**Advanced Matrix theory and Linear Algebra for Engineers**

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**Lecture No. # 31**

**Diagonalization- part 4**

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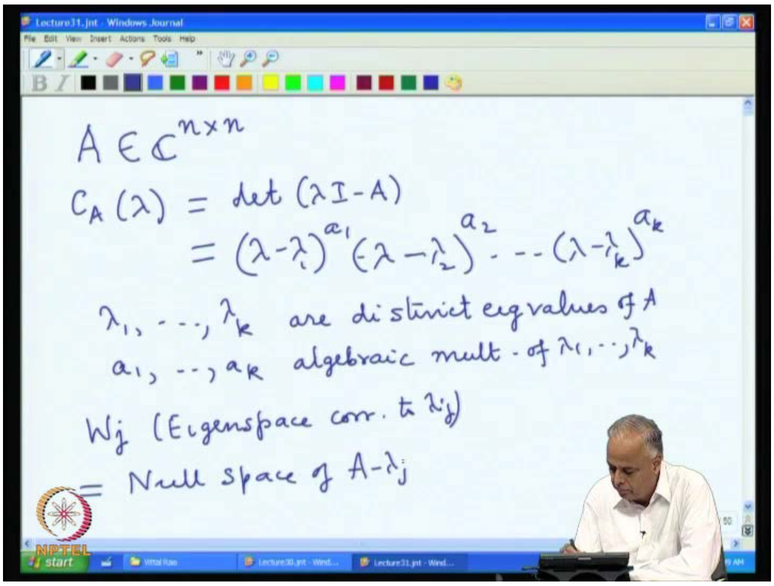


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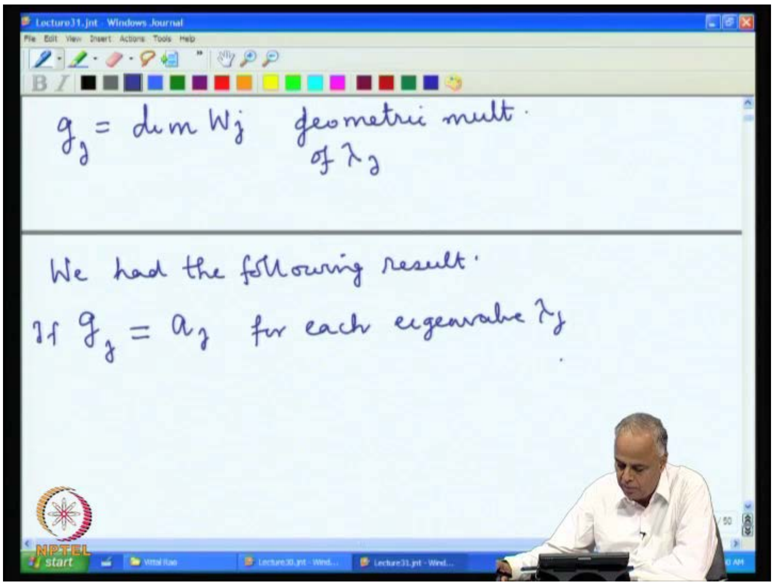

We have been looking at the notion of diagonalizability and we found that, if a is an n by n matrix, then the characteristic polynomial was defined as the determinant of lambda i minus a. We found that this is a polynomial of degree n. Since we have assumed everything of over the field of complex numbers, this can always be factorised as lambda minus lambda 1 into a 1, lambda minus lambda 2 to the power of a 2 and lambda minus lambda k to the power of a k. The lambda 1, lambda 2 and lambda k are the distinct rules of the polynomials and they are the distinct eigen values of a. a 1, a 2 and a k are called the algebraic multiplicity of these eigen values; of these eigen values lambda 1, lambda 2 and lambda k. Then, corresponding to these eigen values, we define the eigen space corresponding to lambda j as the null space of a minus lambda j i and the dimension respond to be all as greater than or equal to 1. This dimension of w j is what we denoted by g j and this was called the geometric multiplicity of lambda j.

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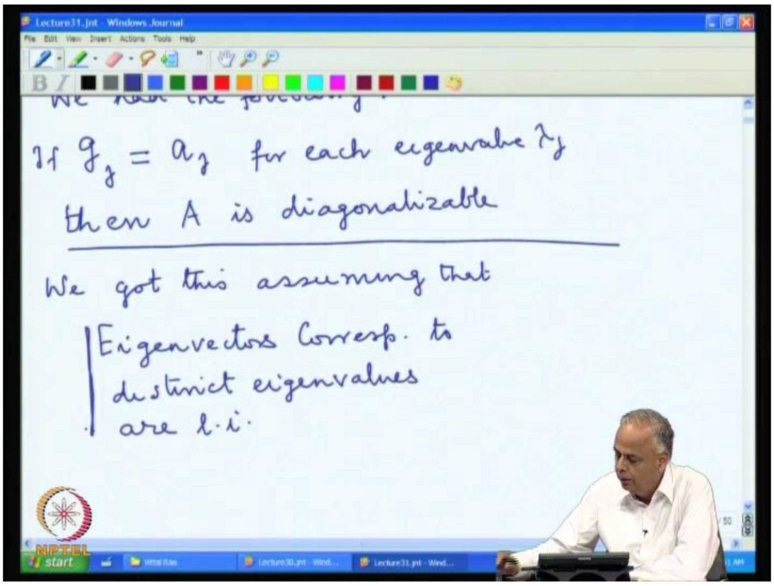

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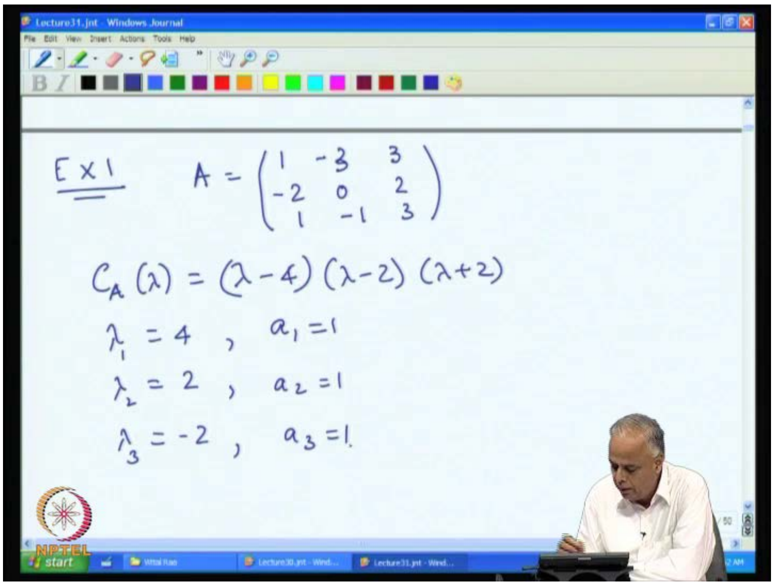


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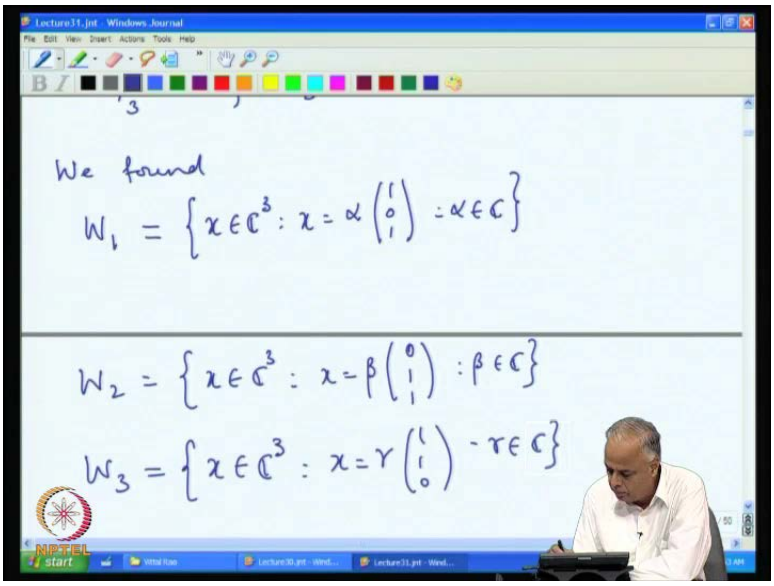


What we observed in the last lecture was, so, we had the following result in the last lecture, which was this. The geometric multiplicity of an eigen value is equal to its algebraic multiplicity and this happens for each eigen value lambda j. So, it is not that g 1 equal to a 1 or g 2 equal to a 2. For each j, g j must be equal to lambda j. For if g j equal to a j for each eigen value lambda j, then a is diagonalizable. We found that the diagonalizing matrix p was made up of all eigen vectors along its columns. This was our main result. We got this assuming another result, which we said will prove later. That is, the eigen vectors corresponding to distinct eigen values are linearly independent. We are not yet proved it. In this lecture, we will eventually get to prove in these results. But, let us look at some examples again to illustrate what we have got. Let us look at the first example. The same matrix which we looked at in the last lecture also, that is 1 minus 3 3 minus 2 0 2 1 minus 1 3. In the last lecture, we found that the characteristic polynomial was lambda minus 4 into lambda minus 2 into lambda plus 2. Consequently, the eigen values where lambda 1 equal to 4, lambda 2 equal to 2, lambda 3 equal to minus 2 and each root occurs only once. Therefore, the algebraic multiplicity is a 1 equal to 1 and a 2 equal to 1 and a 3 equal to 1. Again in the last lecture, we found the eigen space corresponding to lambda 1 was the set of all vectors in c 3 which were of the form x is equal to alpha into 1 0 1 for alpha belongs to c. w 2 is the set of all vectors in c 3, which were of the form x is equal to beta into 0 1 1, for beta belongs to c.

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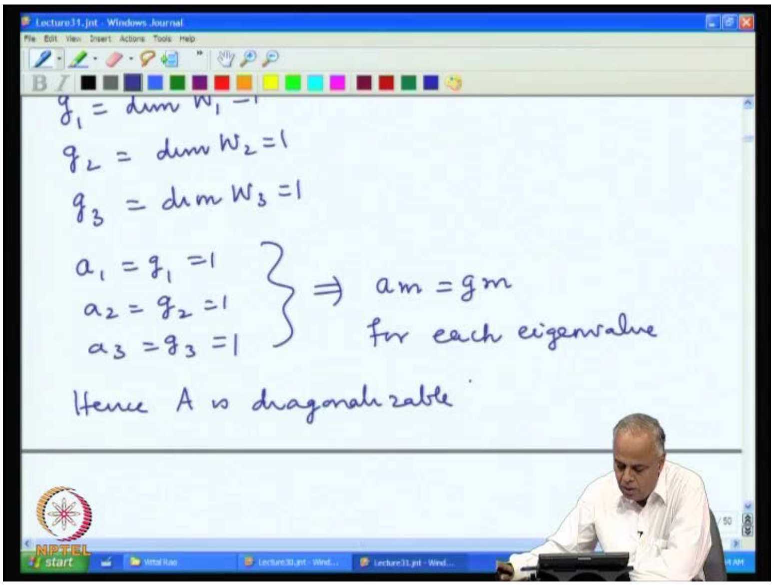

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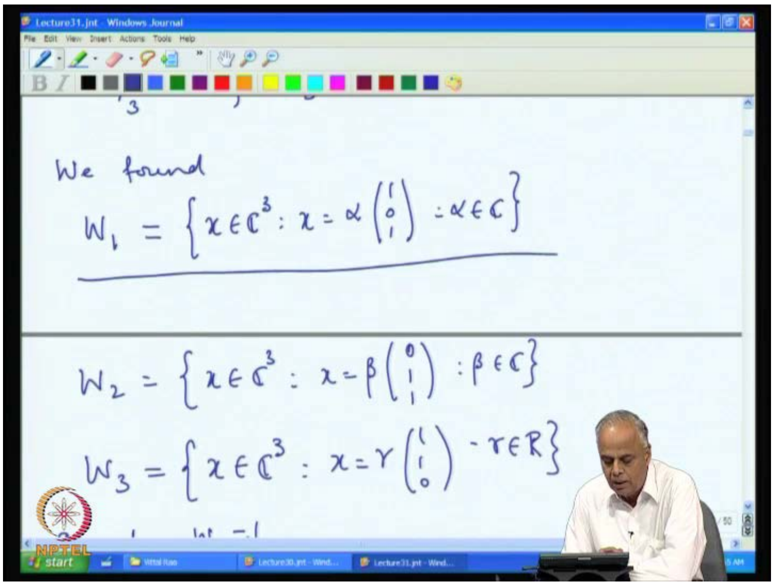


w 3 was found to be the subspace consisting of all those vectors of the form gamma into 1 1 0 and gamma belongs to r. Now, in this case, we have g 1, which is the dimension of w 1 is 1, because 1 0 1 is the basis for w 1. Similarly, 0 1 1 is the basis for w 2. So, the dimension of w 2 is 1. Similarly, 1 1 0 is basis for w 3. So g 3, the dimension of w 3 is 1. So, in this case, we find that a 1 is the same as g 1 and a 2 is the same as g 2 and a 3 is the same as g 3. So, the algebraic multiplicity is equal to the geometric multiplicity for each eigen value am equal to gm. The algebraic multiplicity is equal to the geometric multiplicity for each eigen value. Hence, a is diagonalizable. Now, observe that any eigen vector of lambda 1 is from w 1. If it takes the eigen space, w 1 corresponding to lambda 1, every non zero vector from this place is an eigen vector. So, any eigen vector of w 1 is of the form alpha 0 alpha and alpha not equal to 0.

