Advanced Matrix Theory and Linear Algebra for Engineers
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Lecture No. # 03
Prologue-part 3

In the last two lectures, we discuss various problem and questions that you are connectivity this problem that will be addressing this course. The basic fundamental question was about liner system of equations.

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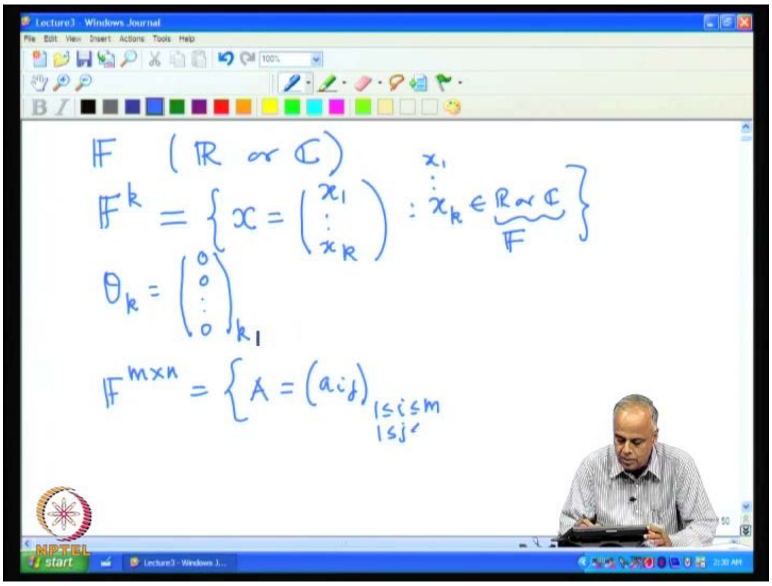

We shall begin our study of linear systems in a formal way. We shall use the notations as follows, we call the notation that we introduces in the last lecture. We will always in considering, either may go with see real entries or may to see with complex entries, most of the statement that will stay will be valued whether the real entries or with complex entries. So, in general we denoted by F, either the set of real numbers or the set of complex numbers. Whenever it is necessary to indentify whether you are using real numbers or complex numbers, we will make this expulse and such vocations. Then, by F k for any positive integer k, by F k we shall denote the set of all column vectors with n entry; with k entries and all the entries are real numbers x 1, x 2 etc are all real numbers or complex numbers and therefore, we will use the symbol F.

And theta k will denote the specific vector namely zero vector with k entries. Then, for any two positive integers m and n by F m cross n, we shall denote this set of all matrices m cross n. So, 1 less than or equal to i less then equal to m, 1 less than or equal to j less than or equal to n, said that all these entries are from F. The real matrix k they are all from R, in the complex k is they are all from C. In particular, we will use the symbol 0 m cross n to denote the m by n zero matrix.

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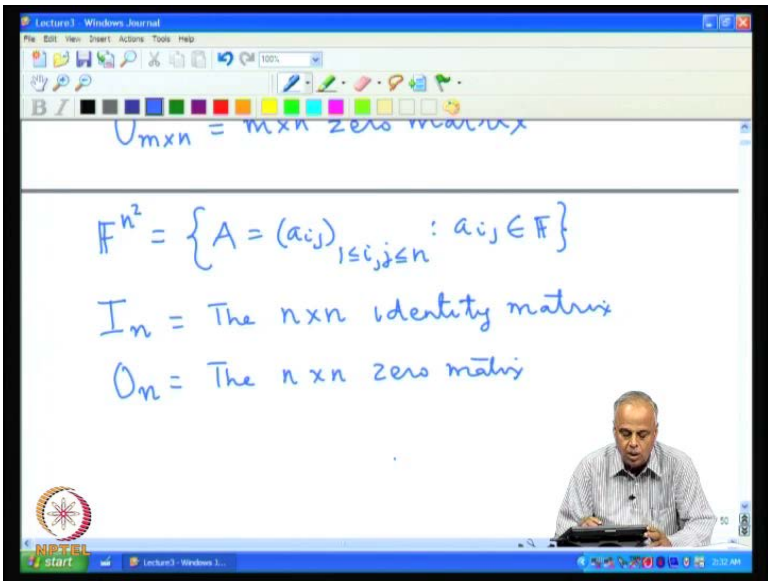

That is the m by n matrix all of this entries are 0. Then for the square matrix, we shall use the notation F n square to denote the set of all matrices which as square that is, 1 less than or equal to i comma j less than or equal to n and all the entries are from the F. Again F equal to R we are talking about real square matrices and F equal to C when we are talking about complex square matrices. In particular, we shall denote by I n, the n by n identity matrix that is, the identity matrix with n rows and n columns. And by 0 n, we shall denote the n by n zero matrix that is, the n by n matrix all of its entries are 0, these are the standard notations that we shall be using as and then required we shall introduced the other rotation.

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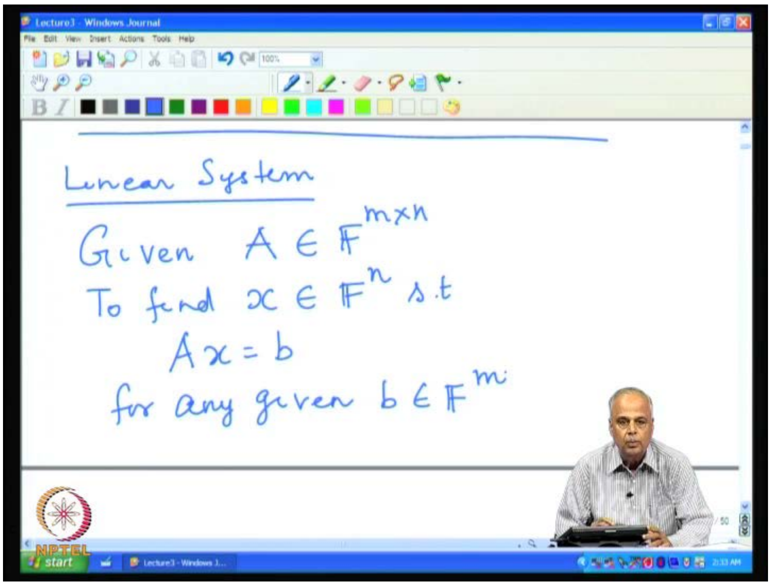

With this the liner system problem, can be return as follows. We have given an m by n matrix A so, A is in F m cross n. And then, we want to find an x which in F n such that, A x equal to b for any given b in F m. For A is the given matrix and for varies b in F n, you would like to find x such that A x is equal to b, this is the fundamental problem of liner systems of equations.

