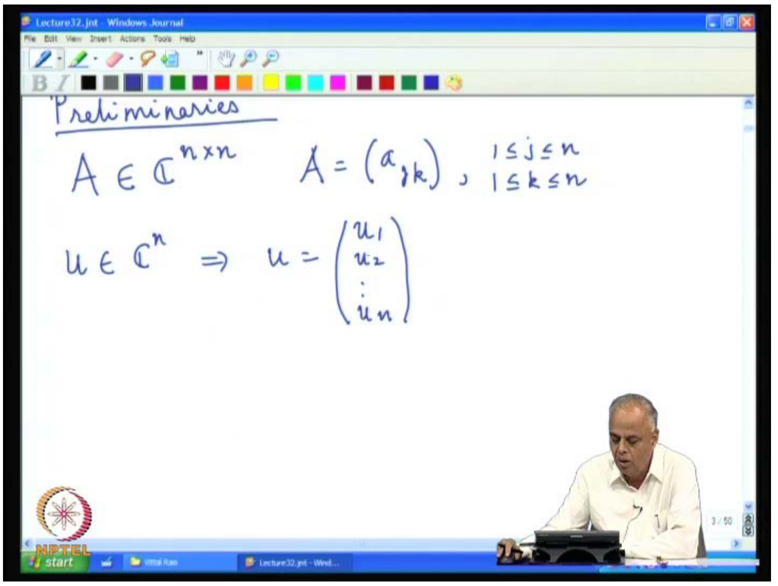

So, for that we will look at some preliminary ideas, simple calculations, which will give us the right notational frame work to work with. So, let us, look at a matrix A, which is n by n complex. Then, let us denote it as the entries, as a jk, where j is the row index, which goes from 1 to n and k is the column index, which goes from 1 to n. So, we have a complex matrix A, whose entries are a jk; a jk denotes the entry in the jth row and the kth column and it is a complex number.

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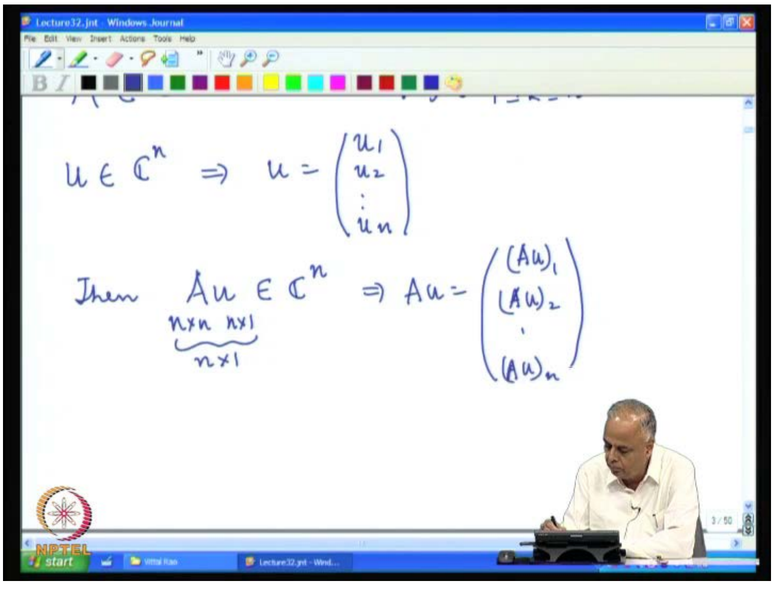

Now, take any vector u in C n, and then u is of the form, u equal to u 1, u 2, u n, where the u j are all complex numbers. Now, if u is a vector in C n, then Au, if you multiply the vector by u the matrix A, that is also a vector in C n, because A is n by n and u is n by 1, so the product is going to be n by 1; it is also going to be vector in C n. If, you take a vector in C n, Au is also going to be in C n. So, we can write Au, as it is 1st component, we will denote by Au 1, 2nd component by Au 2 and so on, Au n.

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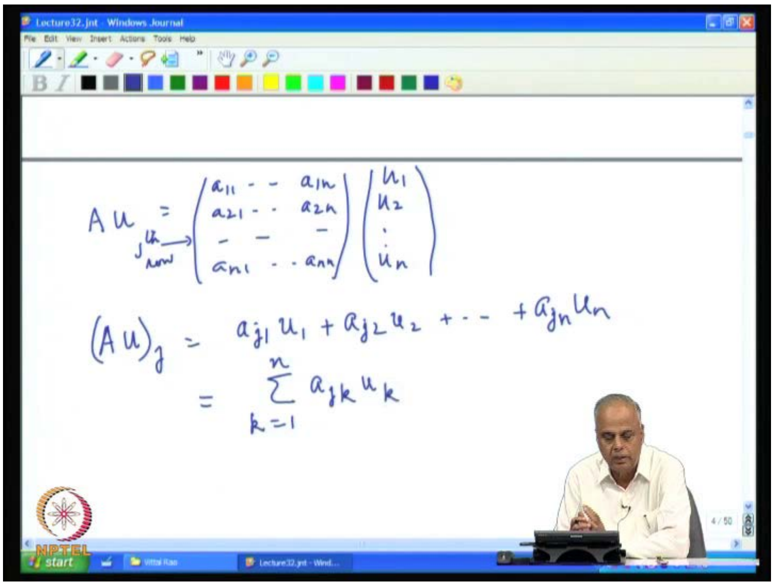

Now, how do we get the jth component? A, look at the vector Au, A is untrained by multiplying them matrix a 11, a 1n, a 21, a 2n and so on, a n1, a nn with the vector u 1, u 2 and u n. In order to get the jth component of this product, we have to look at jth row and multiply it with the vector u 1, u 2, u n. So, we get the jth component of Au to be a j1 u 1 plus a j2 u 2 plus extra, a jn u n, which we will write in summation notation as summation k equal to 1 to n, a jk u j. So, the j, the component of the product A u, it was given by a jk u j.

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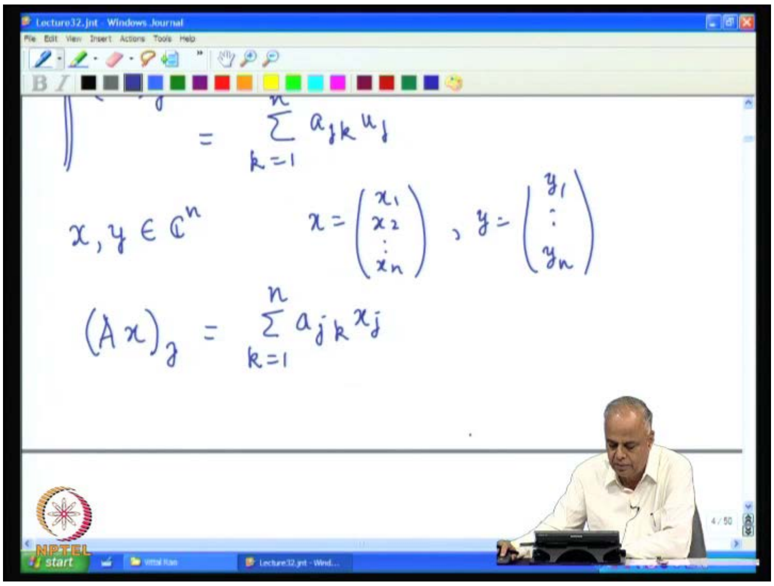

Now, consider two vectors x, y epsilon C n. x is x 1, x 2, x n; y is y 1, y 2, y n. Now, applying the above logic we get A x is the matrix and it is component, jth component. By the above calculation, in this we replace u by x, we get, that is equal to k equal to 1 to n, a j k x j. That is our A xj.

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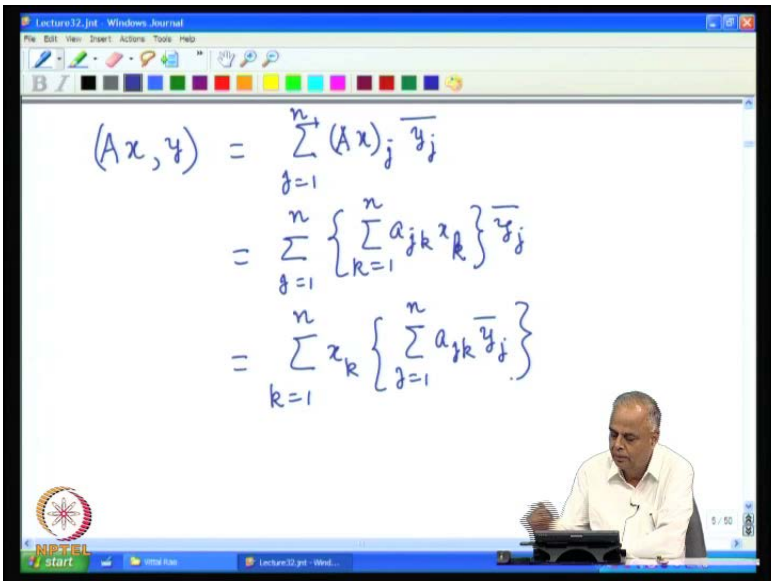

If you look at the inner product of A (x, y) by definition, that is summation j equal to 1 to n, the jth component of x multiplied by the jth component of y with the conjugate because we are dealing with the complex vector space. Now, A xj we have calculated here and if we substitute that, that becomes j equal to 1 to n summation k equal to 1 to n, a j k x k times y j y.

Now, we have two sums, one is on the index j and the other is on the index k and both are finite sums, and therefore, with an interchange in order of the sum. So, we will take the k sum first and then x, this should be x k, the x k comes out, the remaining all are dependent on j, so they all go inside as a j k y j y.

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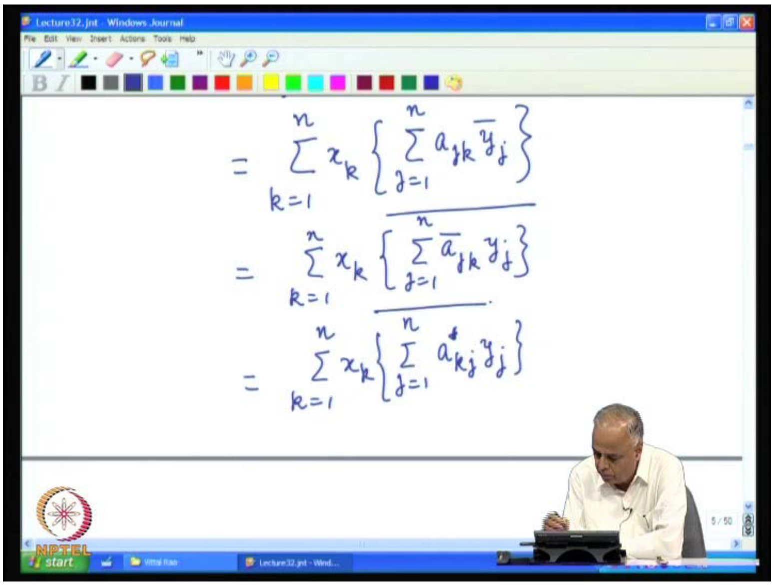

We can now write this as summation k equal to 1 to n, x k, j equal to 1 to n a jk bar y j, this quantity bar. We have taken the conjugate twice and for simple notation we write this as summation j equal to 1 to, summation j equal to 1 to n A star k j y j bar.