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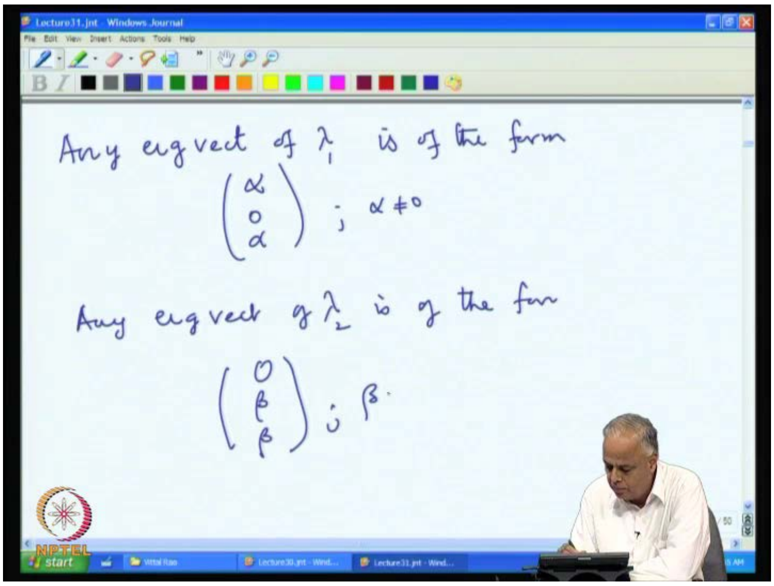


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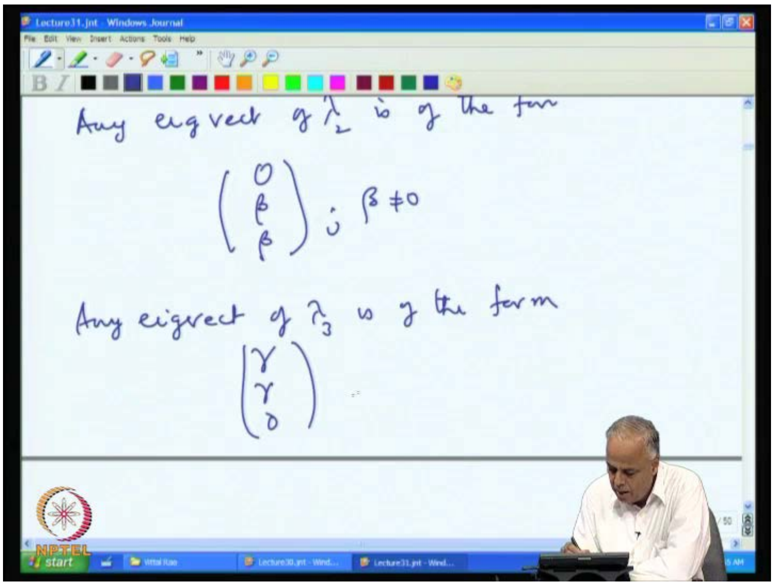

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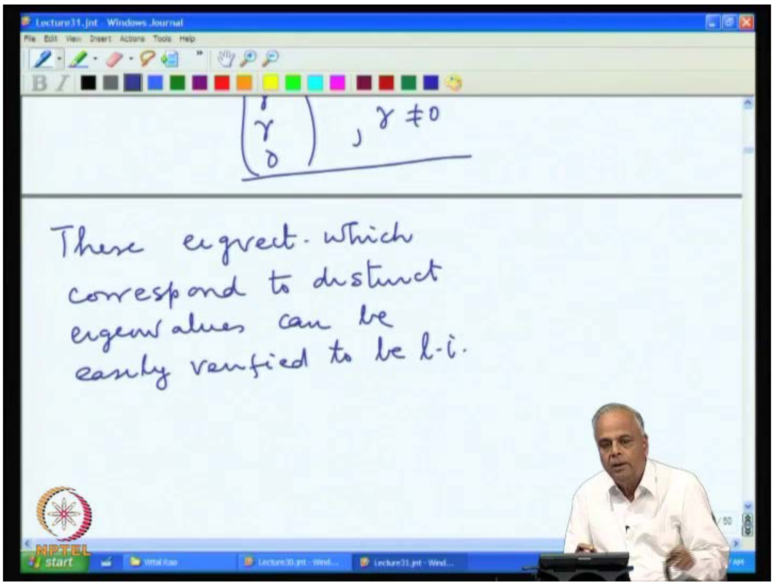


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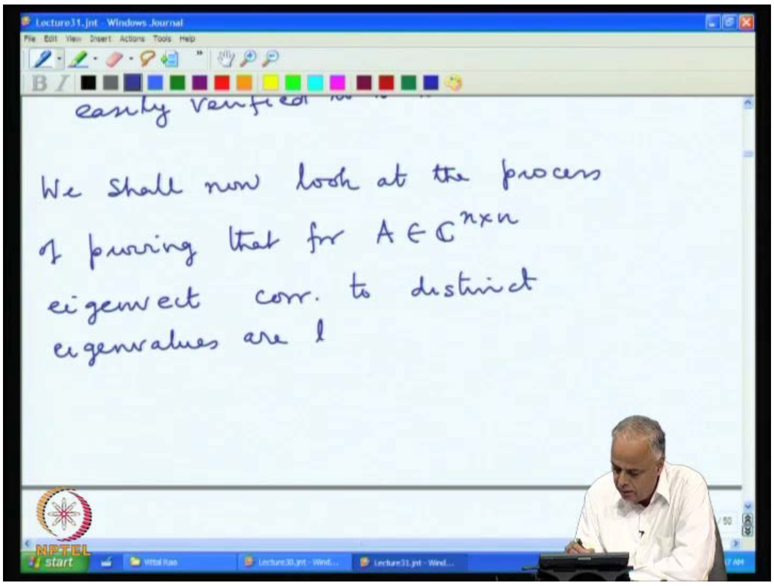

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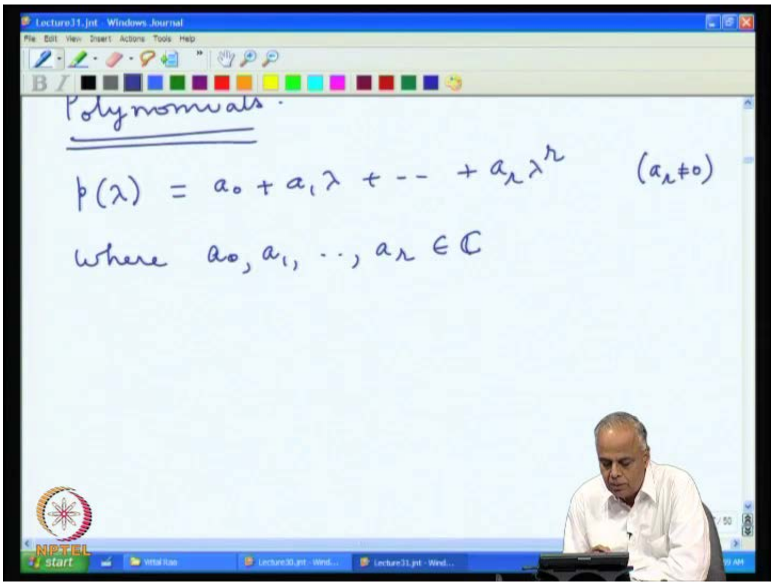


Similarly, any eigen vector of lambda 2 is of the form 0 beta beta and beta not equal to 0. Any eigen vector of lambda 3 is of the form gamma gamma zero and gamma not equal to 0. Now see, these three eigen vectors, this eigen vector corresponding to lambda 1, this eigen vector corresponding to lambda 2 and this eigen vector corresponding to lambda 3, these are eigen vectors corresponding to the distinct eigen values lambda 1 and lambda 2 and lambda 3. We find they are linearly independent. These eigen vectors which correspond to distinct eigen values can be easily verified to be linearly independent. We know how to verify something is linearly independent. Only linear combination that gives a zero vector is a zero linearly combination. So, easily verified to be linearly. This is what we had in mind when we said that distinct eigen vectors corresponding to distinct eigen values are linearly independent.

Now, this is a statement with we have not yet proved. So, we shall embank on proving this fact. So, we shall now look at the process of proving that for a belonging to c n cross n, eigen vectors corresponding to distinct eigen values are linearly independent. Now, for this purpose and also for the other purposes, that we may study or you may study and up in advance courses in linear algebra, there are several polynomials that play a crucial role in analysing a matrix. So, the polynomials play very important role in a matrix. There are several polynomials associated with given matrix. We have already seen one polynomial, namely the characteristic polynomial. Analogous to that, there are several other polynomials associated with the matrix. So, we look at in general, to start with some facts about polynomials. These facts are very important in a complete analysis of the structure of matrices and linear transformations in general.

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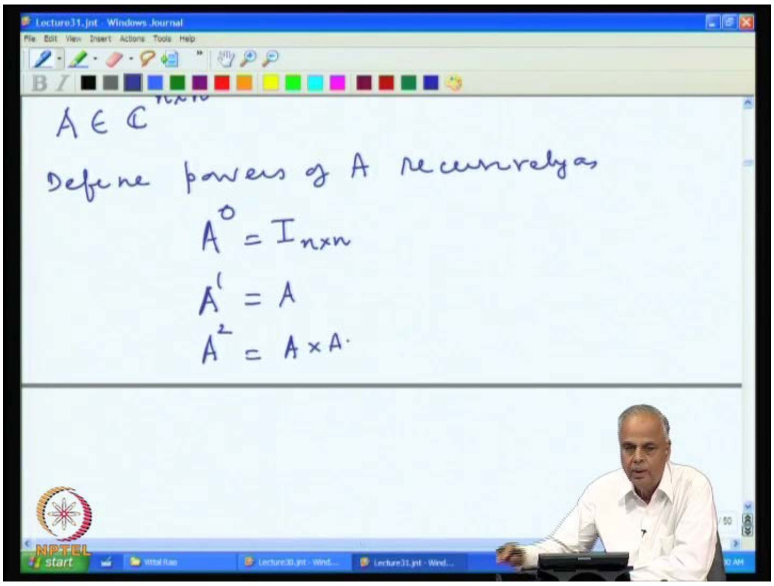


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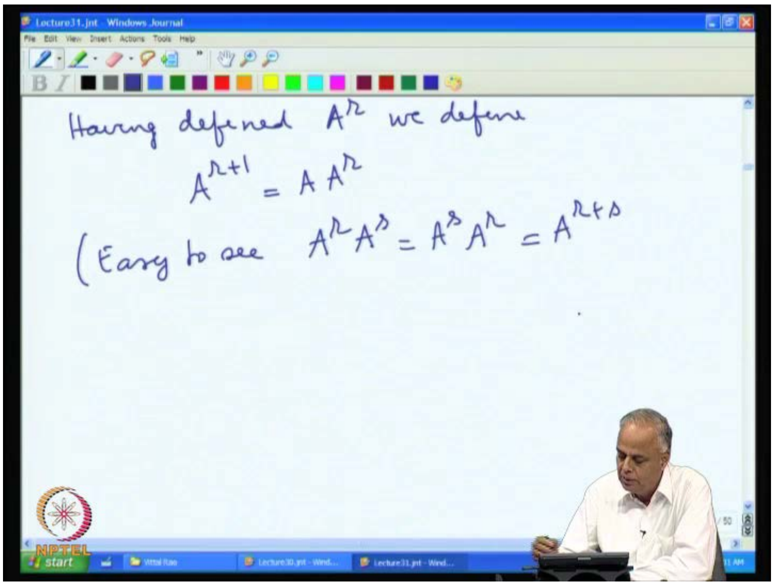


So, we will now consider only polynomials over the complex numbers. So, we always look at a polynomial p lambda, which is of the form a naught plus a 1 lambda plus a r to the lambda to the power of r. We will assume a r is not equal to 0, so that the degree is r. If only the r is only the constant terms is there and all other terms are missing, then we will call it the constant polynomial. So, we consider in general polynomials of this type, where the coefficients a naught, a 1, a r are all complex numbers. So, we are considering polynomials with complex coefficients. Now, the simplest polynomials are like lambda, lambda squared, lambda cube, namely the powers of lambda. Now, suppose a is n by n matrix, we all know that how to define powers of a. We define powers of a recursively, as we define the $0^{th}$ power to be the identity matrix, the first power a power 1 to be a, a squared to be a into a. Now, having defined a power r, we defined a power r plus 1 as a into a power. Now, we can easily see that, if a power r and a power s are two powers of a, they communicate with each other. So, easy to see, a power r into a power s is a power s into a power r is equal to a power r plus s for any positive integers r and s.

