Advanced Matrix Theory and Linear Algebra for Engineers

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Module No. # 07

Lecture No. # 26

Inner Product and Orthogonality- Part 5

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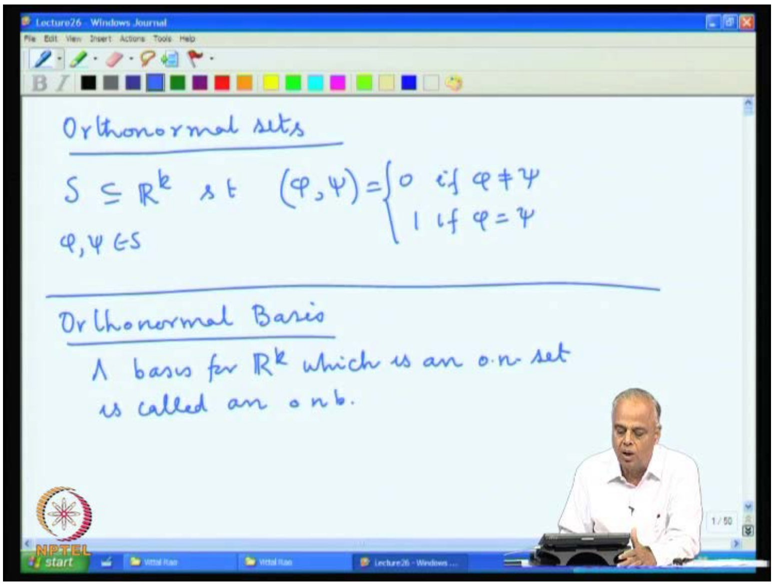

We have been looking at orthonormal sets, and the notion of the orthogonal compliment of a given subspace. So, let us recollect some of the basic ideas, suppose we have a set S in R k, such that phi psi is equal to 0 if phi not equal to psi and 1 if phi equal to psi where phi and psi are in S, then we say S is an orthonormal set. And then we had the notion of an orthonormal basis, so your base is which is an orthonormal set is called an orthonormal basis, so your base is for R k which is an orthonormal set is called an orthonormal basis. This is very important notion that we have introduced and using this orthonormal basis, we showed that any vector can be expanded in terms of the orthonormal basis.

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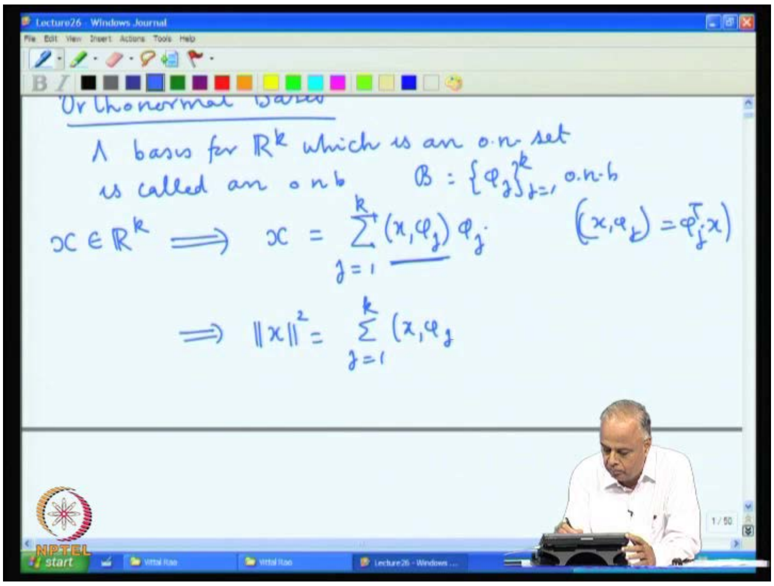

So, if we have x in R k the expansion becomes easy, the vector x can be written as j equal to 1 to k x phi j phi j, where phi j is the ortho normal basis j equal to 1 to k.

Suppose, we had an orthonormal basis, then any vector in R k can be expanded in terms of these phi j and the coefficients in the expansion are nothing but the inner product of x with a vector phi j, recall x phi j is only another notation for phi j transpose x. Now, once we have this from this we also have that the length of x square is given by the sum of these x phi j square.

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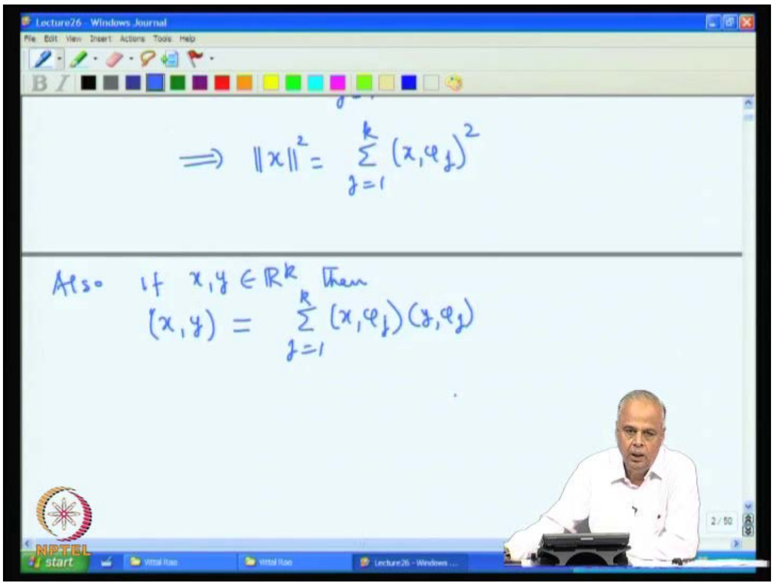

Also, we have if x and y are in R k then the inner product of x and y or the dot product of x and y just in terms of these coefficients is just x phi j comma y phi j. So, the expansion in terms of orthonormal sets is akin to the expansion with respect to the i j k vectors which you might have learnt at an earlier class.

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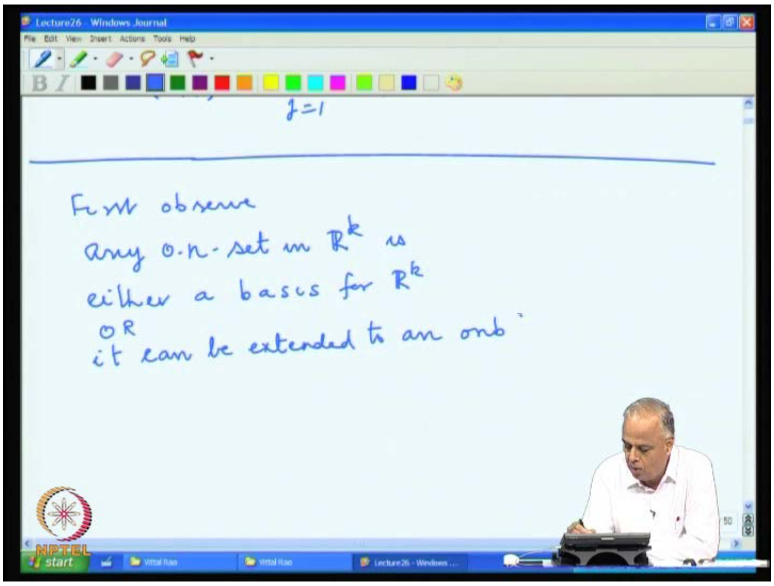

Now, having got this orthonormal set then we introduced the notion of a orthogonal complement for a subspace. Now, we observed first we if you have any orthonormal set in Rk is either a basis for Rk or it can be extended to an orthonormal basis for Rk. We use this notion in analyzing what is known as the orthogonal complement of a subspace.

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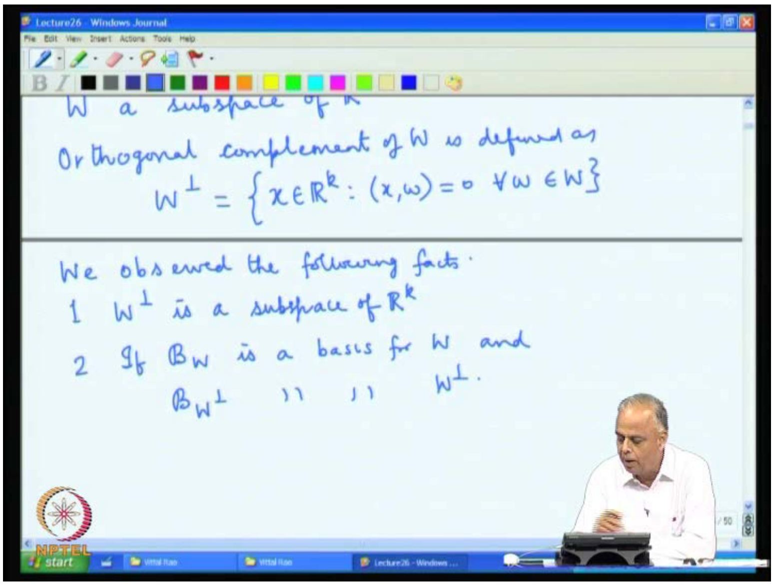

So, then we introduced the notion which we looked last time namely the W is a subspace of Rk then, the orthogonal complement of W is defined as W perp that is the notation for the orthogonal complement. It consist of all those vectors in Rk which are perpendicular which are orthogonal that is the dot product is 0 for every vector in W that is these are vectors which are orthogonal to everyone of the vectors in W.