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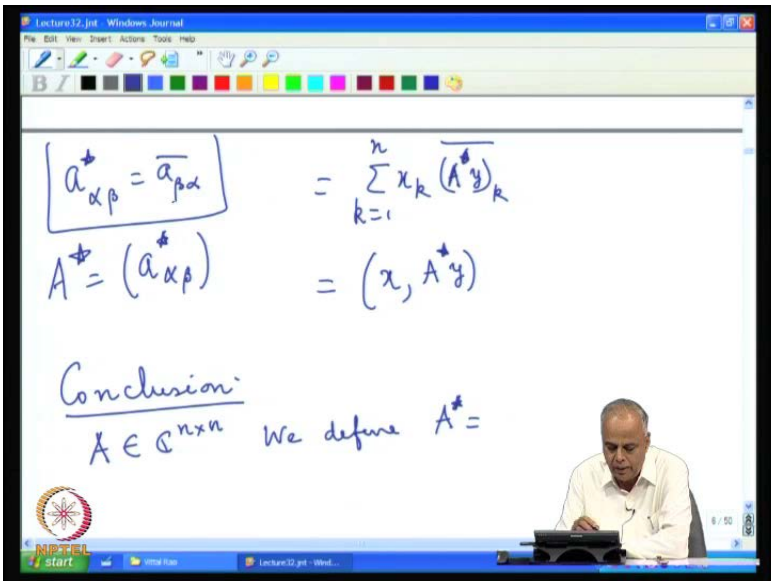

Where A star alpha beta is a beta alpha conjugate, from, from the above definition, for any alpha beta between 1 and n. Therefore, this becomes, if you now, I define the matrix A star to be the matrix, whose entries are A star alpha beta, which is equal to A beta alpha, then this becomes k equal to 1 to n x k A star y k y, which is the same thing as the inner product between x and A star. This is explicatively seen, this competition of A x, y equal to x, A star y.

Therefore, what is the conclusion? The conclusion is that if A belongs to C n by n, we define A star. How do we obtain as A star? We interchange the row index and the column index and then conjugate it. Interchanging the row index and the column index be among s to transposing the matrix, so it is A transpose conjugate.

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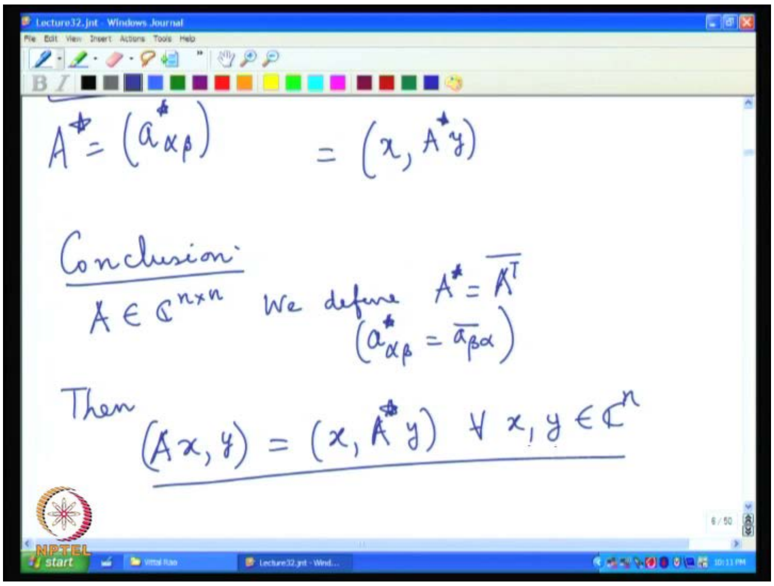

So, A star alpha beta is equal to a beta alpha bar. If we now define A star as A transpose **b**ar, then A x, y is equal to x, A star y for every x, y in C n. This is a very important identity, which will be used repeatedly.

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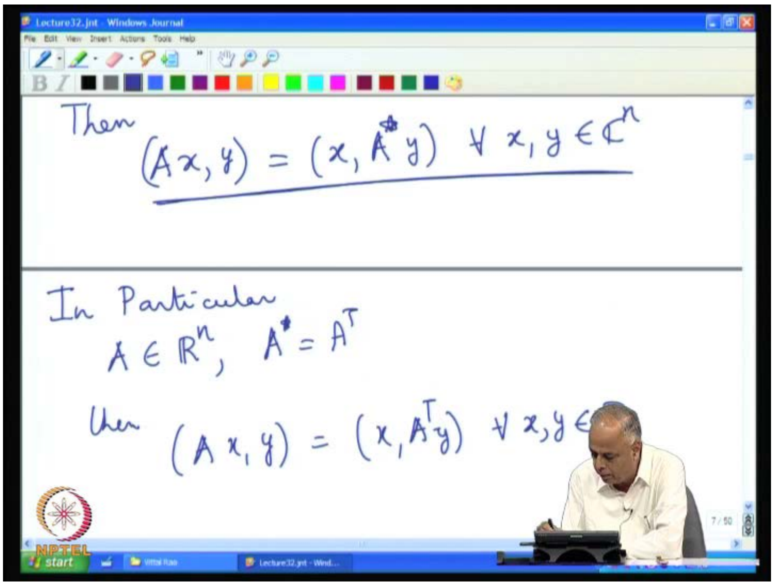

In particular, in particular if we take everything real, if A belongs to R n, then there is no more conjugation involved. So, A star will be defined as A transpose only. Then, then we have A (x, y) is equal to x comma, in place A star we have a transpose A transpose to y for every x, y. Now, in R n this two are important observations. For A matrix, very important identity A (x, y) equal to x, A star y, that is, if you move A in the inner product, from one factor to another factor it moves as a star. Here, A was in the first factor, now we wanted it to move to the second factor, it moved as a star. Similarly, if has comes back from second to here, will come back again with the star note, that A star is equal to A.

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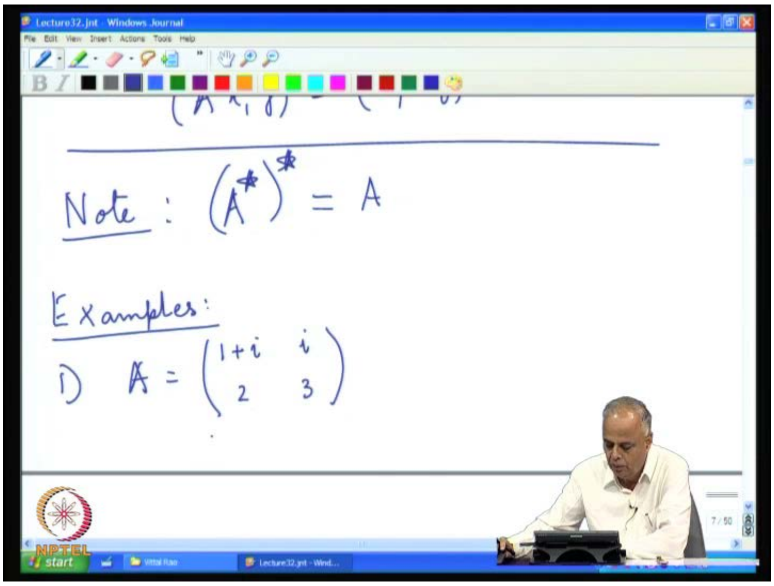

If the matrix again, transpose once you transpose a conjugate again, you transpose a conjugate you get. So, these two identities are going to be very useful identities for us. So, let us look at some examples. Let us take a very simple matrix, which is 1 plus i, i, 2, 3, which is now in C 2 2. So, it is a 2 by 2 matrix, is a complex matrix.

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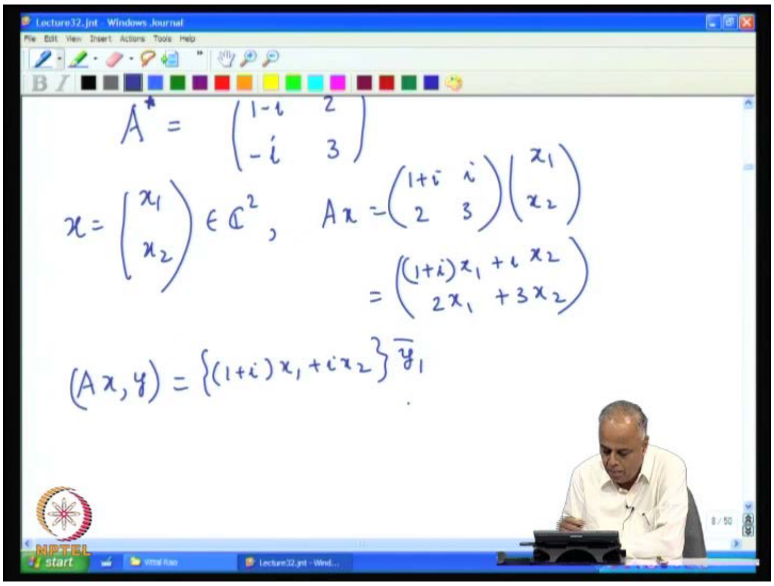

Now, what is A star? In this case we have to transpose and conjugate. So, this conjugation will make this i, transposition will bring i here and conjugate will make it minus i and 2 was there and 3 was here. This is what A star is, suppose x is a vector x 1, x 2, which is in C 2, then what is A x? A x is in 1 plus i, i, 2, 3 into x 1, x 2, which is 1 plus i into x 1 plus i into x 2, 2x 1 plus 3x 2. This is what A x is. So, now, if we take the inner product of x with y, the inner product is taken by taking the components of the product of the components with the second one coming as a conjugate.