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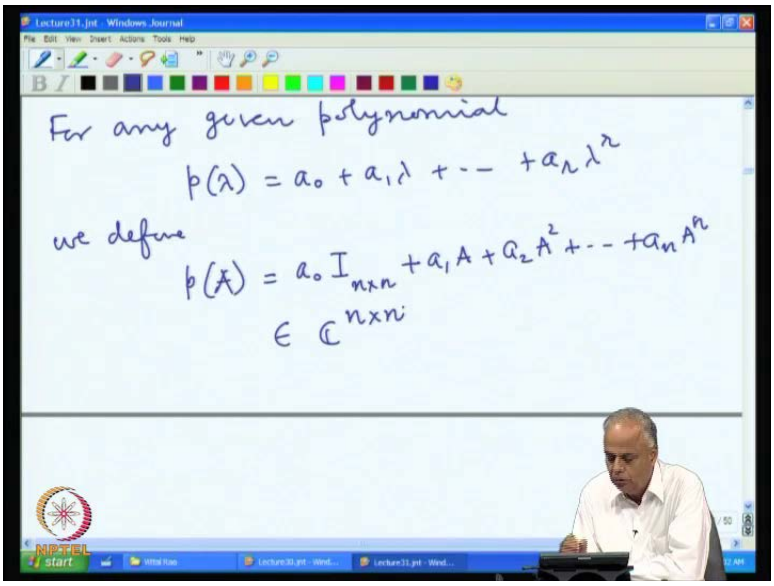

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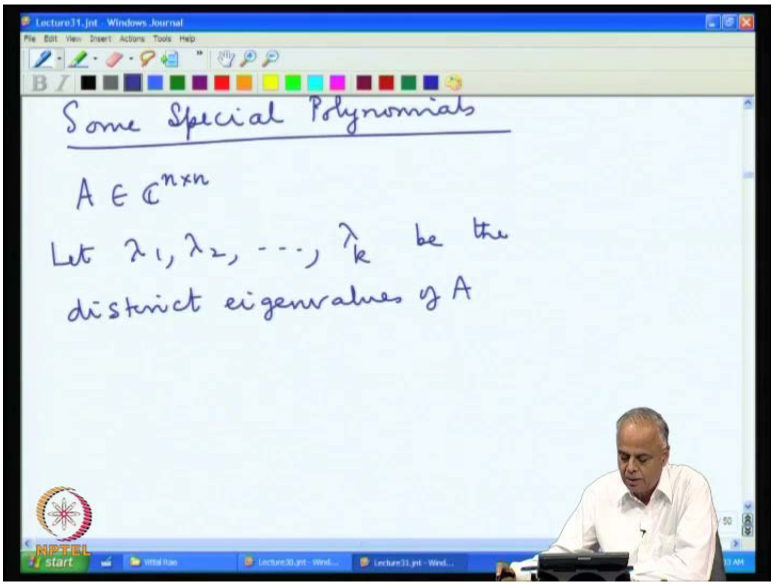


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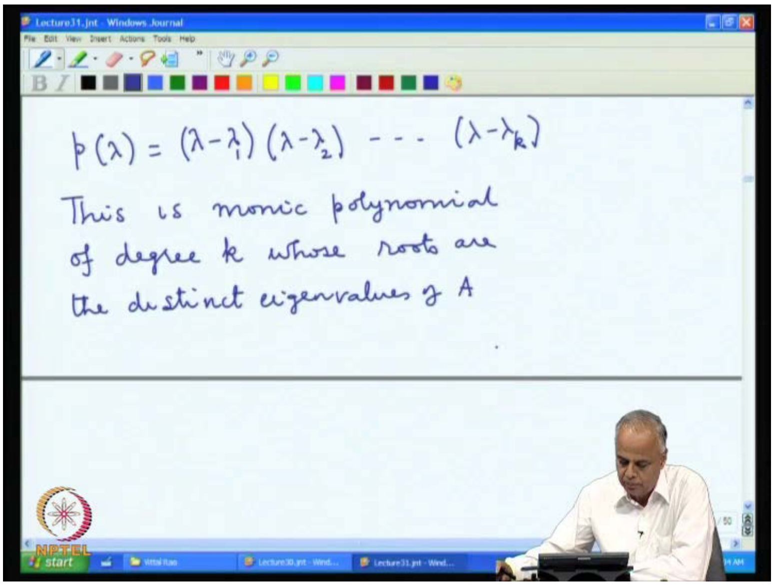


So, we first define the power of a and then, we define for any given polynomial p lambda equal to a naught plus a 1 lambda plus r lambda to the power of r. We defined p a. Now, we are defining a polynomial of the matrix a as a naught i plus a 1 a plus a 2 a squared plus a r a. In other words, the given polynomial replace each power of lambda by the corresponding power of a. The first term is lambda power 0. So, we replace it by a power 0, which is identity. Subsequently, lambda, lambda squared, we replace them by a, a squared and so on. So, given any polynomial p with complex coefficients, we associate with it a polynomial in the matrix a, with complex coefficients. This is again a matrix which is complex. Start with the complex matrix, complex coefficients, the sum of all the matrices again a matrix and again it could be complex.

So, given any polynomial with complex coefficients, we can always associated with it, the matrix polynomial p a. Now, we are going to look at some special polynomials. Some special polynomials, that we will work corresponding to the matrix a. So, we are given the matrix a. So, we know, we can write on the characteristic polynomials and then, find out all the distinct eigen values. So, let lambda 1, lambda 2, and lambda k be the distinct eigen values of a. That means, these are the distinct roots of the characteristic polynomial. Now, consider this polynomial p lambda, which vanishes at all these points, that is lambda 1, lambda 2 and lambda k. So, it has a root lambda 1. So, it is lambda minus lambda 1 must be a factor. It vanishes at lambda 2. So, lambda minus lambda 2 is a factor; lambda minus lambda k.

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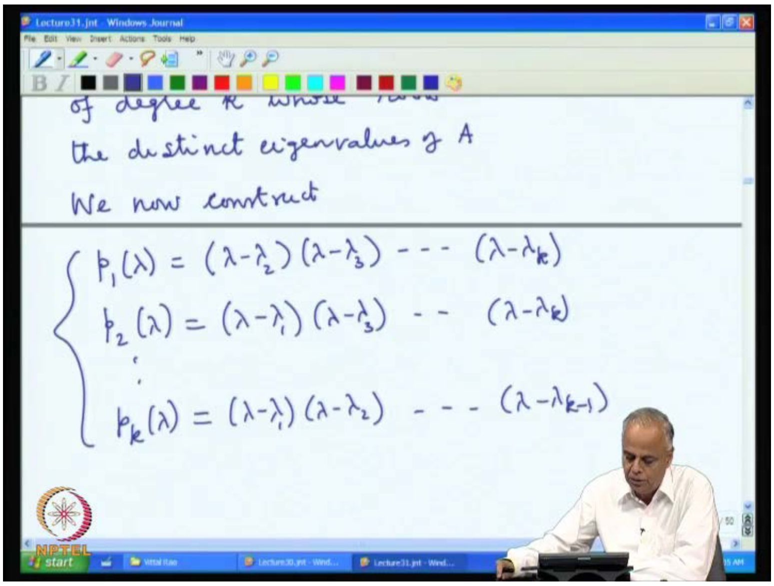

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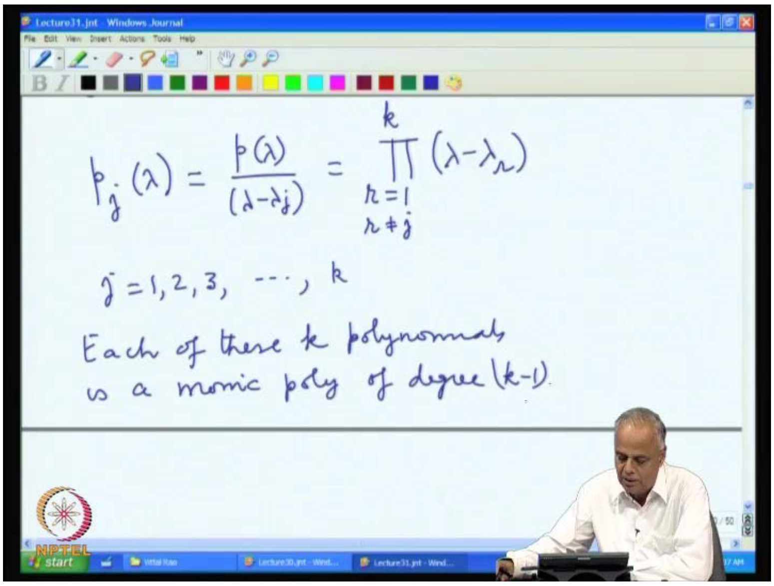


So, this is a polynomial of degree n, whose roots and with polynomial degree of k, whose roots are lambda 1, lambda 2 and lambda k. So, this is a monic polynomial, because the highest power has coefficient 1. This is the monic polynomial of degree k, because the k factors, whose roots are the distinct eigen values of a. Lambda 1 is a root, lambda 2 is a root and lambda k is a root. Now, what we do is, we construct a lower degree polynomial by removing one factor of this k factors one at a time.

For example, we now construct p 1 lambda. When I say one, I mean from p lambda remove the first factor. So, what we get? We do not get lambda minus lambda 1. So, we get lambda minus lambda 2, lambda minus lambda 3 and lambda minus lambda k. To get the p 2 lambda, from p lambda we remove the second factor and we look at only lambda minus lambda 1, lambda minus lambda 3 and lambda minus lambda k. We continue this process to get k polynomials. The k th polynomial, we remove the k th factor and retain all the other factors.

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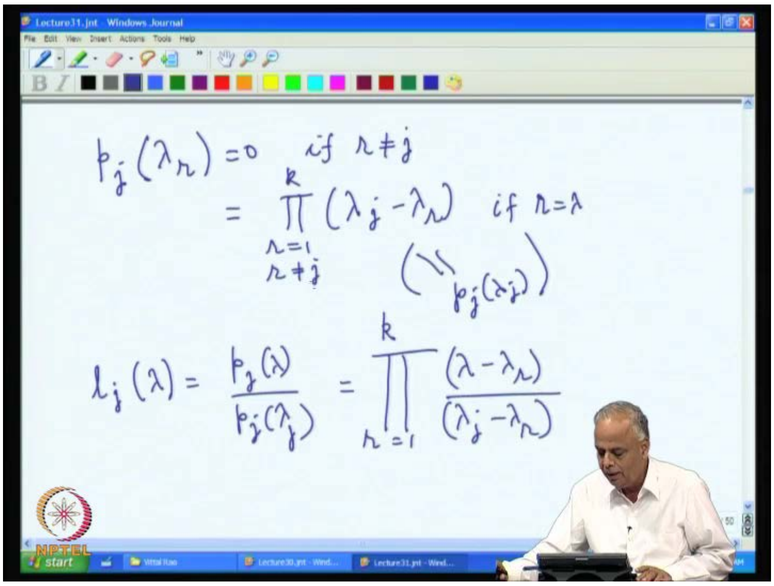


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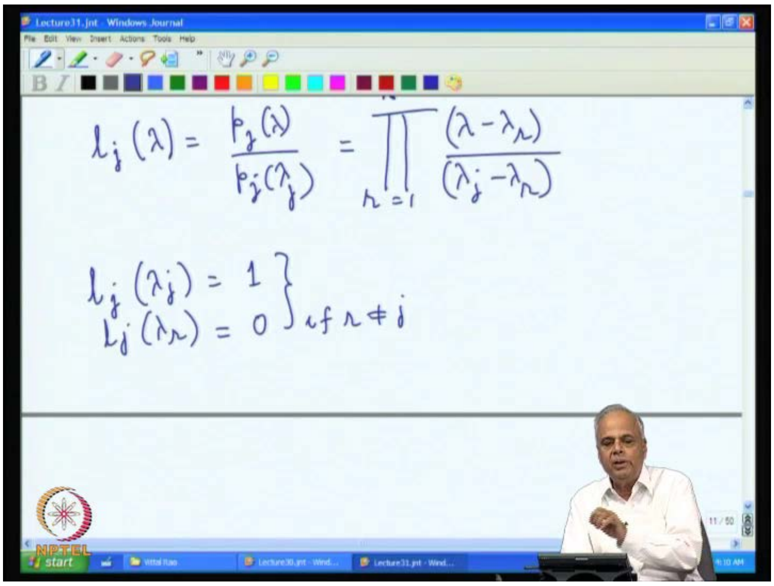

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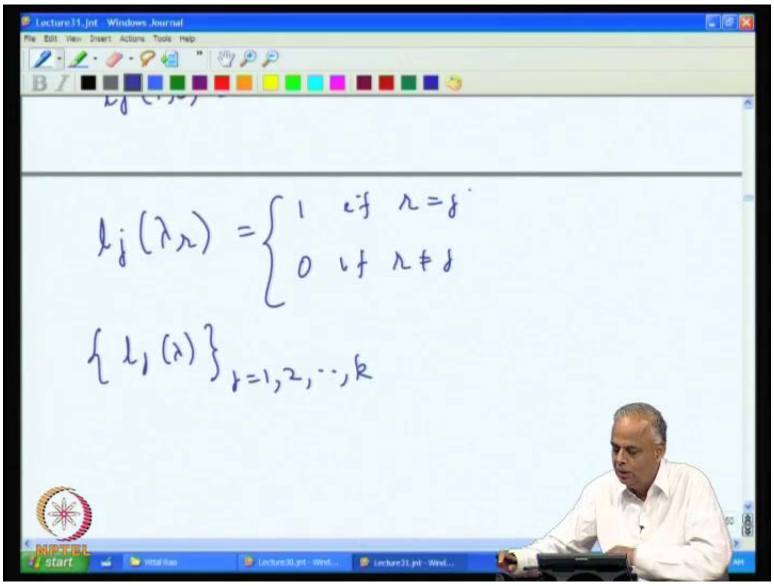


So, in short form, we will write p j lambda, if the polynomial obtained from p lambda by removing the j th. That means, p lambda by lambda minus lambda j. That factor is cancelled out. We can write this in product notation as, r equal to 1 to k r naught equal to j because, the j th factor has been removed; lambda minus lambda r. So, it takes all factors lambda minus lambda r, except the factor lambda minus lambda j. Then, we get the polynomial p j lambda and we do this for j equal to 1, 2, 3, k. Thus, we get k polynomials.

