Advanced Matrix Theory and Linear Algebra for Engineers
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Module No. # 08

Lecture No. # 30

Diagonalization – Part 3

In the last lecture, we found that if A is the matrix, which we treat it may be a real matrix, but we still take it as a complex matrix.

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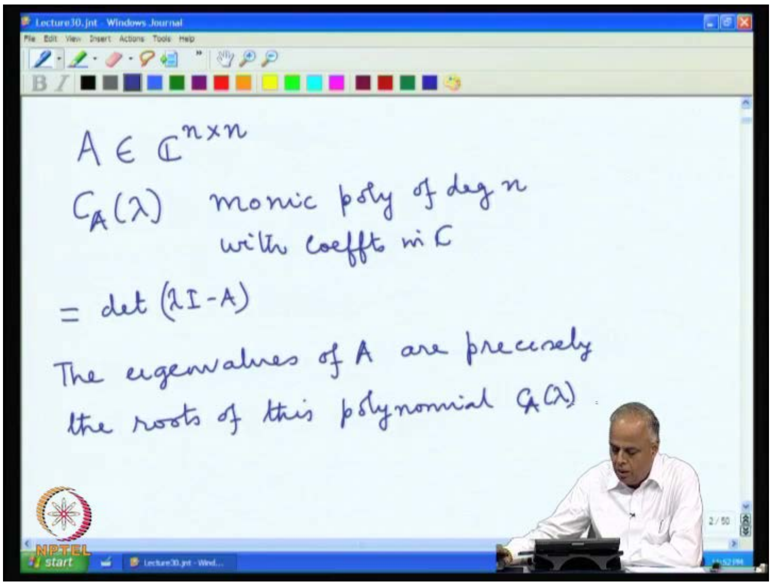

Suppose, A is the (()) then the characters polynomial C A lambda is a polynomial, is a monic polynomial of degree n, with coefficient in C. And it is defined be, the determining of lambda I minus A, and the Eigen values of A, which are now allowed to be complex also are precisely the roots of this polynomial of this polynomial C A lambda.

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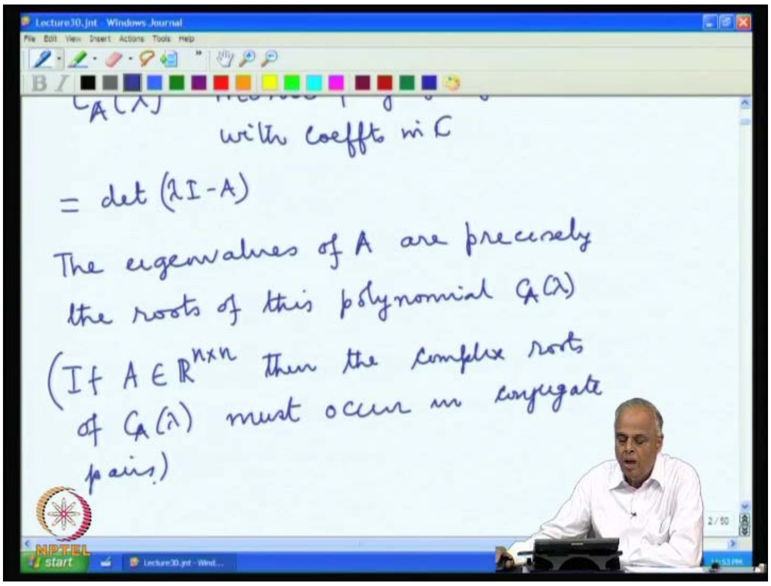

The again recall, but if A is a real matrix, then the complex roots of C A lambda must occur in conjugate pairs. And since, the C A lambda the polynomial of degree n.

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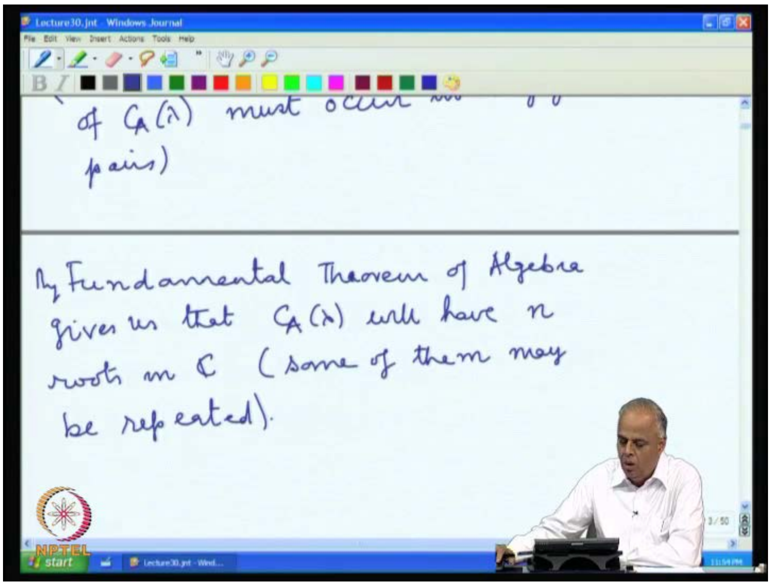

Then the fundamental theorem of algebra is that the fundamental theorem of algebra gives us that, C A lambda will have n roots in C, some of them may be repeated, may or may not be, but the reputation is allowed, some of them may be repeated.

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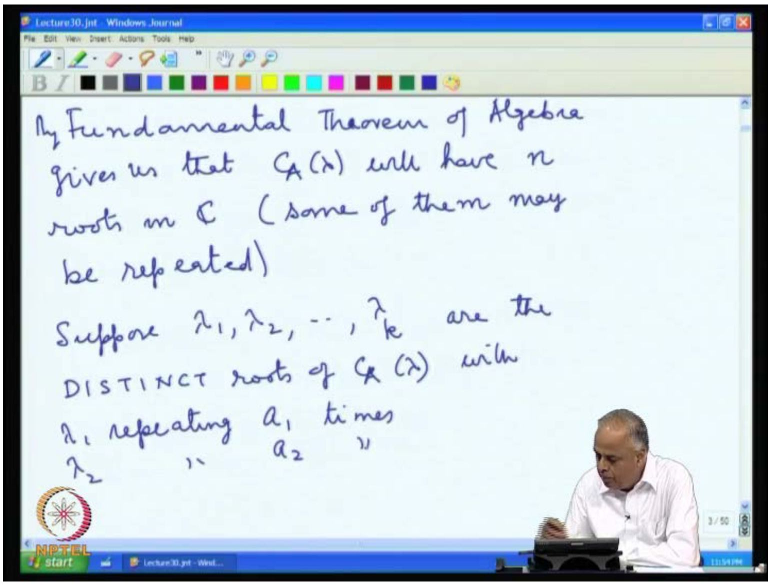

So, suppose the k distinct roots, suppose lambda 1, lambda 2, lambda k are the distinct root of C A lambda with lambda 1 repeating a 1 times.

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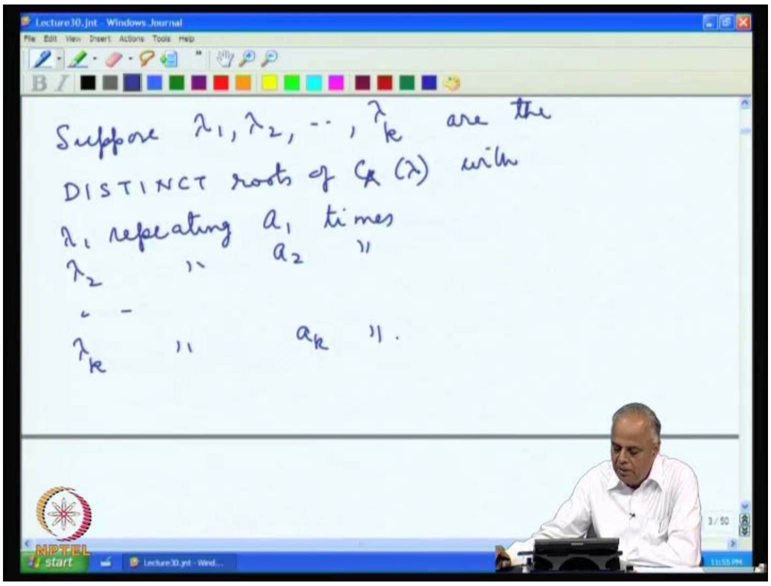

Lambda 2 repeating a 2 times and so on, lambda k repeating a k times, what does this mean?

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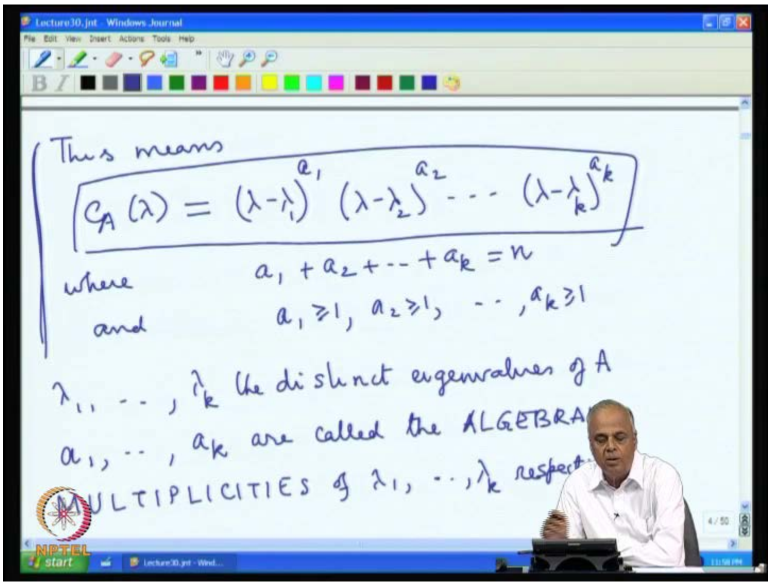

This means, the polynomial C A lambda can now we a factor, at the root lambda 1 in appearing a 1 times, so lambda minus lambda 1 to the power of a 1 is the factor, lambda minus lambda 2 to the power of a 2 is the factor, lambda minus lambda k to the power of a k is the factor. And this exact all the factors, because lambda 1, lambda 2, lambda k are the only roots, and since the polynomial of degree and n roots, where we have a 1 plus a 2 plus a k is equal to n.

And since lambda 1 is the roots, lambda minus lambda 1 must be a factor of C A lambda, so a 1 must greater than or equal to 1, a 2 must be greater than equal to 1, a k must be greater than or equal to 1. Therefore, if the lambda 1, lambda 2, lambda k are the distinct roots, and the multiplicity are a 1, a 2, a k are the reputation a 1, a 2, a k, then the characteristics polynomial has the standard factorization.

And this lambda 1, lambda 2, lambda k are now distinct Eigen values, they are the distinct Eigen values of A, and this reputation are called the algebraic multiplicity of this Eigen values, a 1, a 2, a k are called the algebraic multiplicity. That is the multiplicity root of the polynomial, algebraic multiplicity of lambda 1, lambda 2, and lambda k respectively.

So, therefore, given the matrix A we have our complete picture of this Eigen values, we first construct the characteristics polynomial. Then we find the distinct roots, then we find the multiplicity, then we have all the Eigen values, lambda 1 will be an Eigen value of (()) k 1 times, lambda 2 would be an Eigen values occurring a 2 times, lambda k will be an Eigen value occurring a k times. a 1 plus a 2 plus a k will be n and each one of the (()) greater than or equal to 1. So, this is the standard structure of characteristics polynomial that we will (()) consider. So, will follow the notation whenever the life C A lambda in this form, the really mean that this are lambda 1, lambda 2, lambda k are distinct and this are the multiplicity and so, and so forth. This is the standard notation that will follow from no what, let look at some symbols examples.

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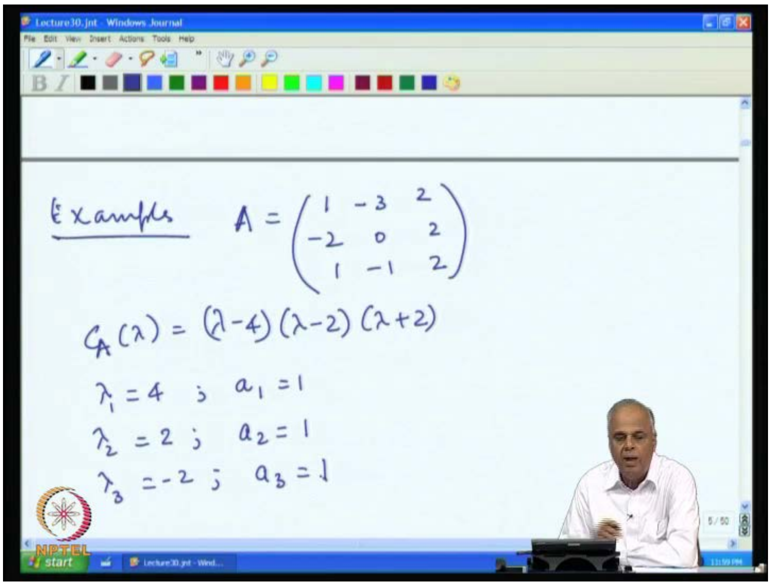

This are the same example, that we have seen before in this context, will now again look at it, let us take the matrix A to be 1 minus 3 2 minus 2 0 2 1 minus 1 2. In the last lecture we found that C A lambda is lambda minus 4 into lambda minus 2 into lambda plus 2. So, what are the Eigen values here, lambda 1 equal to 4, lambda 2 equal to 2, lambda 3 equal to minus 2, and multiplicity of 4 is 1, because the power lambda minus 4 to the power of 1 is the factorization. Similarly, the algebraic multiplicity of lambda 2 is 1 and algebra multiplicity lambda 3 is 1.

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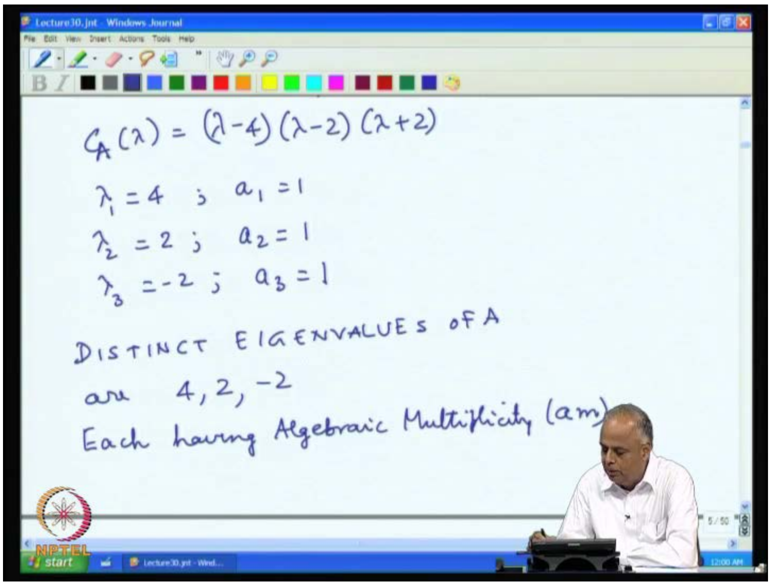

So, this distinct Eigen value values are 4 2 minus 2, so the distinct Eigen values of A or 4, 2, minus 2, each having algebraic multiplicity 1, each having algebraic multiplicity from now 1, we will write a m for algebraic multiplicity 1, so each us algebraic multiplicity 1.