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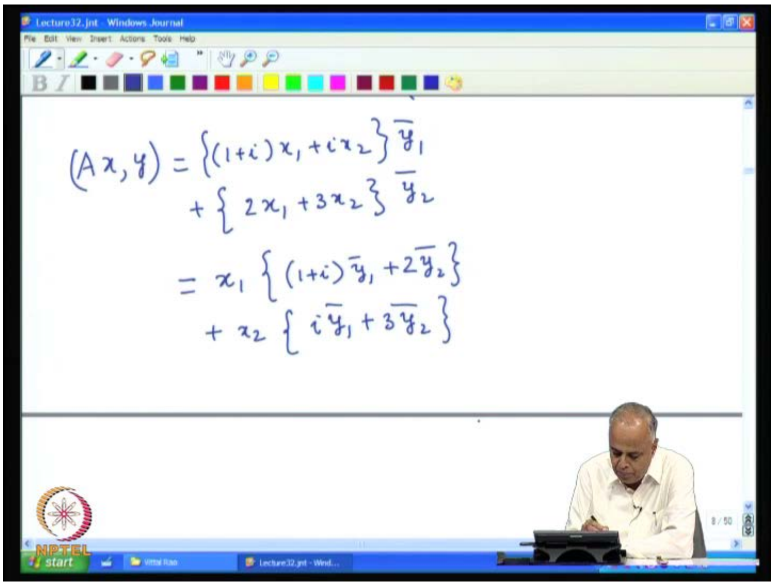

So, the first component of A x times the first component of y conjugated plus the second component of A x times the second component of y conjugated. We will make a slight rejudgement of this, we will write this as x 1, collect all the x 1 terms, which is 1 plus i into y 1 bar and x 2 plus and x 1 comes from 2y 2 bar, then we have plus x 2 into i y 1 bar plus 3y 2 bar. This is what A (x, y) is.

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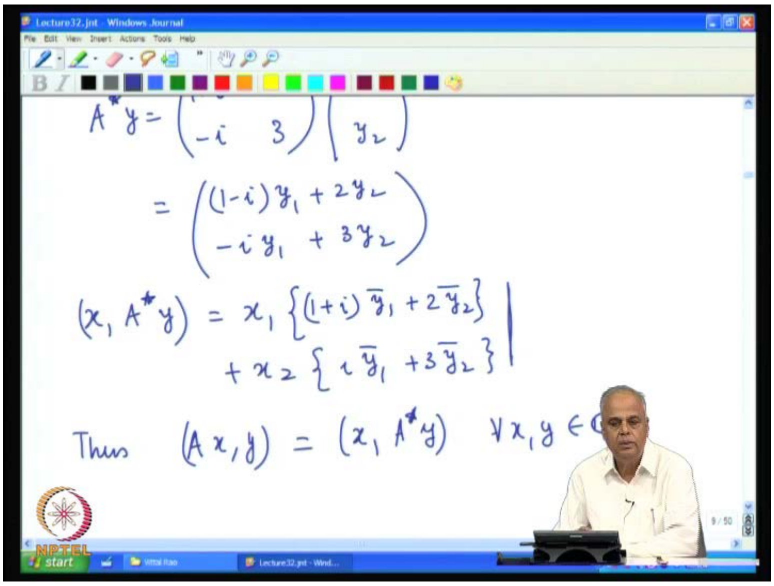

Let us compute A star y. A star y equal, A star is the matrix 1 minus i, 2, minus i, 3, so 1 minus i, 2, minus i, 3. Let us look at this 1 minus i, 2 and minus i, 3 is A star; y is y 1, y 2. So, if we now take this product, this becomes 1 minus i into y 1 plus 2y 2, minus i into y 1 plus 3y 2. And therefore, x, A star y, that is the inner product of x with A star y will be the first component of x 1 times the conjugate of the first component of A star y, which is 1 plus i into y 1 bar plus 2 into y 2 bar. Similarly, second component of x into the conjugate of the second component of y. Now, compare it with A (x, y), which we got here and we see, that it is the same as, thus A (x, y) is equal to x, A star y for every x, y in C. This is the identity that we have been discussing above.

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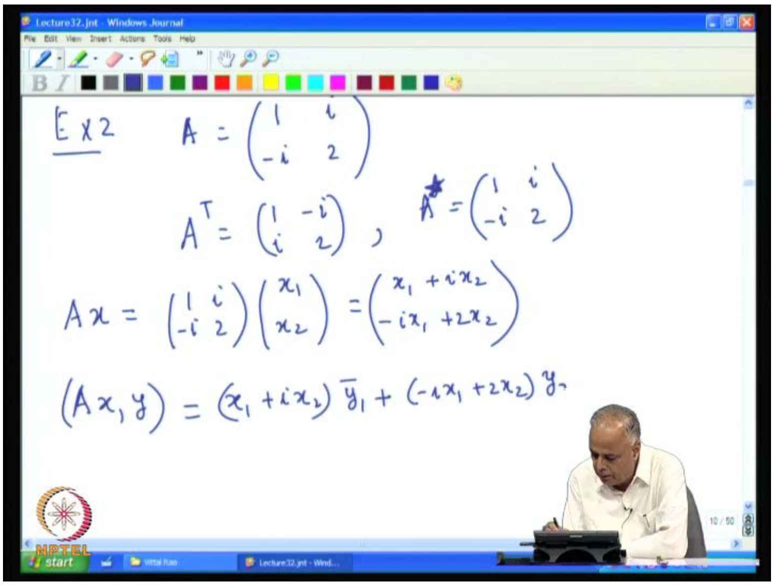

Let us look at another example. Let us take A to be 1, i, 2, minus i. Now, in this case what is A transpose? It is 1, i, minus i, 2 and A star is the conjugate of the transpose. So, it is 1, i, minus i, 2. We have to conjugate the A transpose to get A star. So, again, what is A x? A x is 1, i, minus i, 2, that is, A, x is x 1, x 2. If we now take a product, I get x 1 plus i x 2 minus i x 1 plus 2 x 2. Now, if I take A (x, y), which is inner product of A x with y, I have to take the first component of A x multiplied with the conjugate of the first component of y plus second component of A x multiplied by the conjugate of the second component of y.

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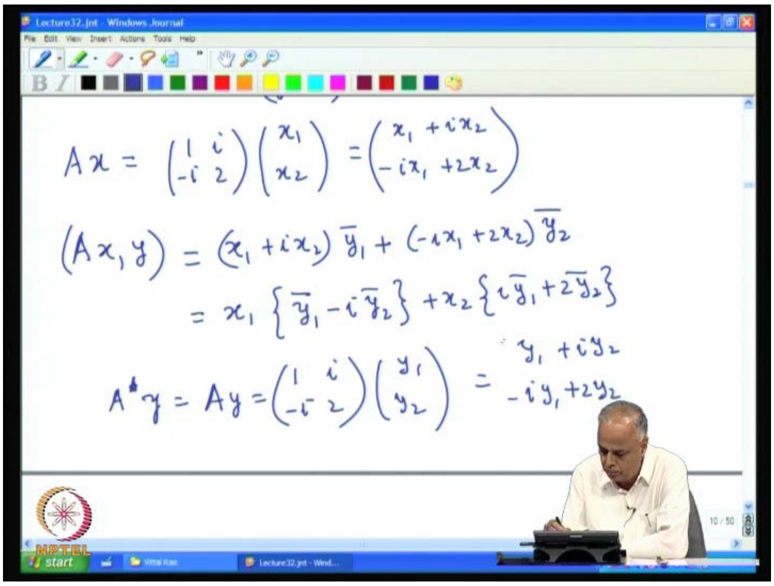

When we do this product we rearrange these terms again as before. Collect the x 1 terms, I get x 1 into y 1 bar minus i y 2 bar plus x 2 into i y 1 bar plus 2 y 2 bar. Now let us, compute x, A star y. First of all, what is A star y? A star y is same as A y because we observe here, that A star is equal to A, A star is equal to, is same as A y.

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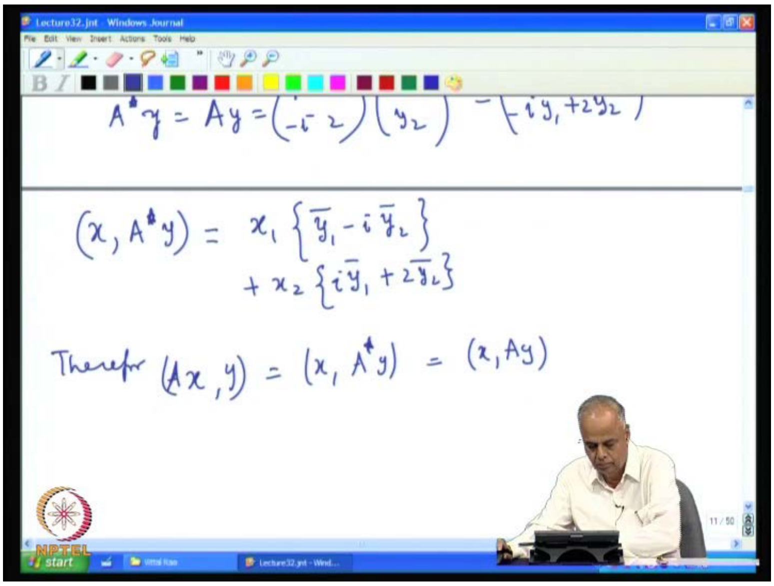

So, it is again 1 i minus i 2 into y 1 y 2, which is y 1 plus i y 2 minus i y 1 plus 2 y 2. Now, therefore, if i take (x, A star y), the inner product of x with A star y, I am (( )) x 1 times the first component of, we have to put the first component of A star y with the conjugate, so it will be y 1 minus i y 2 y 1 bar minus i y 2 bar plus x 2 into the second component of A star y with conjugation i y 1 bar plus 2 y 2 bar, which is precisely what we got here for (A x, y) and. Therefore, (A x, y) is equal to x, A star y. In this case, A star is…

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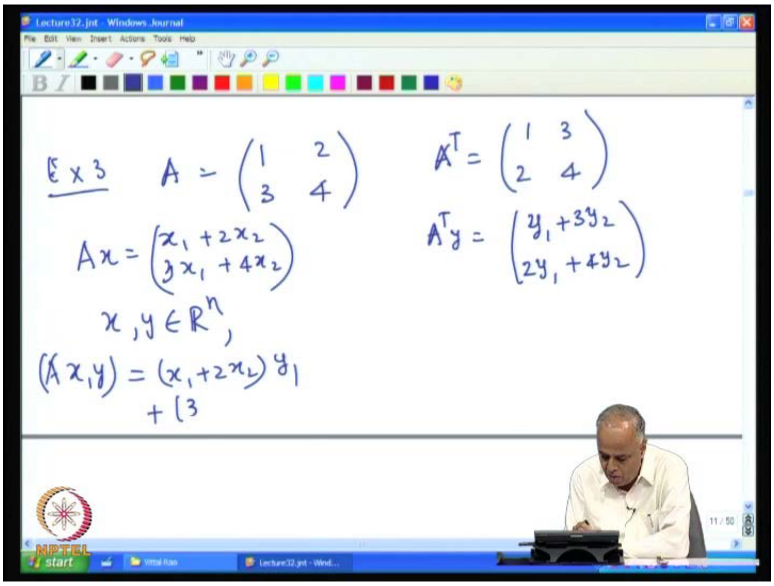

Let us look at another example, the real case. Consider the matrix 1, 2, 3, 4, what is A transpose? That is 1, 2, 3, 4, Ax is x 1 plus 2x 2, 3x 1 plus 4x 2, A transpose y is x y 1 plus 3y 2, 2y 1 plus 4y 2 and therefore, (A x, y). Now, we know, we do not have any conjugation because now we are looking for x, y in R n.