So, each of these k polynomials is a monic polynomial of degree k minus 1 because, p was of degree k. Now, we removed one factor, so the degree becomes k minus 1. Now, let us look at this polynomial p j lambda. The p j lambda has factors, lambda minus lambda r, except lambda j factor. So, if I put r equal to 1, lambda minus lambda 1 is a factor. So, lambda 1 will be a root and similarly, lambda 2 will be a root and lambda 3 will be root. So, all these lambdas will be root, except lambda j. So, we have p j lambda r is equal to 0, if r is not equal to j. If r is not equal to j, that factor will appear here and therefore, p j lambda r will become 0. But, if r is equal to j, what will happen here? We will have lambda j minus lambda r. None of these factors vanish and we get r equal to 1 to k r naught equal to j. It is actually p j p lambda j. So, it is actually p lambda j. Now, we have these polynomial p j, which vanishes at all these points except at the lambda j, where it takes this value. So, this is what p j lambda j. So, this is p j lambda r is 0. If r is not equal to 0, so this is p j lambda j. Now, we normalise these polynomials. Suppose, I define l j lambda to be p j lambda by p j lambda r, what is this polynomial? p j lambda is this polynomial, except the j th term. p j lambda r is just this polynomial, where we evaluated at the point, which p j. So, what does this become?

We know that these have all the factors. So, let us write it in the factor form. This will be just nothing but, the polynomial r equal to 1 to k. Let us look at this as, to make this, sorry, we will make this lambda j. Then, this becomes all the factors. Lambda minus lambda r and all the factors lambda j minus lambda r. From these two definitions, except the lambda minus lambda j, factor does not appear in the numerator and the lambda j minus lambda j factor, do not appear the denominator.

We find that, l j lambda j is p j lambda j by p j lambda j, from the definition is 1 and l j lambda r is p j lambda r by p j lambda j by p j lambda r is 0, if r naught equal to j. l j lambda r equal to 0, if r naught equal to j. So, in other words this l j is a polynomial. If you keep moving along the lambda 1, lambda 2 and lambda case, it lights up at lambda j and takes the value 1 and it just dormants at other lambda j, 0 at lambda 1, 0 at lambda 2, 0 at lambda j minus 1. The moment it hits lambda j, it peaks up to 1 and again at lambda at j plus 1 goes to 0, and lambda j plus 2, it is 0 and lambda k, it is 0. So, l j lambda r is equal to 1, if r equal to j, 0, if r not equal to j. So, we have polynomials l j lambda j equal to 1, 2, k. So, we have got k polynomials, which are such that each polynomial lights up at one particular eigen value and dice of at other eigen value.

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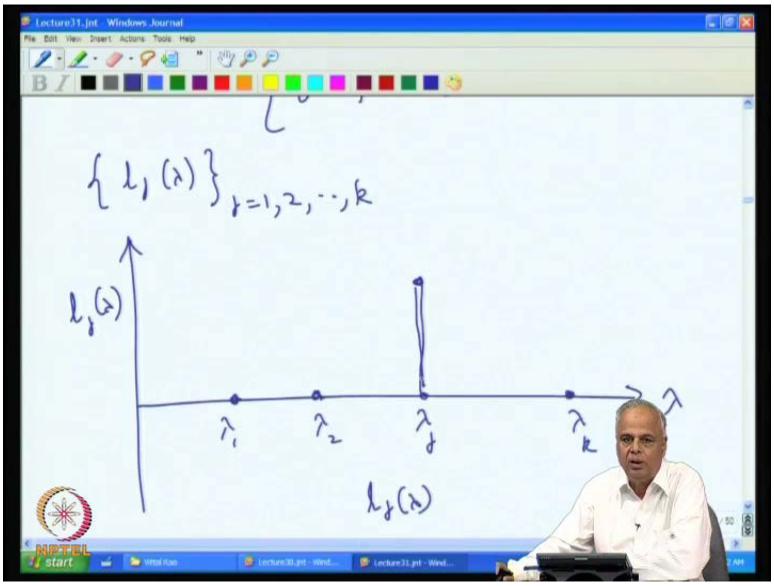

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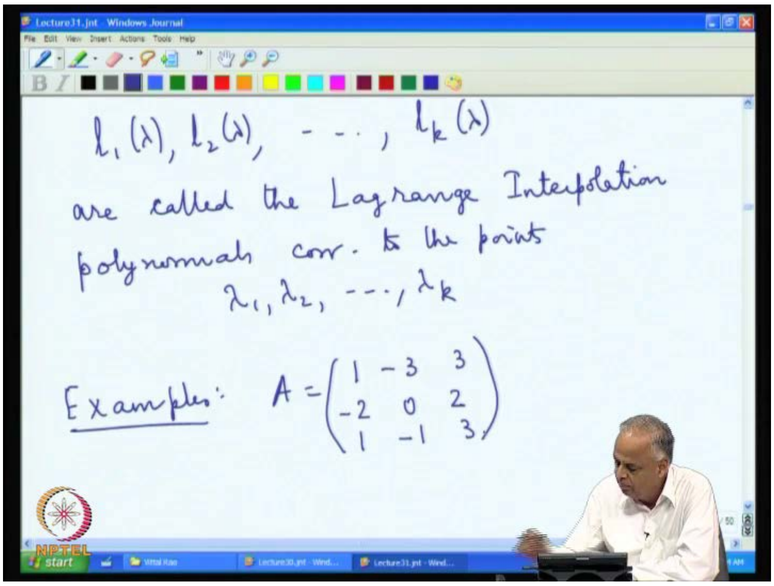


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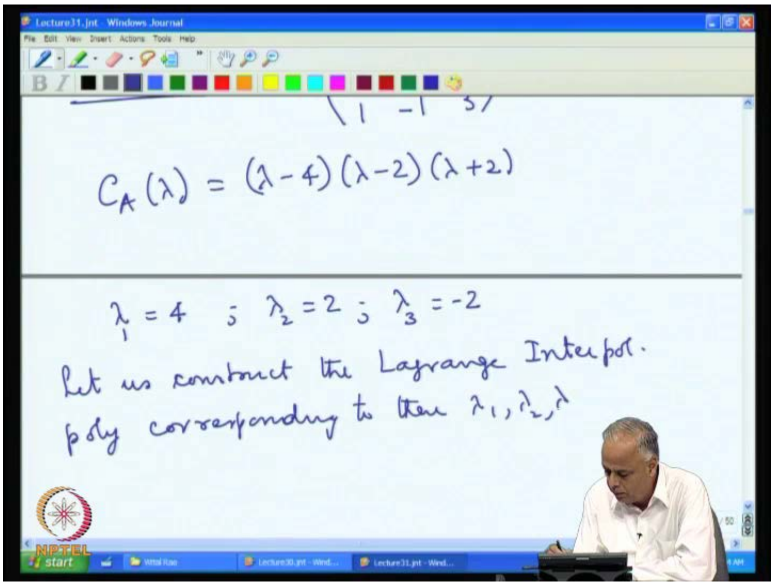

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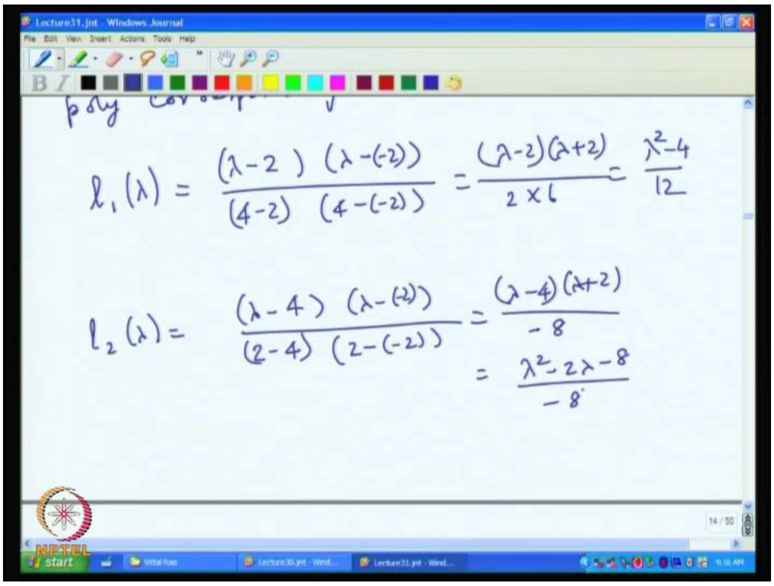


l 1 takes the value 1 at lambda 1 and 0 at all the other lambdas. l 2 takes the values 1 at lambda 2 and dice of at all this. So, if you, if we just loosely plot, assuming that, suppose lambda 1, lambda 2 and lambda k were all real numbers and if we are plotted this l j, so, let us say we are going to plot l j lambda along the y axis and this is the lambda axis. Then, l j lambda will take the value 0 here, 0 here and it will be 0 here, except at lambda j, it will take the value 1. So, that is what we mean, when we say that this lights up at lambda j and dice of at all other places. It looks like lambda j is a switch for the polynomial lambda j, l j lambda.

So, we have these polynomials, these k polynomials l 1 lambda, l 2 lambda and l k lambda are called the lagrange interpolation polynomials. Corresponding to the points, lambda 1, lambda 2 and lambda k, these are the distinct points corresponding to lambda 1, lambda 2 and lambda k. So, we have this k lagrange interpolation polynomial. Let us look at some examples. Let us look at the matrix again, 1 minus 3 3 minus 2 0 2 1 minus 1 3. Again, this is the same matrix, which we have seen at the beginning of the class, as well as the previous lectures. We have found that the characteristic polynomial is lambda minus 4 into lambda minus 2 into lambda plus 2. Now, what are the eigen values? The distinct eigen values are lambda 1 equal to 4, lambda 2 equal to 2, and lambda 3 equal to minus 2. Now, let us construct the lagrange interpolation polynomials corresponding to these lambda 1, lambda 2 and lambda 3. How do I construct the first polynomial?