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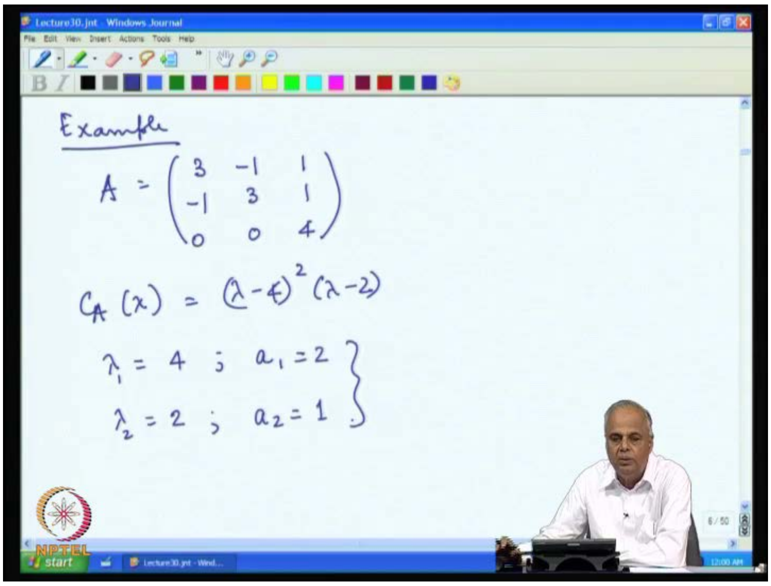

Let us look at another example, A to be 3 minus 1 1, minus 1 3 1, 0 0 4, in the last lecture again we found that the characteristics polynomial was lambda minus 4 square into lambda minus 2. Now, we find that there are 2 distinct Eigen values, lambda on equal to

4 and lambda 2 equal to 2, and the multiplicity of the Eigen value is 4 is 2, because lambda minus 4 to the power of 2, and the multiplicity of the Eigen value lambda 2 is 1. So, thus we have 2 distinct Eigen value here, one of the one of them has algebraic multiplicity 2 and the other one algebraic multiplicity 1.

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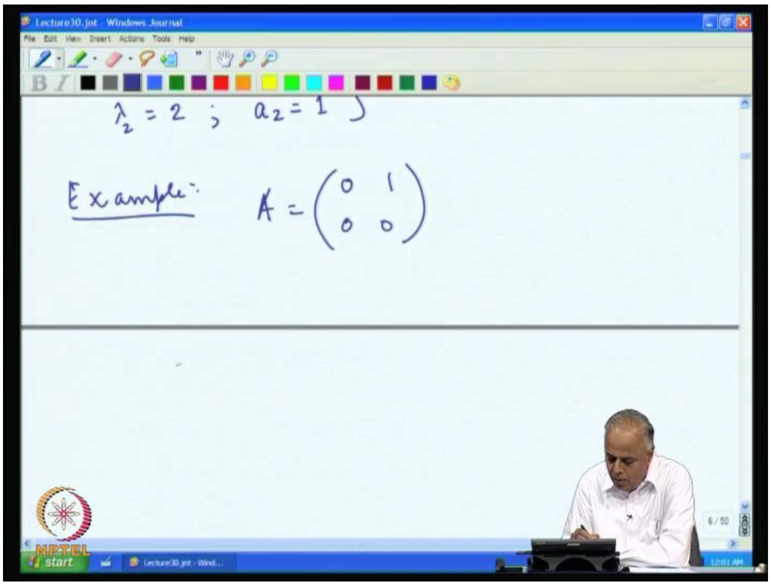

Let us look at another example, a simple example A equal to 0 1, 0 1 we treat all these as complex matrixes is remember.

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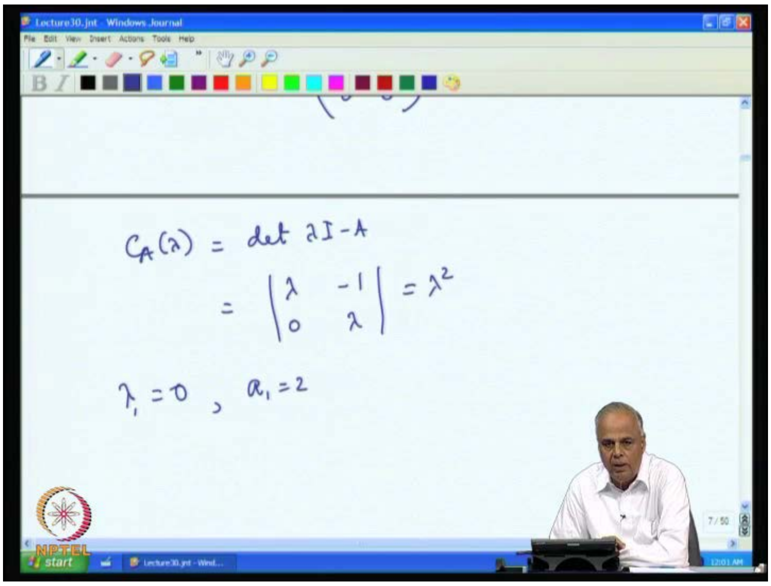

In this case we have the C A lambda between the determine the lambda I minus A, which is lambda minus 1 0 lambda, which is lambda square. And therefore, there is only on Eigen value lambda 1 is 0, and it is multiplicity is 2. So, here is an example, there we have only 1 Eigen value and this multiplicity is 2, algebraic multiplicity is 2.

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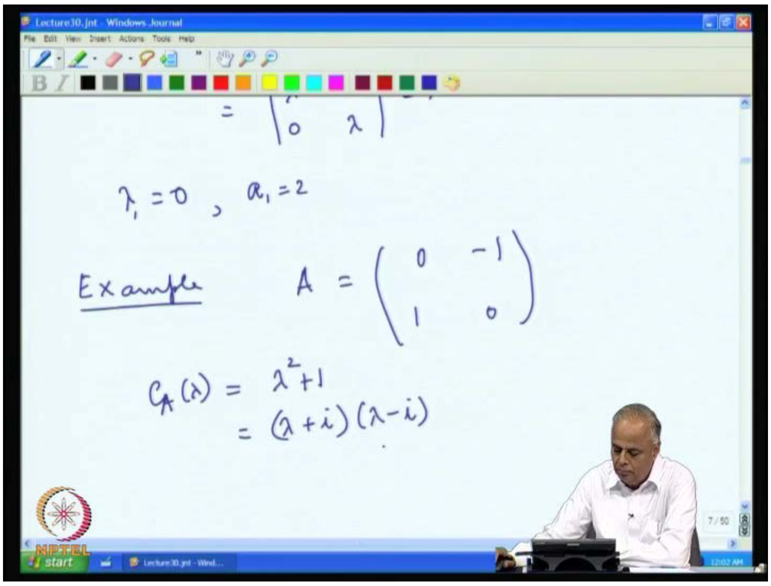

Another example simple again, which we are seen before, 0 minus 1, 1 0, we have characteristics polynomial, as we saw in the last lecture is lambda square plus 1, which can factor as lambda lambda plus i into lambda minus i. We find now, even through the matrix is real; we end up with complex root.

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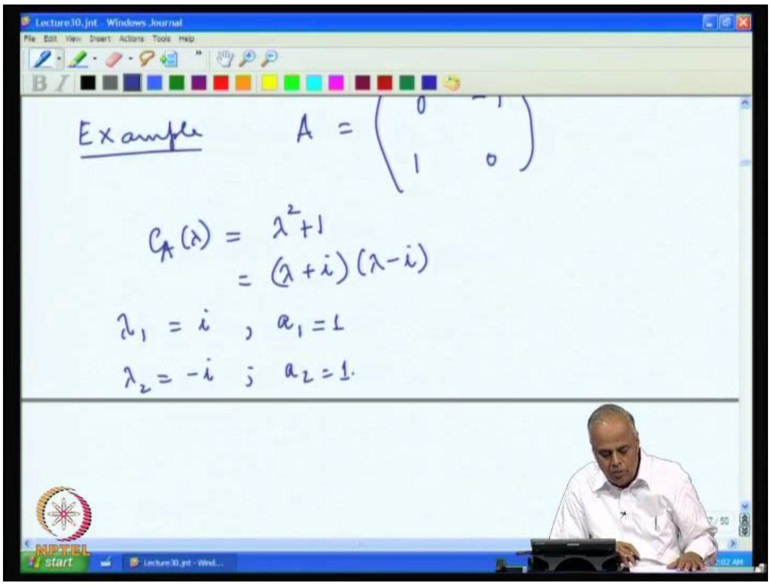

The 2 roots are the 2 Eigen values i and minus i, wrote they are in the conjugate text, because the matrix is real, whenever the complex roots occur, they must occur in conjugate pairs. In the algebraic multiplicity of the Eigen values i is 1, and Eigen value minus i is 1, so thus we have to complex roots both algebraic multiplicity 1 and 1 (()).

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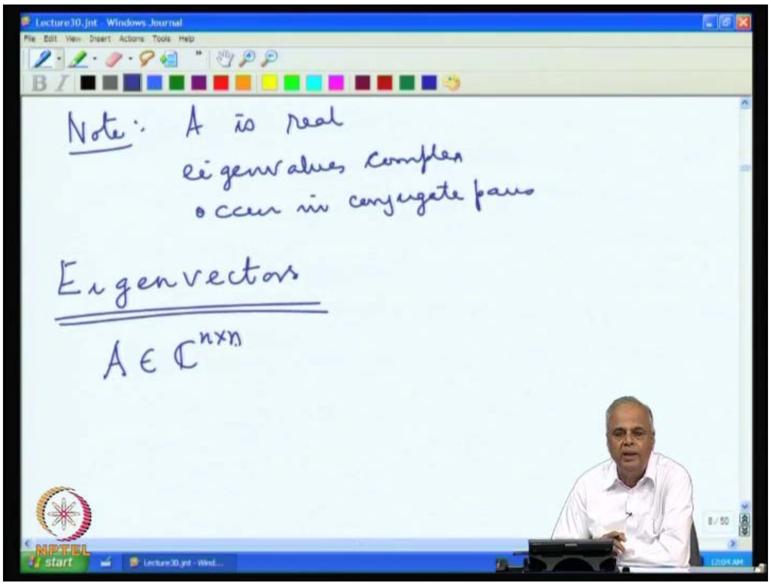

So, here note A is real, Eigen value is complex occur in conjugate pairs, so the Eigen value occur in conjugate pair. Whenever, we are matrix real matrix and it has a conjugate it has the Eigen value, which is complex, the complex Eigen value must always appear in conjugate pairs. So, now we have a fair idea of the Eigen values, remember such for answer to the question of the diagnosable, depending finding this Eigen pair n of there.

Now, in the Eigen pair, the pair two things involve, the first part of the pair is number, which is the Eigen value; now we are seen the analysis of Eigen value, in order to such for this Eigen values, you construct the characteristic polynomial, which determine lambda n minus a.

Then we go find it roots, then including the multiplicity they are provide you are n Eigen value that you are seeking part, may be this Eigen value are complex, and the matrix is real, and if the by chance it by complex Eigen values, they will at occur in conjugate pairs. So, now having got fair idea of this Eigen values, we now go and look at what and where we should such for Eigen vectors. So, our next search or next analysis will be the

Eigen vectors search. So, let us start with the matrix A, which may real or complex, so general we write in C n n, it could be real also, because any real matrix (()) of complex matrix, so consider n the n matrix.

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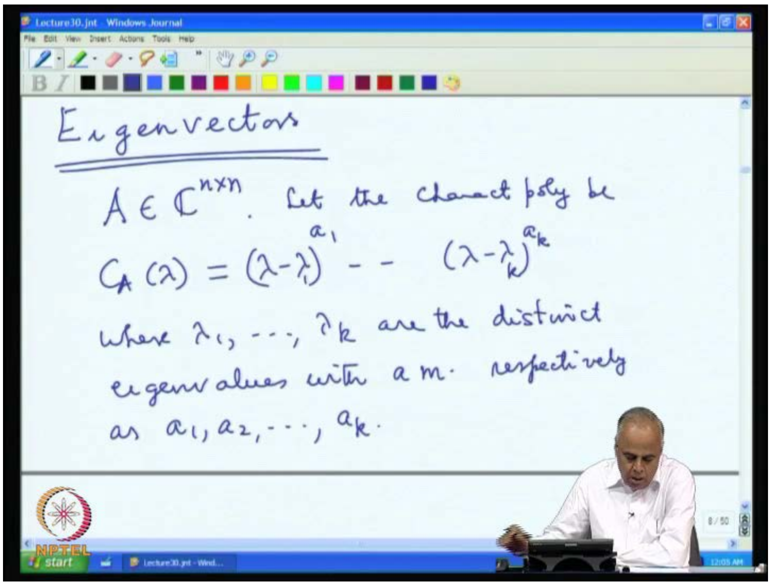

And look at its characteristics polynomial, as the explain above, if lambda 1, lambda 2 lambda k are the distinct roots with algebraic multiplicity a 1, a 2, a k, then the characteristic polynomial can be factor of this. So, let the characteristic polynomial, this where lambda 1, lambda 2, lambda k are the distinct Eigen values. Now, with algebraic multiplicity, will write a m for algebraic multiplicity, respectively us a 1, a 2, a k, now all are search for Eigen vector should be, to find Eigen vector for each one of this Eigen values.

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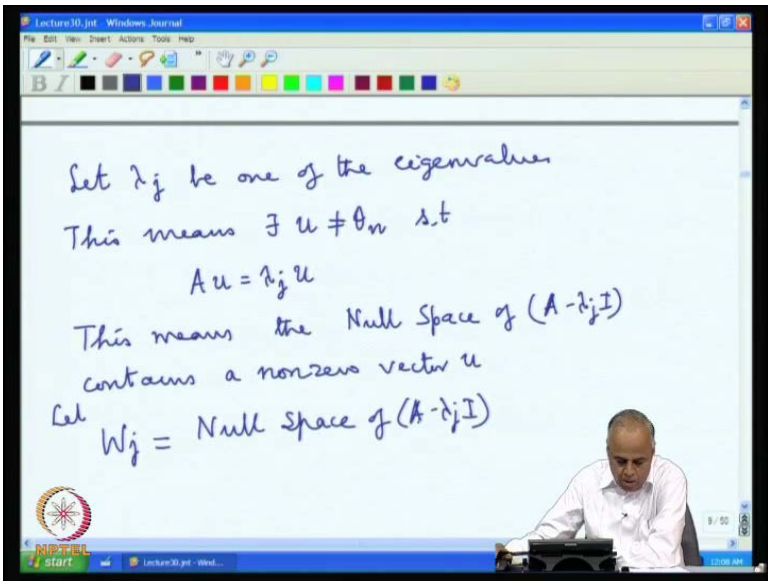

Now, let us consider any one of them, so let lambda by j be one of the Eigen value, now what does it mean to say that, it is an Eigen value, it means it should have a vector u associated with, which is different from 0 for that A u is lambda j u. This means, there excites u not equal to theta m such that, A u equal to lambda j u, what is mean by saying that something is Eigen value, because it is an Eigen value means, determine of lambda j minus a is the 0, the determine 0 lambda j minus a is not in (()).