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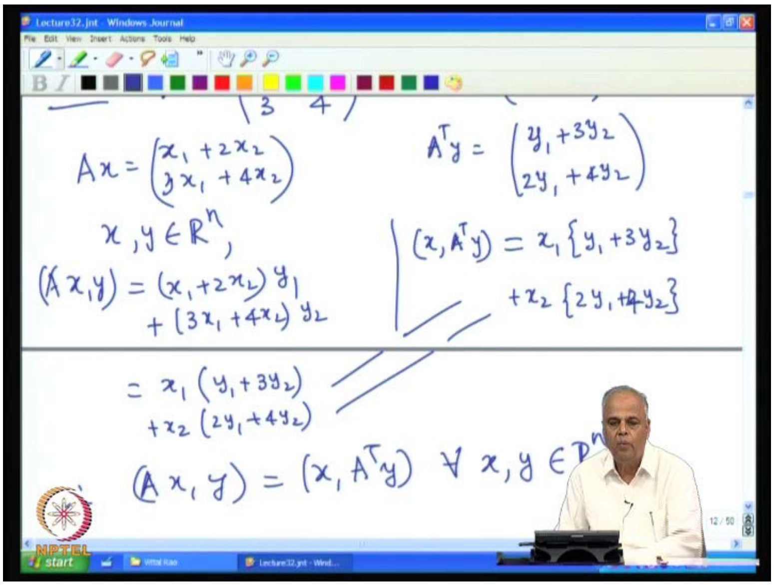

So, if you now take x, y in R n, all real, then (Ax, y) is first component of Ax into the first component of y plus the second component of Ax into the second component of y, which we will rearrange again as before, x 1 into y 1 plus 3y 2 plus x 2 into 2y 1 plus 4y

2. On the other hand, we have x comma A transpose y is equal to x 1 into the first component of A transpose y, which is y 1 plus 3 y 2 plus the second component of x into the second component of A transpose y. You, you compare these two and be sure, that these two are equal and therefore, (Ax, y) is equal to x comma A transpose y for every x y in R n.

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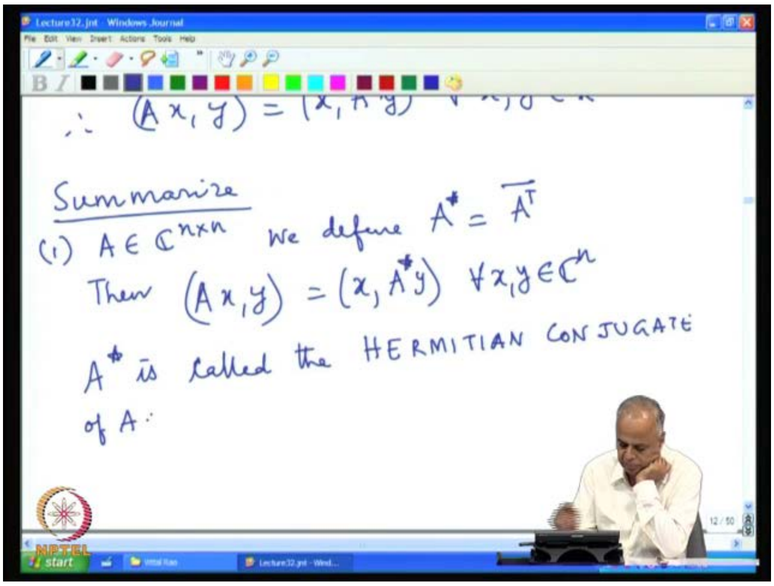

So when we have, so let us again summarize with the examples and… So, first A belongs to C n n, we define A star to be transpose the matrix and then conjugate and then (Ax, y) is equal to x comma A star y for every x, y (( )). A star is called the Hermitian conjugate, is called the Hermitian conjugate of…

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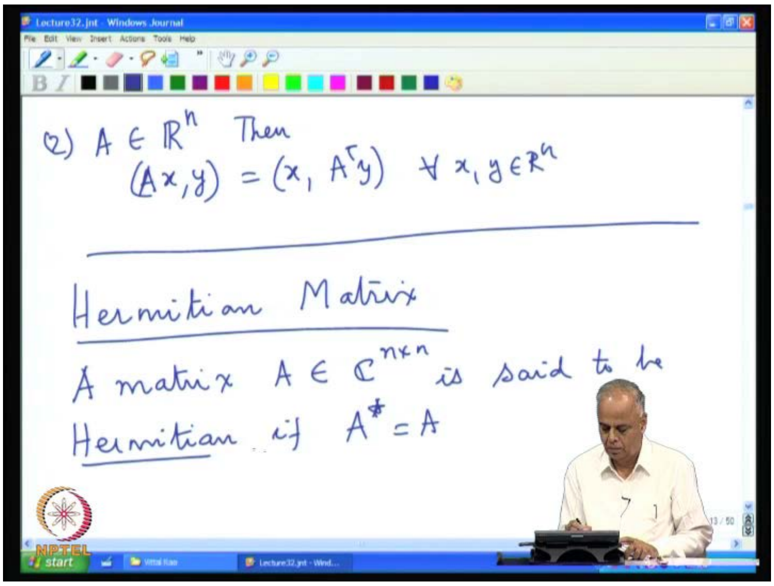

Then, the second thing is, we observe is A is in R n, then (Ax, y) is equal to x comma A transpose y for every x, y in R n. So, now, we observed, that when A star is equal to A in this example two in the above, particularly if I look at this identity when A star is equal to A, then we get A x equal to (x, Ay), that is, we can freely move A from one factor to the other factor without any change. If A star equal to A, when we move this A to be second factor, it will still move as A 1 and that makes things work much nice.

We now make a special name for such matrixes, so we now introduce the notion of a Hermitian matrix. A matrix A, which is complex and n by n is set to be Hermitian if A star equal to A. So, the conjugate Hermitian conjugate is itself, so it is self conjugate matrix; so, it is self conjugate matrix in the sense of Hermitian conjugation, the Hermitian conjugation transpose conjugate. If you transpose the conjugate, the matrix, if you get back the original matrix, then it is called a Hermitian matrix.

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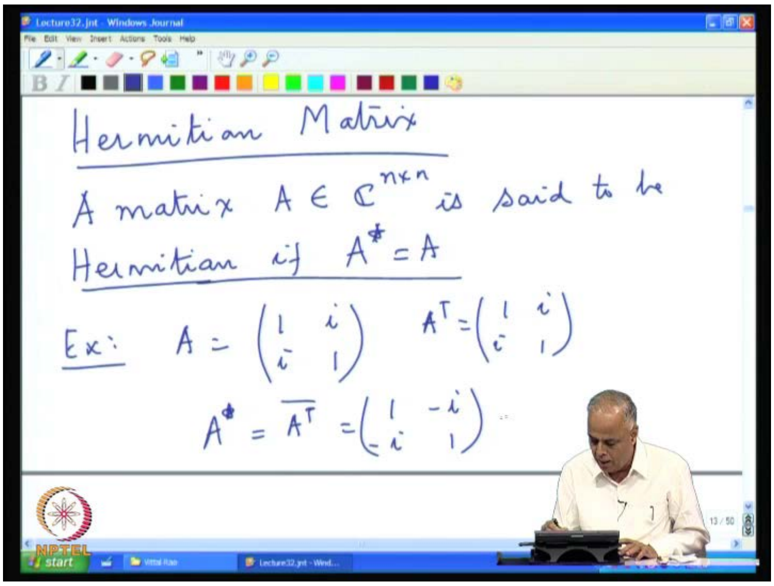

For example, if A equal to 1, i, i, 1; A transpose is 1, i, i, 1. Therefore, A star is equal to A transpose conjugate is 1, minus i, minus i, 1 and this is not equal to A.

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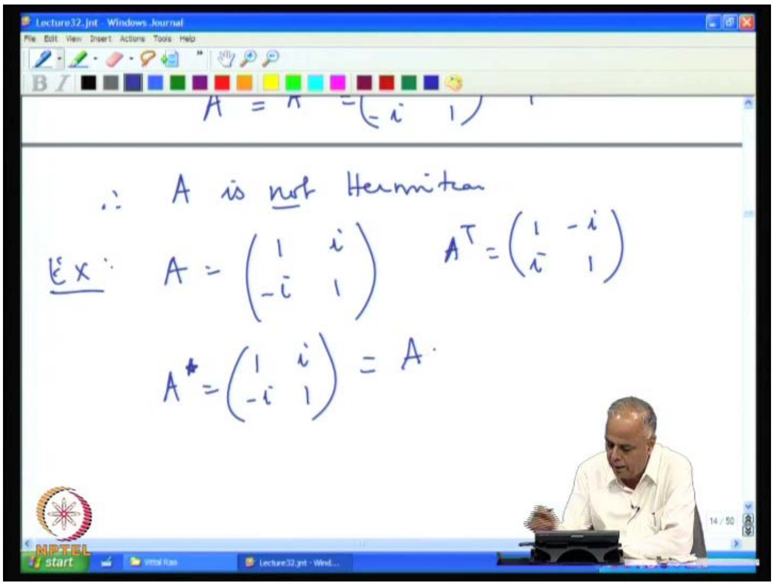

And therefore, A is not Hermitian. On the other hand, look at this example, A equal to 1, i, minus i, 1, then A transpose is 1 minus i, i, 1, rows are written as columns and columns as rows. Therefore, A star, which is the conjugate of A transpose is 1, i, minus i, 1, which is equal to A and therefore, A is Hermitian.

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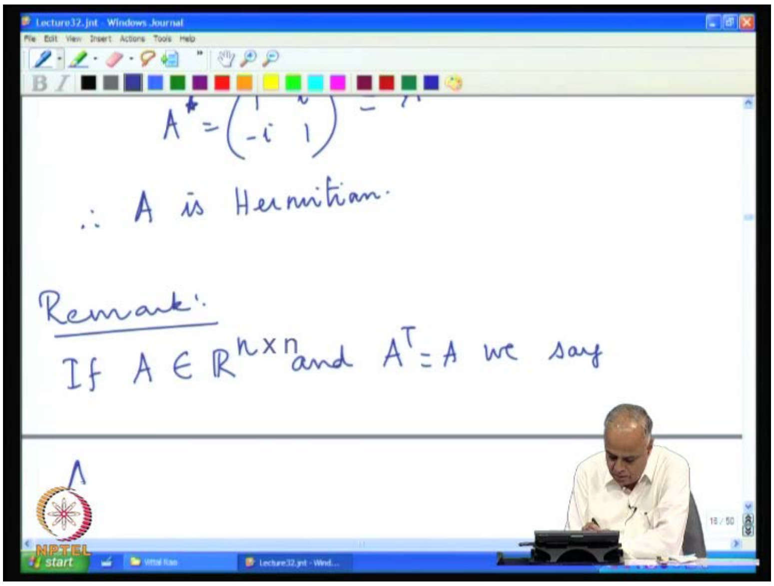

Therefore, A is Hermitan, if and only if its star is itself. In particular, when we are dealing with real matrix is conjugation, is no effect, A star means, is same as transpose.