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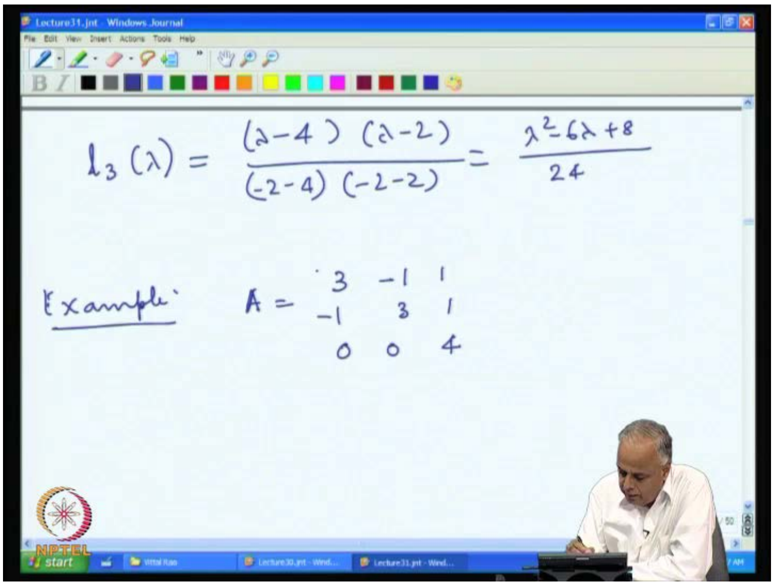


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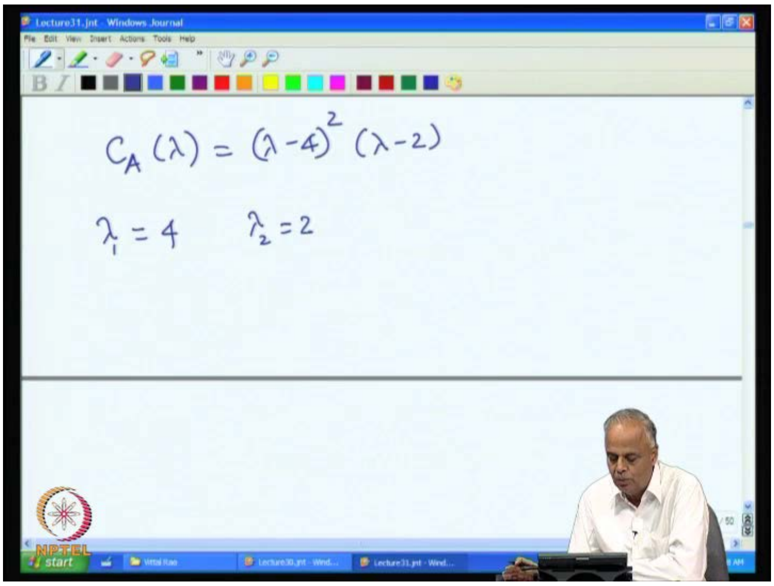

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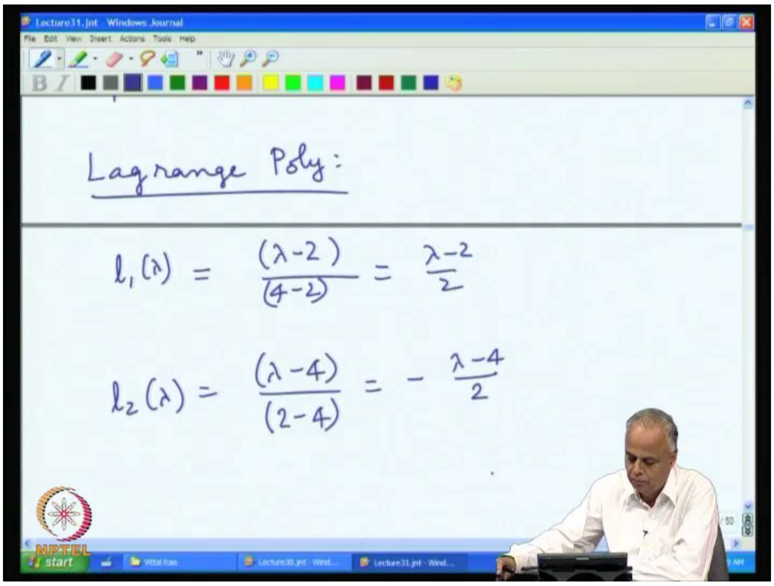


The first polynomial is constructed by the writing the terms, lambda minus lambda minus and we put the value, except the first eigen value. So, lambda minus 2 into lambda minus minus 2 and then, divide it by evaluating the numerator at the first eigen value. So, it will be 4 minus 2 into 4 minus minus 2, which is lambda minus 2 into lambda plus 2 divided by 2 into 6, which is lambda squared minus 4 by 12. That is the first lagrange polynomial.

The second langrange polynomial is, again you write lambda minus lambda minus and then, you skip the second eigen value and put the remaining two eigen values. Then, in the denominator, write the numerator with the lambda replaced by the second eigen value now. Second eigen value is 2. So, it is 2 minus 4 into 2 minus minus 2, which is lambda minus 4 into lambda minus 2 divided by minus 8, which can be written as lambda squared, this plus 2, lambda squared minus 2 lambda minus 8 by minus 8.

The third lagrange interpolation polynomial is, again you write lambda minus and lambda minus and skip the third eigen value. So, you put the first one, 4 and 2 and then, in the denominator, replace the lambda by the third eigen value, which is minus 2, which is equal to lambda squared minus 6 lambda plus 8 divided by minus 6 into plus 4 minus 6 into minus 4, which is 24. So, whatever is the simplification. So, we have got these three lagrange interpolation polynomials. Let us look at another example. a is 3 minus 1 1 and then, minus 1 1 1and 0 0 4. This is another example, which we have seen in the previous lecture. We found that the characteristic polynomial is lambda minus 4. The quantity squared into lambda minus 2. Therefore, eigen values were lambda 1 equal to 4 and lambda 2 equal to 2. There are only two distinct eigen values now.

The eigen value 4 repeats twice. So, the distinct eigen values are 4 and 2. So, now, the construction of the lagrange polynomials is much easier now. The lagrange polynomials are, now since there are two distinct eigen values, there are going to be two lagrange interpolation polynomials. How do I get the first one? Now, I have only, out of two, I have to remove one of them. So, I get lambda minus, I should not take the first eigen value. I should take the second one. That divided by, in the denominator you replace lambda by the first eigen value, which is lambda minus 2 by 2.

To get the second one, skip the second eigen value and write only the factor involved in first eigen value. In the denominator, replace lambda by the second eigen value we get minus lambda minus 4 by 2. So, these are the only two lagrange interpolation polynomials for this matrix, because only there are two distinct eigen values. So now, we are seeing that given any k distinct numbers, lambda 1, lambda 2 and lambda k, in particular given the k distinct eigen value of the matrix a, with these we associate the k lagrange interpolation polynomials l 1, l 2 and l k.

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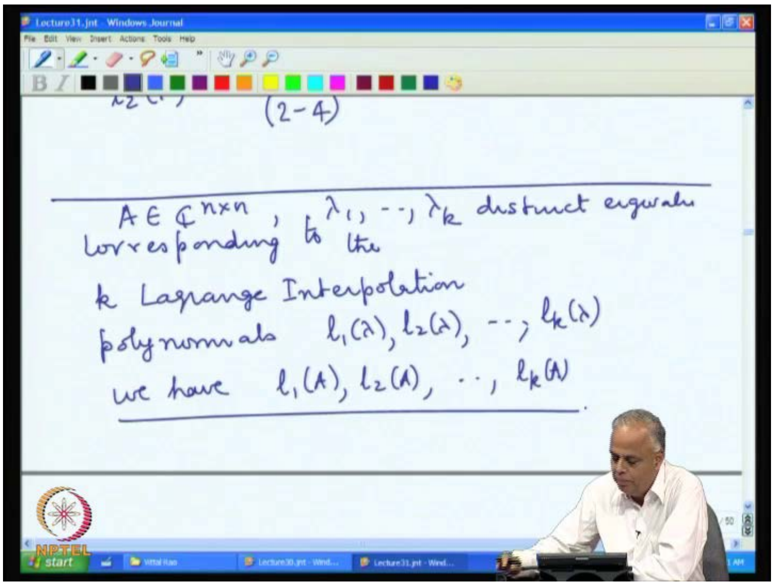


This special properties of these lagrange interpolation polynomials is, the j th polynomial vanishes except at the j th eigen value, At the j th eigen value, it takes the value 1. So, l j lambda j is 1, l j lambda r is 0, if r is not equal to j. This is the special property of the lagrange interpolation polynomials. Now, let us see, how we will use this lagrange interpolation polynomials. Now suppose, we have this polynomials corresponding to the matrix a. We have seen that, corresponding to every polynomial in lambda, we have the corresponding polynomials matrix l j lambda. So therefore, corresponding to, let us take a general matrix a c n n and then, lambda 1, lambda 2 and lambda k distinct eigen values. Now, once I have distinct eigen values k of them, we have k lagrange interpolation polynomials. Corresponding to the k lagrange interpolation polynomials, what are they? l 1 lambda, l 2 lambda etcetera, and l k lambda.

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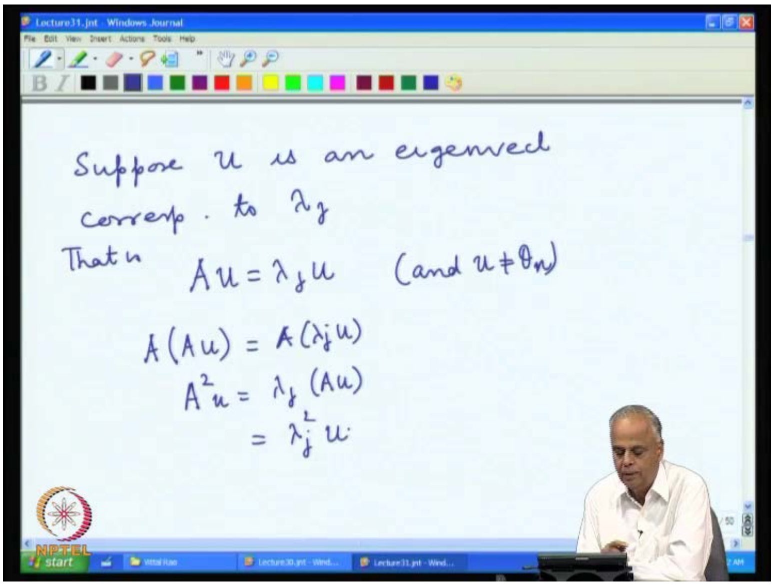

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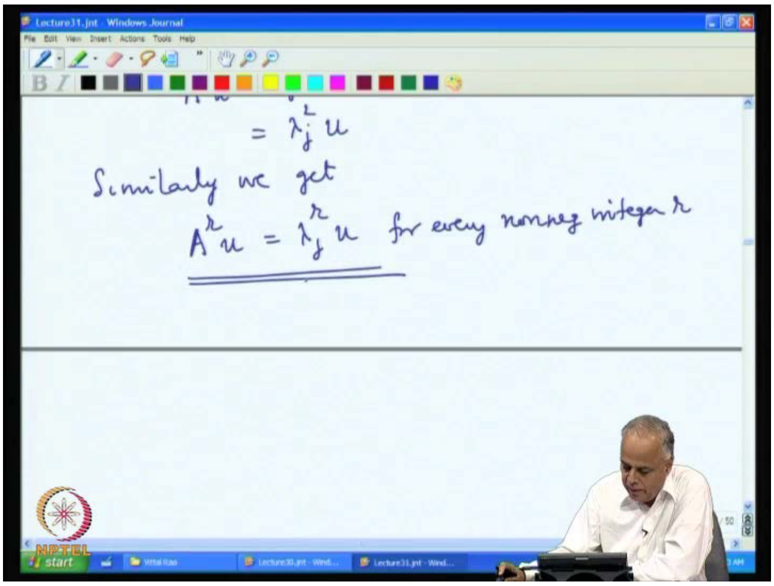


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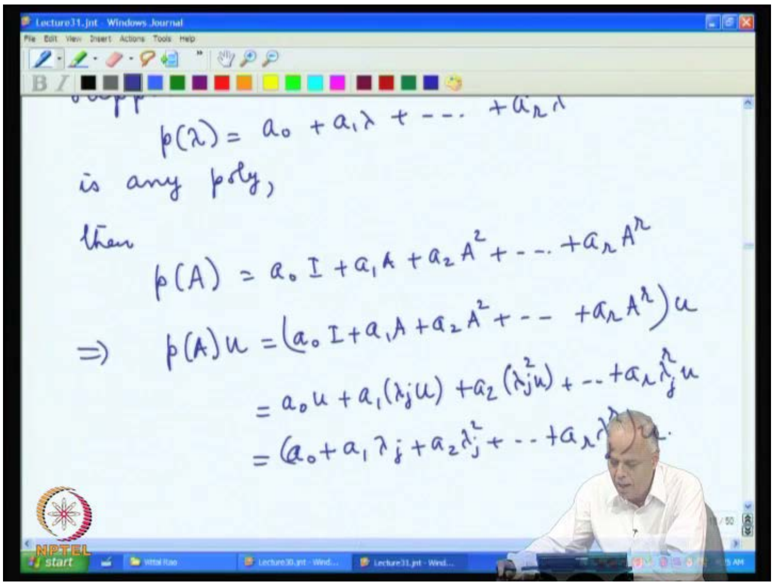

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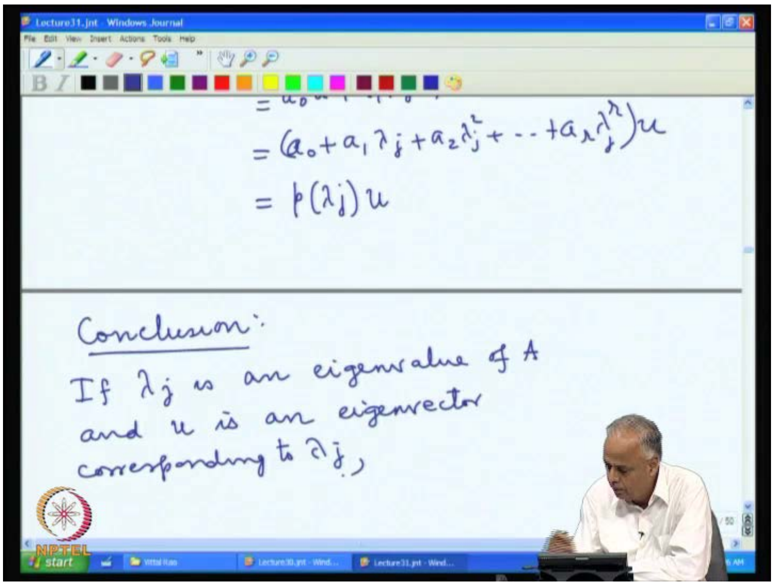


We have the corresponding matrices, l 1 a, l 2 a etcetera and l k a. We will see these matrices. Now, suppose u is an eigen vector, corresponding to eigen value lambda j. What does this mean? This means, since it is an eigen vector, that is, au must be equal to lambda j u and u is not equal to theta n. So, au equal to lambda j u, that is the eigen equation. Since it is an eigen vector, it must be non zero. If that is the case, we get, multiplying both sides by a, we get a of lambda j u, that is a squared u is lambda j au. But, au is lambda j u. So, lambda j squared u. So, au is lambda j u, a squared u is lambda j squared u and similarly, recursively, we get, for any power r, a power r u is lambda j power r u, for every non negative integer. a cube u will be lambda j cube u, a power 25 u will lambda j to the power of 25 u and so on.