And therefore, this homogenous system must have a (()) real solution, all this we have specify previous lecture. So, therefore, there is the vector u, it is different 0 for that A u equal to lambda j u, this means the null space A minus lambda j I this matrix, A is an n by n matrix, I is the n by n matrix. And therefore, A minus lambda j I is the n by n matrix, the null space of the n by n matrix A minus lambda j I contains a non 0 vector u; this means the null space is not prevent, let us denote by W j, so let W j the null space of A minus lambda j I.

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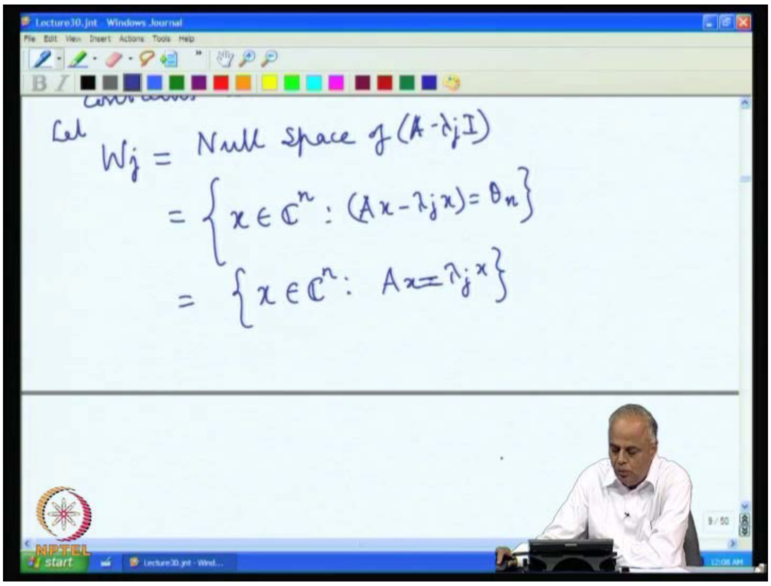

What does this mean, this consists of all those vectors in C n such that, A x minus lambda j x equal to theta n, that is the set of all vectors in C n such that, A x equal to j x. And the important thing is that this W j has a non 0 vector u and therefore, W j is non trivial, dimension of the W j is the greater than or equal to 1.

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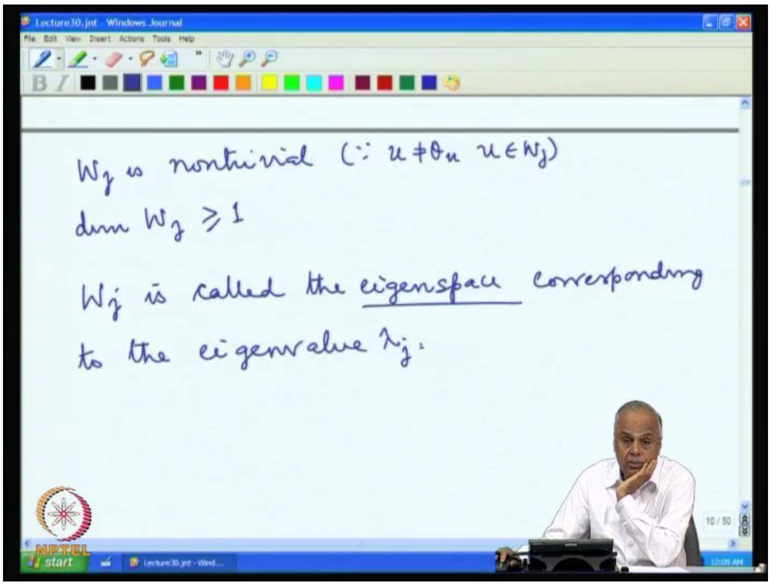

W j is the non trivial, because u is not equal to theta m u belongs to W each other, the u that we observed here, the excite the u that A u equal to lambda j u. Now, W j is not prevail and therefore, dimension of W j is greater than or equal to 1, this W j is called the

Eigen space, corresponding to the Eigen value lambda j u. So, W j is called the Eigen space corresponding to the Eigen value lambda j; if you look at W j it contains, because is the sub space, the null space of the any matrix sub space, and because it has sub space contain zero vector, that we are seen that will contain the vector non zero also, every non zero vector in W j is an Eigen vector corresponding to lambda j.

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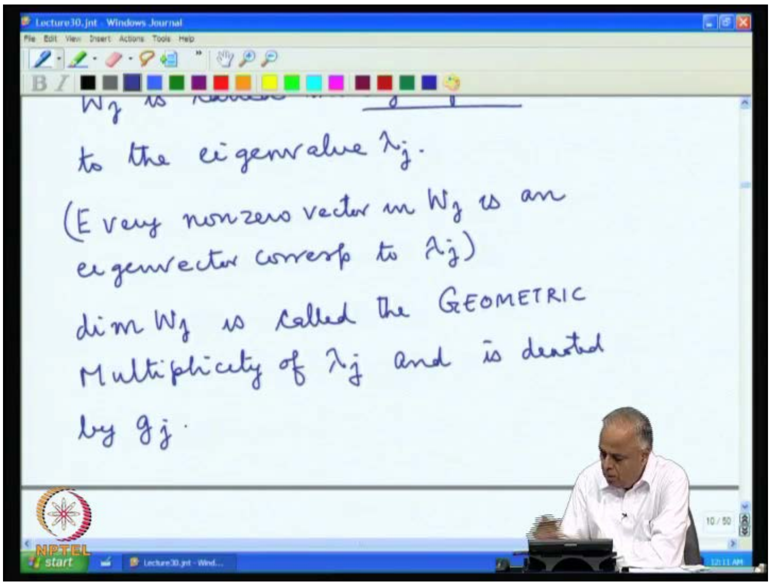

Every non zero vector in W j is an Eigen vector corresponding to the Eigen value lambda j. So, now we have for every Eigen value a corresponding Eigen space called W j, and this Eigen space every non zero vector, in this Eigen space is an Eigen vector corresponding to W j. And dimension of W j is called the geometric multiplicity geometric multiplicity of the Eigen value lambda j and is denoted by g j.

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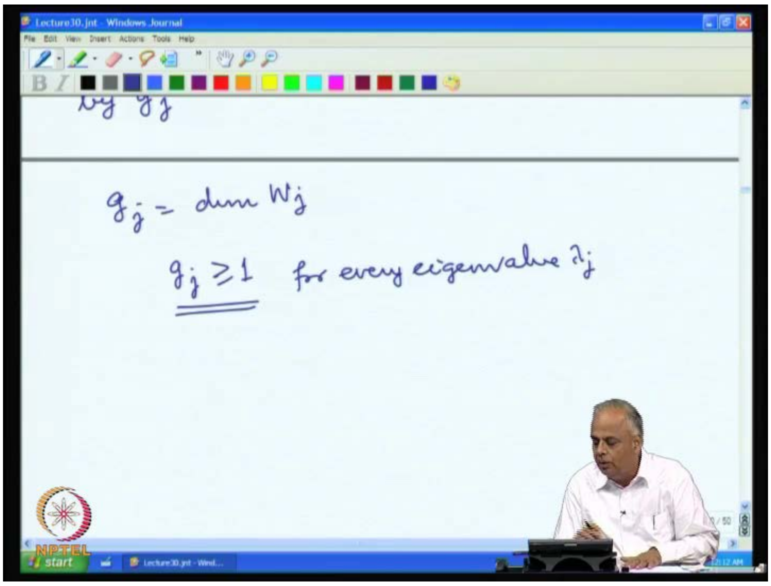

So, g j is the dimension know W j, now W j contain non zero vector, we are observed above the W, the dimension of W j is greater than or equal to 1, so g j is greater than or equal to 1 for every Eigen value lambda j. Will denote the geometric multiplicity from know on has g m for a m will denote algebra multiplicity, g m will mean geometric multiplicity.

So, put every Eigen value now, we have two numbers, two integer, positive integer associated one is a j, which is algebraic multiplicity, it is multiplicity of the root of the characteristics polynomial. And the g j, which is geometric multiplicity, it is dimension of the Eigen space corresponding to Eigen value lambda j.

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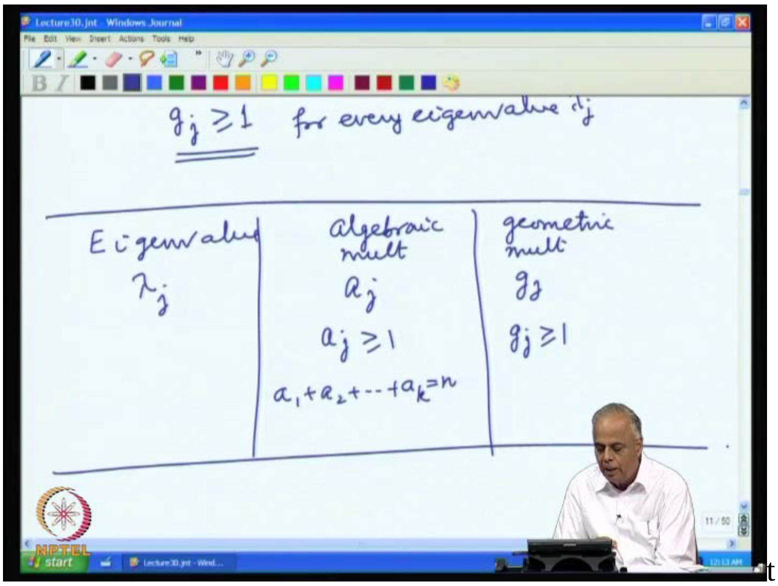

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So, therefore, if you have Eigen value, lambda j corresponding to that we have algebraic multiplicity a j, corresponding to that also have geometric multiplicity g j, what we know is a j is greater than or equal to 1, because we must appear at least 1 of root the characteristics polynomial. What we are observe now, g j is greater than or equal to 1, what we also had was the some of all this multiplicity has root must add up to n.

a 1 plus a 2 plus a k n, we do not know what our g 1 plus g 2 plus g k is equal to n, all the know no so for, the g j is must be greater than or equal to n, greater than or equal to 1, each 1 of them must be at least of dimension one. Now, the question is at the movement we make the remark.

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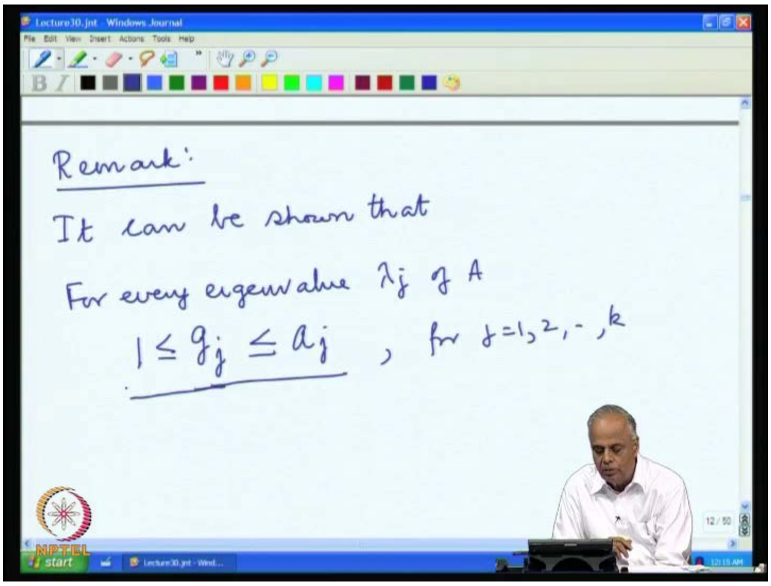

We will prove this statement little later, we will need a little more material for that at to be develop, but we shall now observe, it can be shown of the movement that not prove it, we will prove it little later. It can be shown that for every Eigen values lambda j of A, the geometric multiplicity the corresponding to lambda j, we know it is at least 1, we are just observe that g j is grater than or equal to 1, so is less than or equal to g j, and this will be at most the algebraic multiplicity.

For j equal to 1, 2 k, we are assuming the lambda 1, lambda 2, lambda k for the distinct Eigen value, a 1, a 2, a k, are the algebraic multiplicity, g 1, g 2, g k, are the geometric multiplicity. Then any Eigen value, the geometric multiplicity is at least 1 at most the algebraic multiplicity, at the movement we are not going to prove the statement, we know this part that 1 is less than or equal to g j, the part that g j less than or equal a j, we shall look at the (()).

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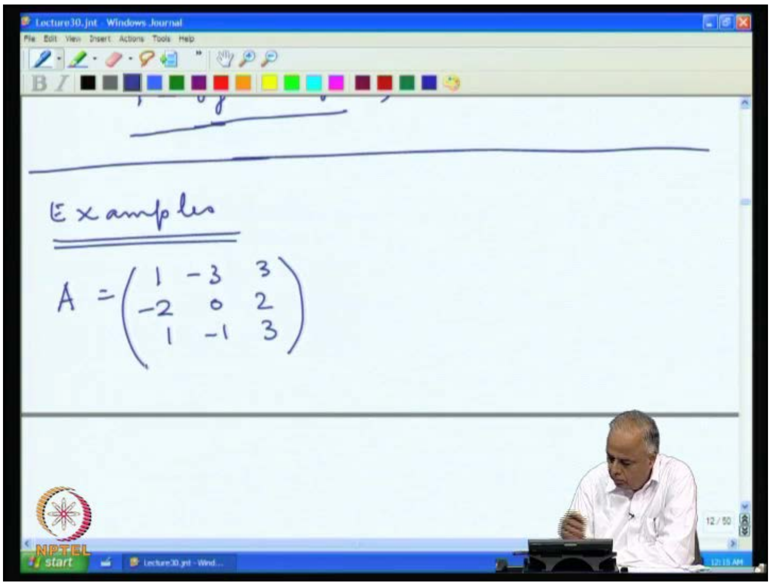

Let us now, look at some examples (No audio from 23:41 to 23:50), take the matrix A this is again we keep look at the same example, which we seen before minus 2 0 2, 1 minus 1 3.