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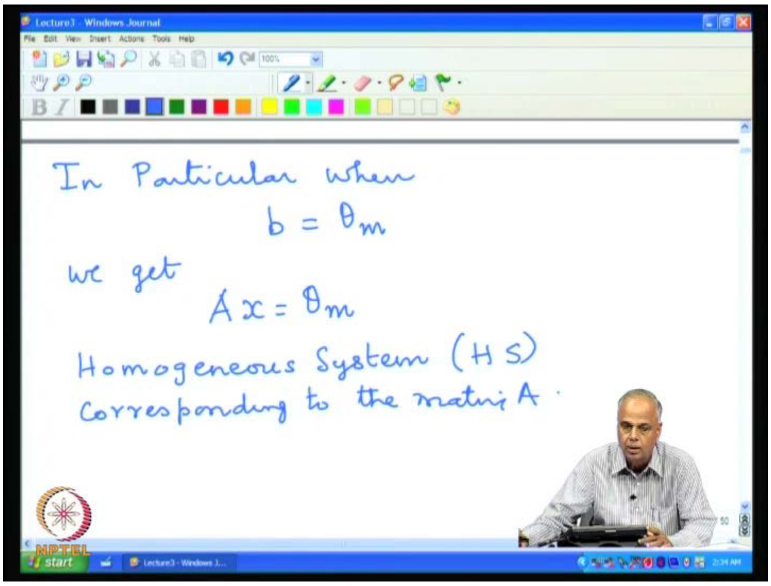

In particular, when b is the zero vector we get the system A x equal to theta m and we call this the homogeneous system corresponding to the matrix A so, this is called the homogeneous system. From now on, we will write short HS for homogenous system, corresponding to the matrix A when b is not theta m, then the system A x equal to b is called a non-homogeneous system. We shall see that in the analyses of non-homogeneous system, the homogeneous system place a very important role. We shall look at some very simple properties of the homogeneous system.

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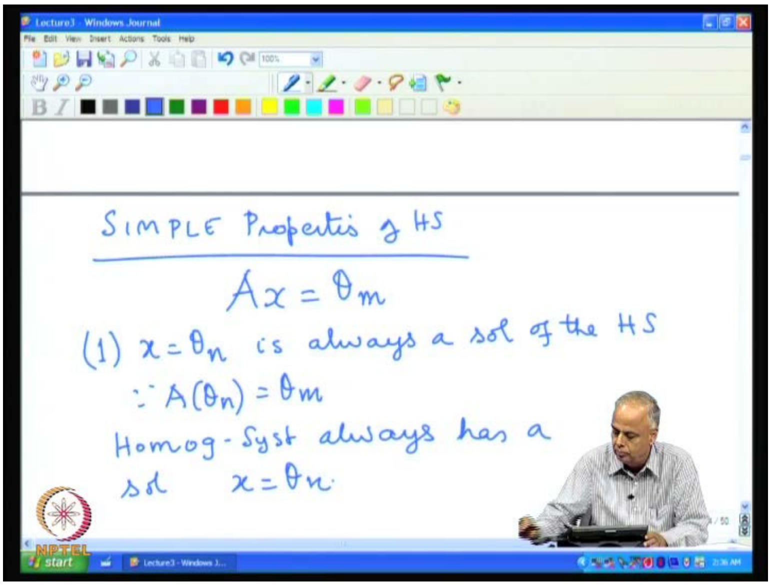

Simple properties of homogenous system. So, we are looking at the matrix A and we are looking at the homogeneous system A x equal to theta m. The first think we observe is, that x equal to theta m is always a solution of the homogeneous system, because A times theta n, we settled with theta m. Because A is m by n, theta is n by 1, theta is the zero vector so, the resultant to the n by 1 zero vector. So therefore, the homogeneous system always has a solution, always has a solution x equal to theta n. This solution is called the trivial solution. This is called the travail solution.

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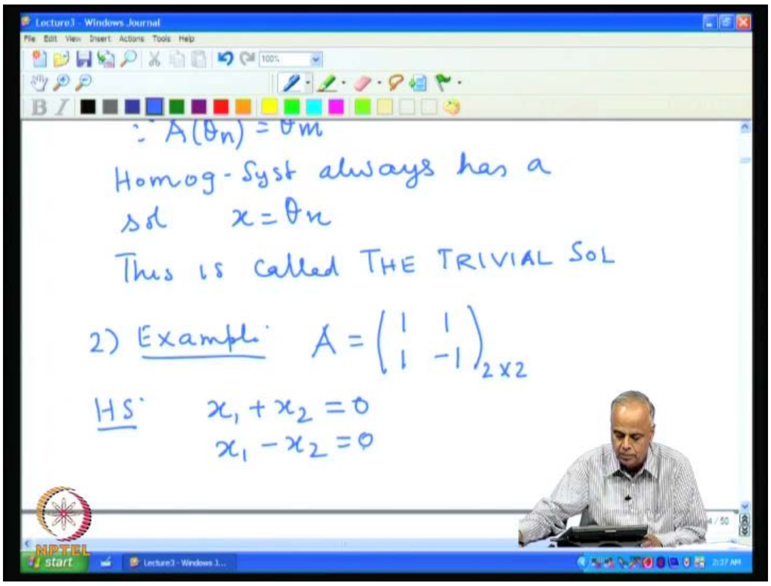

Let, you look at one or two examples. (No Audio from: 08:37 to 08:43) consider the matrix A 1 1 1 minus 1 so, here we have a numbering system m is 2, m is 2 so, it is called system. And what is the homogenous system corresponding to this, the equation becomes the first equation is x 1 pulse x 2 equal to 0 and the second equation is x 1 minus x 2 equal to 0.