In general, a power r u will be equal to the lambda j to the power of r u. So, therefore, if u is an eigen vector, then to calculate the power of the matrix a acting on u, all we have to do this take the power of the eigen value and multiply it to with the vector. Now, this has an additional simplification. Suppose, now p lambda is equal to a naught plus a 1 lambda plus r lambda power r, is any polynomial with complex coefficient, any polynomial, then we know, p a is a naught i plus a 1 a plus a 2 a squared and so on plus a r a power r. This is how it is defined, pa corresponding to any polynomial p lambda. Therefore, what is p a u. P a u is a naught i plus a 1 a plus a 2 a squared and so on, plus a r a power r into u. Now, we multiply a naught i times u is u, plus a 1, a times u is lambda j u, because u is an eigen value. a 2, a squared u is lambda j squared. We have just now seen that and so on, a r lambda j power r u.

We see that, this can be written as a naught plus a 1 lambda j plus a 2 lambda j squared plus a r lambda j power r into u. What is in bracket? It is nothing, but the polynomial p evaluated at point lambda j. So, p a of u, if p lambda j of u. So, what is our conclusion?

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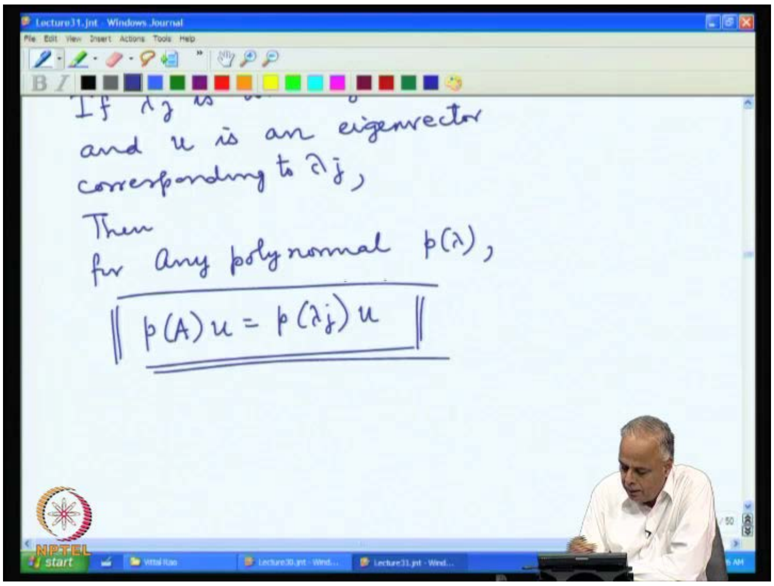


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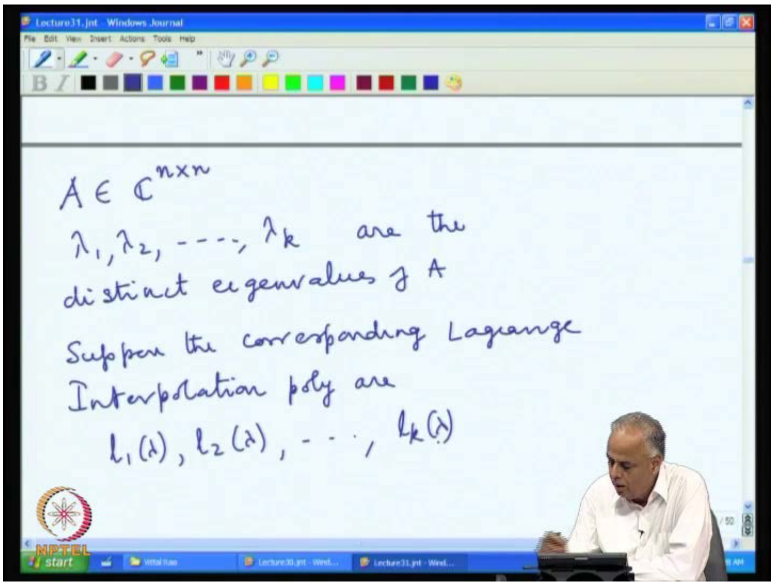


The conclusion is that, if lambda j is an eigen value of a and u is an eigen vector, corresponding to the eigen value lambda j, then for any polynomial p lambda, we are assuming everything over complex, where any polynomial p lambda, p a u is equal to p lambda j. This is a very important identity, because it says, the computation over the matrix, calculating the polynomial matrixes action on u is a very simple action. We have to simply calculate the value of the polynomial at the eigen value and multiply with the vector u. So, this is a very nice and simple identity regarding eigen values, eigen vectors and polynomials. How do we use this? Therefore, now let us look at a matrix a and then, say lambda 1, lambda 2 and lambda k are the distinct eigen values of a. Suppose, the corresponding, the moment we have the eigen values, distinct eigen values, we have seen, we can construct the corresponding lagrange interpolation polynomial.

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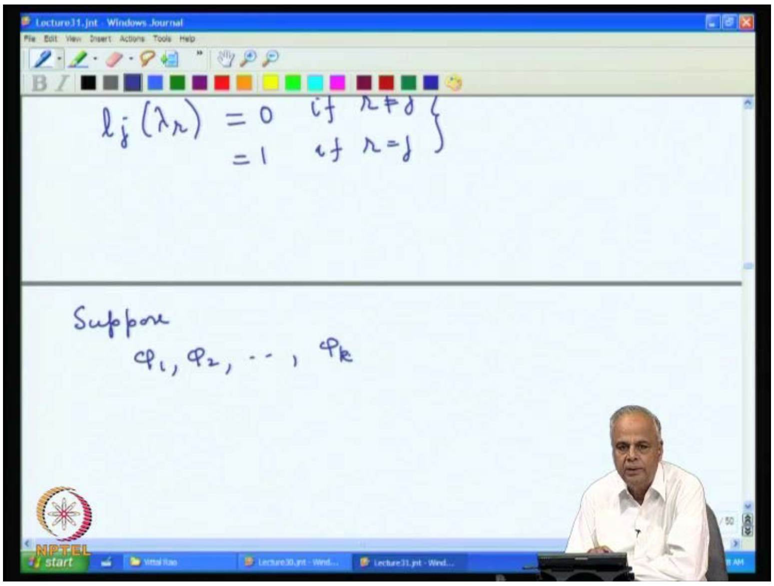

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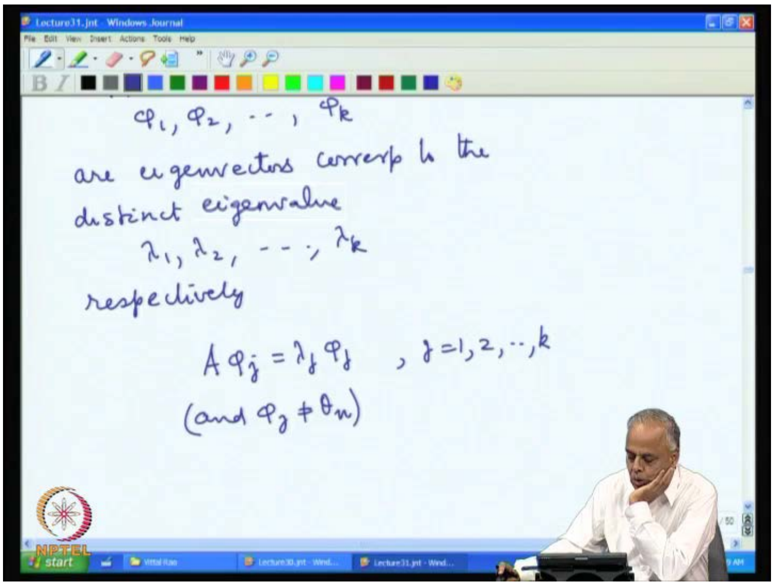


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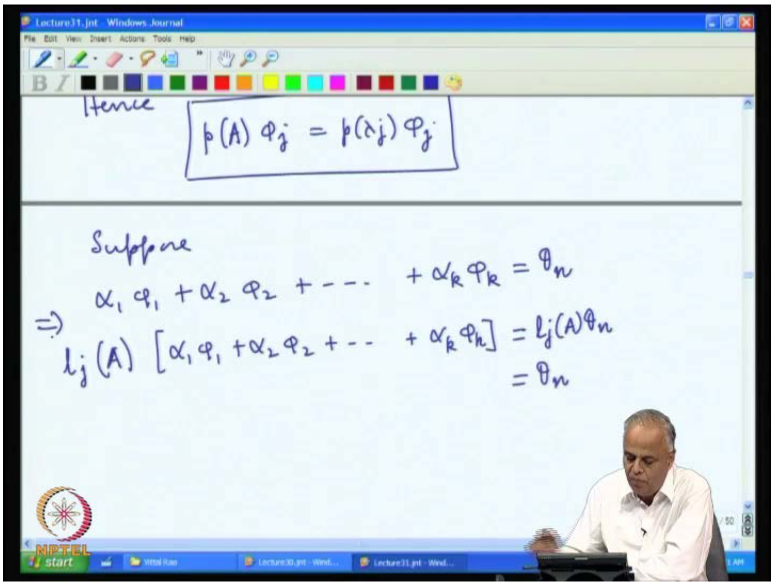


So, the corresponding lagrange interpolation polynomials are l 1 lambda, l 2 lambda etcetera, and l k lambda. Then, what do we know about them. Remember, l j lambda r is 0, if r is not equal to j and equal to 1, if r equal to j. This is the characteristic property of the lagrange interpolation polynomial. The j th polynomial lights up at the j th eigen value. Now, we have these k distinct eigen values. Suppose, phi 1, phi 2 and phi k are eigen vectors corresponding to these distinct eigen values lambda 1, lambda 2 and lambda k respectively. So, phi 1 is the eigen vector corresponding to lambda 1, phi 2 is the eigen vector corresponding to the lambda 2 and so on. So, what does that mean? a phi j is equal to lambda j phi j for j equal 1 2 k and the phi j r not 0. They are non zero vectors because, they are present to be eigen vectors.

So, we have a phi j equal to lambda j phi j and we have just now observed that, when we have an eigen vector, the polynomial evaluation is easy. Hence from our above calculation, p a phi j is equal to p lambda j phi j. This is what we have calculated and found about that the polynomial matrix evaluation acting on the eigen vector is polynomial evaluation, the eigen value multiplying the eigen. Now, how we are going to use? Now, were claiming that the eigen vectors corresponding to distinct eigen values are linearly independent. So, what we were claiming is, over that phi 1, phi 2 and phi k must be linearly independent. So, let us see how we get it. How do we prove that phi 1 phi 2 phi k are linearly independent. We have to show that, if any linearly combination is 0, then coefficients must be 0.