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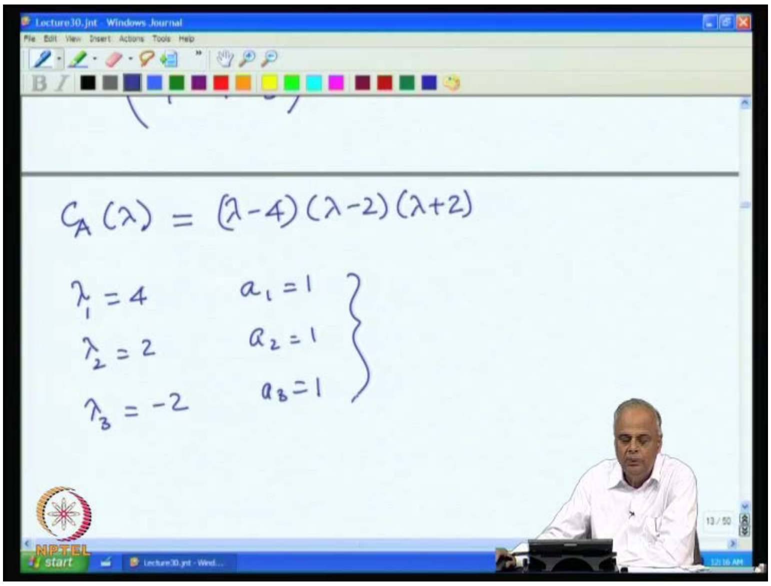

Now, what we have seen in before, that the characteristic polynomial is lambda minus 4 into minus 2 into lambda plus 2. And therefore, there are three Eigen values lambda 1 equal to 4 with algebraic multiplicity 1, lambda 2 equal to 2 with algebraic multiplicity 1, lambda 3 equal to minus 2 with the algebraic multiplicity 1; these are the three distinct Eigen values.

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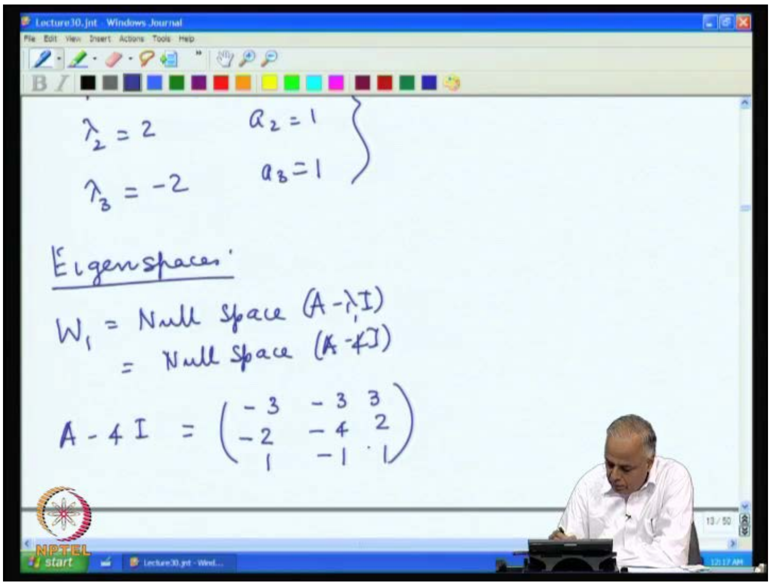

Now, what are the Eigen spaces, the first one W 1 is the null space of A minus lambda 1 I, which is the null space of a minus 4 I. Now, what let us find this out, so what is A minus 4 I, A minus 4 I from the matrix A in the diagonal, we have to subtract minus 4.

Then we do that, we get the matrix A minus 4 I as minus 3 minus 3 3, minus 2 minus 4 2, 1 minus 1 1, that is you take this matrix A and subtract 4 from the diagonal, because taking minus 4 I, this diagonal become minus 3, this diagonal become 0, the third diagonal will become minus 1 third diagonal become minus 1, you call you subtract a 4 from it.

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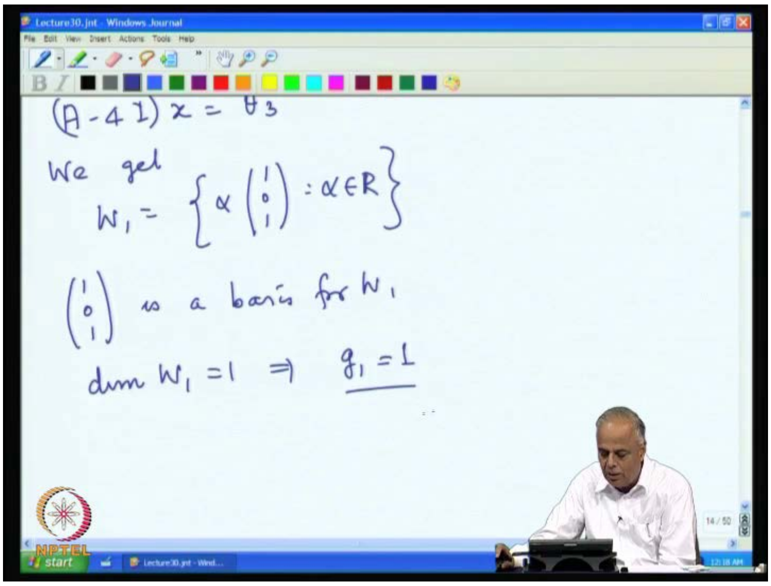

And if we solve the null space A minus 4 I x equal to theta 3, we get W 1 consists of all vectors of the form alpha in to 1 0 1, said that alpha belongs to R. Therefore, 1 0 1 is the basis for W 1 and therefore, dimension of W 1 is 1, and that is what the geometric matrix g s. So, the geometric multiplicity of the Eigen value is 1, and move that g 1 must be at least 1, because we said the g is greater than or equal to 1 in the also (()) the g j cannot be more a j, in this case a 1 is 1. So, it is cannot be more than 1, it cannot be less than 1 and therefore, it has turned out to the 1.

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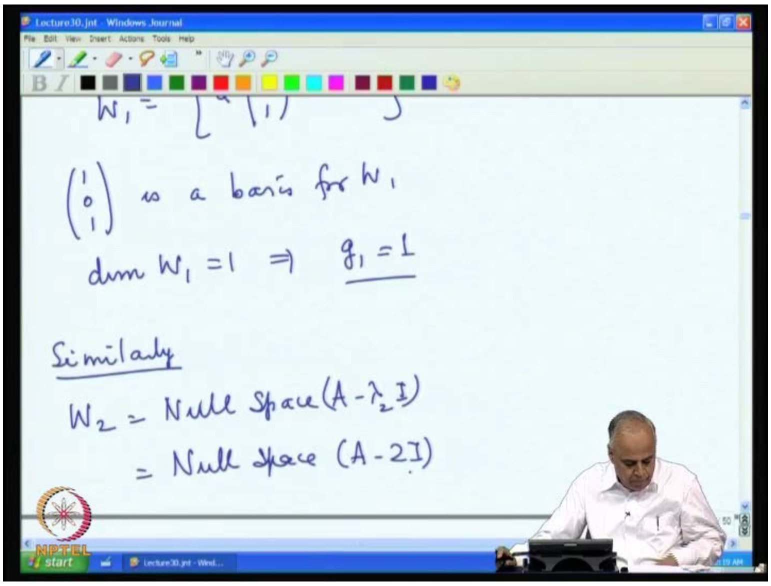

Similarly, W 2 if the null space of A minus lambda 2 I, which is the null space of A minus 2 I.

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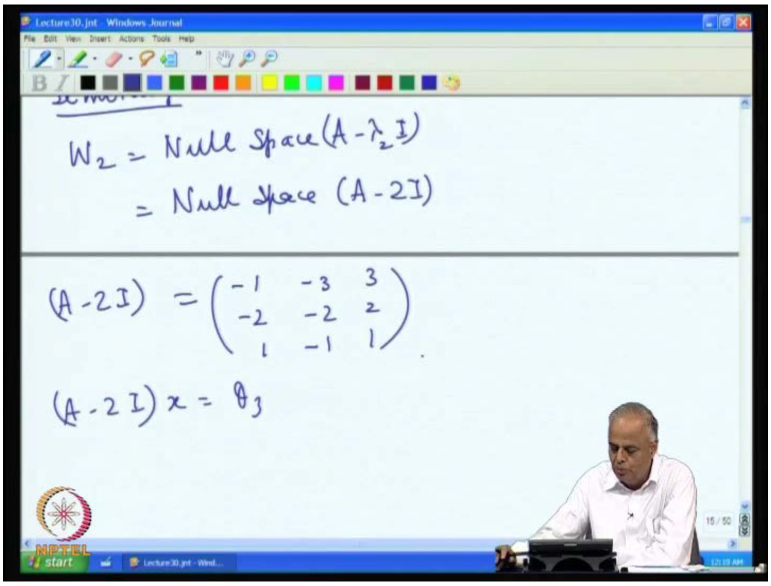

Now, again we have subtract 2 from the diagonal, and so A minus 2 I again we take the matrix A we had here, and subtract 2 from the diagonal, this is the matrix A, minus 2 I we have to subtract 2 from the diagonal, you get the matrix 1 minus 1 minus 3 3, minus 2 minus 2 2, 1 minus 1 1, and we solve the system now, A minus 2 I x equal to theta 3 (Refer Slide Time: 27:45).

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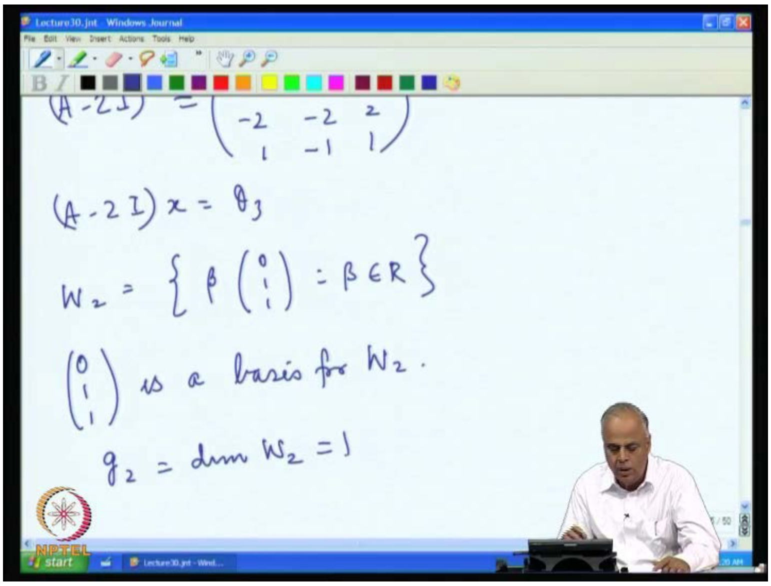

Where a minus 2 I is this, you get the W 2 consists of all vectors of form beta into 0 1 1 where beta belongs to R. And therefore, 0 1 1 is a basis for W 2 therefore, g 2 which is dimension of W 2 this 1, because there is basis consists exactly one vector.

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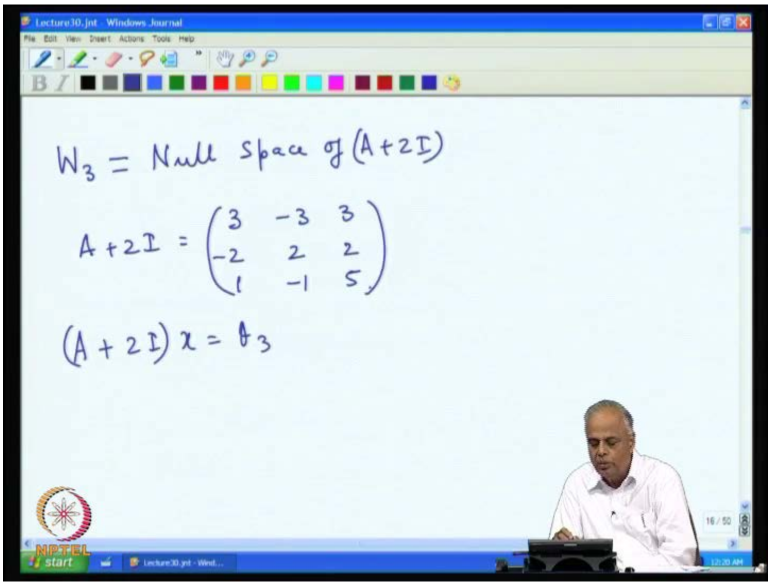

Finally, we find W 3 which is the null space of A plus 2 I, because it is A plus A minus lambda 3 I, lambda 3 is minus 2, so we all A plus 2 I, so A plus 2 I again with the given matrix, it turns out to be, we have to just add 2 to the diagonal, you get this matrix. And therefore, we want to solve A plus 2 I x equal to theta 3, where A plus 2 I this matrix.

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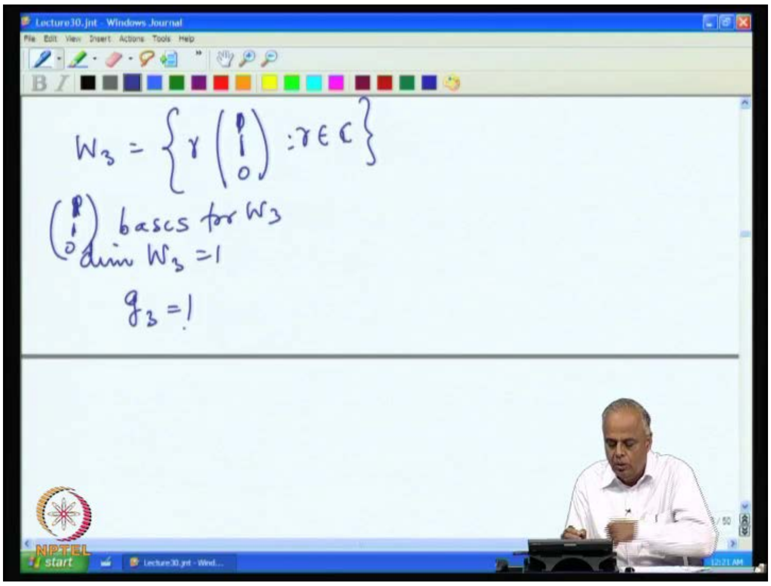

And when we solve this, we get W 3 to be all the solution (()) form gamma into 0 1 of the 1 1 0 gamma belongs to C or r since, we are dealing with r, we can take it us real number also. And therefore, dimension of W 3 is 1, because 0 1 1 0 is the basis, for W 3 and therefore, g 3 equal to 1.

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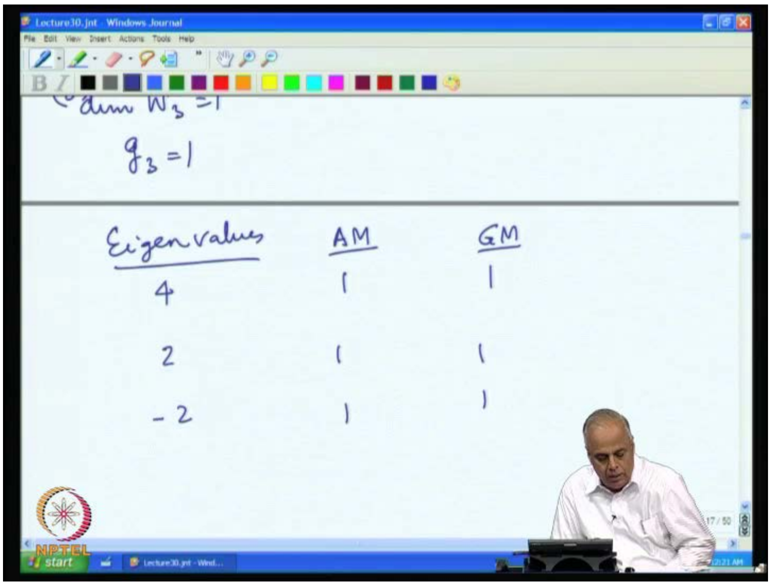

So, in this case, we have the Eigen values 4 2 and minus 2, their algebraic multiplicity 1 1 1, and the geometric multiplicity 1 1 1. So, 3 Eigen values each one of them as algebraic geometric multiplicity equal to 1.