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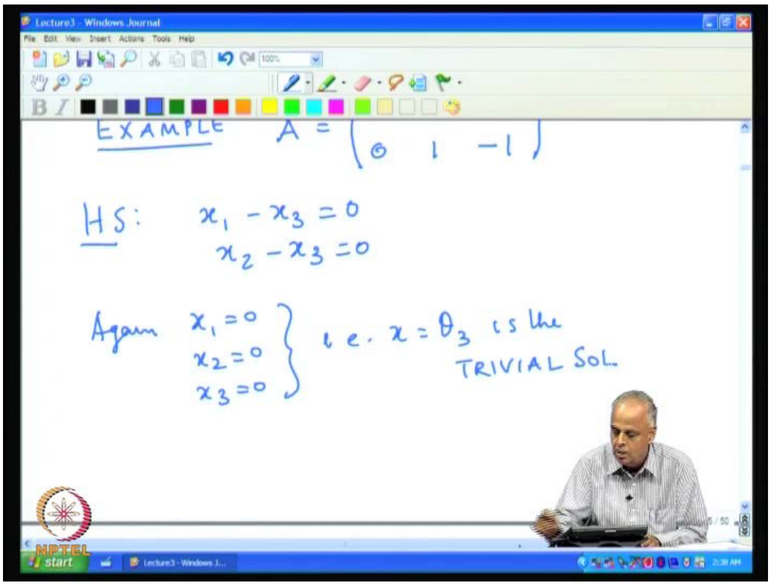

And as up their earlier, the zero solution x 1 equal to 0, x 2 equal to 0 gives x equal to theta 2 the trivial solution. And from the system we easily see, that this is the only solution for the homogeneous system, this is the only solution for the HS, for here is an example of the homogeneous system where the only solution is the trivial solution. Let us now look at another example, consider the matrix A equal to 1 0 minus 1 and 0 1 minus 1. What is the homogeneous system corresponding to this matrix, which x 1 minus x 3 is 0 and x 2 minus x 3 is 0. Here, again x 1 equal to 0, x 2 equal to 0, x 3 equal to 0 that is, x equal to theta 3 is in the trivial solution.

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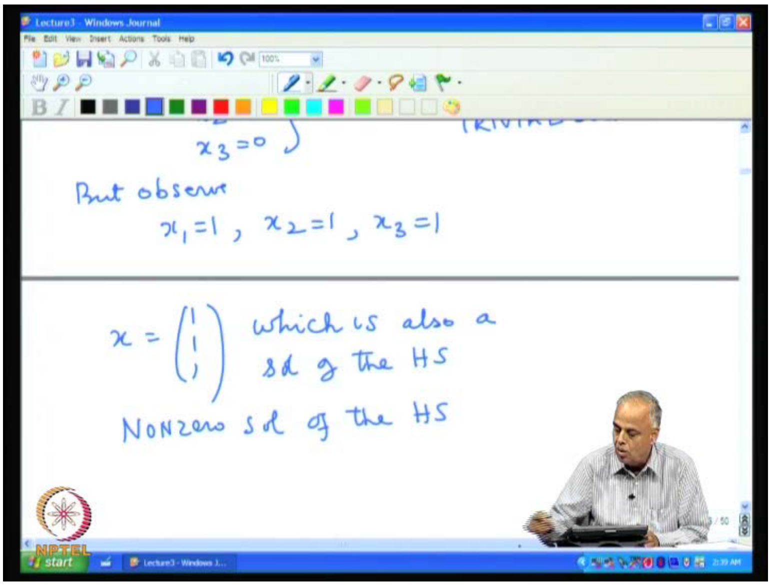

But now observe, if we take x 1 equal to 1, x 2 equal to 1 and x 3 equal to 1 that we get the vector x equal to 1 1 1 which is also the solution of the system, which is also a solution of the homogeneous system. This is not the zero solution which we always that, this is a nonzero solution; this is nonzero solution of the homogeneous system. So, here is the example of the homogeneous system which has not only the trivial solution, but also has nonzero solution, the nonzero solutions are called nontrivial solutions. So, a homogeneous system in general can have trivial solution only or it may have trivial, as well as non-trivial solution.

So, homogeneous systems may have only trivial solution or may have non-trivial solutions. The fact, whether it has only trivial solutions or whether it has non-trivial solutions, if going to have an impact on the solution of the non-homogenous system which we shall see shortly.

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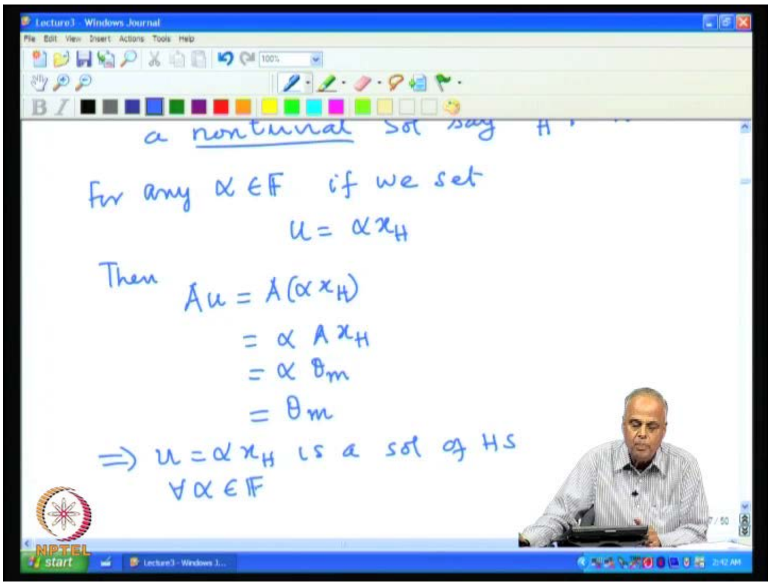

Now suppose, the homogeneous system A x equal to theta m has a non-trivial solution say x H and seeming to it is non- trivial, we call it not equal to theta n. Suppose, I have a matrix A such that, the homogenous system corresponding to this matrix has a non-trivial solution. In that case, for any alpha in F, remember F is the real numbers of complex numbers if A is real matrix and we are dealing with everything real, then F is real. When a complex matrix and we are dealing everything complex, then F is complex. So, it take any alpha which is the real number in the case of real and complex numbers is they have complex F. For any alpha, if we set u equal to alpha x H, then A of u is equal to A of alpha x H.