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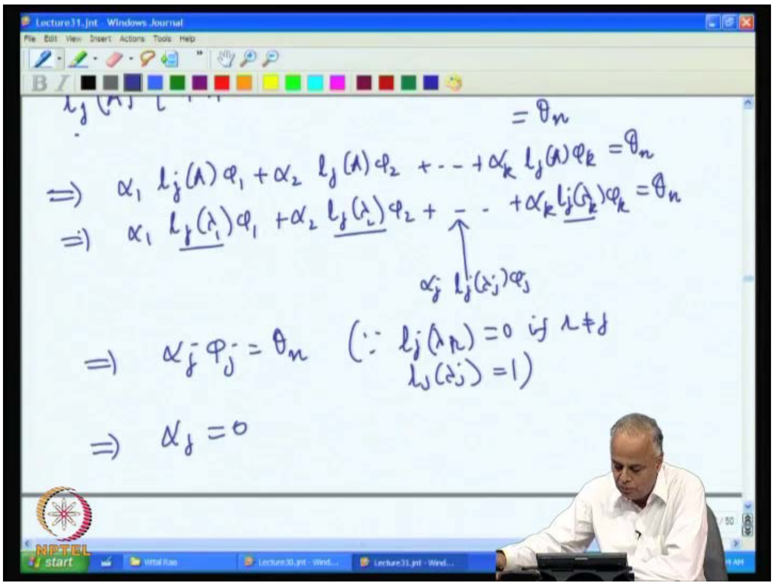

So, we start with linear combinations. Suppose alpha 1 phi 1 plus alpha 2 phi 2 plus alpha k phi k is equal to theta n. We want to show that alpha 1 must be 0, alpha 2 must be 0 and alpha k must be 0. That will establish linear independence. Now, what we do is, we multiply both sides by the matrix l j a, where l j is the j th lagrange interpolation polynomial. Now, that will be l j a into theta n. But any matrix multiplying 0 vectors gives the 0 vector. So, if alpha 1 phi 1 alpha 2 phi 2 alpha n phi n is 0, then l j a of this must be zero 0.

That says, alpha 1 l j a of phi 1 plus alpha 2 l j a of phi 2 plus etcetera and alpha k l j a of phi k is 0. Now, phi is an eigen vector and phi 1 is an eigen vector corresponding to lambda 1. We have seen from this fact that for any polynomial pa phi 1 will be p lambda 1 phi 1. So, l j a phi 1 will be l j lambda 1 phi 1. So, this will be l j lambda 1 phi 1, and l j lambda 2 phi 2 and so on. Then, there will be an alpha k and l j lambda k lambda k must be 0. Now, what is the property of these l j s'? The l j s' are such that, they light up only at lambda j and at lambda 1, lambda 2, lambda j minus 1 that takes the value 0.

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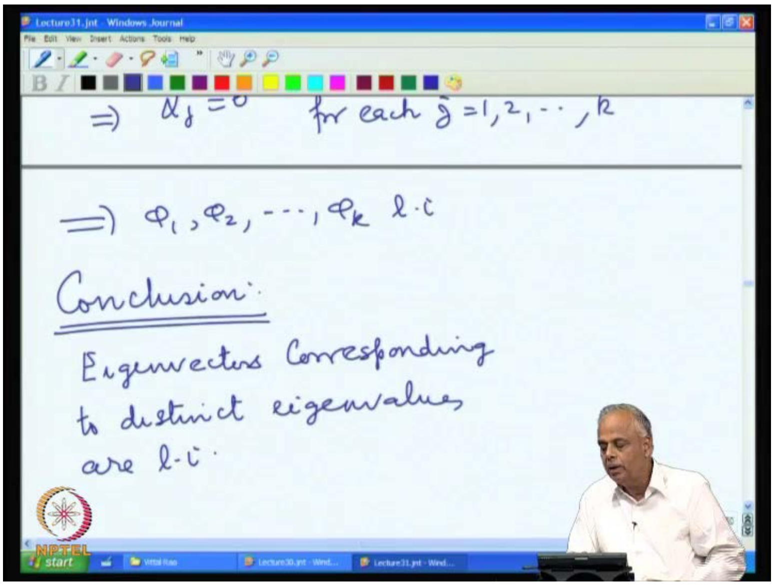


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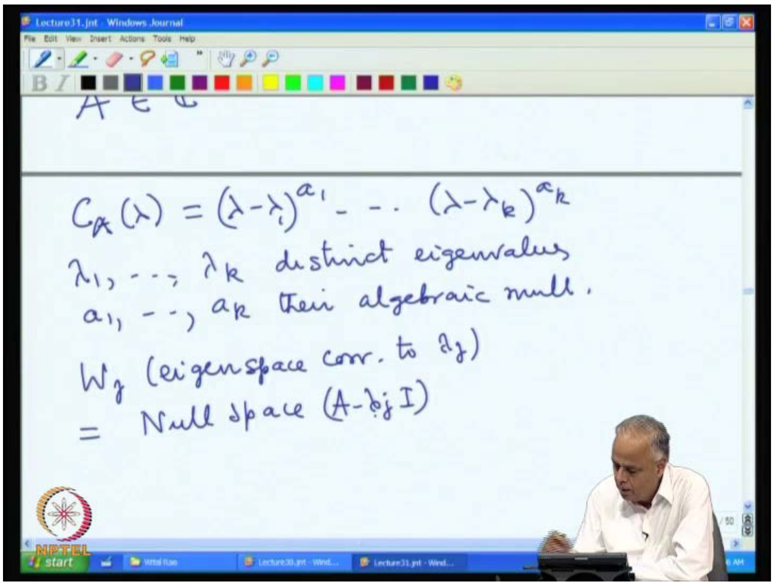

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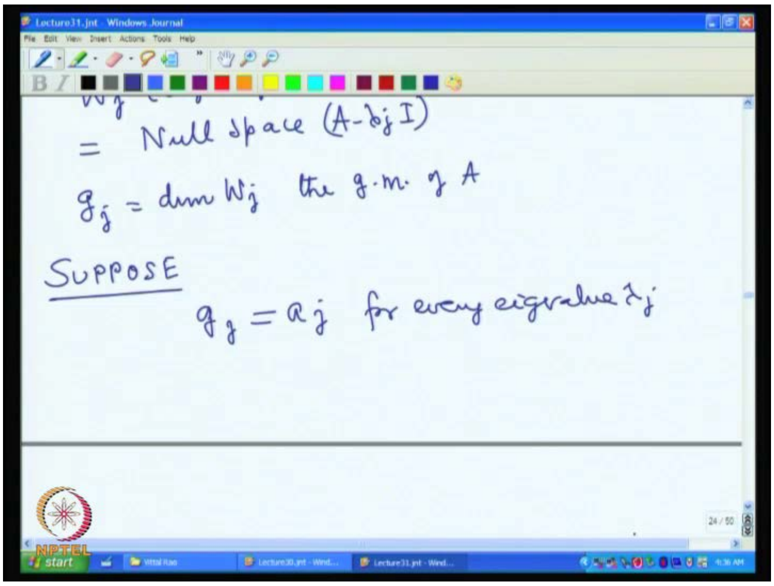


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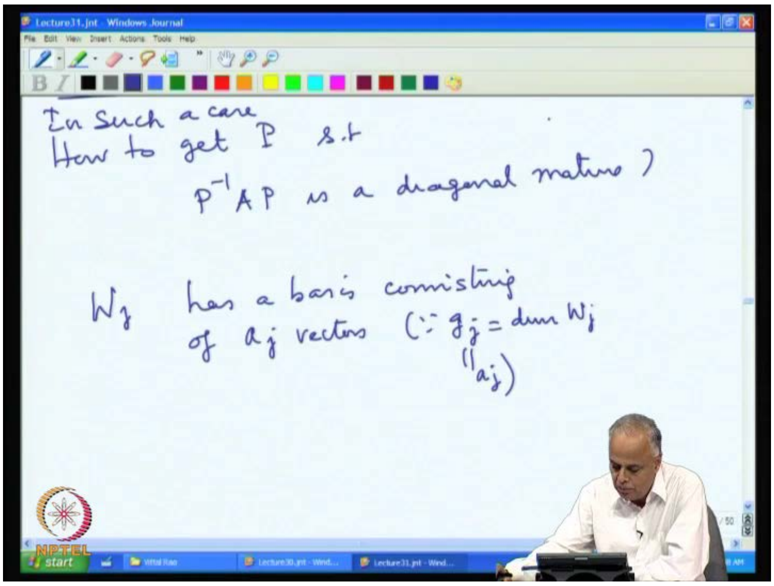


So therefore, l j lambda 1 will be 0, l and j lambda 2 will be 0. The only term that will survive will be l j lambda j. That term, because l j lambda j is 1 and this will become simply alpha j phi j equal to theta n. Because, l j lambda r equal to 0, if r not equal to j, and l j lambda j is equal to 1. If use those, you get this. But now, phi j was an eigen vector. Therefore, phi j was an non zero vector. So, since phi j is a non zero vector, alpha j phi equal to theta n implies alpha j is 0. This we can do for every j. If we start with l 1, we get alpha 1 is 0. If we start with l 2 here, we get alpha 2 is 0. So therefore, each j equal to 1, 2 till k. So, that says phi 1, phi 2 and phi k are linearly independent.

This is the result that we said in last lecture that we will prove later. So, the conclusion is that, eigen vector corresponding to distinct eigen values are linearly independent. So, this completes our argument that we started in the last lecture about matrixces, for which algebraic multiplicity was equal to geometric multiplicity. So, in summary, we have the following result. Namely, suppose I have a matrix a, which belong to the complex n by n and its characteristic polynomial is this lambda 1 minus lambda to the power of a 1 lambda minus lambda k to the power of a k. Lambda 1, lambda 2 and lambda k distinct eigen values and a 1, a 2, a k their algebraic multiplicity. So, we have these eigen values and eigen multiplicity. Then, w j, the eigen space corresponding to lambda j, which is equal to null space of a minus lambda j i. Then, we have g j equal to dimension of w j. the geometric multiplicity of a. Suppose, g j equal to a j for every eigen value lambda j. So we have, if the matrix a is such that the algebraic multiplicity is equal to the geometric multiplicity for every eigen value, then a is diagonalizable. So, how do we? Then, how to get p, such that p inverse a p is a diagonal matrix. In such a case, how do we get the p matrix, that p inverse a p is diagonal matrix.

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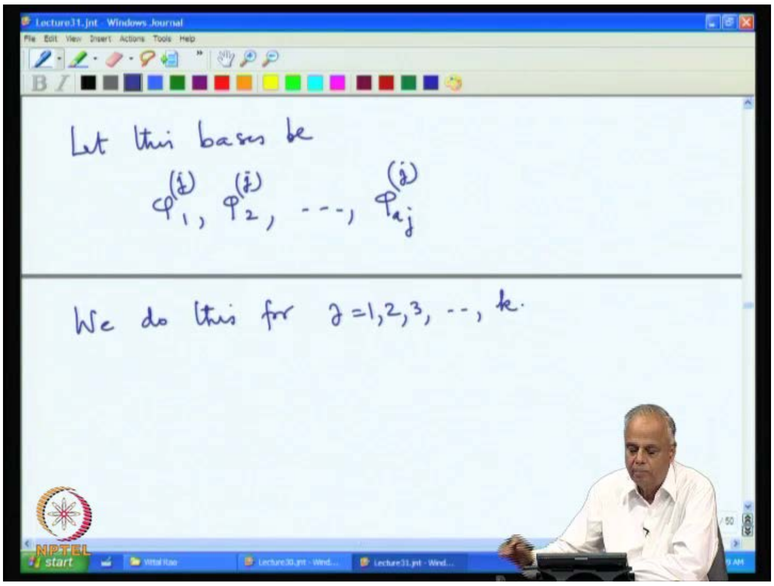

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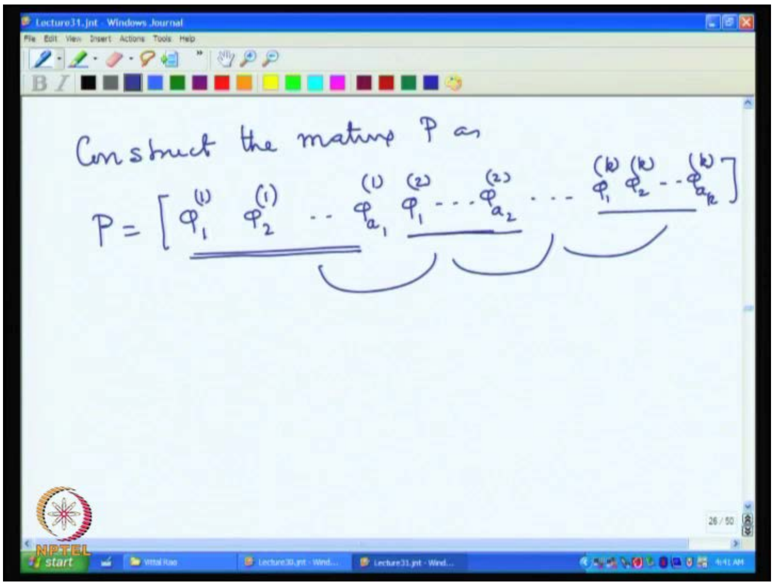


What we do if the following, we look at w j has a basis, null space, the sub space, the basis, now its dimensions in the geometric multiplicity because the geometric multiplicity is algebraic multiplicity which dimension is a j. Since, its dimensions is a j, a basis will consist of a j vectors, because g j equal to a j w j and g j is equal to a j. This is what we are assuming. We are assuming that suppose the geometric multiplicity is equal to algebraic multiplicity. So, it has a basis consisting of a j vectors. Let this basis be. So, we always find the null space of, first we find an eigen value and then we find then null space.