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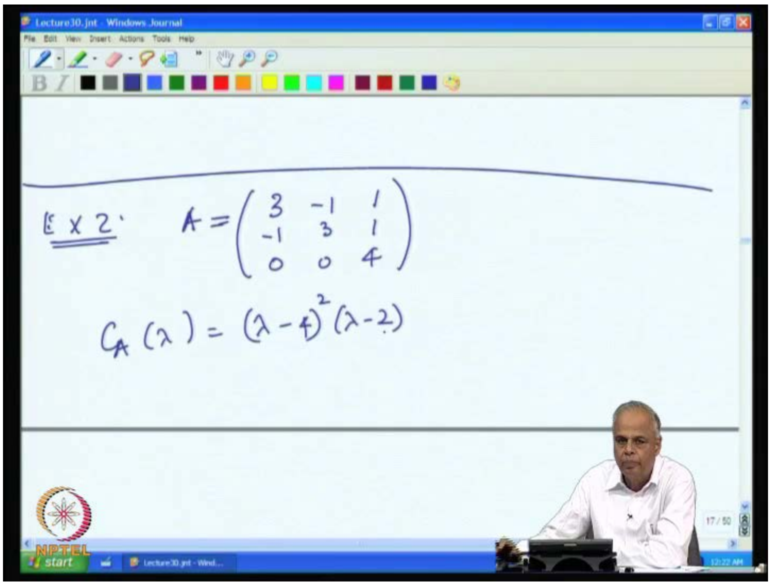

Let us now look at another example, A 3 minus 1 1, minus 1 3 1 and 0 1 4, this is again the matrix (()) we consider in the last lecture. And we found, that the characteristic polynomial was lambda minus 4 squares into lambda minus 2.

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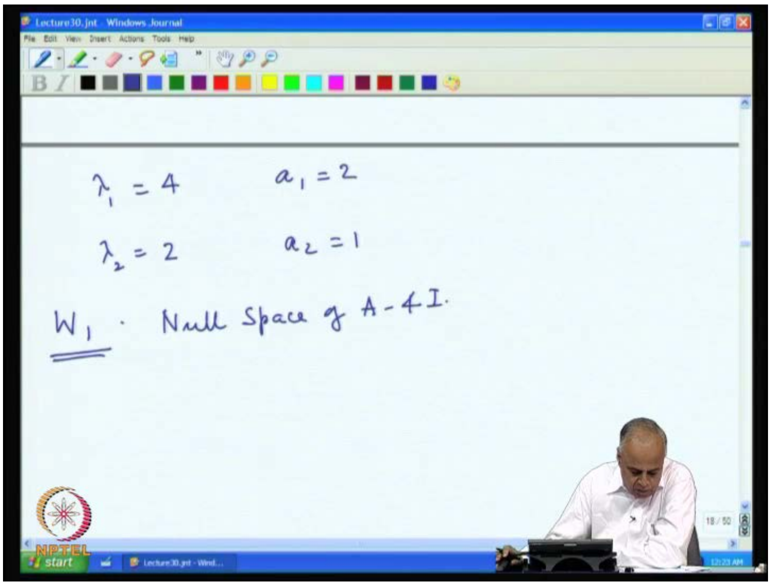

And therefore, there the two distinct Eigen value, lambda 1 equal to 4, lambda 2 equal to 2, but the algebraic multiplicity as 2 and 1 respect, lambda minus 4 square therefore, the algebraic multiplicity is 2. So, now let us find the Eigen space W 1 is the null space of A minus 4 I. So, you have to subtract 4 from the diagonal, and when we do that you get W 1.

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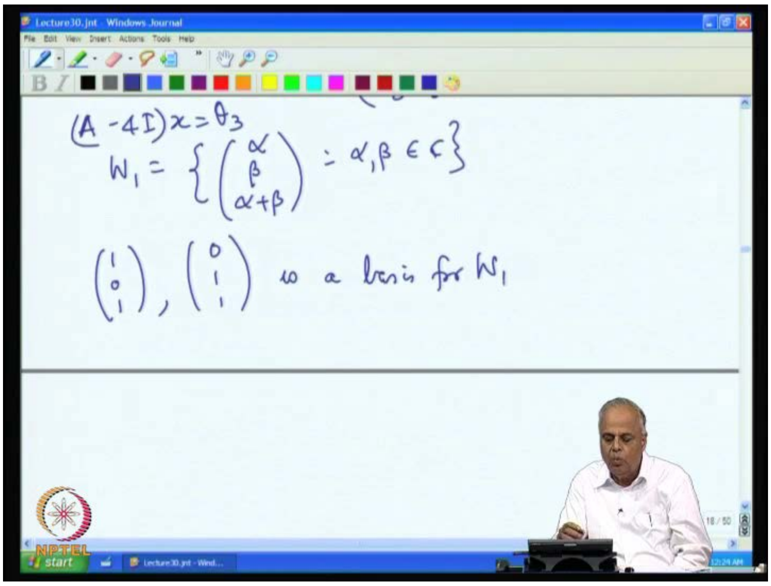

Let us first write a minus 4 I is subtract from the matrix A fore along the diagonal, so get alone the diagonal minus 1 minus 1 and 0, so the matrix becomes minus 1 minus 1 1, minus 1 minus 1 1, 0 0 0. And then we find that, W 1 if you solve A minus 4 I x equal to theta 3, which is the homogenous equation, which can easily solve, we find that the all the solution can be express in the form, alpha beta alpha plus beta, where alpha and beta they have to see.

And now, we find that 1 0 1, and 0 1 1 is basis for W to 1, 1 0 1 is obtain taking the alpha equal to 1, beta equal to 0, 0 1 1 is obtain by taking alpha equal to 0 and beta equal to 1. Since, there is the basis consisting of two vectors dimensionally W 2 is 2 therefore; g 2 is g 1 is 2.

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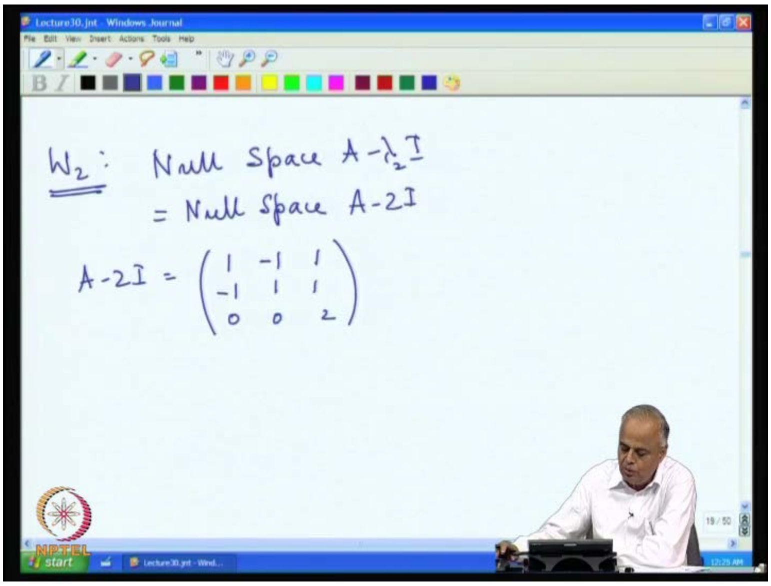

The next Eigen value the next Eigen gives the the Eigen space this is the null space of A minus lambda 2 I, so the lambda 2 is 2 this is the same as null space of A minus 2 I. Now, we have subtract 2 from the diagonal A minus 2 I is the matrix, 1 minus 1 1, minus 1 1 1, 0 0 2, now if we solve this we usually see that the third equation gives x 3 equal to 0, then the first 2 give x 1 equal to x 2.

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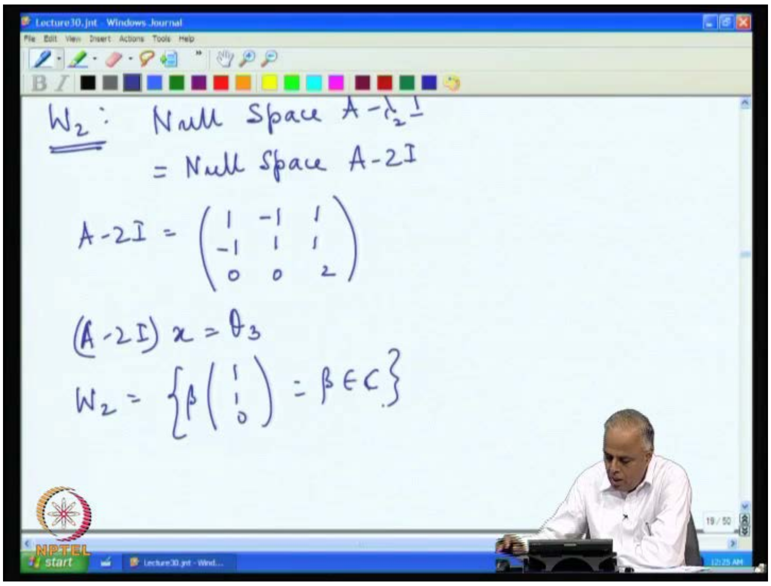

So, therefore, if you solve this we have solve this we get W to be the set of all vectors, which are of the form beta in to 1 one 0 beta belongs to c.

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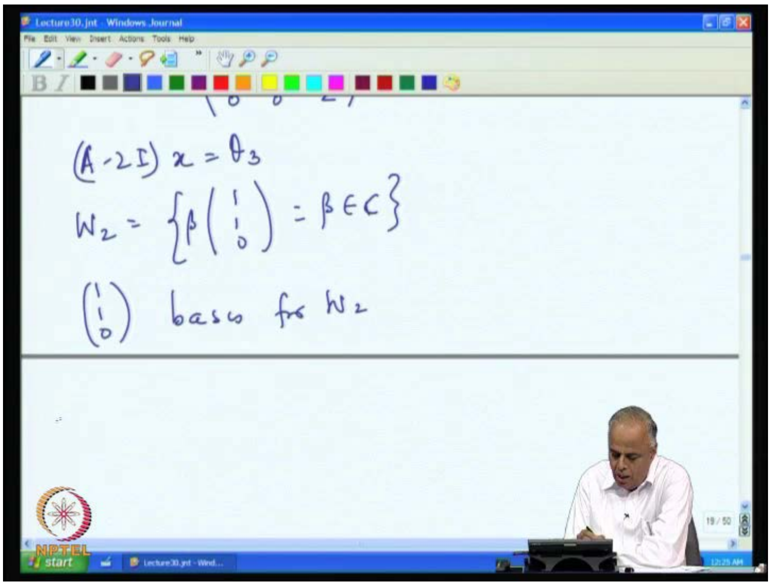

And therefore, 1 1 0 is the basis for W 2.

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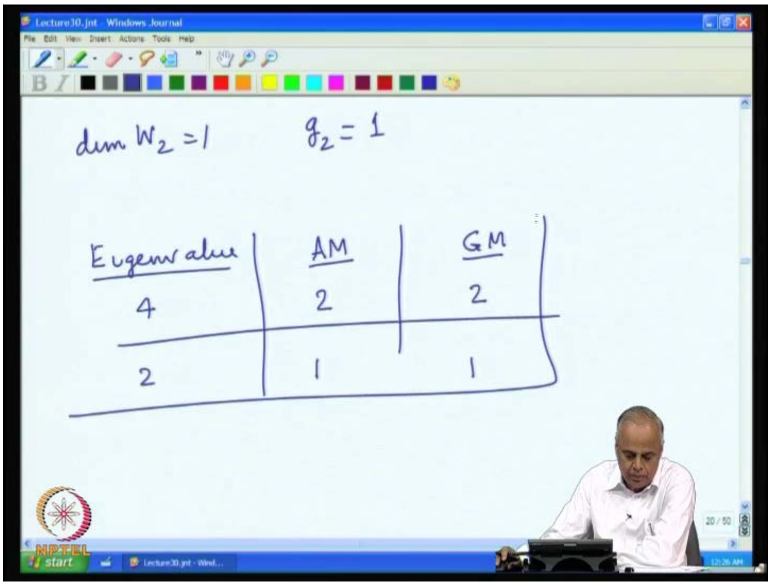

And hence, dimension of W 2 is 1, because we have a basis consisting of 1 vector and therefore, g 2 which is the dimension of W 2, which is geometric multiplicity Eigen value lambda 2 is 1. So, in this case we have 2 Eigen values are 4 and 2, the algebraic multiplicity, the Eigen value 4 as algebraic multiplicity 2; the Eigen value 2 as the algebraic multiplicity 1, the geometric multiplicity where again 2 and 1; so we have this sample.

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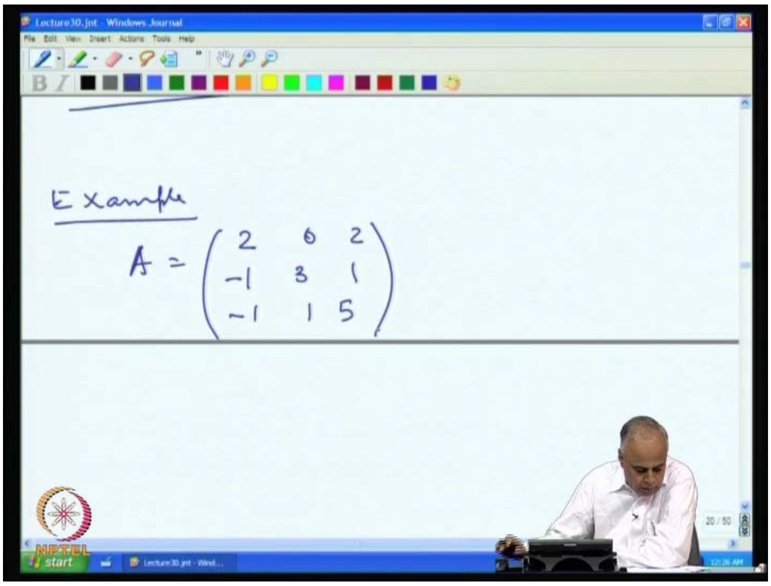

Let us now look at, one more simple example to illustrate, what we are going adding towards examples, consider the matrix A 2 0 2 minus 1 3 1 minus 1 1 5.