Since, alpha is a number we can pull it out, it is the alpha A x H, but since x H is the solution of the homogenous system. A x H must be theta m so, that is theta m and alpha theta m is theta m. So, that says u equal to alpha x H is a solution of the homogeneous system for every alpha belong to F. So, the moment the homogeneous system has the non-trivial solution x H, we have now alpha x H has alpha various over F gives different solution for the Homogeneous system. This implies since, x H is not theta m, there are infinite number of solutions for the homogeneous system obtained by varying alpha over F. So, best we see that the moment one non-trivial solution x is for the homogeneous system, there exist the infinite number of solution.

So, what is the conclusion we can draw from this? The conclusion that we can draw from this is the homogenous system A x equal to theta m either has only trivial solution or has infinite number of solutions. Obviously, infinite of them must be not Trivial, because the only trivial solution if seen. So therefore, at the homogenous system swings from one end to the other end, it can how only the trivial solution namely the zero vector or you can have in infinite number of solutions. Of which one of this is true for the matrix A is going to have a impact on the nature of the solutions of the non-homogeneous system A x equal to b.

Now having how, that the Homogeneous system has this varying properties of either only trivial solution or infinite number of solutions. We shall see, what it has got to do with the non-homogeneous system that we are trying to solve. So, let us now look at non-homogeneous systems.

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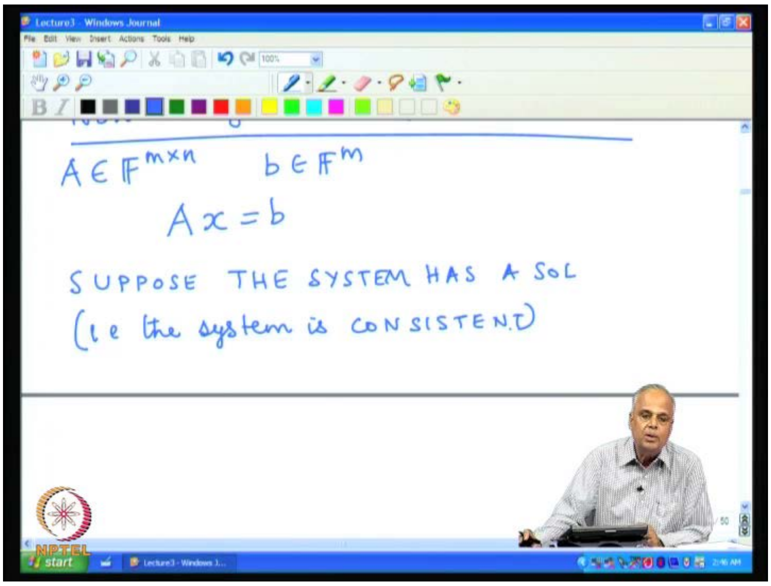

As I pointed out, from now on we will simply write it as NHS. So, we are given a matrix A so, we belong to F m n and we are given b in F m and we are looking at the system A x equal to b. Now, supposing we assume that the system is consistent, what do you mean by the system is consistent, that mean the system has the solution. That mean b satisfies the criterian required that the system to have a solution. So, we say suppose, the system has the solutions, whenever such a thing we say this system is consistent. So, when you say the system is consistent, we mean that there is the solution for the system. Suppose, we have a non-homogeneous system corresponding to the matrix A and suppose the system is consistent, that is it has the solution.

The question is how many solutions? In their be 1 or many more or 2 or 3 or how many solution will their b? So, let us investigate this question. Suppose, the non-homogeneous system has two different solutions that is, solution to NHS is not unique. Suppose, the system has more than one solution, the solution to the system is not unique. Therefore, there exist u and v such that u is a solution and b; v is also a solution and u and v are different. To suppose, you could find two solutions that is, what is mean by saying that is the solution is not unique.

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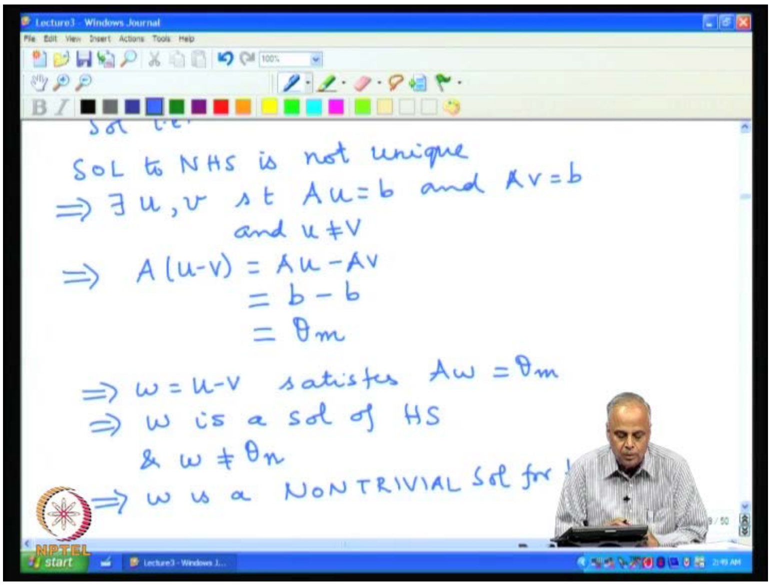

Then, since A u is b and A v is b, we get A u minus v equal to A u minus A v, but since A u is b and A v also b, we get this equal to theta m. Now, that says if you said w equal to u minus v satisfies A w equal to theta m which means, w is a solution of the homogeneous system. Here that, since u is not equal to v, we assume u is not equal to v since u is not equal to v, w is different from and w is not equal to theta n. And hence, w is a non-trivial solution for w is the non-trivial solution for the homogeneous system.