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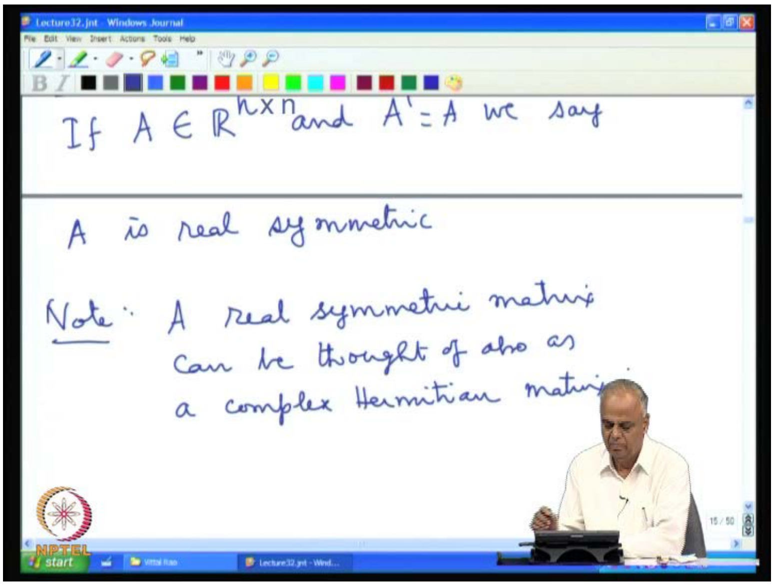

If A belongs to R n and A transpose equal to A, we say, A is a real symmetric matrix. Note, that a real symmetric matrix can be thought of, is the complex Hermitian matrix because the real numbers can be thought of as complex. So, starring again does not affect, A transpose conjugate will still be A. So, note, a real symmetric matrix can be thought of also as a complex Hermitian matrix.

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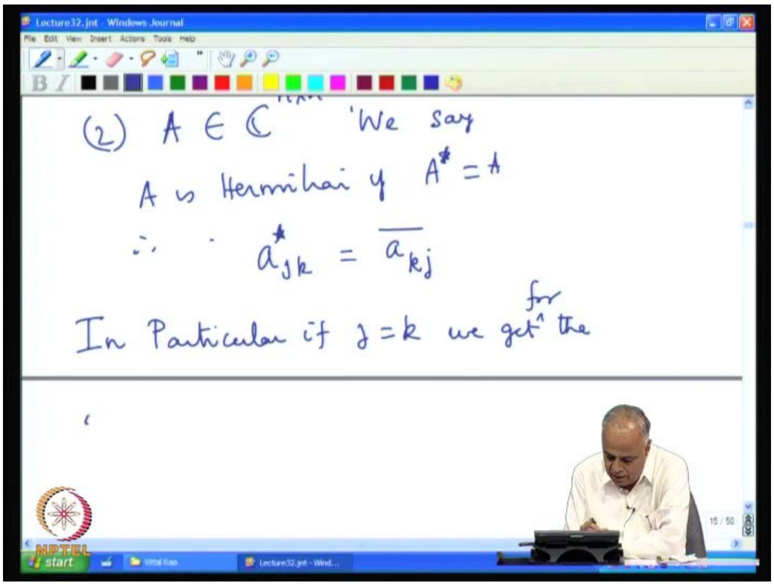

Now, suppose, A is Hermitian and we denote, we, we say, A is Hermitian if A star is A. Therefore, if you look at the diagonal, what does that mean? This mean a jk bar let us, using the following notation, correct notation…

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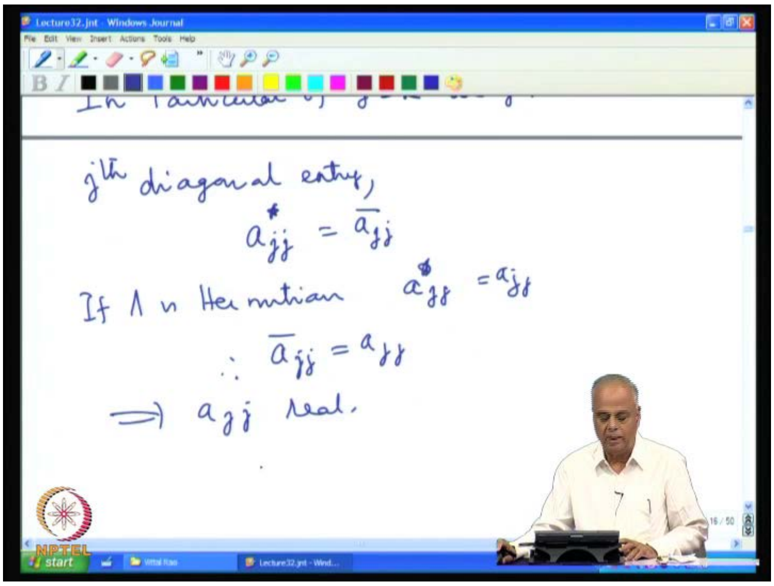

So, the, the jkth entry of the starred matrix is obtained by the kjth entry of the original matrix with conjugate. In particular, if j equal to k, we get the diagonal entries. We get the jth for, for the jth diagonal entry, a jj star must be equal to a jj bar. Now, if A is

Hermitian, a jj star is the same as a jj because A star is equal to A and therefore, a jj bar is equal to a jj, which says a jj is real.

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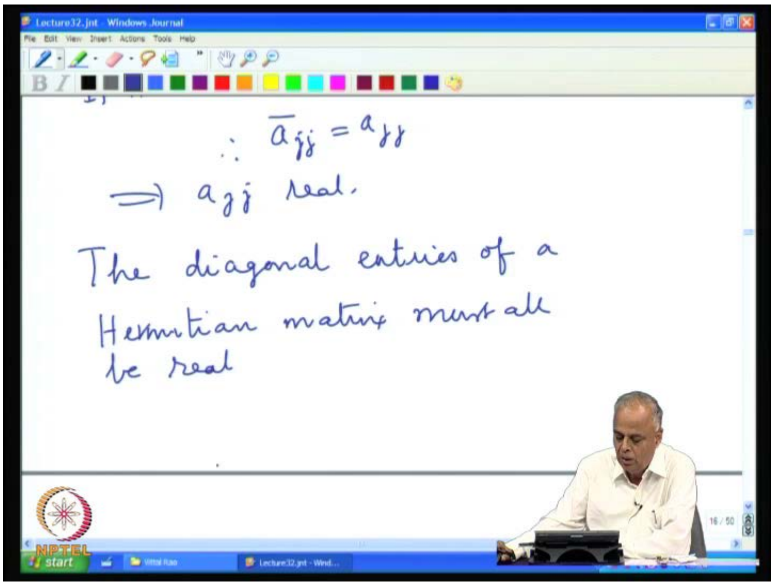

Therefore, for a Hermitian matrix, all the diagonal entries must be real. The diagonal entries of a Hermitian matrix, Hermitan matrix, should all be real. The matrix may be complex, but when the matrix has to be Hermitian conjugate, the diagonal entries are forced to be real numbers. So, you cannot have a complex Hermitian matrix with complex diagonal entries, all the diagonal entries must be real.

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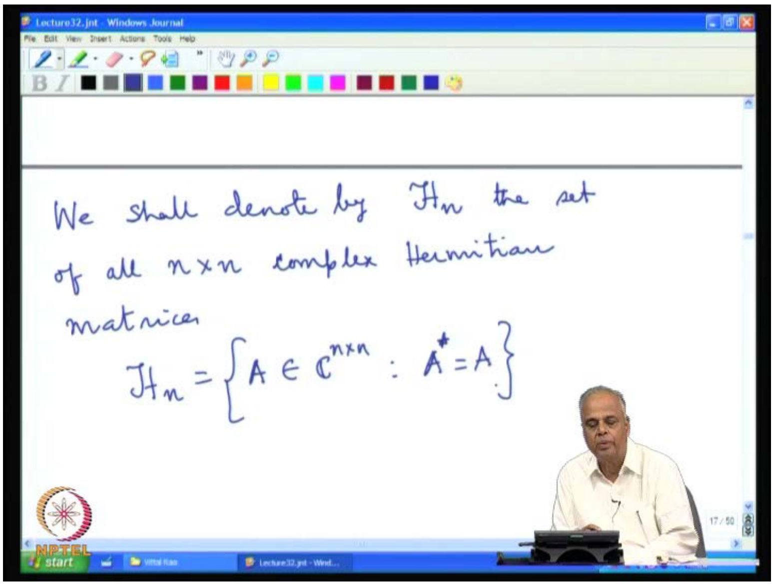

Now, we shall denote by H n, the set of all n by n complex Hermitian matrices. So what is H n? H n is all those matrices in C n n, the complex Hermitian complex matrix is n by n such that A star is equal to A. So, this is the collection of all Hermitian matrices. Now, it is this class of matrices, which are having a very nice set of properties as for the eigen values in eigen vectors are concerned and it is this class, that it will be very useful in all our computations and answering many of your questions, that we rise in the beginning of the course.