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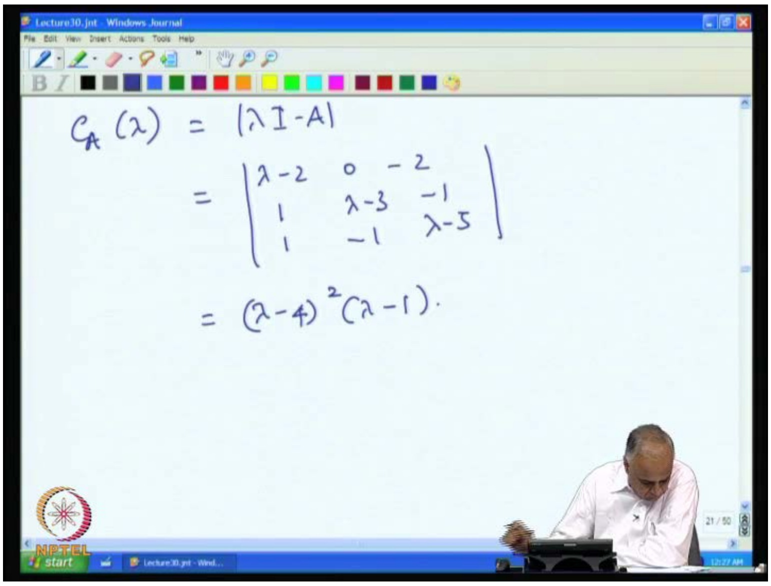

If we now find the characteristic polynomial, it is determine of lambda I minus A, which is lambda minus 2 0 minus 2 1 lambda minus 3 minus 1 one minus 1 lambda minus 5, when we expand this determinant, we get lambda minus 4 square into lambda minus 1 (()).

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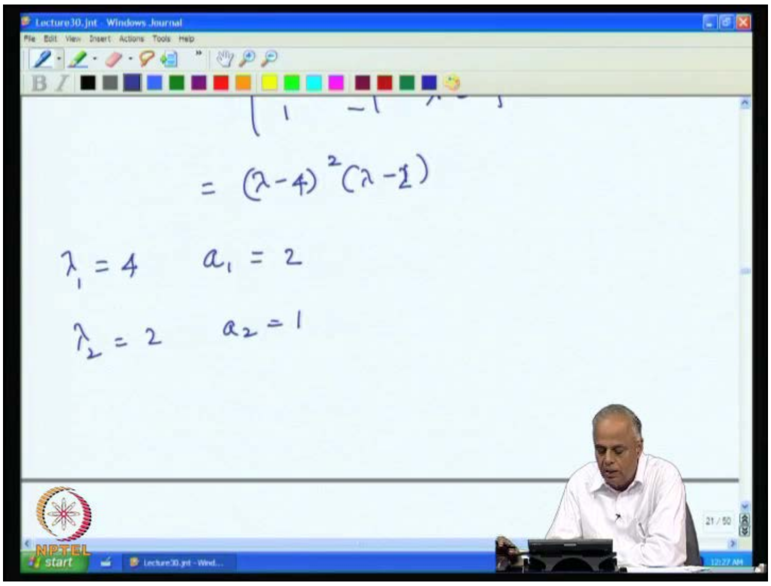

Now, if you look at Eigen values, lambda 1 is 4 is multiplicity 2 lambda 2 is 2 and the multiplicity is 1, the multiplicity is 2 here for lambda 1 equal to 4, because lambda minus 4 square term. So, the root lambda equal to 4 appears twice therefore, algebraic multiplicity is 2.

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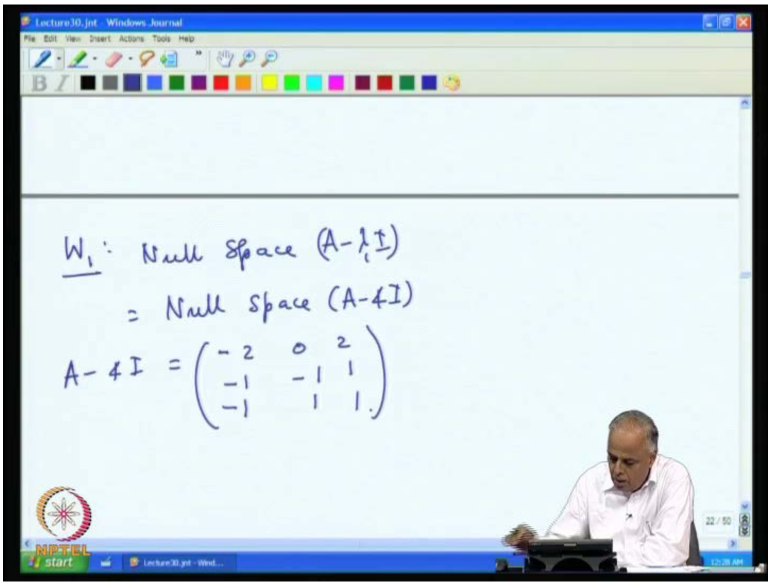

Now, let us again find Eigen spaces as before W 1 will be the null space of A minus lambda 1 I, which is null space of a minus 4 I. That now, what is A minus 4 I, we must remove 4 from the diagonal entry of the given matrix, the given matrix is here, so if you subtract 4 from the diagonal, the diagonal change 2 minus 2 minus 1 and 1. So, we get A minus 4 I as minus 2 0 2 minus 1 minus 1 1, then minus 1 1 (Refer Slide Time: 36:48).

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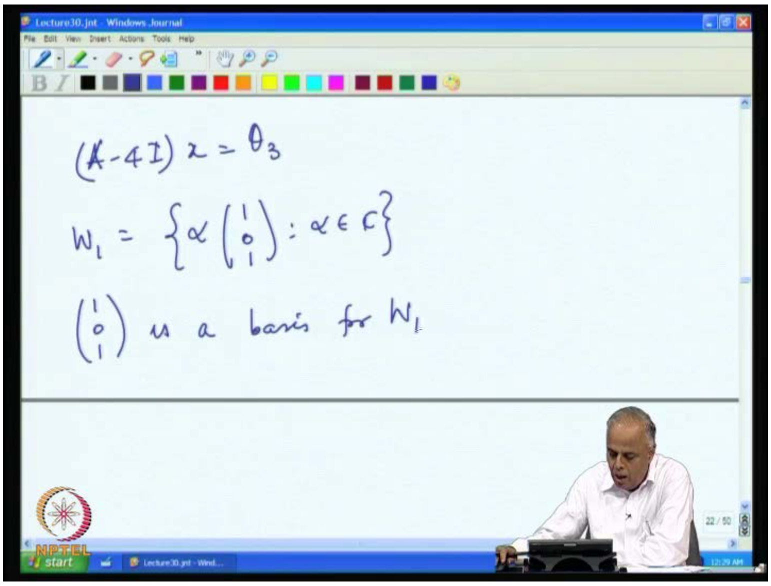

So, if you now solve, the system A minus 4 I x equal to theta 3 we find that, W 1 is consists of all vectors of the form and therefore, 1 0 1 is a basis for W 1.

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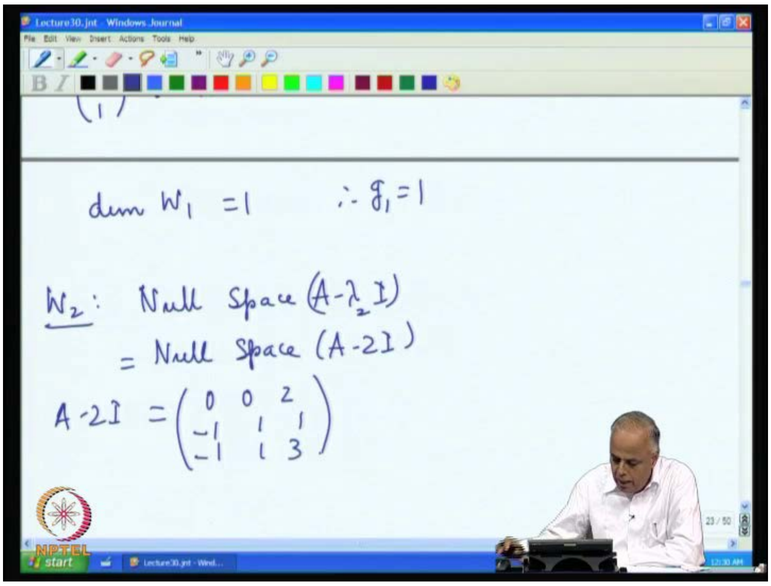

And therefore, the dimension of W 1 is occurring 1 and therefore, g 1 is 1, the geometric multiplicity, the dimensional W 1 and therefore, it is equal to 1. Let us now find W 2, the null space is the Eigen space, corresponding to the Eigen value lambda 2, which is the null space A minus lambda 2 I, since the lambda 2 is 2, this is the null space of A minus 2 I. Now, I have to subtract to find A minus 2 I, have to subtract 2 from the diagonal of the diagonal given matrix, when I do that I get this matrix.

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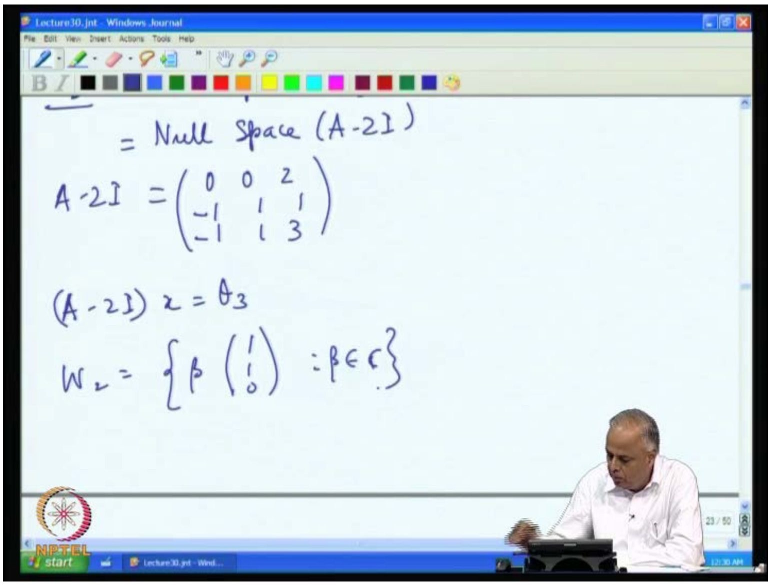

And then now, we solve for the system A minus 2 I x equal to teta 3 to get W 2, as the set of all vectors of the form beta in to 1 1 0, where beta can (()).

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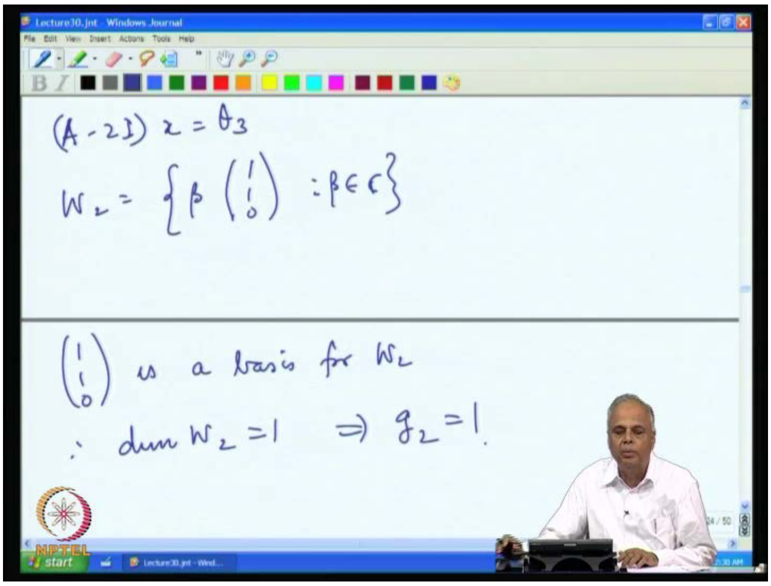

Now, 1 1 0 is a basis for W 2 therefore, dimension W 2 is 1, that means the geometric multiplicity is 1, because geometric multiplicity is the dimension of the null space.

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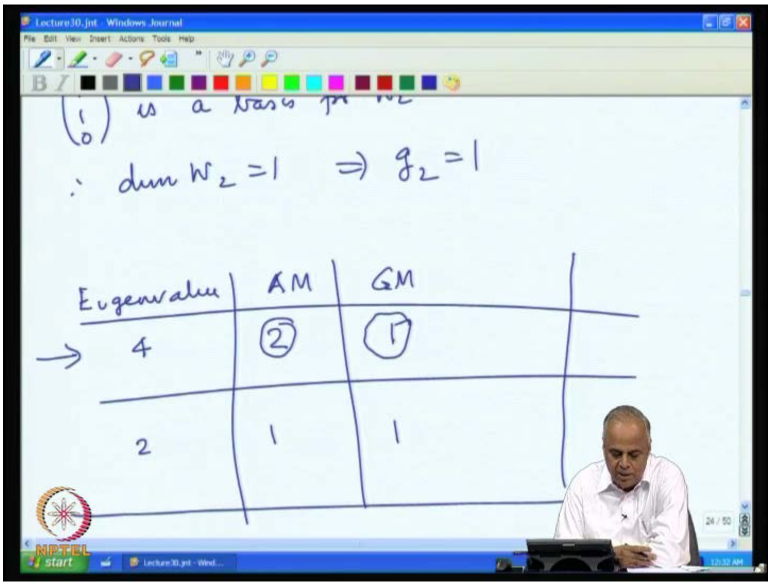

So, what we have in this example, we have in this example, the Eigen values as 4 and 2, the 2 Eigen value vector 4 and 2; their algebraic multiplicity 2 and 1, and the geometric multiplicity all where 1 and 1. Now, where is an example, where one of the Eigen value namely, the Eigen value 4 has algebraic multiplicity 2, but the geometric multiplicity which is smaller than this.

This is what we said that the always have, the geometric multiplicity is either equal to the algebraic multiplicity are smaller than the algebraic multiplicity. In fact, the entire question of the A is diagnosable or not dependents on, weather geometric multiplicity falls short at any stage, if it falls short of the algebraic multiplicity, we will end of the difficulty of the diagnosable. In fact, we put all shorts even for 1 Eigen value even by 1, suppose, there are 4 Eigen value, for 3 Eigen value the algebraic multiplicity is equal to geometrically multiplicity.

But, for 1 of the Eigen value, the geometric multiplicity is 1 just less than algebraic multiplicity then (()) will be fake will see like facts will be later. So, the relationship between the geometric multiplicity and algebraic multiplicity, we know that both are at least 1, the geometric multiplicity at most this algebraic multiplicity; it can never be more than, we have not prove it, we will prove it little later. Now, there is one more property, did will take now, which will not prove, we will again prove this later.

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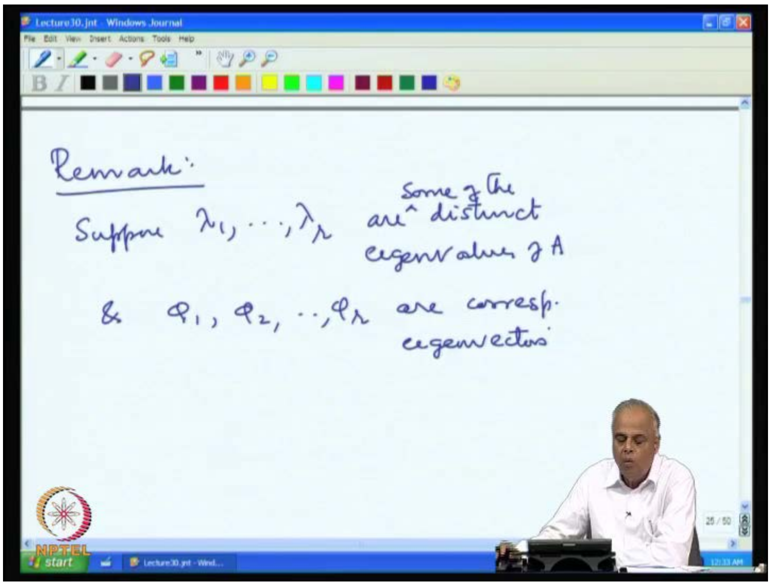

Remark, suppose lambda 1, lambda 2, lambda r are distinct Eigen values some of the, we may not later all of them, some of the distinct Eigen values of A and pi 1, pi 2, pi r are corresponding Eigen vectors.