So, the conclusion is if we assume that the solution of the non-homogenous system is not unique, then it automatically implies that the solution to the homogeneous system must contain non-trivial solution. So, let us write conclusion 1, we will make one more conclusion little later. Solution to non-homogeneous system not unique implies homogeneous system has non-trivial solutions. So, whenever the solution to the non-homogeneous system is not unique, the homogeneous system is automatically has non-trivial solutions. Let us look at the converse, what do you mean, again we start with consistent system A x equal to b and suppose homogeneous system has non-trivial solutions.

Suppose, we shall show that the converse is true, that means the solution now we are assuming that the homogeneous system non-trivial therefore, we will now show that the non-homogeneous system is not unique, the solutions are not unique. So, we will show that, to show that non-homogeneous system solution is not unique. For this, you may show that there are two solutions at least for the non-homogeneous system. We are already, assuming that the system is consistent.

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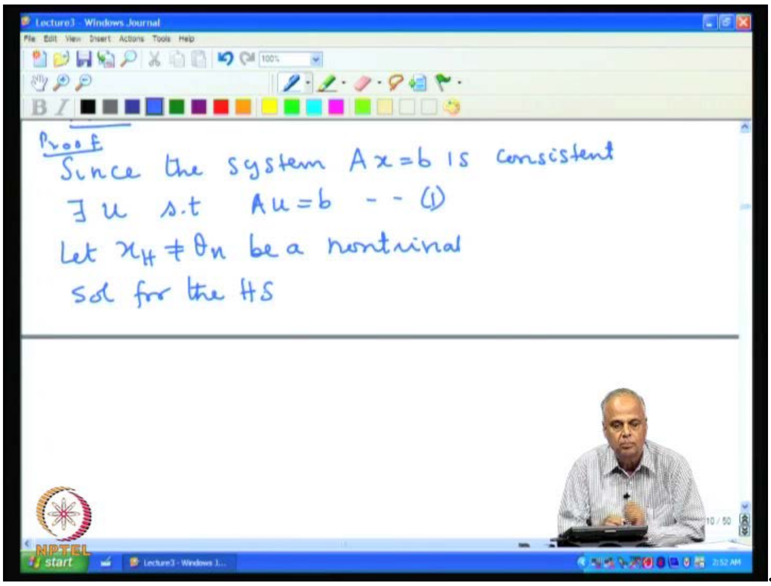

So, how do we show this the proof of that, since the system A x equal to b is consistent there exist a solution u such that A u equal to b. So, there exist a solution for the non-homogeneous system that is our basic assumption, because you are assuming that the system is consistent. Now, we are further assuming that the homogeneous system has non-trivial solutions so, let x H not equal to theta n be a non-trivial solution, for the homogeneous system. We would now like to produce at least two solutions for the non-homogeneous system. We already have one in u, we are now going to produce another solution.

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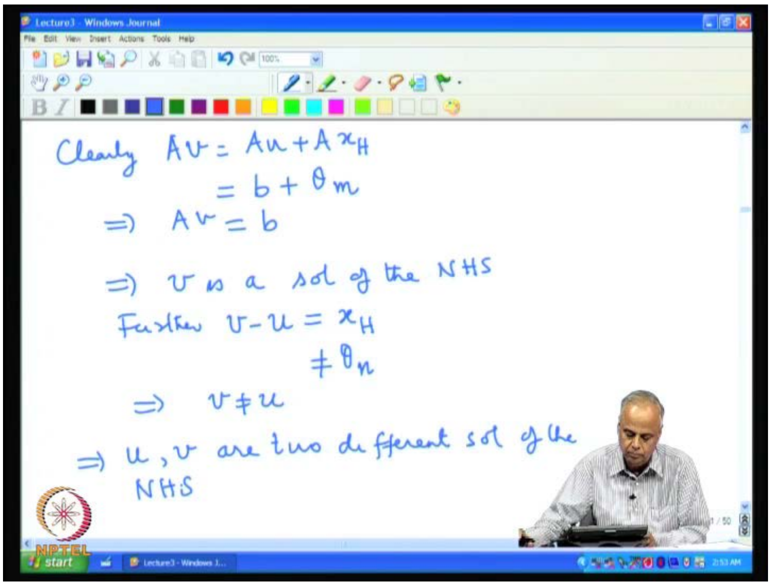

Let v equal to u pulse x H. Clearly, A v is A u pulse A x H, A u is b by equation 1 and A x H is theta m, because x H is the solution of the homogeneous system and therefore, A v equal to b. What does it mean to say? It means to say that v is a solution of the non-homogeneous system. Further, v minus u is equal to x H, because v is u pulse x H therefore, v minus u must be x H. But x H is assume to be non-trivial solution so, x H is not equal to theta n that says, v is not equal to theta m. So, v is the solution of the homogeneous system and is different from the solution u and therefore, u and v are two different solutions of the non-homogeneous system.

So thus, what is the conclusion? The conclusion is this is the second conclusion, when the homogeneous system has non-trivial solution that implies non-homogeneous system, the solution is not unique. So, combining these two conclusions, in the first conclusion we show reverse of this, the non-homogeneous not unique implies homogeneous as non-trivial solution. Now, we are shown that homogeneous is non-trivial implies non-homogeneous is not unique so, combining the two conclusions. (No audio from: 26:51 to 26:59)