Then, we find the basis for the null space. Let this basis be phi j 1, phi 2 and phi j a j. This we can do for j equal to 1, for the eigen value lambda 1, j equal to 2, the eigen value lambda 2 and so on. So, we do this for j equal to 1, 2, 3 up to k. Then, construct the matrix p as p equal to, first you write all the eigen vectors corresponding to the eigen value 1. How many of them are there? a 1 of them are there. Construct, write down all the eigen vectors as columns of this matrix p. First look at the eigen vectors corresponding to lambda 1. They come as the basis of w 1.

w 1 is the basis consisting of a 1 vectors. Take this a 1 basis vectors and put them as the first a 1 columns of p. This decides the a 1 columns of p. We need totally n columns. Then, we write the next a 2 columns, which corresponds to the matrix eigen value lambda 2. We continue this process and the last a k columns will be corresponding to the eigen value lambda k. So, we will have a 1 plus a 2 plus a k columns, but a 1 plus a 2 plus a k is equal to n. Therefore, we have n columns, since all the n columns are linearly independent now, by the theorem that we have just proved. These are all linearly independent and these are all linearly independent and these are all linearly independent.

But, across their linearly independent because, we have just now observed that the eigen vectors corresponding to distinct eigen values are linearly independent. So, p is the matrix, n columns, each column is an eigen vector corresponding to a. The first a 1 columns correspond to the a 1 eigen vectors are the a 1 basis corresponding to w 1. The next a 2 columns correspond to a 2 eigen vectors corresponding to eigen value lambda 2 are same as the a 2 basis vectors for w 2.

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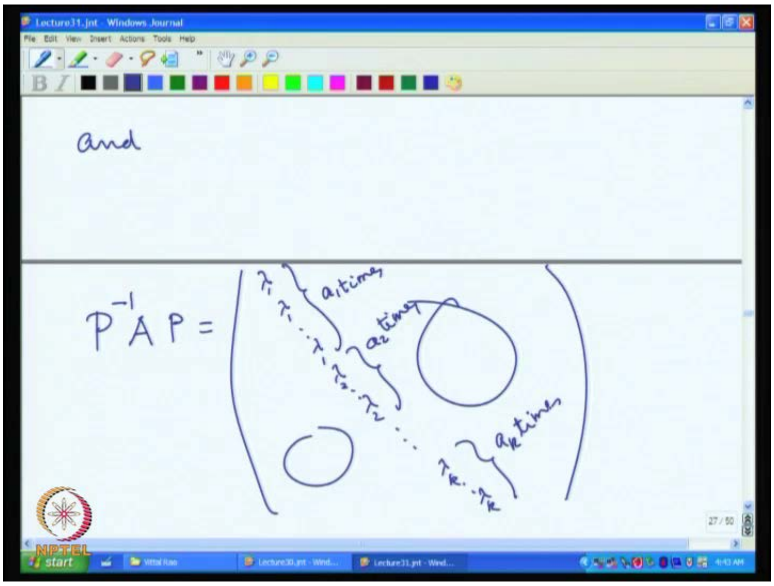

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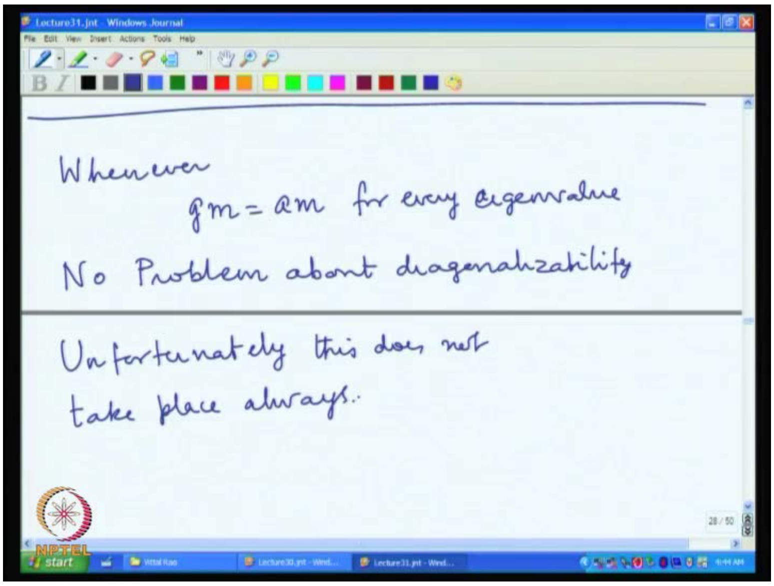


Continuing this process, the last a k columns are the a k eigen vectors corresponding to the eigen value lambda k are same as the a k basis vectors corresponding to the sub space wk. The eigen space wk. Then, all the columns are linearly independent, p is invertible. Since, columns are linearly independent and when we write p inverse a p, we get a diagonal matrix. What is the diagonal matrix? The diagonal entries are going to be, lambda 1 will appear along the diagonal a 1 times, then lambda 2 will appear a 2 times and so on. In the end, lambda k will appear a k times.

So, this will be huge diagonal matrix and n by n lambda 1 will appear a 1 times and lambda 2 a 2 times along the diagonal. Finally, lambda k a k times, and a becomes diagonalizable. In case, all the eigen values with the matrix a is real and all the eigen values are real, then p can also be chosen as real, if j gm equal to am. So, thus we know that, if the geometric multiplicity is equal to the algebraic multiplicity for every eigen value, the diagonalization problems can be completely resolved. But the moment we have shortage of eigen vectors, that is if geometric multiplicity becomes smaller than the algebraic multiplicity, we have problems. So, whenever gm is equal to am for every eigen value, so this is the main moral of our discussion.

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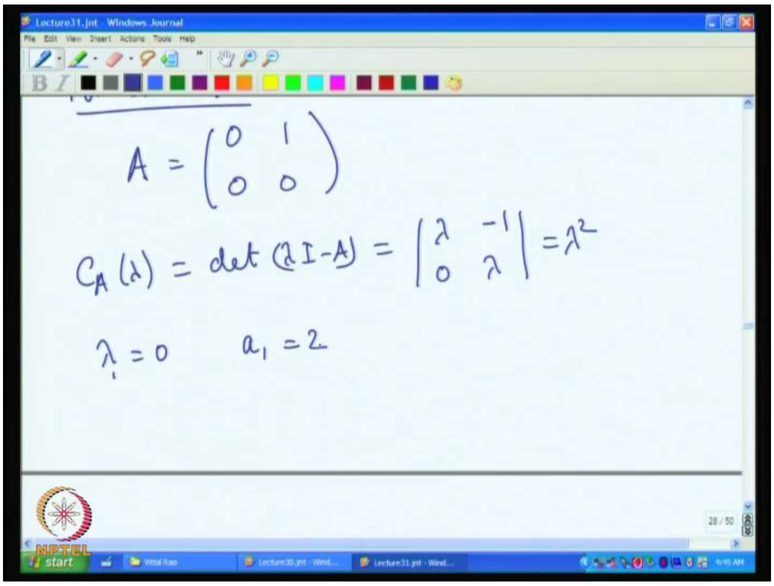


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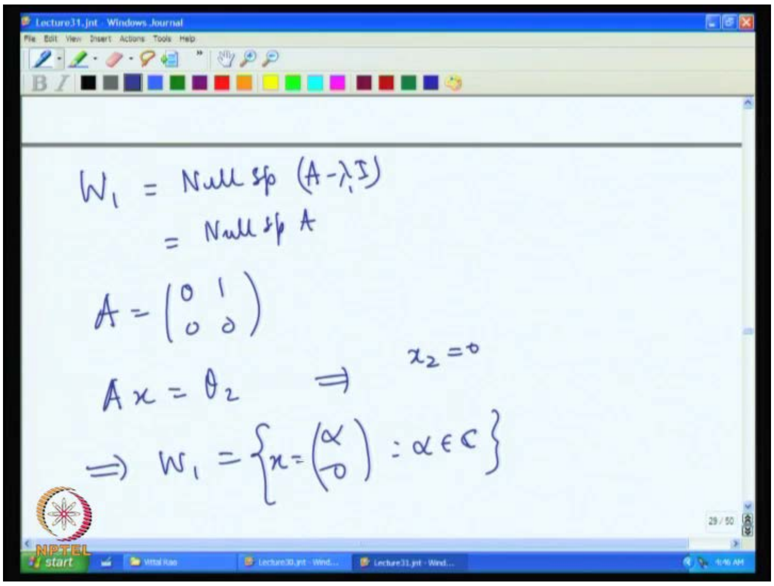

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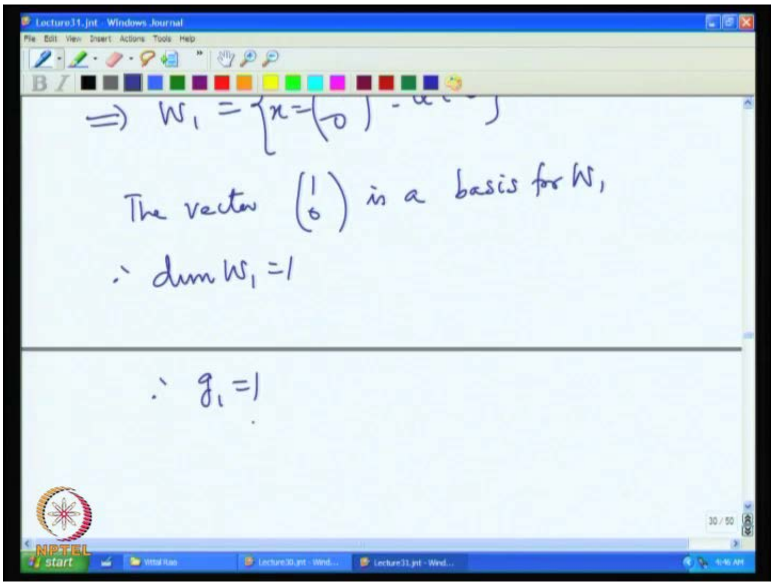


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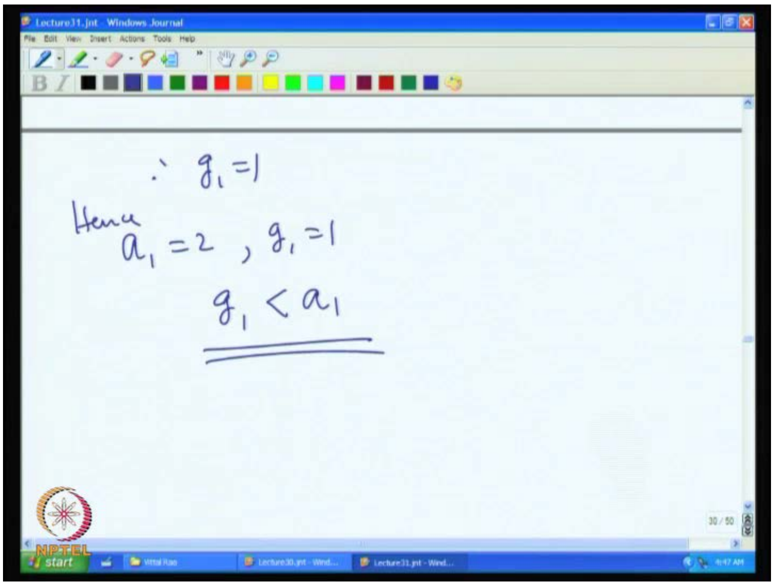


Whenever gm is equal to a m for every eigen value, no problem about diagonalizability. But unfortunately, it is not always the case. Unfortunately, this does not take place always. That is, there are matrices for which gm will not be equal to n. For example, let us look at a simple example. Consider the matrix a equal to 0 1 0 0, then the characteristic polynomial which is the determinant of lambda i minus a, which is the determinant lambda minus 1 0 lambda, which is this lambda squared. We have seen this before again.

So, there is only one eigen value 0, with algebraic multiplicity as 2. So, let us find the eigen space. We have to find the null space of a minus lambda 1 i. Lambda 1 is 0, which is the same thing as null space of a. But, a is the matrix 0 1 0 0. So, to find the null space, we must solve this equation a x equal to theta 2, which gives us x 2 equal to 0. That is the only equation. Therefore, w 1 consist of all vectors for which x 2 is 0 and x 1 can be anything. The vector 1 0 is a basis for w 1. Therefore, dimension of w 1 is 1 and therefore, g 1 is equal to 1. So, in this case, we have a 1 equal to 2, g 1 equal to 1 and we have g 1 strictly less than a 1. So, therefore, there are situations where geometric multiplicity is less than algebraic multiplicity.

Therefore, we look at a class of matrices, for which this is always a guarantee. That is, we would like to have a class of matrix, where by looking at them, we can immediately say these matrices are not going to create any problem. For all these matrices, the geometric multiplicity will be equal to the algebraic multiplicity. These are the so called hermitian matrices, the complex case and the real symmetric matrix is the real case. We shall start looking at them in the next lecture.