We observed in the last lecture the following facts what are they? the first thing that we observed was the W perp is a subspace of Rk. This is the first simple observation the w perp is forced to be a subspace of Rk, the second thing we observed was that if BW is a basis for W and BW perp is a basis for W perp. So, you have a basis for W and you have basis for W perp.

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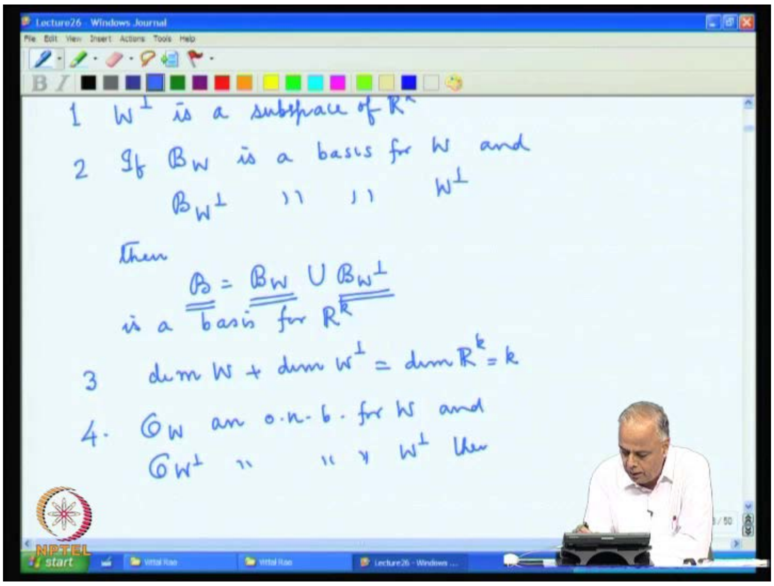

Then if you put them together let us call that B as BW Union BW perp is a basis for the whole space so, we can get a basis for the whole space by getting a basis for W and a basis for orthogonal complement, a consequence of two is the fact that the dimension of W plus dimension of W perp is equal to the dimension of Rk which is k. The dimension of a subspace plus the dimension of its orthogonal complement is always equal to the dimension of the whole space because the number of vectors in the basis W is equal to the number of vectors in the basis W and basis W perp. So, the dimension of W plus dimension of W perp is equal to the dimension of Rk.

Another special version of two is the fact that if you take an orthonormal basis for W so, let us call it as oW an orthonormal basis for W and oW perp an orthonormal basis for W perp, then putting them together we get an orthonormal basis for the whole space.

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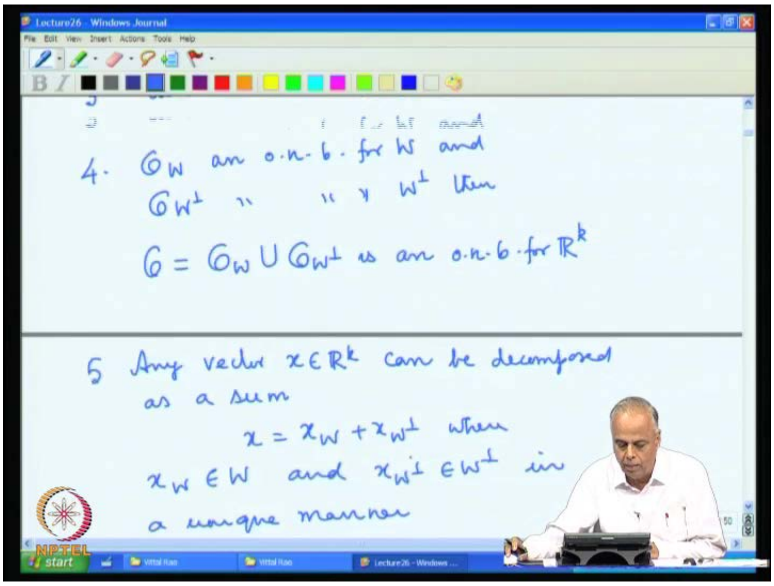

Then o equal to oW Union oW perp is an orthonormal basis for Rk. Therefore, you can put together basis for W and W perp to get a basis for the whole space you can put together an orthonormal basis for W and an orthonormal basis for W perp to get an orthonormal basis for the whole space. We further observed using these facts that any vector x in Rk can be decomposed as the sum of two vectors xW plus xW perp where xW belongs to W and xW perp belongs to W perp and this decomposition is unique in a unique way.

So, what really matters here is that the basis can be split a part in to two pairs that is the basis for the whole space can be split into the basis for W and the basis for W perp the orthogonal basis for the whole space can be split into an orthogonal basis for W and an orthogonal basis for W perp, this allows us to split a vector into two parts one part coming from W and the other part coming from W perp.

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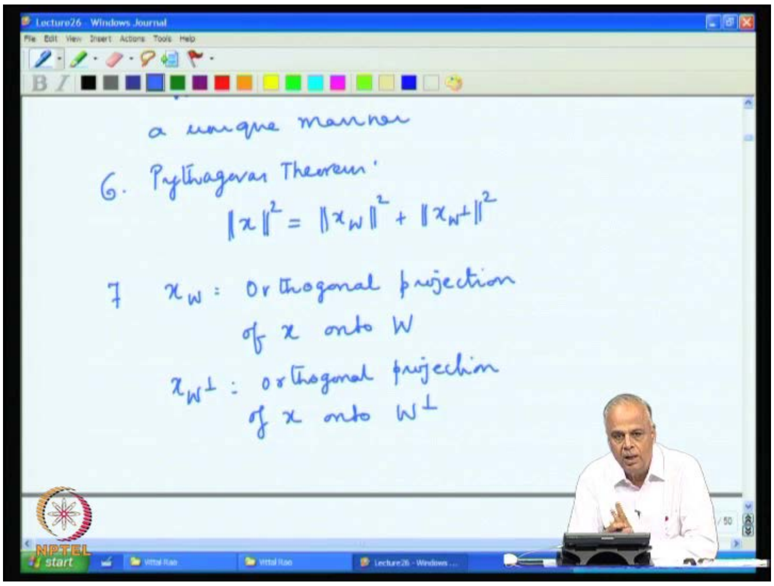

And we had the consequential Pythagoras Theorem which was a at that when we do such a splitting the length of x squared will be the length of xW squared plus the length of xW perp square. The xW is what we called as the orthogonal projections of x onto W and xW perp is called the orthogonal projection of x onto W perp. So, these are some of the important facts about orthonormal sets orthonormal basis the orthogonal complement and their basis and the splitting that we have studied so far, let us look at a simple example to illustrate all these facts.

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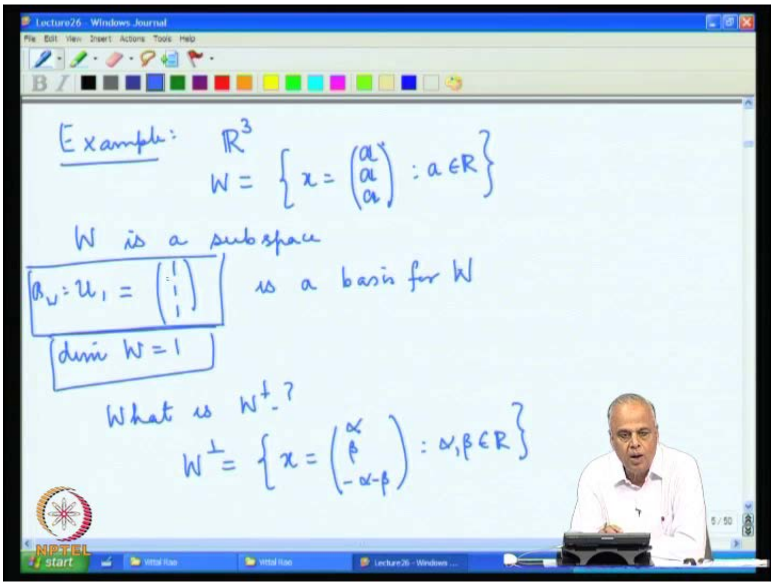

Let us consider R3 and consider the subspace W which consists of all vectors of the form a a a where a belongs to R that is, it is the subspace consisting of all those vectors which have all components equal to each other, the first component is a second component is a and third component a so to say the collection of all vectors which have all components equal to each other.

Now, W is a subspace we have seen this before and if you take Bw to be the simple vector 1 1 1 then, this is a basis for W you see easy to verify that this is a basis for W because 1 1 1 as all components equal and therefore, u1 belongs to W and any vector in W is just a times u1 and therefore, a linear combination of u1 and u1 is linearly independent and therefore, Bw is a linearly independent set in W which spans W and hence, it is a basis for W. Now, what do we get immediately out of this the dimension of W is equal to 1 so this is a basis for W and the dimension of W is equal to 1 because there is only 1 vector in the basis.