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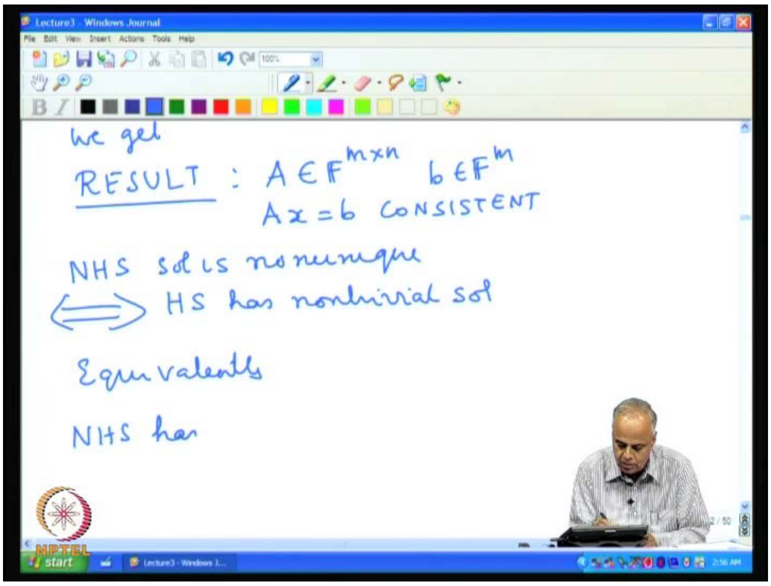

We get the important result that colon, at A b an m by n matrix, b belong to F m, A x equal to b consistent. So, I have a consistent system of m equations in n unknowns, then what we have concluded in these two statements above as the following that the non-homogeneous system solution is non unique iff and only if the homogeneous system has non-trivial solution. In other words or equivalently, we can say that the non-homogeneous system has unique solution iff and only if the homogeneous system has only trivial solution.

Thus we see that, the homogeneous system place an important role in deciding whether the non-homogeneous system whenever its consistent has a unique solution or not. So, the homogeneous system plays a decisive role in determining uniqueness of the solution of non-homogeneous system. And therefore, it become necessary that we analysis the homogeneous system corresponding to a given matrix very carefully. Because this going to have very important varying on the solution of non-homogeneous system.

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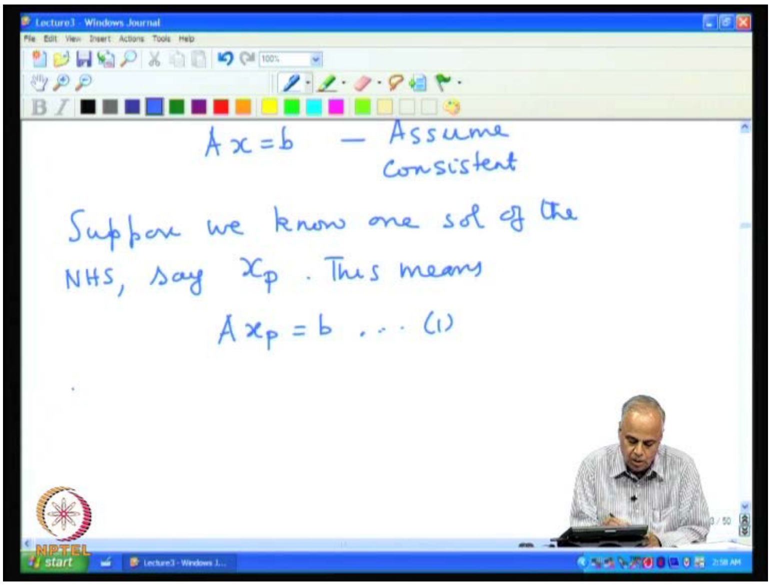

That is now look at, the non-homogeneous system A x equal to b again, assume consistent. So, now we have a consistent non-homogeneous system, that means A is the m by n matrix, b is the n by 1 vector, then m by n system A x equal to b is known to be consistent, that is known to be having a solution. Now, suppose we know one solution, suppose we know one solution of the non-homogeneous system say, x p. We know one solution of non-homogeneous system which we denote by x p. What does that mean? This means A x p is equal to b, the x p is a solution of the system A x equal to b so, A x p must be equal to b, that is the first conclusion, assumption will make. We know that the system have the solution, suppose by (( )) or by inspection we of on one solution namely x p. Now, we would like to look at the various possible solutions that may exist.

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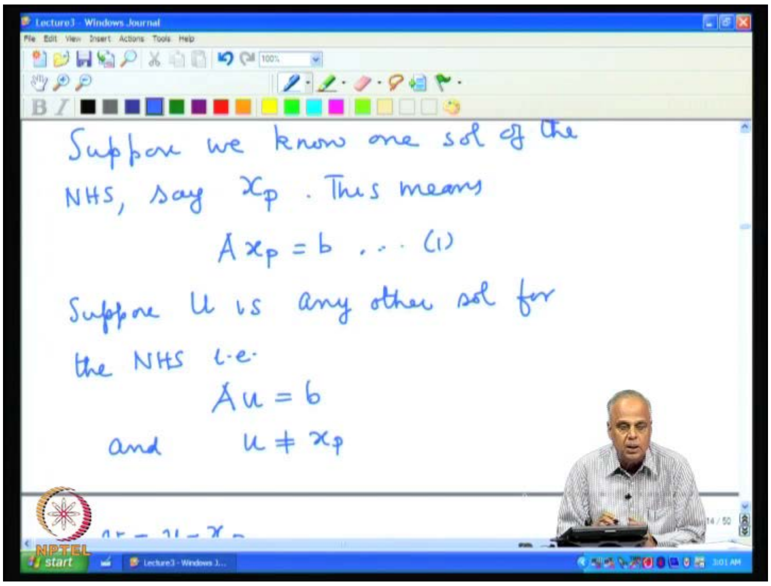

So, suppose u is any other solution for the non-homogeneous system, that is; it is a solution of the non-homogeneous system and u is not equal to x p. Suppose, you look at any other solution of the non-homogeneous system so, it must be satisfying A u equal to b and it is different from x p. Now, let us look at v equal to u minus x p, we have A v is equal to A u minus x p, which is equal to A u minus A x p. A u is b, because u is the solution, A x p is b, because x p is the solution so, that is equal to theta 1. So, that says v is a solution of the homogenous system, because A v is theta m. That says, u is equal to x p plus a solution of the homogeneous system, because v is; u is x p plus v, but v is a solution of homogeneous system therefore, u is equal to x p plus v.

So, what are the proof we say that, suppose one solution x p, suppose one solution x p of the system, then any other solution of the system must be of the form x p plus a solution of the homogeneous system.