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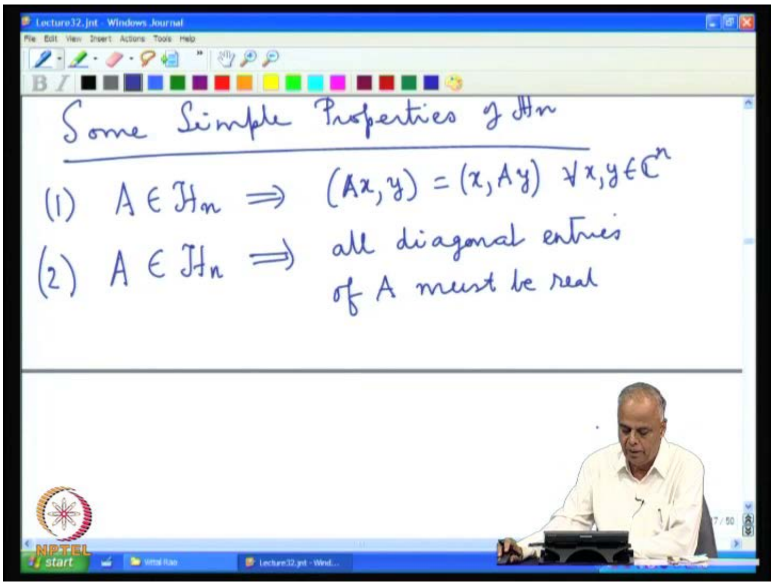

So, we shall study this class a little bit more closely. It is first, some simple properties of H n of this collection. Look at some simple properties of this collection of matrices. First, we have observed, that the moment it is Hermitian, so if A belongs to H n, (Ax, y) must be equal to (x, Ay) for every x, y in C n, because A star is equal to A. We had (Ax, y) is equal to x comma A star y. But since, A star is equal to (Ax, y) must be equal to x comma A star. So, this is the first property, which every matrices in H n possess, that is, in an inner product the factor A can be moved from the first to the second without any change.

The second is, as we have observed above, if A belongs to H n, then all diagonal entries of A must be real; all diagonal entries of A must be real.

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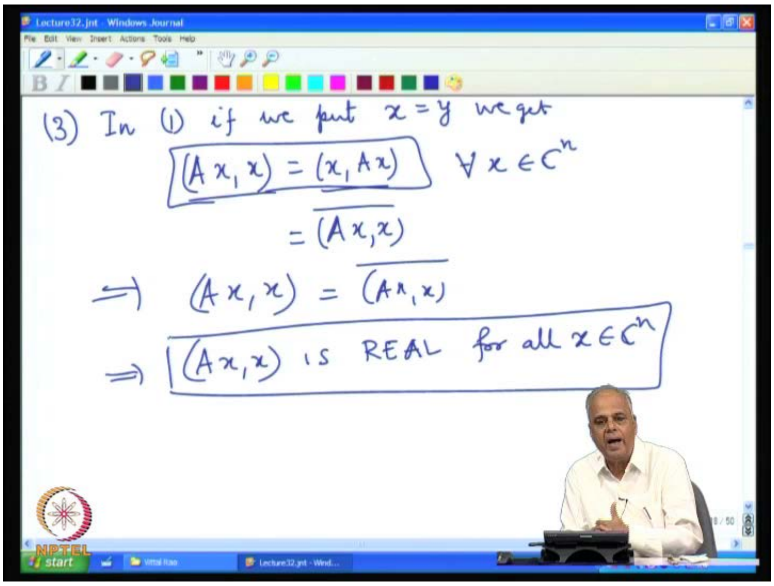

Let us now look at the property 1 as in a special situation. So, in 1, if we put x equal to y, we get (Ax, x) is equal to (x, Ax) y is equal to x. So, (Ax, y) becomes (Ax, x) and (x, Ay) becomes (x, Ax). So, we have in, we have this simple thing when could they, so this is true for every x in C n. y is also taken to be equal to x, so that becomes for every x in C n. Now, but the right hand side, by the inner product (( )), inner product of a vector with itself the conjugate, when the order is reversed, so we have got (Ax, x). So, therefore, (Ax, x) is equal to Ax comma x bar.

A number is equal to its own conjugate means, that number must be real. So, that says, (Ax, x) is real for all x in Cn, so this is the third important property. Not only the diagonal entries are real, the many things are going to be real for a Hermitian matrix; not only the diagonal entries are real.

We now see, that (Ax, x) is real for all x in C n, whatever x, the x may be complex a, A is a complex matrix, only thing we know, it is complex Hermitian matrix. So, there are many non-diagonal entries, which are complex, x could be highly complex matrix vector and x, if A is a Hermitian, (Ax, x) must be real. All the complexity is gone and everything becomes real. So, (Ax, x) is real for all x in R n. It is a very, very important property, so which we will be using repeatedly.

So, we have seen now three properties, this is the first fundamental identity for Hermitian matrices (Ax, y) equal to (x, Ay) for all x, y in C n. All diagonal entries must be real and (Ax, x) must be real for all x in C n.

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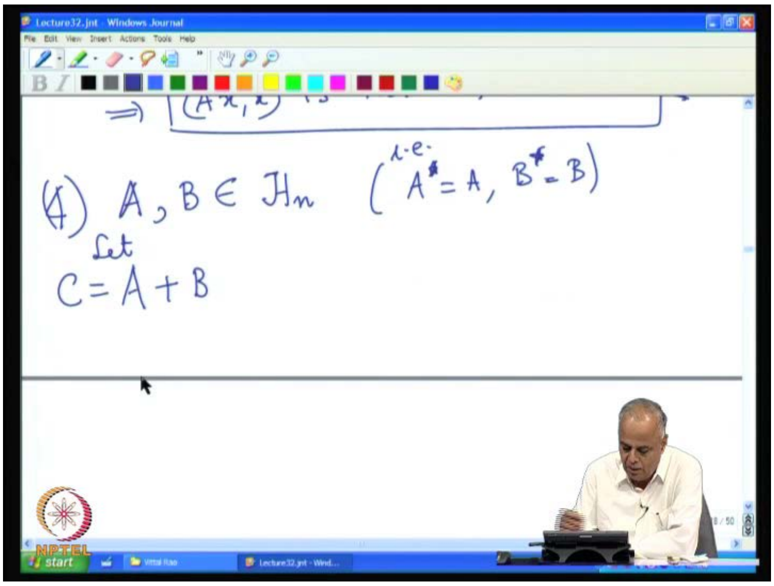

Suppose, A and B are, A and B are Hermitan matrices, suppose A and B are Hermitian matrices, that is, A star equal to A and B star is equal to B, both are Hermitian matrices. Suppose, we take two Hermitian matrices and we look at their sum, call that as C. Let C be equal to A plus B, the sum of these two matrices.

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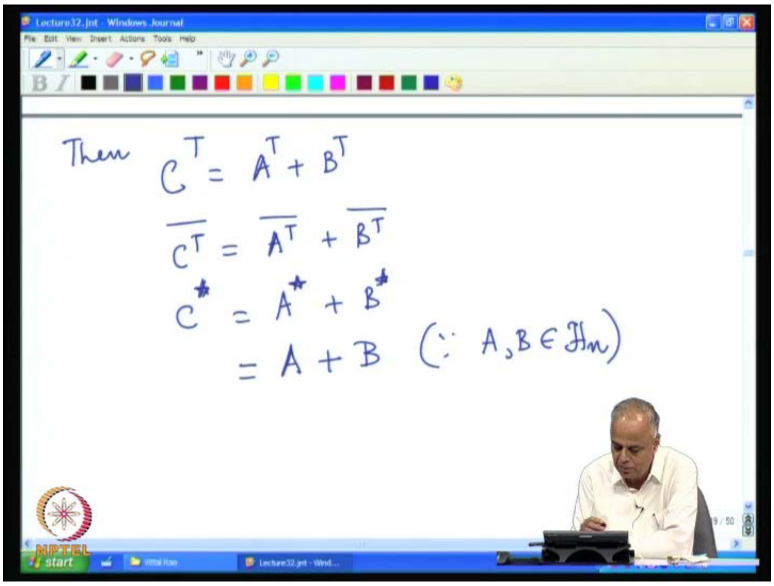

Then, C transpose is A transpose plus B transpose, because the transpose of a sum is the sum of the transpose, thus was C transpose conjugate is A transpose conjugate plus B transpose conjugate. This is C star and this is A star, this is plus B star, that is, the sum of the star is star of the sum. So, C star is equal to A star plus B star. In particular, if A and B are Hermitian, this is the same as A plus B because A and B are Hermitians.

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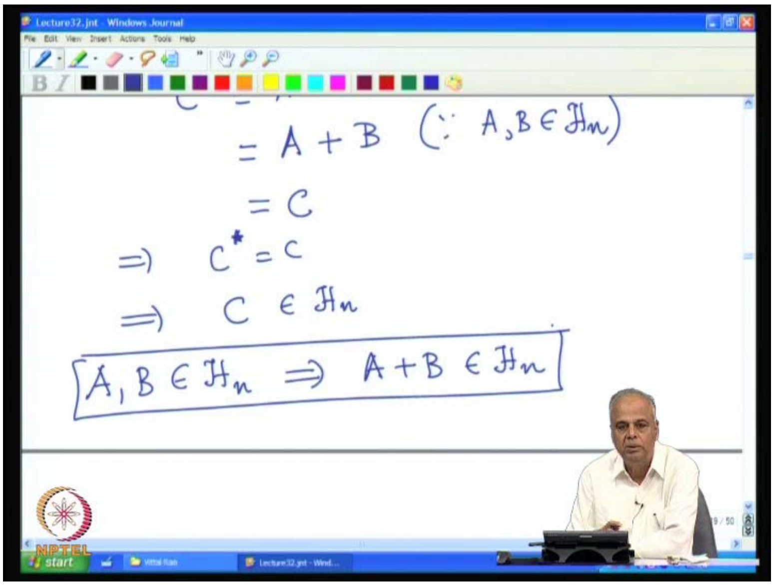

So, if A and B Hermitian, A star is equal to A and B star is equal to B, but A plus B was C, that means, C star is equal to C, that means, C is also Hermitial. So, conclusion is A, B are Hermitian, implies their sum is also Hermitian. This sum of Hermitian matrices is a Hermitian matrix.

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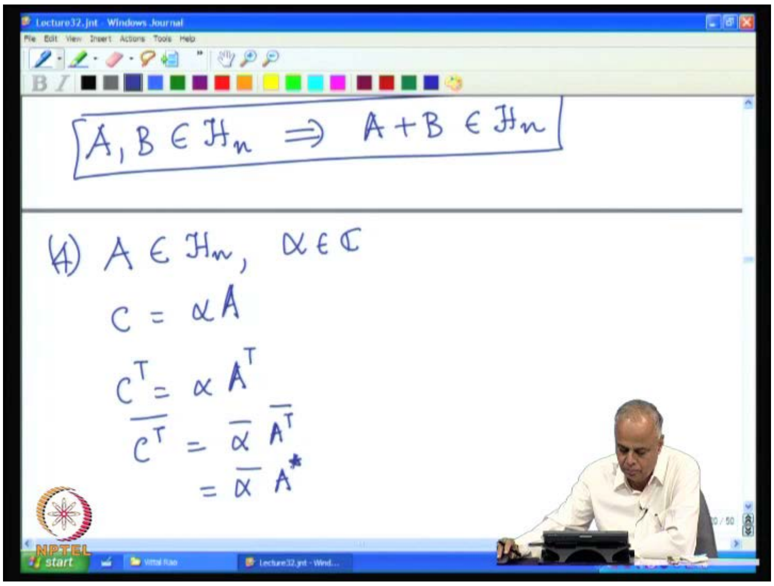

However, there is certain, that is the 4th important property. However, there is a slight problem as far as com product, scalar multiple on products are concerned. Let us look at a Hermitian matrix and take any complex number C. Then, let us define C to be alpha times A, that is, the matrix A is multiplied by alpha, which means, every entry is multiplied by alpha. So, C transpose is alpha times A transpose and therefore, C transpose conjugate is alpha conjugate A transpose conjugate, that is, alpha conjugate A star. Now, therefore, C star is equal to alpha conjugate A star.