What is W perp? we have already seen this in the previous lecture and the W perp turns out to be this set of all vectors of the form alpha beta minus alpha minus beta where alpha and beta are real numbers. That is the set of all those vectors whose sum of the components must be 0 the sum of the components must be 0 because it must be orthogonal to the basis vector of W which says since, all the components of the basis vectors are 1 because or dot product will be the sum of the components so the sum of the components must be 0 and hence, W perp is of this form if it of this form.

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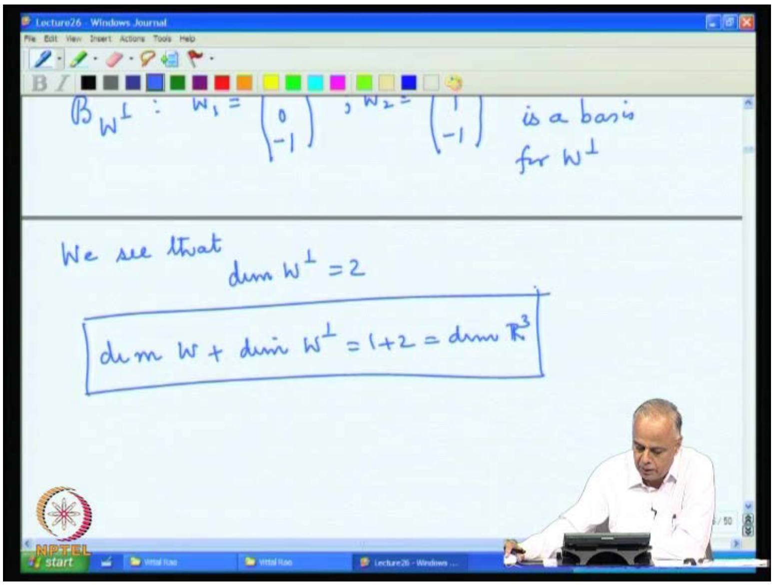

We can easily see that BW perp consisting of these following vectors W1 equal to 1 0 minus 1 W2 is equal to 0 1 minus 1. These two vectors are obtained as follows by putting alpha equal to 1 and beta equal to 0 we get W1 by putting alpha equal to 0 and beta equal to 1 we get W2. Therefore, W1 and W2 are vectors in W they are obviously linearly independent and any vector in W perp is alpha times W1 plus beta times W2 and Hence, the span W perp so these two vectors are in W perp they are linearly independent they span W perp and hence they form a basis for W perp.

Now, we see that since the basis consists of two vectors dimension of W perp is 2 and therefore, we immediately see that dimension of W plus dimension of W perp is equal to 1 plus 2 which is the dimension of the whole space. This is one of the properties which we looked at last time this illustrates the fact that the dimension of a subspace plus the dimension of the orthogonal complement must always be equal to the dimension of the whole space.

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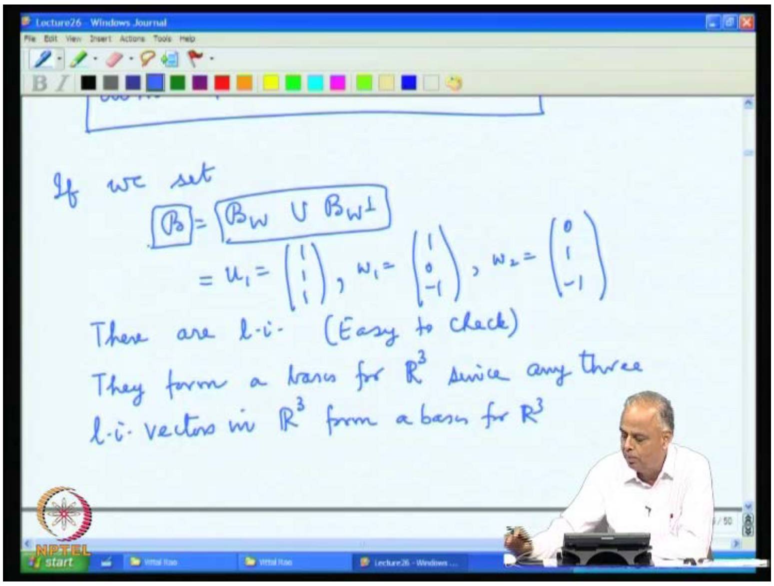

We also see that if we set B equal to BW Union BW perp we get the vectors u1 equal to 1 1 1 comma W1 equal to 1 0 minus 1 and W2 equal to 0 1 minus 1.Now, it is very easy to see that these three vectors are linearly independent so, these are linearly independent I will just put easy to check, these are the linearly independent vectors but we are in the space R3 in R3. since the dimension is three any three linearly independent vectors will form a basis and therefore, they form a basis for R3 since, any three linearly independent vectors in R3 form a basis for R3.

This illustrates the fact that if you put together a basis for W and a basis for Wperp you will get a basis for wv the whole space so, if you put a basis for W and a basis for W perp together you will always get a basis for the whole space which is what gives the fact that the dimension of w plus dimension of Wperp is the dimension of the whole space.

Now, let us look at the orthonormal versions of this basis.

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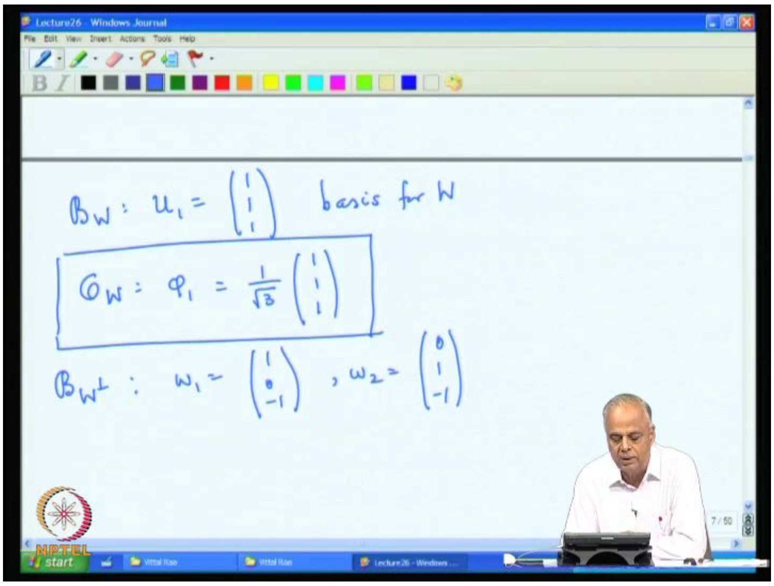

Now, let us look at BW which is a basis for W now, since there is only 1 vector to get an orthonormal basis all we have to do is normalize it to have length 1 and therefore, the orthonormal basis for W will be just dividing by the length the u1 divided by its length gives us ow so this is an orthonormal basis for W.

Now, let us look at BW perp may recall we have got the W perp basis as W1 and W2 this is the basis for the W perp let us take these two basis W1 which is 1 0 minus 1 and W2 which is 0 1 minus 1. In order to extract an orthonormal basis out of it we have to apply the Gram Schmidt process.

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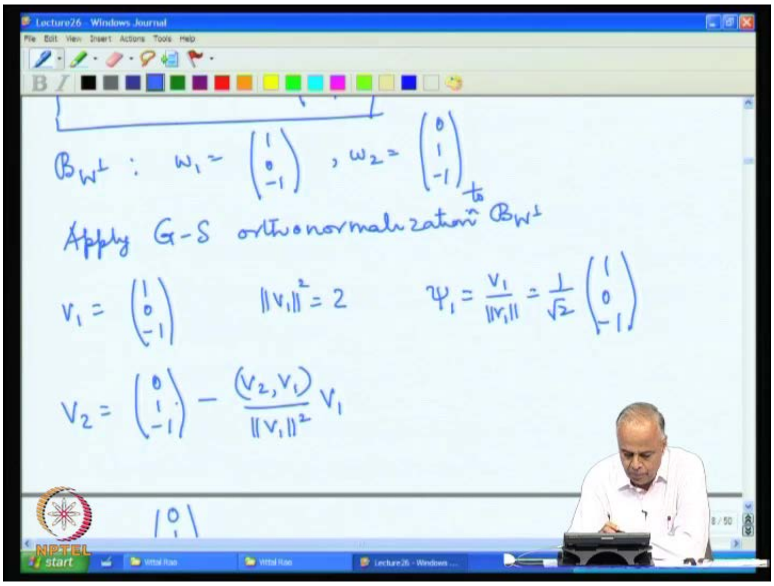

So, apply the Gram Schmidt process orthonormalization to BW perp. We apply the gram Schmidt orthonormalization to BW perp the result will be we will get an orthonormal basis ow perp for W perp. How does the orthonormalization process go first, we start with the first vector w1 and call it as V1 find its length squared which is 2 and then consider the vector psi 1 which is obtained by dividing V1 by its norm which in this case is 1 by root 2 1 0 minus 1.