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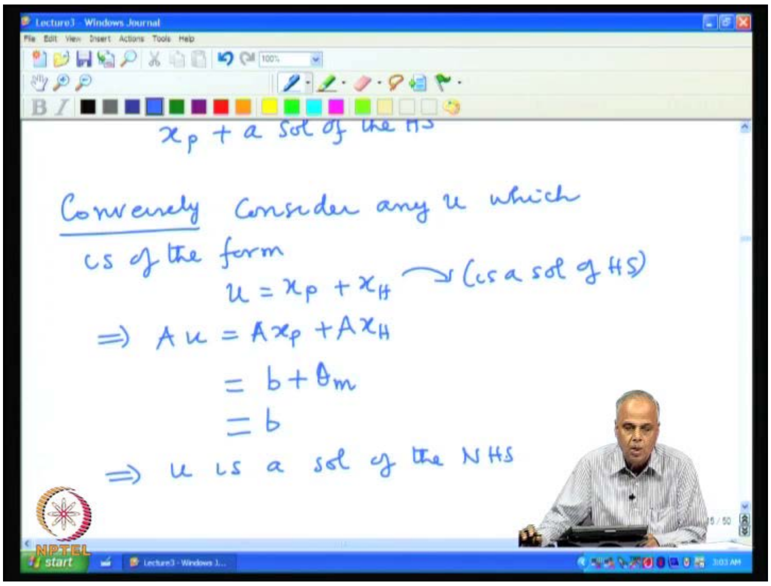

So, the conclusion 1, x p a solution of the non-homogeneous system implies any other solution of the non-homogeneous system must be of the form x p plus a solution of the homogeneous system. Conversely, consider any u which is of the form u equal to x p pulse x H where x H is the solution of the homogeneous system. Then, that implies A u equal to A x p plus A x H. But A x p is b, because x p is solution to the non-homogeneous system, A x H is theta m, because x H is the solution of the homogenous system and that is equal to b, that says u is the solution of the non-homogeneous system.

Now what is the conclusion, the conclusion is that x p a solution of non-homogeneous system and x H a solution of homogeneous system implies u equal to x p pulse x H must be a solution of non-homogeneous system, it is conclusion 2. Comparing to these two conclusions together, we get the result that A x equal to b consistent, x p a solution of the non-homogeneous system implies every solution is obtain; every solution of the non-homogeneous system is obtain in the form x p plus x H by varying x H over all solution of homogeneous system.

Thus we see, that finding the solution a non-homogeneous system involves two parts that we finding x p one solution of non-homogeneous system, that is why we use the symbol p, because is called the particular solution, a particular solution of the non-homogeneous system. And we have to vary x H over all possible solution of the homogeneous system so, we need all the solution of the homogeneous system. So, finding solution of non-homogeneous system involves two parts, the first part is finding all solutions of the homogeneous system and the second part involves finding a solution for the non-homogeneous system. Once we find these two, we can now generate all solutions of the then can let us call this one solution is x p, then you can generate all solutions of the non-homogeneous system by adding to x p the various solutions of the homogeneous system.

So, there are two part involve in the study of non-homogeneous system. Then, I repeat finding all solution of the homogeneous system and number two finding a particular solution of the non-homogeneous system. So, we shall now look at the first major part namely finding all solution of the homogeneous system. So, first we develop strategy for finding all solutions of the homogeneous system. So, the next major topic of our discussion will be homogeneous systems. (No Audio from: 39:39 to 52) So, we are given A again in F m n, we look at the homogeneous system A x equal to theta m, problem find all solutions. Recall that we have said that this may have only trivial solution or it may have infinite number of solutions.

So, therefore, we may have to find an infinite number of solutions for this, in case A x equal to theta m as the infinite number of solution so, it may be a huge problem finding all the solutions. So, we are now going to develop a strategy for this finding all solution of the homogeneous system. The main tool, in handling homogeneous system or what are known as elementary row operations for short we will write EROs. What are these EROs? The EROs are some operations, the O stands for operations and they are where are they perform, the R stands for the rows perform on the rows of a matrix. They are call elementary, because they really elementary operations, they do not involve very highly complicated.

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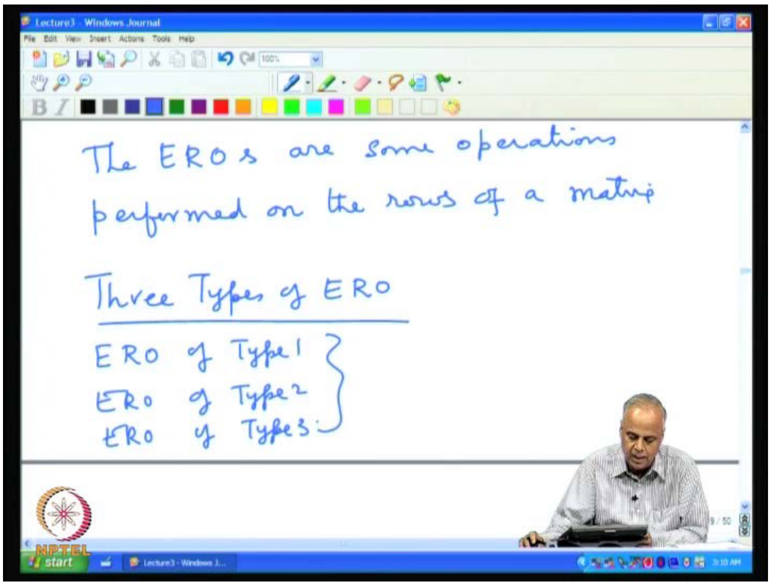

There are three types of elementary row operations that we will be using, we will call them as ERO of type 1, ERO of type 2 and ERO of type 3 is are the three types of they are going to three types of elementary row operations which we will be using. And we give them the names as ERO of type 1, ERO of type 2 and ERO of type 3. So, let us, look at first briefly describe what they are and we look at the implications later.