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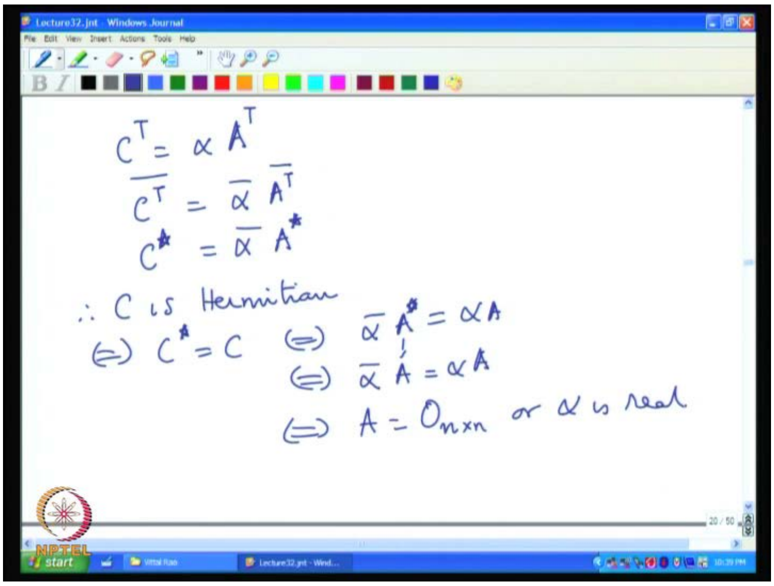

Therefore, C is Hermitian if and only if C star is equal to C, if and only if alpha bar A star, that is, the C star must be equal to alpha A, if and only if alpha bar A is equal to alpha A, because A star is A. We have assumed that A is in H n. Since A is in H n, A star can be replaced by A. So, C will become Hermitian, if and only if alpha bar A equal to alpha A. Now, this is satisfied if A is zero matrix. If A is not the zero matrix, then alpha must be equal to alpha bar, if and only if A equal to 0 n cross n or alpha is real and therefore, if we have a non-zero Hermitian matrix, its scalar multiple is also Hermitian if and only if the scalar is real.

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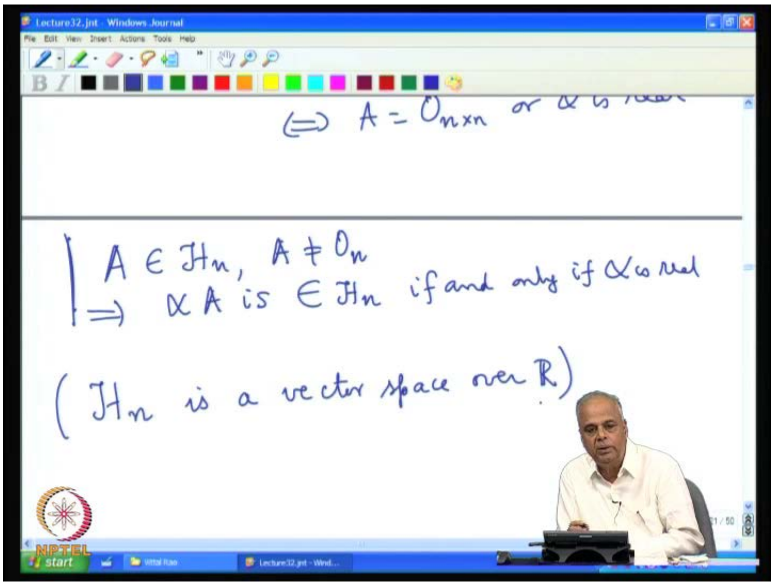

So, therefore, A belongs to H n, A not equal to 0 n. then, implies alpha A is Hermitian, is also in H n if and only if alpha is real. So, this is the scalar multiple of a Hermitian matrix, will become Hermitian only if the scalar, which is multiplying is real. This is same thing as saying H n.

The class of all Hermitian matrices is a vector space, not over the field of complex numbers, but over R because addition of two Hermitian matrices is Hermitian. So, addition no problem, scalar multiple in order, that it be close with respect to scalar multiple, we have to take only scalars to be real. That is the problem with R, the constraint with respect to scalar multiplication.

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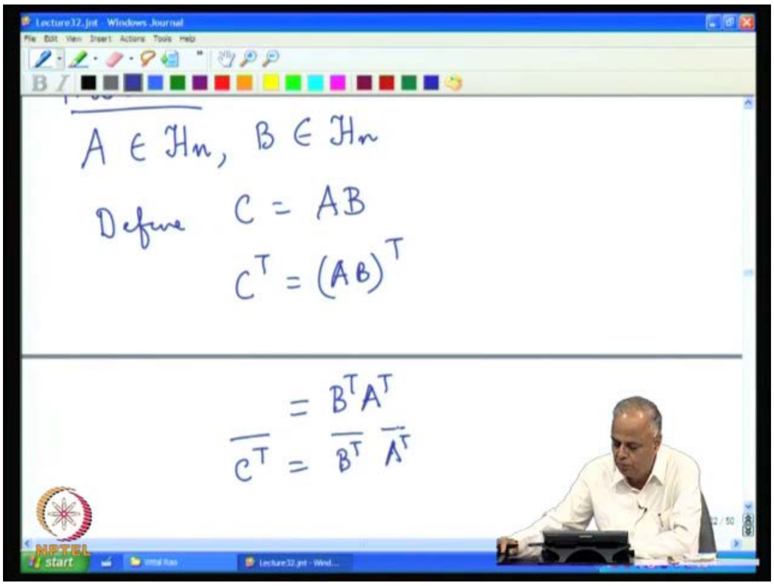

The next property is look at the product of two. This is the product. Suppose A is a Hermitian matrix and B is also a Hermitian matrix. Let us define C to be the product, define C to be the product AB. So, we have two Hermitian matrices, we are looking at their product. What is C transpose? It is AB transpose, but AB transpose, the transpose of the product is the product of this transpose in the reverse order and therefore, C transpose conjugate is B transpose conjugate into a transpose conjugate.

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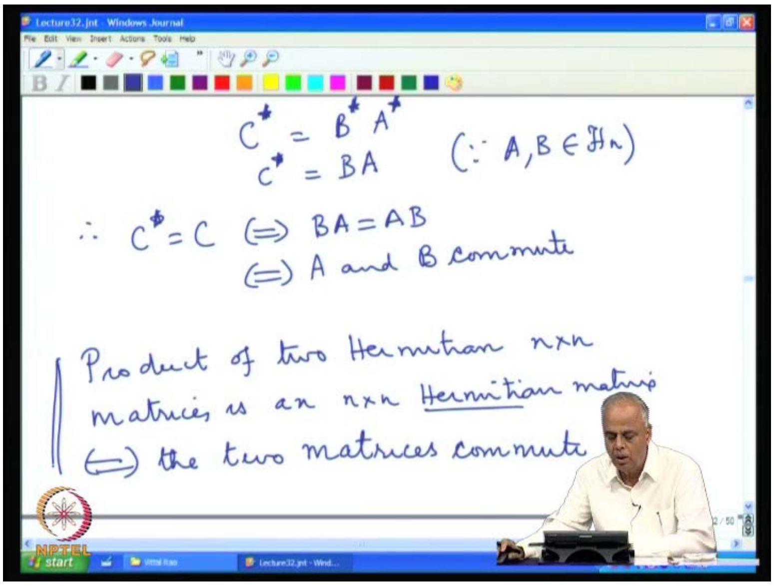

And this is C star and this is B star and that is A star and that is equal to BA, because B and A and B are Hermitian. Since A and B are Hermitian, B star is, B and A star is A. Now, therefore C star is equal to C, that is, C will be Hermitian if and only if C star is BA, C is AB, that is, if and only if A and B are commutable. Therefore, that is the next property.

Product of two Hermitian n by n matrixes is an n by n Hermitian matrix, Hermitian matrix, if and only if the two matrixes commute. These are some of the important properties of a, herm, the collection of Hermitian matrix. Let us go over them.

The first property we have was, that we must have (Ax, y) equal to (x, Ay) for every x, y in C n. Then, we must have, that all the diagonal entries must be real. Then, (Ax, x) is always real, that is the 3rd property and the 4th property is that the product, the sum of the two Hermitial matrixes is Hermitian always and should be the 5th property, the number in this problem. The 5th property is that A is Hermitian, then the scalar multiple is again Hermitian, if and only if all the scalar is real. And this is the 6th property is about the product. The 4th and the 5th properties together give us, that H n is a vector space over R. It is not a vector space over C; is not a vector space over C. Then, the product of two Hermitian matrixes is again Hermitian if and only if the two matrices is commute.

So, now, we have this fundamental properties of Hermitian matrices and we again stress the two of the most important properties, which will repeatedly use is the fact, that (Ax, y) equal to (x, Ay) for all x, y and (Ax, x) is real for all x. This is a, these all are two characteristic properties of Hermitian matrixes (Ax, y) equal to (x, Ay) for all x, y and (Ax, x) is real.

So, now, we are going to look at this class of matrices, which is closed and under addition, which is closed and real scalar multiplication, which is not closed under multiplication because the product of the two Hermitian matrices need not be a Hermitian matrices. The product becomes Hermitian if and only if the two matrices commute with each other. Commutatively is an important property of matrices, which has a lot of things to say about between the two matrices what happens.

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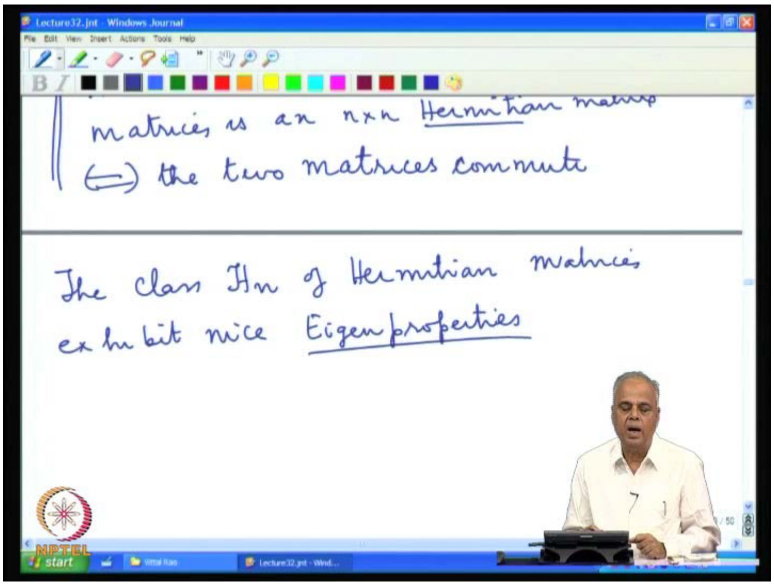

The class H n of Hermitian matrices exhibit nice Eigen properties. That is the reason why we look at this and deal with these matrices so very often because as for this Eigen properties, Eigen values, Eigen vector properties and their structure, they are very nicely built-in, which makes them automatically diagonalizable, not only diagonalizable, but nicely diagonalizable.