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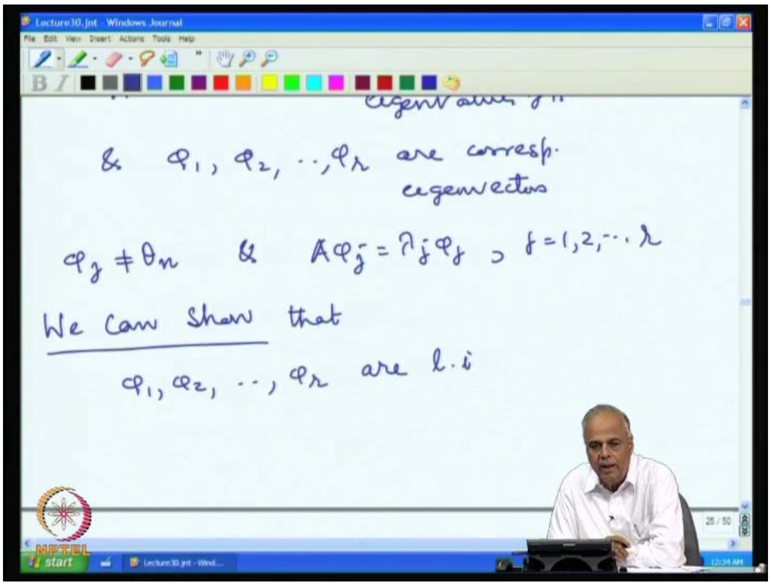

What does that mean, this means pi j are not 0 and A pi j is equal to lambda j pi j for j equal to 1, 2, so we are considering some r distinct Eigen values and we are looking at Eigen vectors corresponding to them. Then we can show again at the movement we are not going to prove it, we shall prove it every later, we can show that pi 1, pi 2, pi r are linearly independent.

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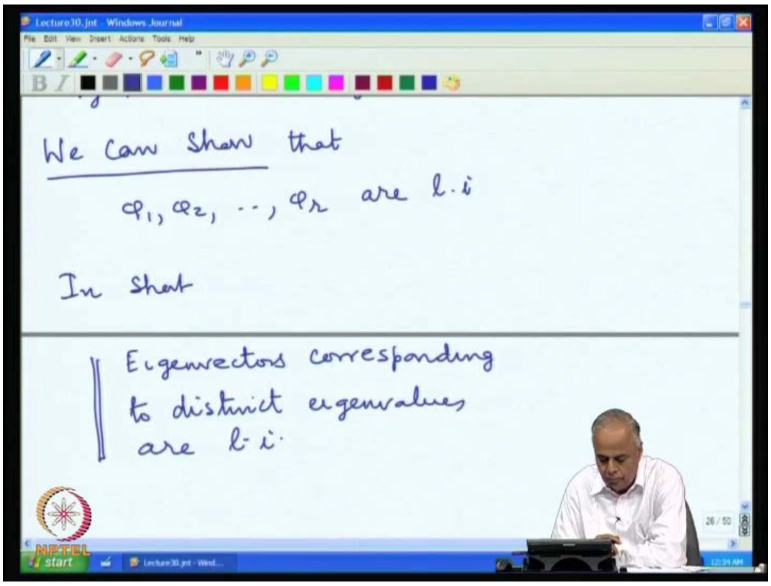

What is that mean, it can be simply state at that Eigen vectors corresponding to distinct Eigen values are linearly independent. So, in short what we are claiming is this Eigen vector, corresponding to distinct Eigen values are linearly independent Eigen vector corresponding distinct Eigen values these are linearly independent, now this is what you going help us to see whether, we are going to have (()) are not.

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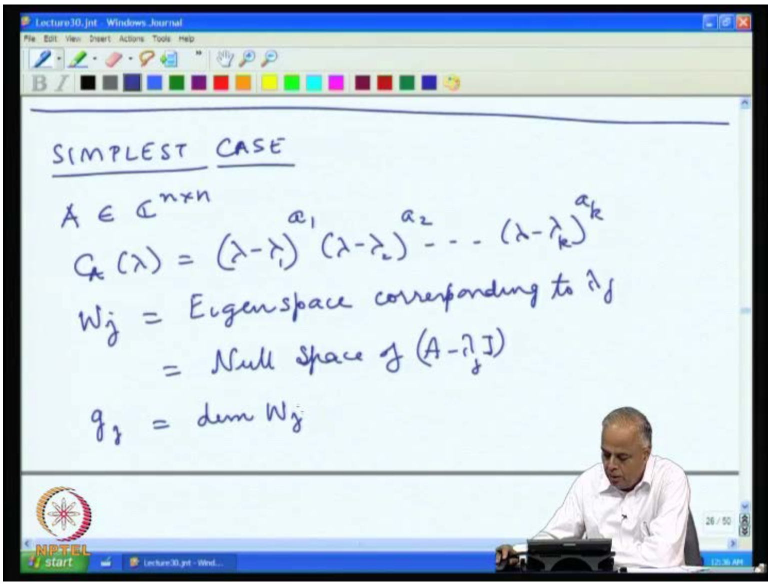

So, now let us case this simplest case, why we will see why this is simplest case, the simplest case is A is complex matrix, we are the characteristics polynomial lambda minus lambda 1 to the power of a 1 lambda minus lambda 2 to the power of a 2 lambda minus lambda k to the power of a k, with the usual notation, that lambda 1 lambda 2 lambda are the distinct Eigen values, a 1, a 2, a k are the algebraic multiplicity is.

And then W j to be the Eigen space corresponding to Eigen value lambda j, what is this, is nothing but, the null space of the matrix A minus lambda j I, so even the matrix A, we have use ingredients, and g j the algebraic multiplicity of the geometric multiplicity is thus the dimension of W j.

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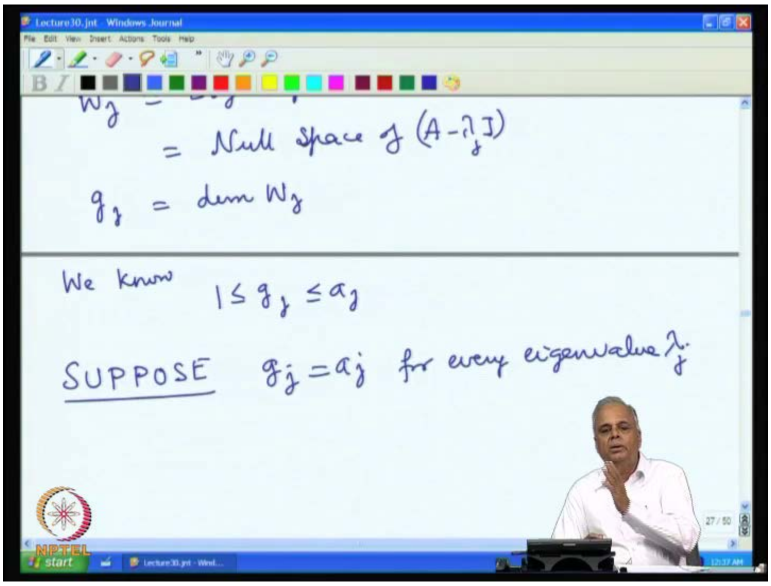

So, we know 1 less than are equal to g j less than are equal to a j, we have accepted, we have not prove it, but we will say we prove it later, the geometric multiplicity will be always less than are equal to 0. So, suppose g j is equal to a j for every Eigen value lambda j that is the algebraic multiplicity is the same of the geometric multiplicity for every Eigen value.

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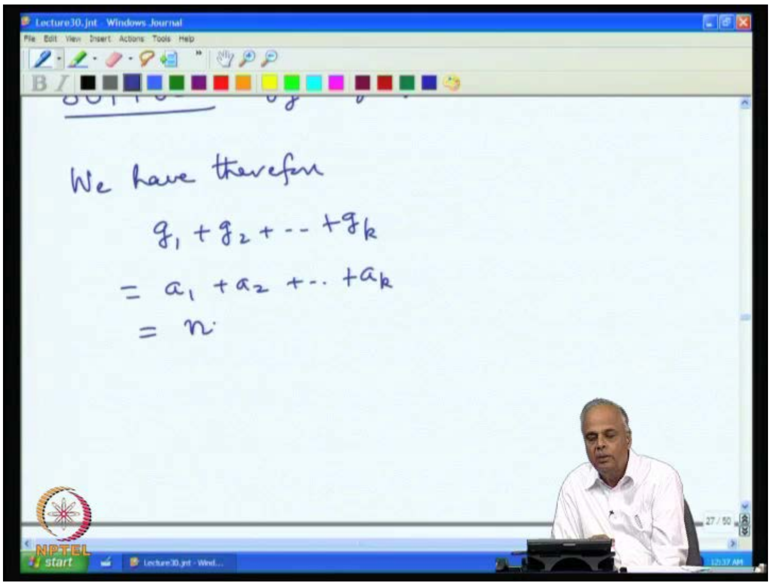

Then we have therefore, g 1 plus g 2 plus g k is the same as a 1 plus a 2 plus a k, (()) a 1 plus a 2 plus a k is (()) n, so how does it help us.

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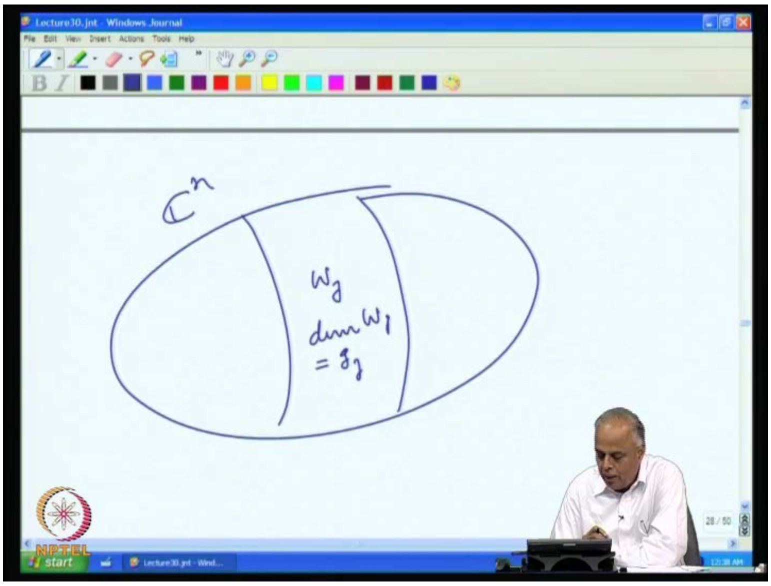

So, now look at this, we have this sales space (()) which is C n and W j is we will see, here we will the sub space. Now, the dimension of W j is equal to g j, what is the dimension W j is equal to g j mean, the dimension of W j is equal to g j means, that we can find a basis for W j consisting of g j vectors.

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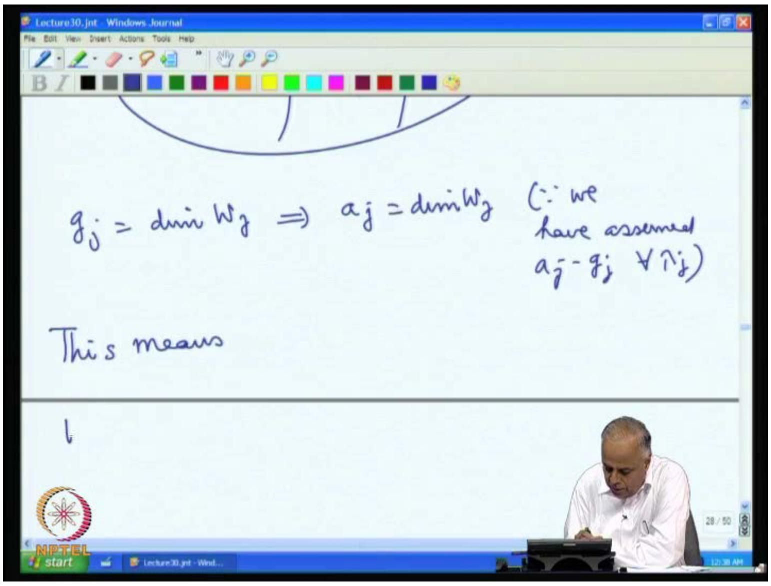

So, let us first g j is equal to dimension of W j, but we have assuming implies a j equal to dimension W j, because we have assume a j equal to g j for every lambda j. Since, we are assuming the geometric multiplicity is equal to algebraic multiplicity for every Eigen value we have, but the dimension W j if a j.

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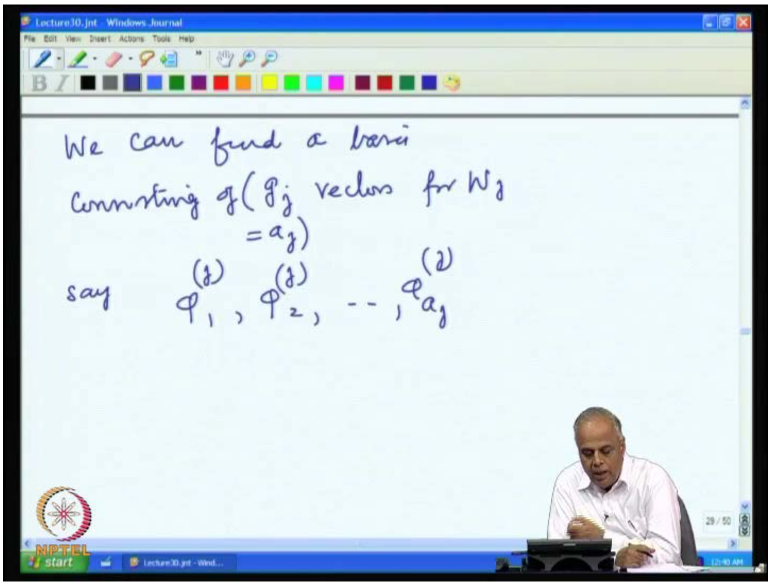

This means, we can find a basis consisting of g j vectors, which is the same as a j vector, g j equal to a j vector, g j equal to a j vector for W j. Say, let us call them us pi j 1, pi j 2, pi j g j; now we super script now the script the j tells as that it, we are talking about the j th Eigen space and subscript tells will the Eigen value Eigen vector numbering, I basis vector numbering, there are a j vector basis, so there are 1 2 3 a j subscript and super script j, say this are all basis vectors for W j.