So, that is our first vector in the orthonormal basis in order to get the second vector we start with the second vector which is 0 1 minus 1 the second vector in the basis BW perp is W2. We start with that vector and then, we subtract the dot product of V2 with V1 divide by its length and multiply by V1 this is what the first step of orthogonalization in the Gram Schmidt process. What does this give us 0 1 minus 1 dot product of V2 and V1 is 0 times 1 plus 1 time 0 plus minus 1 into minus 1 which is plus 1 so, V2 V1 is 1 norm V2 V1 squared is 2 into V1 which is 1 0 minus 1 which gives me minus half 1 and minus half.

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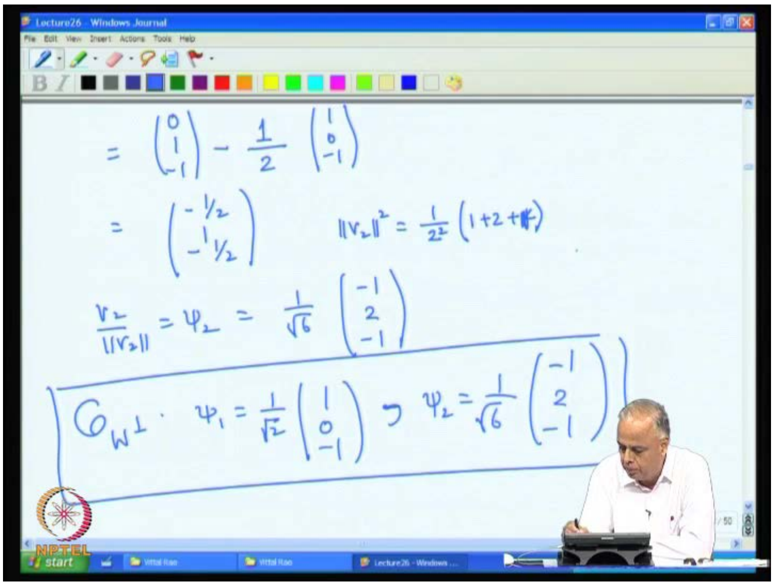

Therefore, next we calculate norm V2 squared which is 1 by 2 into 1 by 1 plus plus 2 plus 1 and then 2 squared so, the 1 by 2 squared has been pulled out now, if you divide V2 by norm V2 we get psi 2 which is the second vector in the orthonormal base. So, we get ow perp to be psi 1 which is equal to 1 by root 2 into obtained psi 1 here 1 by root 2 into 1 0 minus 1 1 by root 2 into 1 0 minus 1 and then psi 2 which is 1 by root 6 into minus 1 2 minus 1 so, thus we have the orthonormal basis for W perp so, phi 1 was our orthonormal basis for this is the orthonormal basis for W and now we have the orthonormal basis for W perp.

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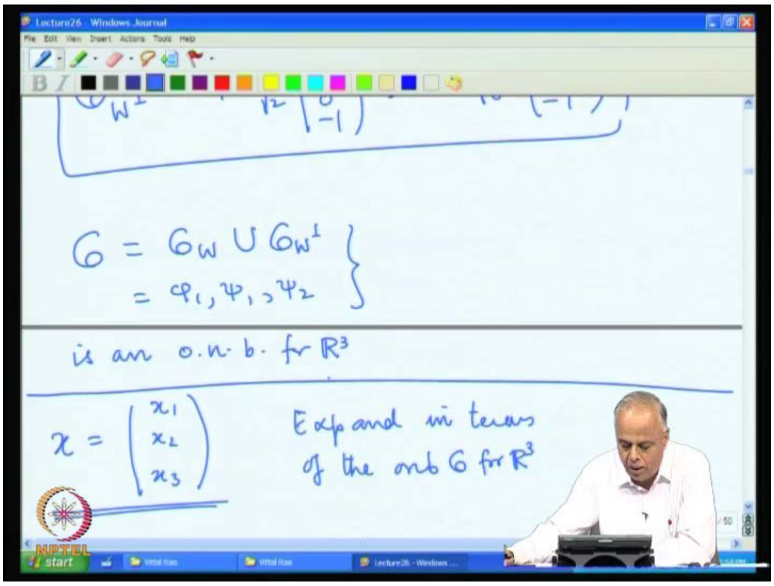

Now, if you put these two together o equal to ow union ow perp which gives me phi 1 psi 1 and psi 2 this is an orthonormal basis for R3. So, it is easy to check they are orthonormal and there are three vectors and therefore, they will form an orthonormal basis thus, we see that if you put together an orthonormal basis for W and an orthonormal basis for W perp you always end up with an orthonormal basis for the whole space.

Now let us take any vector x we are now going to look at the splitting of the vector we has now seen the splitting of the basis, we have seen the splitting of the orthonormal basis and now, we are going to look at the splitting of the x. The basis was split into the part in W and the part in W perp the orthonormal basis was split into the orthonormal basis part of W and the orthonormal basis for W perp. Now, we are going split the vector x into the part in W another part in W perp. Let us look at that splitting so, we have x we can expand this in terms of this basis o so, expand in terms of the orthonormal basis o for R3. Now, how do I get the orthonormal basis the expansion is we know that whenever we have an orthonormal basis the expansion is x comma phi 1 phi 1 the coefficient of the phi 1 vector is at the dot product of x with phi 1.

Similarly ,the coefficient of the psi 1 vector is the dot product with psi 1 the coefficient of the psi 2 vector is the dot product with psi 2. Let us find each 1 of these numbers so let us call this as 1 what is x phi 1? x phi 1 we have phi 1 is this vector 1 by root 3 into 1 1 1 so, if a dot product x with that I get x1 plus x2 plus x3 by root 3 next, what is x psi 1? we have psi 1 is this vector 1 by root 2 into 1 0 minus 1.

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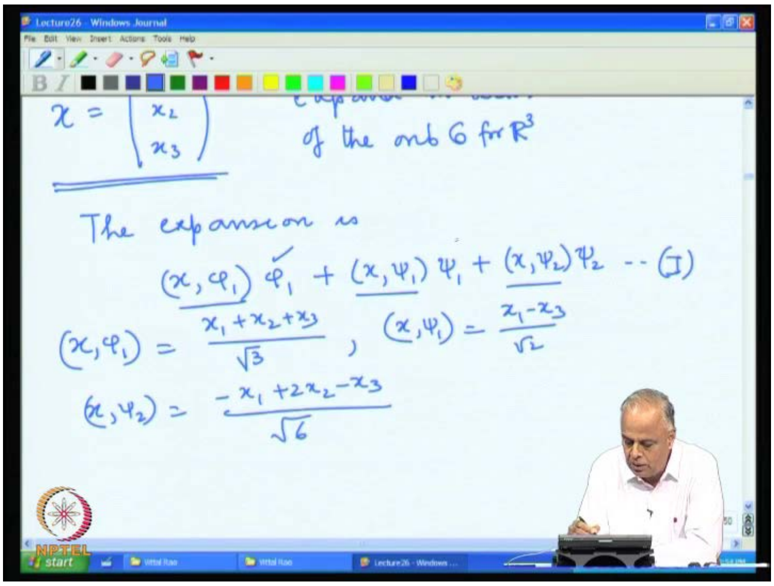

And therefore, if a dot product with that we will get x1 minus x3 by root 2 and what is x psi 2? again with dot product with that vector psi 2 1 by root 6 into minus 1 2 1 so, we get minus x1 plus 2 x2 minus x3 by root 6. If you substitute these values here x phi 1 x psi 1 x psi 2 and we know what phi 1 is and what psi 1 is and what psi 2 is.

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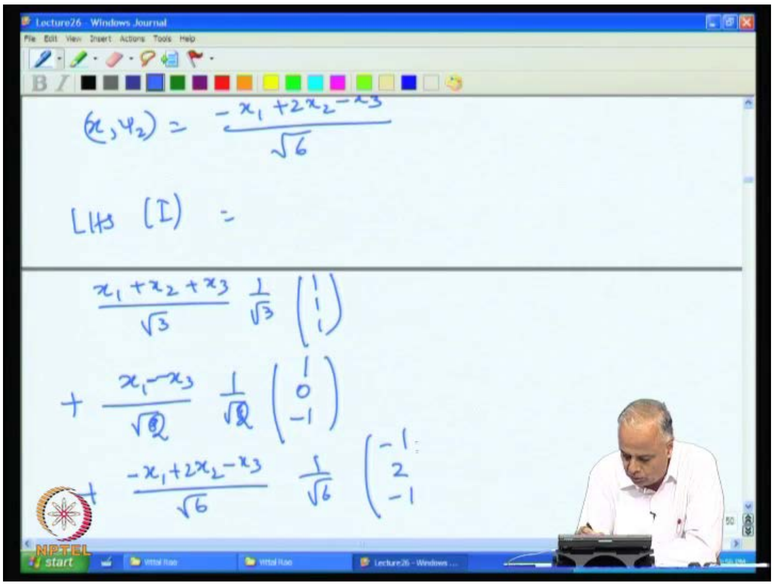

So therefore, the LHS of 1 becomes x by 1 which is x1 plus x2 plus x3 by root 3 into phi1 1 1 1.Then plus the x psi 1 which is x1 minus x3 by root 6 into 1 by root 6 into minus 1 2 minus 1, which is the x psi 2 psi 2 the last one is x psi 3 x psi 2 psi 2 this will be 1 0 minus 1 and this will be 1 by root 2 1 by root 2. Then, the last 1 is x psi 2 psi 2 which gives me the root 6 which is minus x1 plus 2x2 minus x3 by root 6 into 1 by root 6 into minus 1 2 minus 1. If you simplify this becomes I will write this part separately and this part separately because, the first part here belongs to W and this part belongs to W perp because psi 1 and psi 2 this psi 1 and psi 2 belong to W perp this linear combination belong to W perp.