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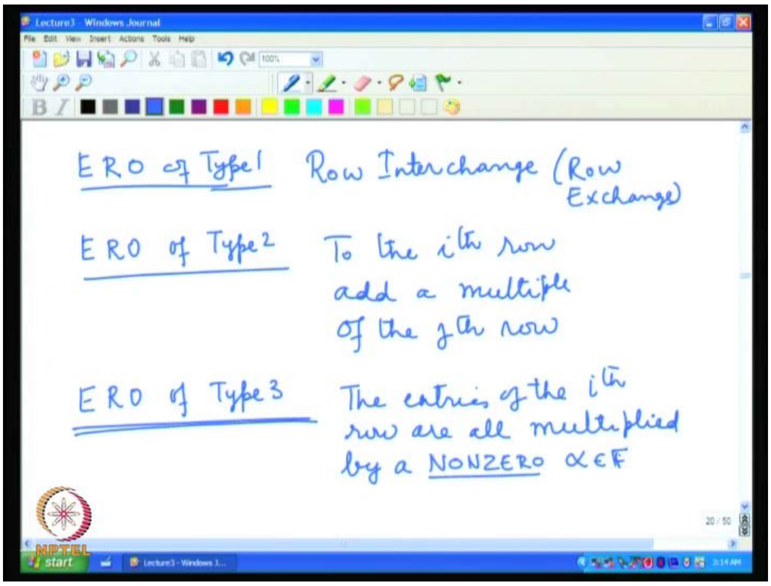

ERO of type 1, this simply changes as the position of two rows by interchanging. In other words, we can write the first row of third row and third row of first row so, this is called row interchange will describe them in detail shortly row interchange. Then, the second type, this is we what do you mean by; let us let me repeat again, then I say row interchange. Suppose, I have a matrix with 10 rows, I may pick the second row and put into the fourth place that is bit that the second row become fourth row and I will take the fourth row and go on put it back in the second row. So, the second and fourth row exchanges positions so, they are also called row exchanges, two rows exchange their positions.

The ERO of type 2, again the operation is done only in one row now, in the previews ERO of type 1 all rows remain unaltered two rows decided to exchange their positions. Now, in the ERO of type 2 all rows remain unaltered except one row where we are going to make a change, what are we going to do? We take a row and somewhere along the line we pick up another row and take a multiple of that row and add it to this row. So, to the i th row add a multiple m of some other rows say the j th row. So, here all the rows remain the same except the j th row where every entry is stamped with, how is it stamped with? To its value some multiple of corresponding value in the i th row is added so, that is called the elementary row operation of type 2.

The elementary row operation of type 3 is the fallowing here, again all the rows remain unaltered, only one row we are going to make a change, what kind of the change we are make to change? We are scale the entry of the matrix. So, what we do the entries of the i th row are all multiplied by a nonzero alpha in F. We will see why you want a nonzero alpha in F, because if we multiply by 0, the entire row will become 0 and all the information content of the row will be lost. We will see very precisely, why we want to multiplied by a nonzero constant.

So, we have the three types of elementary row operations which we shall describe in detail shortly, why do we want this? What are the reasons for this? The idea, what is the idea? Use elementary row operations on the given matrix A repeatedly take the matrix A go on the plying all kinds of by applying ERO of 1, ERO of 2, ERO of 3 and keep on varying the rows, varying the nonzero numbers varying what multiples you are going to add so on so forth. Keep on performing this elementary row operations and keep on getting ((.)) you around ER matrices repeatedly to get the new matrix B. The first make the idea and then we will work out the details.

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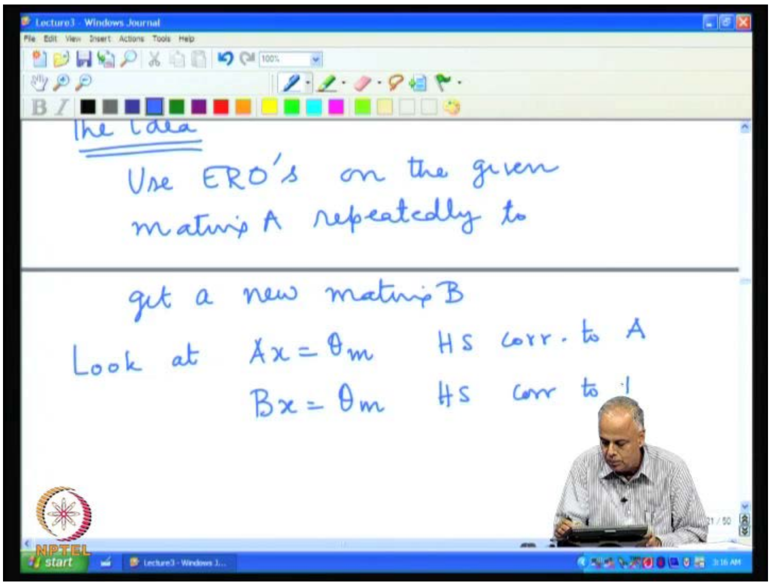

So, use the EROs on the given matrix A repeatedly to get the new matrix B, what is big deal. Now you look at, A x equal to theta m, the homogeneous system corresponding to A and B x equal to theta m, the homogeneous system corresponding to B. We shall see that, whenever something is the solution of the homogeneous system corresponding to A, it is also a solution corresponding to the homogeneous system B and vice versa. So, what you mean is, it turns out that the homogeneous system for A has the same set of solutions as the homogeneous system for B.

What does it mean? It means therefore, instated of solving the homogeneous system A x equal to theta m, we can solve the homogeneous system B x equal to theta m. What is the advantage? After all instated of solving one homogeneous system A x equal to theta m, we are now reduce only solving into another problem B x equal to theta m, is there any advantage by solving this system d x equal to theta m? Now, it turns out if we are cleaver, if we are cleaver then we can choose our EROs such that, the matrix B is simple enough to make solving the system B x equal to theta m easy, that is the advantage. If, we are cleaver, then we can choose our EROs such that, the matrix B is simple enough to make the solution of the system easy.

So, instead of solving the general system A x equal to theta m, we will be solving an easier system which is B x equal to theta m. For this to be achieved we must 1, know what simple systems B can be? 2, how can we use EROs effectively to reduce a given A to a simple plan B? We shall look at EROs and the simple questions of reduction in the next lecture.