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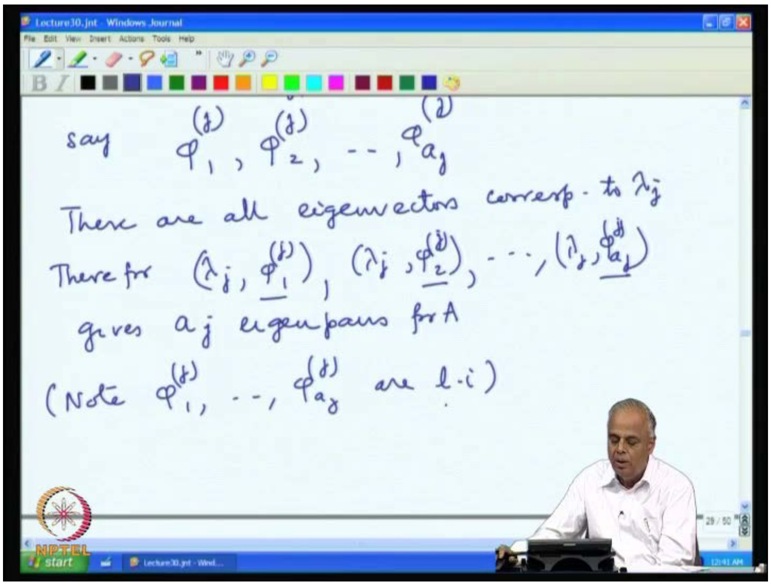

Now, we observe that every non 0 vector in W j is an Eigen vector for a corresponding to the Eigen value lambda j, and this pi 1, pi 2, pi a j are non 0 vector, because the form a basis and there for must be Eigen vector, these are all the Eigen vector corresponding to the Eigen value lambda j. And therefore, lambda j pi j 1 is an Eigen pair, lambda j pi j 2 is an Eigen pair, lambda j pi j a j is an Eigen pair, gives as a j Eigen pairs for A.

So, therefore, the sub space W j, the Eigen space corresponding to the lambda j is already generated a j Eigen pairs. Notice that the vector pi 1, pi 2, pi j appearing in this Eigen pair are already linearly independent, because they forming basis for W j, note pi j 1 etcetera, pi j a j are linearly independent.

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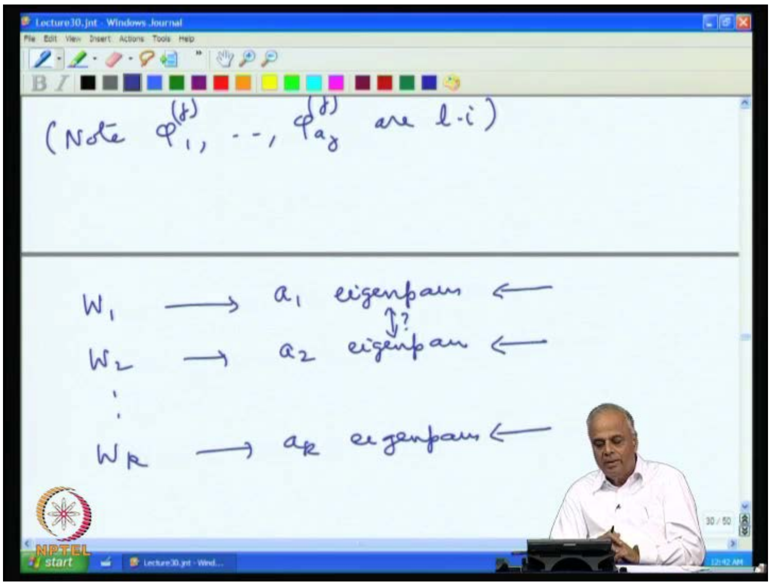

So, thus we have W j alone gives rises to a j Eigen vectors, so we have W 1 gives rise to a 1 Eigen pairs, W 2 gives rise to a 2 Eigen pairs and so on, W k gives rise to a k Eigen pairs. Now, the Eigen vector appearing in the Eigen linearly independent, the Eigen vector appearing in the these are linearly independent, the Eigen vector appear in the these are linearly independent. But, we do not know weather, the Eigen vector appearing in this to this gather are linearly independent; suppose they are then got a 1 plus a 2 plus a k which is n Eigen pair and you would have diagonalizable.

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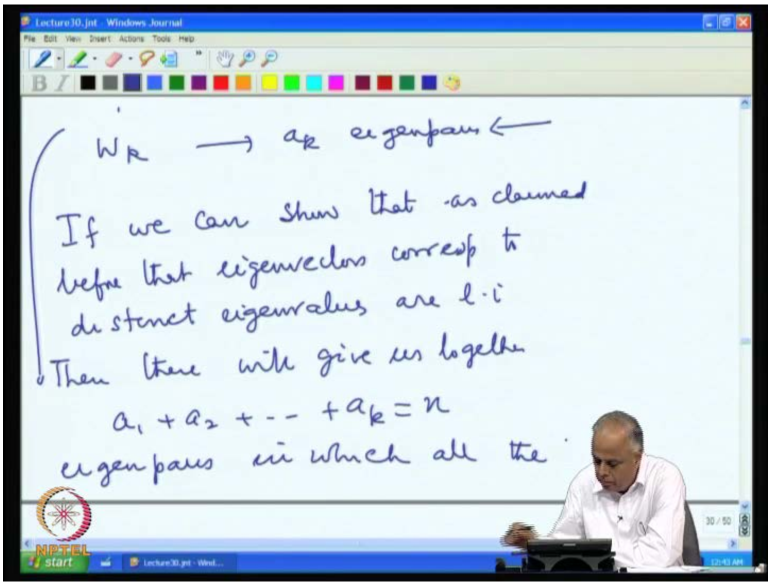

If, we can show that, which we already claim that is true, as claim before, that Eigen vectors corresponding to distinct Eigen values are linearly independent. Then these will give to gather, when I say this and this will give us to gather a 1 plus a 2 plus a k equal to n Eigen pair.

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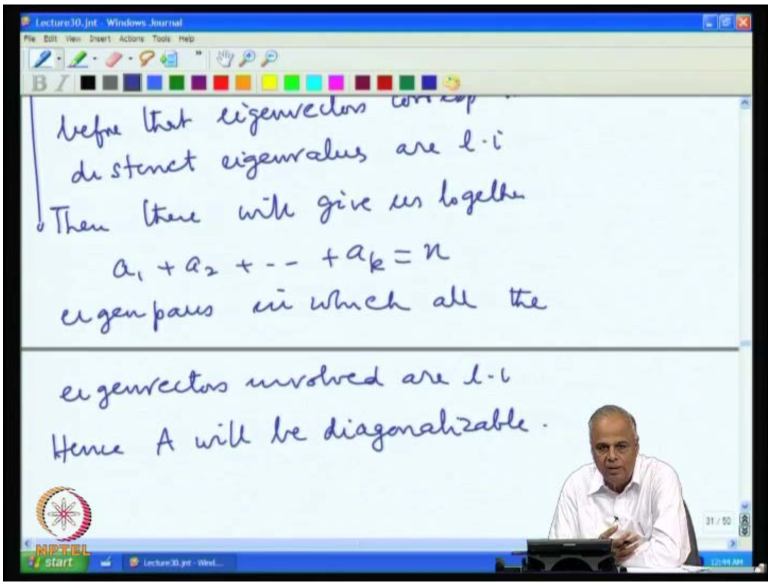

in which, all the Eigen vector involved linearly independent, and thus we would have had n Eigen pair as we are looking for, and hence A will be diagonalizable. Therefore, we have shown that, if we can show what, Eigen vectors corresponding to distinct Eigen values are linearly independent number 1. And number 2, if we assume, that the geometric multiplicity is equal to the algebraic multiplicity for every Eigen value, and then A is diagonalizable.

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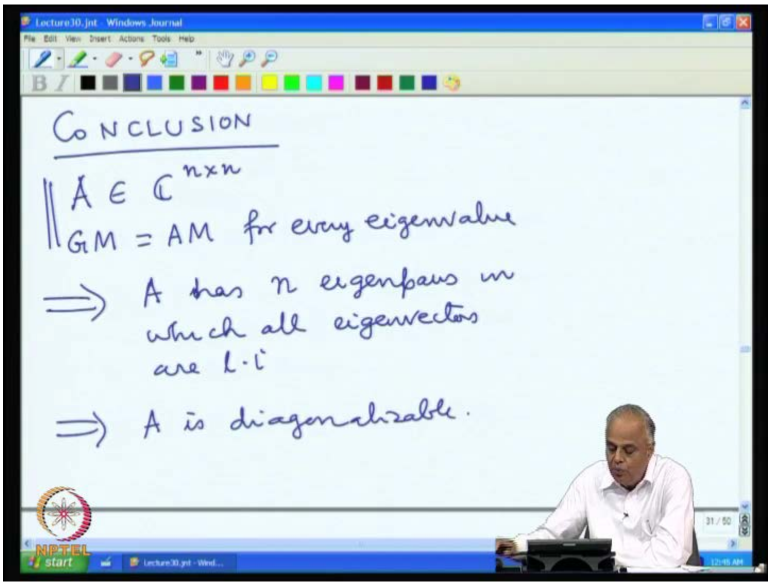

So, let us what is the conclusion therefore, let us summaries all other discursion, include conclusion suppose, we have matrix A for which, the geometric multiplicity is equal to algebraic multiplicity for every Eigen value, implies A has n Eigen pairs, in which all Eigen vectors are linearly independent, implies A is diagonalizable, but this was provided the following holds.

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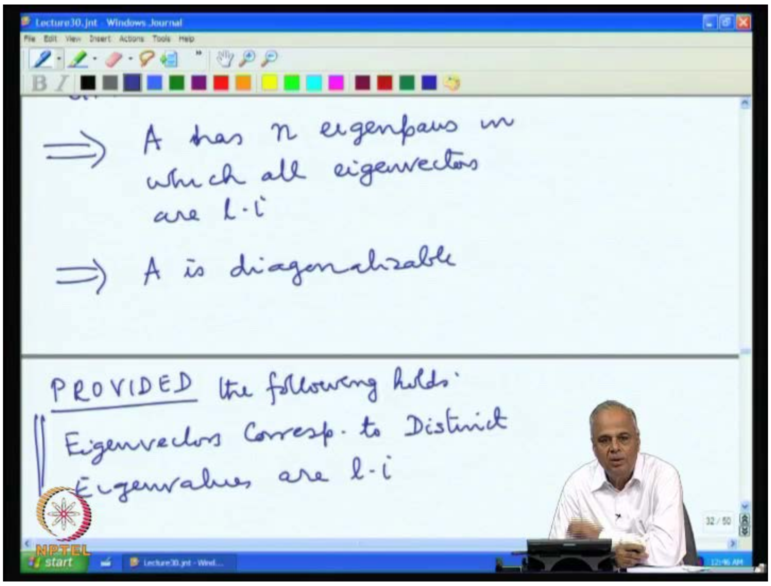

Eigen vectors corresponding to distinct Eigen values are linearly independent. So, this is an important property, which we have to prove, if we can prove this property, then what we are observe is that, if the geometric multiplicity is equal to algebraic multiplicity, for every Eigen value, then the matrix is necessarily diagonalizable.

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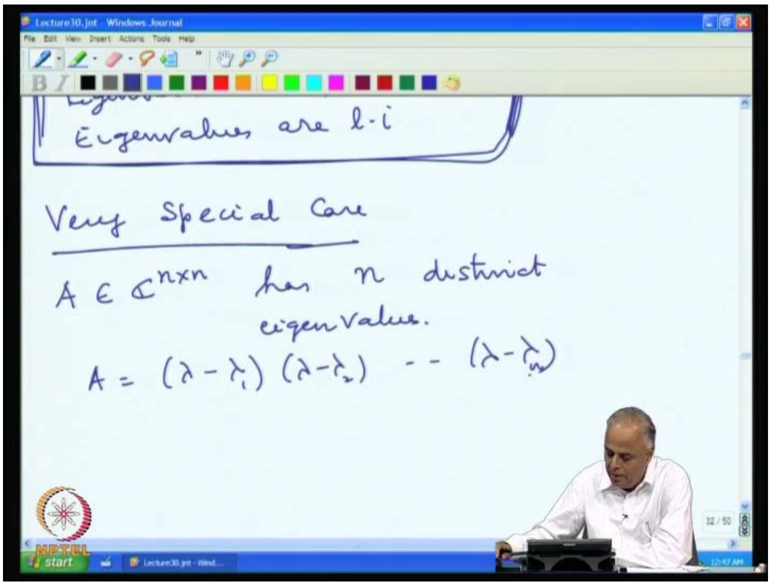

Now, let us look at the very special case, before we do that (()) therefore, this property that we are listed here is an important property, which we have prove and we shall look at the property in the next lecture. But, now let us look at a very special case a very special case is A has n distinct Eigen values, that is all the Eigen values are distinct. So, A has n distinct Eigen value and in that case, we have A equal to lambda minus lambda 1 lambda minus lambda 2 into lambda minus lambda n.

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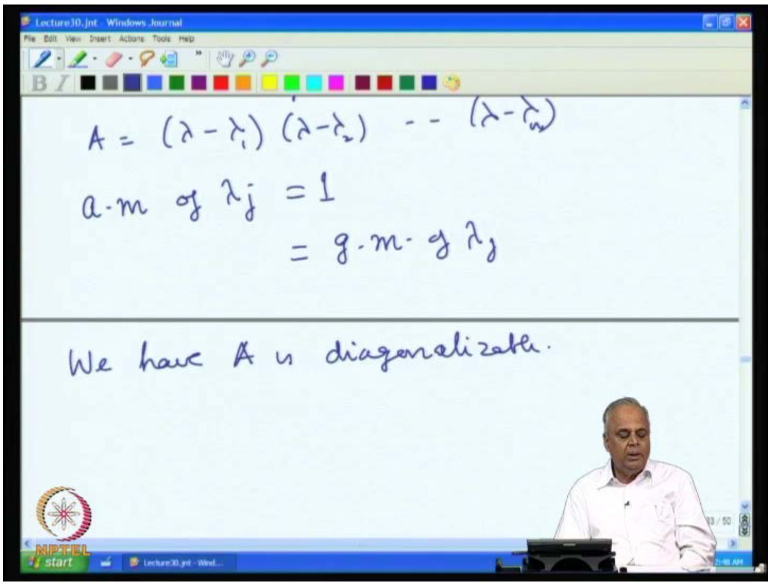

And therefore, the algebraic multiplicity of lambda j is equal to 1, but since the geometric multiplicity has to be at least 1, and it cannot be more than geometric, algebraic multiplicity, the also get this is equal to the geometric multiplicity of lambda j. And since, a m equal to g m and for every lambda j, we have A is diagonalizable, and therefore, a special case is that if matrix A has n distinct Eigen value, then the matrix A is necessarily diagonalizable (Refer Slide Time: 56:38).

And as we observed this is the crucial point, that we have to now look at, whether the Eigen vector corresponding to the distinct Eigen value as linearly independent, if you can prove it, we have a achieve the long go namely. In the case where the geometric multiplicity is equal to algebraic multiplicity, for every Eigen value to be guarantee the diagonalizable. We will further see details of the diagonalizable, once this property is prove, and that will be the goal for our next lecture.