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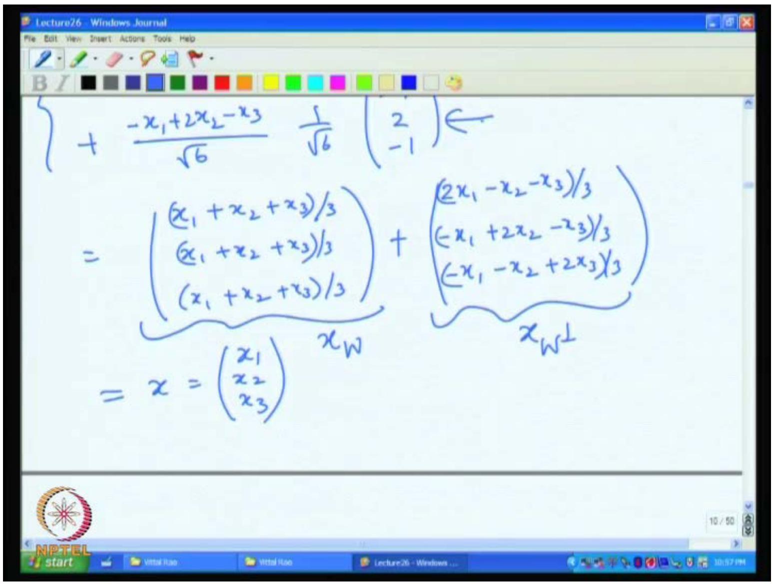

And since this vector belongs to W that part will belong to W so, we will just write it as x1 plus x2 plus x3 by 3 x1 plus x2 plus x3 by 3 x1 plus x2 plus x3 by 3 plus, If you now, simplify the other part you are going to obviously get 2 x1 minus x2 minus x3 by 3 minus x1 plus 2x2 minus x3 by 3 and then, minus x1 minus x2 plus 2x3 by 3 and when you add you get equal to x. So, thus we have the vector x has been split into two parts this part is the xW and this part the xW perp so, every vector x can be split into two parts one part belong to the W space the other part belonging to the orthogonal complement.

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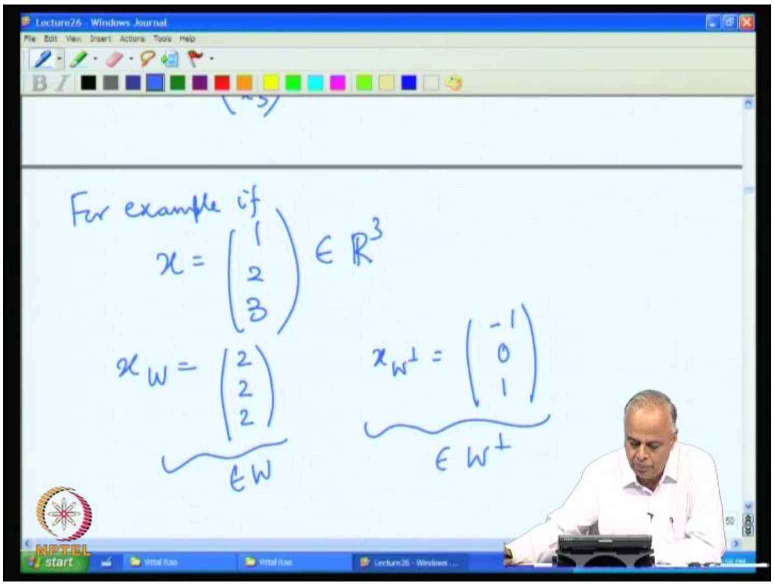

For example, if x is equal to 1 2 3 is the vector in R3 then, what is xW the splitting now, the splitting simply takes the average of the three components and you get 1 plus 2 plus 3 by 3 which is 6 by 3 6 by 3 6 by 3 and xW perp is just minus 1 0 1.

Where these two you observe that this belongs to W because all the components are equal recall the space W consists of all the vectors whose all components are equal and this belongs to W perp because it is orthogonal to all the vectors in W.

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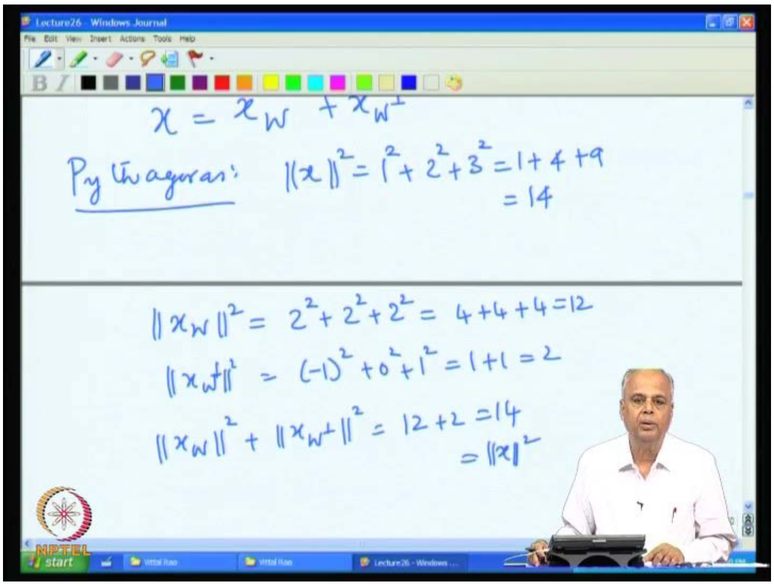

Thus, we see that the vector x equal to 1 2 3 has been split into xW plus xW perp. Let us check the Pythagoras Theorem for this, what is norm of x squared? Since, the vector x is 1 2 3 the norm x squared is 1 squared plus 2 squared plus 3 squared which is 1 plus 4 plus 9 which is 14. On the other hand norm of the length of the xW squared xW the vector whose components are 2 2 and 2. So, this length squared will be 2 squared plus 2 squared plus 2 squared which is 4 plus 4 plus 4 which is equal to 12 and the length of xW perp squared is minus 1 squared plus 0 squared plus 1 squared which is 1 plus 1 which is 2 and therefore, xW squared plus xW perp squared is equal to 12 plus 2 which is 12 which was the normal x square this is the one that now verifies the Pythagoras Theorem.

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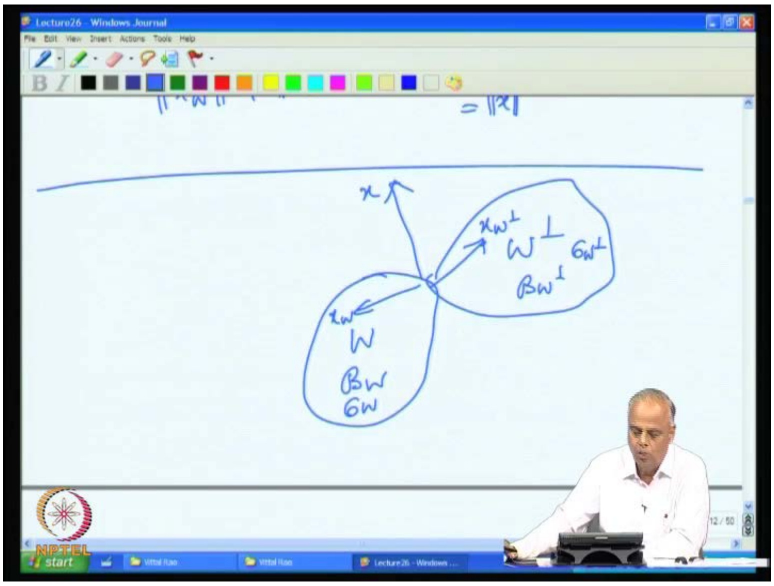

So, we have therefore, suppose we have a subspace W in Rk then there is a subspace W perp which is orthogonal to it and any vector x can be split into a portion here and a portion there this is the xW and this is the xW perp and that is x any vector x can be split into two parts and that splitting is unique and the basis can be split into a basis here and a basis there and an orthogonal basis can be split into orthogonal base, which means that in order to study the whole space we can split the study into two parts and the study of W and the study of W perp because now every vector is split into two parts every basis is split into two parts every orthonormal basis is split into two parts.