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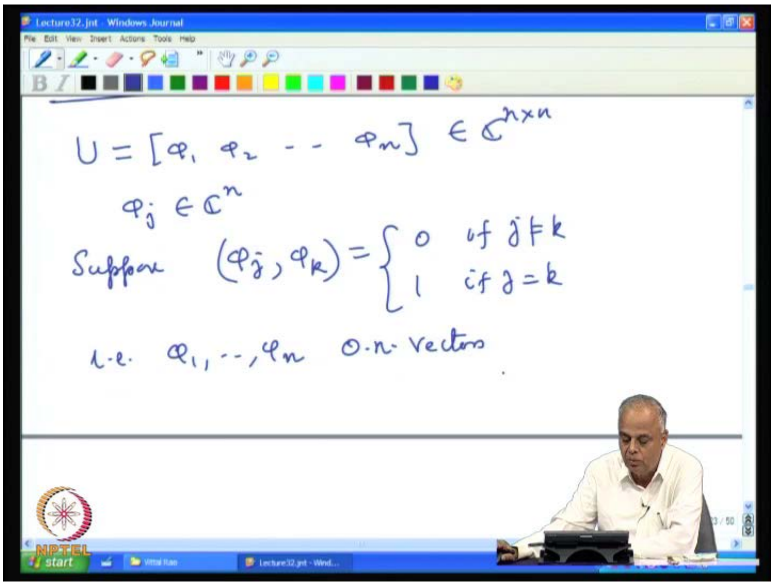

So, in this context, before we get to study the Eigen properties of these matrices, we shall introduce certain notations and terminologies. Let us say, U is a matrix whose columns are phi 1, phi 2, phi n. So, U is a matrix, therefore phi j belongs to C n. Each column is n component vector. So, we have a matrix U whose columns are phi 1, phi 2, phi n and suppose, what does this mean? This means, that phi j, phi k are orthogonal to each other and each vector has length one, which means phi j are ortho-normal vector, that is, phi 1, phi 2, phi n are ortho-normal vectors. So, a matrix U in which the column forms ortho-normal set of vectors.

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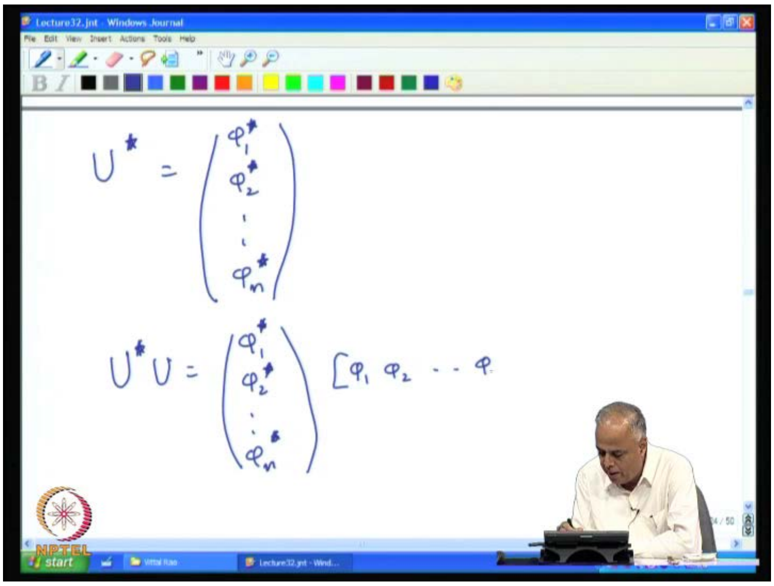

Then, we have U star is transpose conjugate, so it will be phi 1 star, phi 2 star, phi n star and when we multiply U star and U we get phi 1 star, phi 2 star, phi n star, that is, U star into U is phi 1, phi 2, phi n. We now multiply, first we get phi 1 star, phi 1, which is 1 because phi 1 star, phi 1 is the inner product of phi 1 with phi 1.

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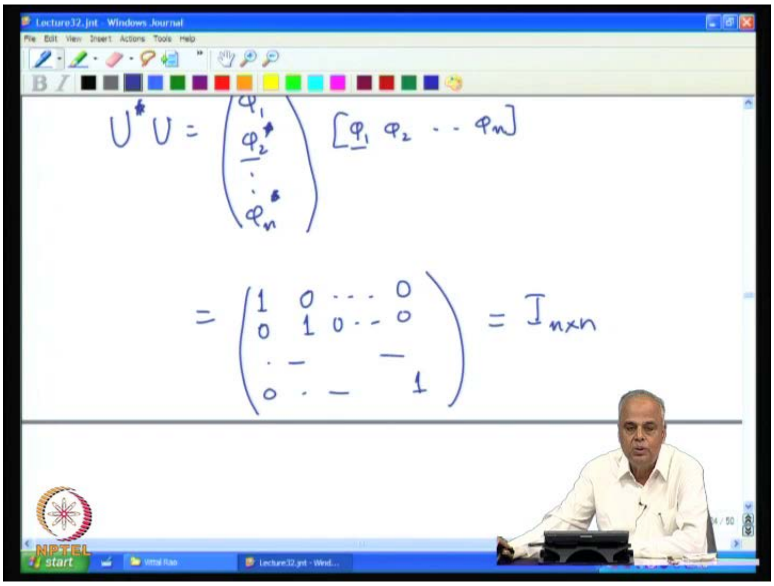

So, that is 1, phi 1 star, phi 2 is 0 because phi 1 and phi 2 are orthogonal and we go on getting 0. And in the second row we get phi 2 star, phi 1, which is 0 again because the inner product within phi 2 and phi 1 is 0, phi 2 star, phi 2 is 1, again we get this. So, we see, that we get the matrix I n by n, which means U star is the inverse of U and y (( )).

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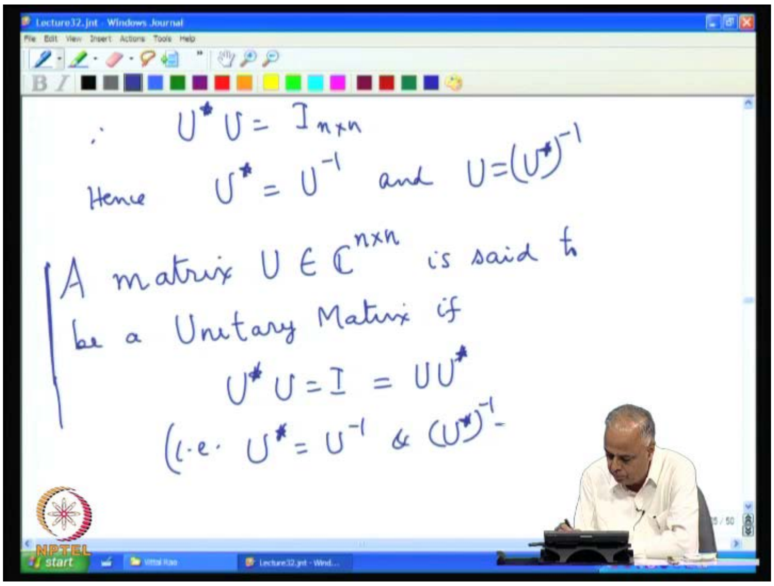

Therefore, U star U equal to identity I n by n, hence U star is equal to U inverse and U is this U star inverse. Any matrix, which has this property, is called a unitary matrix. So, A complex matrix U in C n by n, is said to be is the definition, is said to be a unitary matrix.

If U star U equal to I and since U and U star become inverses of each other, this is the same as the (( )) U star. That is, U star is equal to U inverse and U star inverse is equal to U, then we say it is a unitary matrix. We see, that if the columns are ortho-normal vectors, and then automatically U is a unitary matrix. In the real case, we have to replay, there is no conjugation, so U star is same as the transpose.

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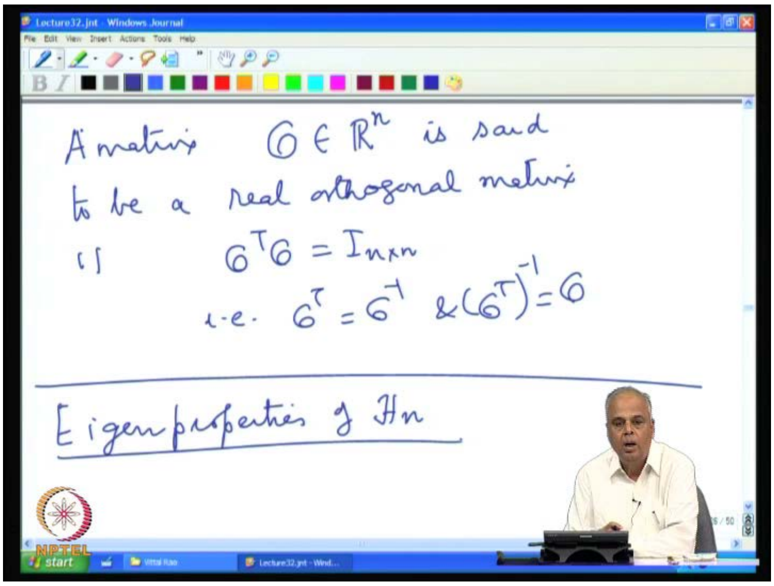

So, we have, we define A, a matrix, let us call it as O, let us use a different symbol, let us put this O belonging to R n is said to be a real orthogonal matrix. If O transpose O is equal to identity n cross n, that is O transpose is O inverse and O transpose inverse is equal to O. So, in the real case, we have an orthogonal matrix notion and the complex case, we have the unitary matrix notation.

The commonality is, that if you are in the unitary case in the complex situation, that columns form an ortho-normal set of vectors. In the real case, when we are having an orthogonal matrix with real in a product, the columns form an orthogonal matrix. So, now, we are going to look at the Eigen properties of H n, this is the most convenient class of matrices for which the Eigen properties are very nice and this will form the subject matter for the next lecture.