Therefore, the vector space is made up of these two pieces the W and the W perp knowing W and W perp we can analyze the whole space now, given the vector space which is the W that we choose in order to analyze which subspace we will choose and who whose orthogonal complement we will choose so, that the splitting is useful for us and therefore, the splitting of the vector space in terms of a subspace and it is orthogonal complement depends on the problem that we have in hand. Now, we shall focus mainly on the problems connected with matrices and therefore, we shall look at matrices and the orthogonal complements and the subspaces and the orthogonal complements in order to do that, Let us first look one more simple property of orthogonal complements..

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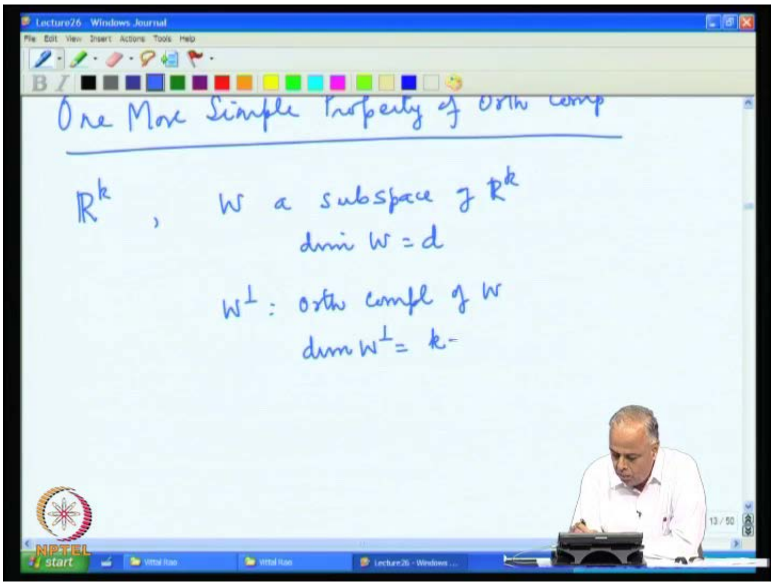

So, let us look at one simple property of orthogonal complements so, let us take Rk and W a subspace of Rk say dimension of W is equal to is equal to d let us say the dimension of W is equal to d. Then W perp is the orthogonal complement of W then we have seen the dimension of W plus dimension of W perp is the dimension of the whole space so, the dimension of W perp must be the k minus d.

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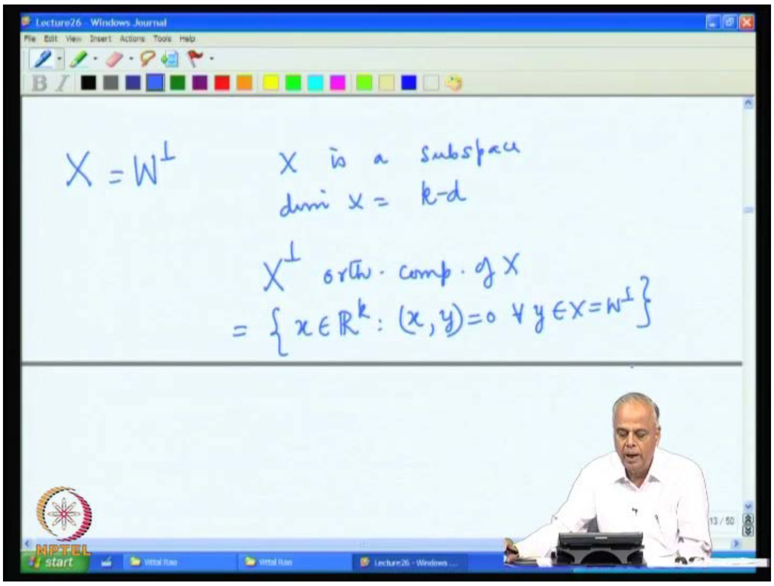

Now, look at the subspace W perp let me call X as W perp. Now, X is a subspace dimension of X is k minus d, since X is a subspace we can talk about X perp; X perp orthogonal complement of X. This is the set of all vectors X in W X in R k such that x y is equal to 0 for every y in X which is X is W perp. So, in other words X perp is the set of all the vectors which are orthogonal to all the vectors in W perp.

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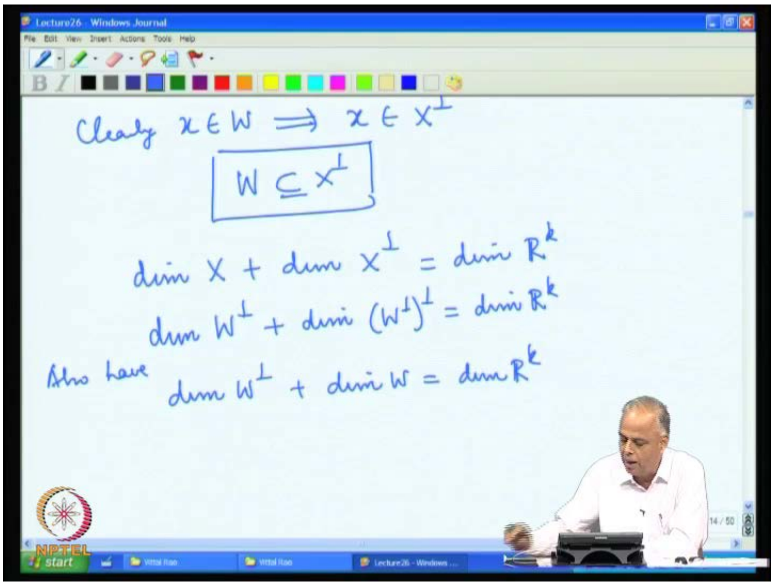

Clearly x belongs to W implies x belongs to X perp because, every vector in W is orthogonal to every vector in W perp and therefore, we have W is contained or equal to X perp. So, that is one thing we observe.

Also the dimension of X plus dimension of X perp whenever you take a subspace and its orthogonal complement the dimensions add up to the dimension of Rk so, this says dimension of W perp because X is W perp plus dimension of W perp perp X is W perp so, X perp perp is W perp perp is equal to dimension of Rk. We also have dimension of W perp plus dimension of W is equal to dimension of Rk because, again the dimension of subspace plus its dimension, of its orthogonal complement will be equal to the dimension of the whole space.

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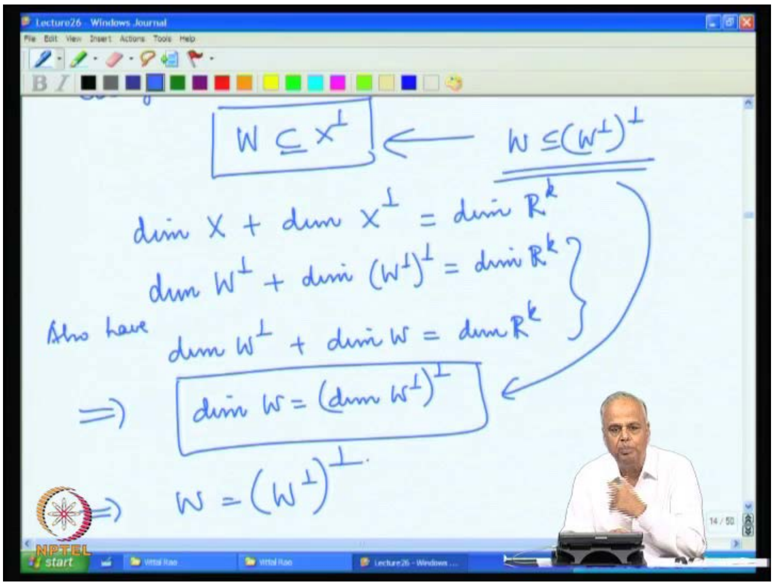

Now, compare these two we find that the dimension of W is equal to the dimension of W perp perp. Now, W is a subspace is contained in X perp, X perp is nothing but W perp. perp. So, W is contained in W perp perp, but its dimension is the same as W perp perp so, the two together says W equal to W perp perp, so the second perp is the original space.

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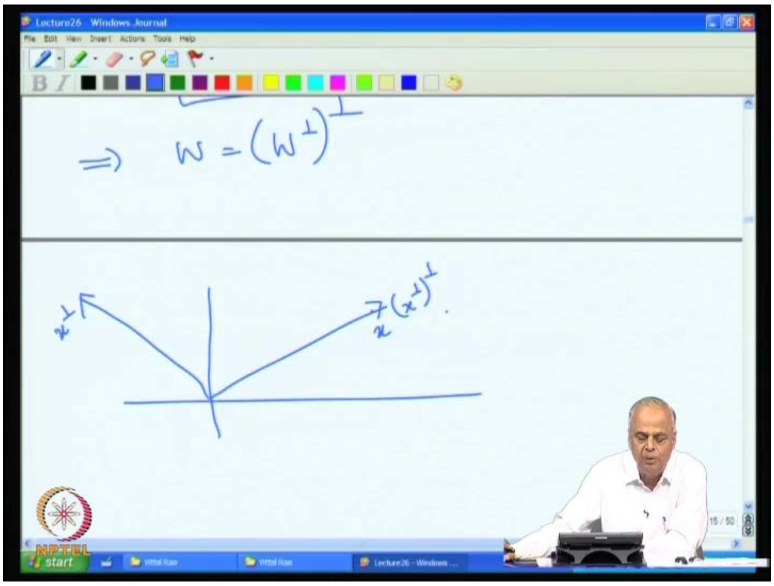

This is an akin to the fact that suppose I take L vector and I take a vector which is perpendicular to it again I take the perpendicular to this I get back to this original vector so, this is x this is x perp and this is x perp perp so, we do double perpendicular you get back to the original one and that is exactly what is being observed here in the general contact the W perp perp is equal to W. Now, as we mentioned that all these decompositions you will have to study with respect to a particular problem given the problem what is the W ,we must choose and therefore, the corresponding W perp then split the problem into two parts by splitting the space, by splitting the basis, by splitting the orthonormal basis, and by splitting the vectors, and therefore, a priori thing is what is the subspace that we should choose in order to splitting and now, we will analyze this in the context of matrices.

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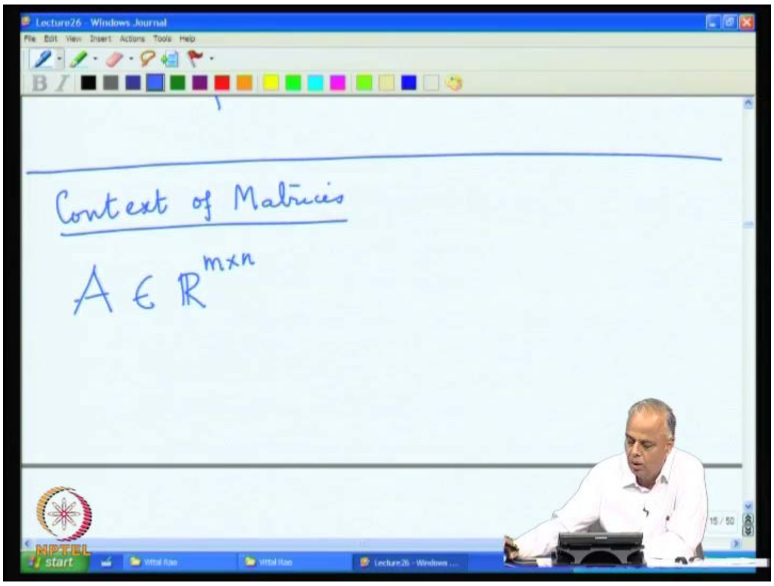

Hold the splitting has to be analyzed in the context of matrices so, let us now consider A to be an m by n matrix. We have seen that with every matrix we can associate four important fundamental subspaces.

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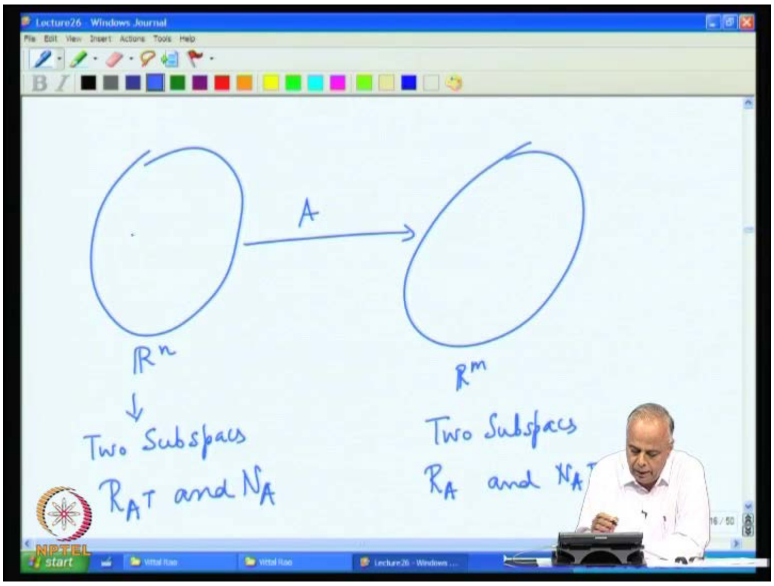

The four fundamental subspaces, there are two subspaces of Rn and two subspaces of R m if A is an m by n matrix they will be two subspaces of Rn so, we have Rn on this side and we have Rm on this side and A it takes n vectors to m vectors.

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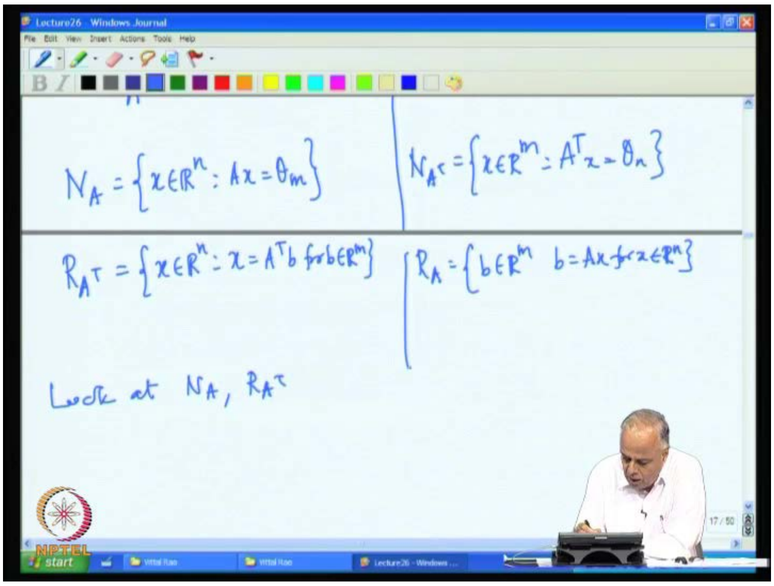

Now we are going to get two subspaces on the Rn side the two subspaces that we had where the range of A transpose and the null space of A. The range of A transpose and the null space of A where subspaces of Rn and similarly, there were two subspaces on the Rm side they were the range of A and the null space of A transpose.

What are these subspaces? the null space of A is the set of all vectors in Rn which get mapped with the 0 vector under this transformation, and RA transpose is the set of all vectors in Rn such that x is of the form A transpose b for b in Rm. It is the image of some vector the range is it consider a vector x which is the image of some vector under the transformation A transpose. On the other hand in the Rm side they have NA transpose which is the set of all vectors in Rm such that A transpose x is equal to theta N and RA will be set of all vectors b in Rm. Such that b is equal to Ax for x in Rm. these now, the image of A. So, RA transpose is the image of the function A transpose as a function from Rm to Rn RA is the image of the function A as a function from Rn to Rm.

Now let us look at the pair NA RA transpose which is in Rn whatever we do analogous results we will get on the Rm side.

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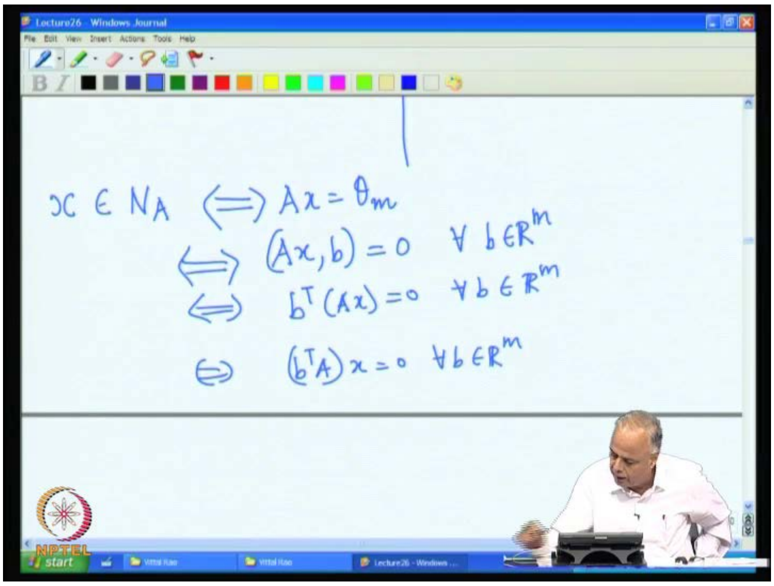

Now therefore, we are going to look at the subspaces NA and RA transpose on the Rn side. Suppose A vector x belongs to the null space of A this can happen if and only if Ax equal to theta m this is what is meant by saying that x is the null space of A that means x get annealed by the matrix A.

Now if Ax is the 0 vector the only way something can be 0 vector is it is orthogonal to all the vectors. Since, it is 0 vector on the Rm side it must be orthogonal to all the vectors on the Rm side that means its dot product with all the vectors on the Rm side must be 0. So, Ax comma b must be equal to 0 for every b in Rm but what is the definition of the dot product? it is b transpose Ax must be equal to 0 for every b in Rm this means b transpose Ax equal to 0 for every b in Rm because, matrix multiplication is associative we can group them as we want, we can write this as a transpose b transpose x equal to 0 for every b in Rm.

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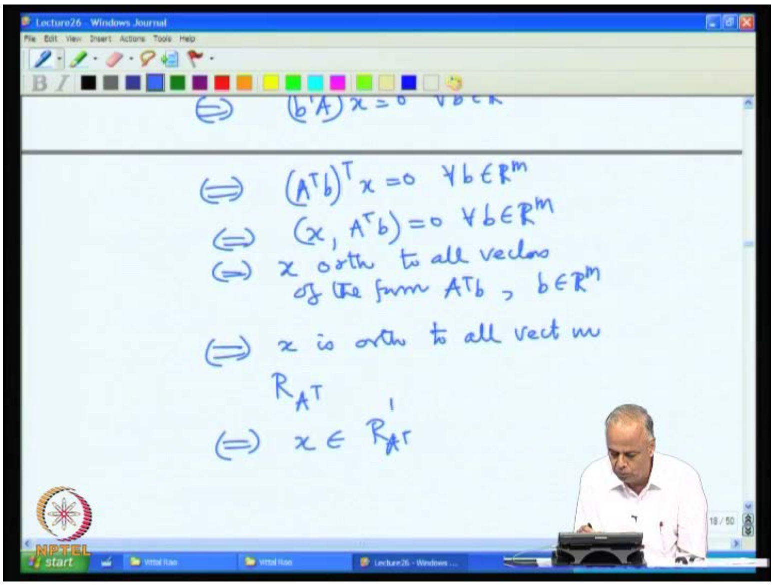

So, what this says is x is orthogonal to all the vectors of the form A transpose b so, x comma A transpose b is equal to 0 for every b in Rm. That is if x is orthogonal to all vectors of the form A transpose b, b belonging to Rm but if you now look at the definition of the range of A transpose that is precisely the definition the vectors in the range of A transpose are all vector to the form A transpose b and therefore, this means x is orthogonal to all vectors in range of A transpose which means x belongs to range of a transpose perpendicular thus we see that the null space of A is the same as range of A transpose perpendicular.

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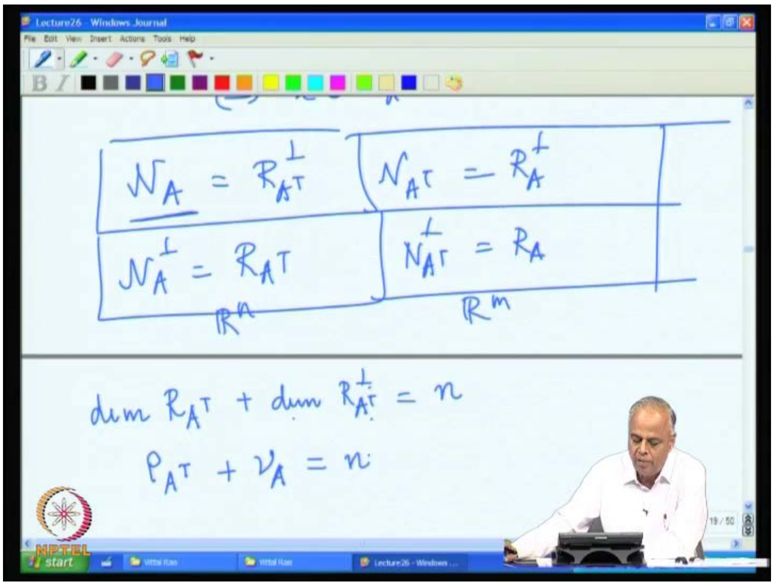

Since, the perp of the perp is itself we get NA perp is RA transpose perp perp which is equal to RA transpose analogous to this, on the other side we get NA transpose is the range of A perp and NA transpose perp is the range of A so, this is on the Rn side and this is the Rm side.

Now, what does this mean ? we have dimension of range of A transpose plus dimension of range of A transpose perp. Whenever we take a subspace and its orthogonal complement and we look at the dimensions and their sum it must give me the dimension of the whole space these are all subspaces in Rn therefore, the dimension of the whole spaces n that must be equal to n. Now, the dimension of range of A transpose is what we call as the rank of A transpose RA transpose perpendicular is NA and therefore, dimension of RA transpose perpendicular is dimension of NA dimension of NA is what we call as a nullity of A and therefore, rank of A transpose plus nullity of A is equal to n.

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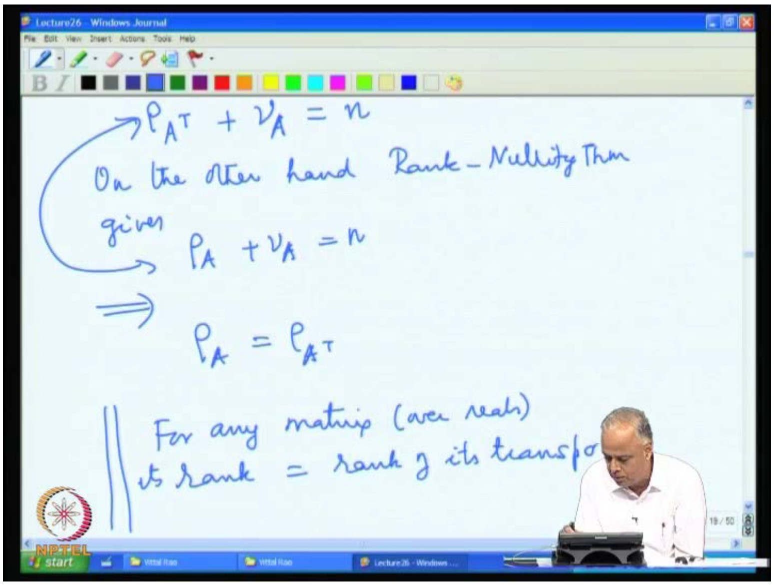

On the other hand Rank Nullity Theorem gives the rank of A plus nullity of A is equal to n now, compare these two we get rank of A is equal to rank of A transpose and therefore, we have an important conclusion that for any matrix over the real numbers rank its rank is equal to the rank of its transpose. So, the matrix and its transpose have the same rank and secondly, the spaces are all orthogonal to each other therefore, the structure is as follows.

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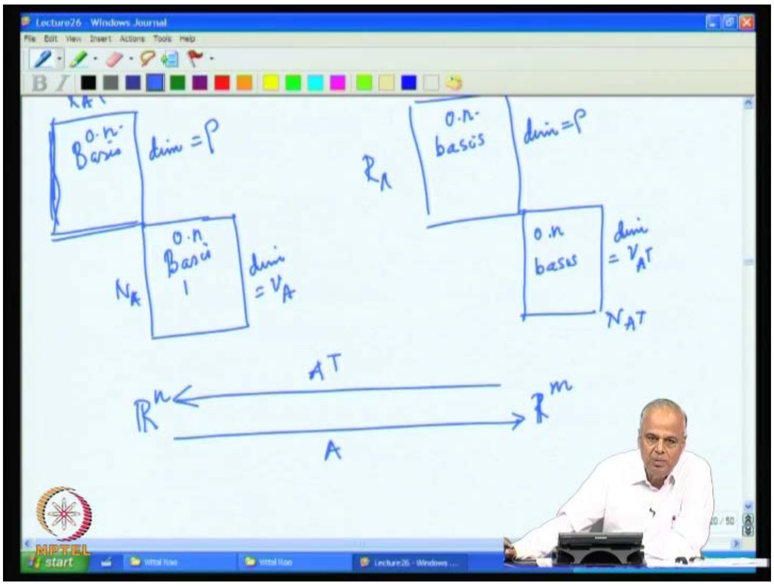

We have R n on the one side Rm on the other side A takes these vectors n vectors to m vectors A transpose takes m vectors to n vectors on the Rn side we have two subspaces the range of A transpose and the null space of A and they are orthogonal to each other that is the thing we have found so far, on this side we have the null space of A transpose at and the range of A which is the and these two are orthogonal to each other.

So, we have four subspaces two pairs, one pair in Rn and this pair is called orthogonal complements of each other another pair in Rm and this pair is also orthogonal complement of each other. Therefore, we have two pairs of orthogonal complements of each other and the main idea in analyzing a matrix is to get a suitable basis for RA transpose, a suitable basis for NA. Now, by our analysis we know that if we get a basis for these two orthogonal fellows one being orthogonal complement of the other by putting them together we can get basis for the whole space.

So therefore, finding a basis for Rn is converted into the problem for a suitable basis for RA transpose and a suitable basis for Na and we would even look for orthonormal basis and similarly, we would look for orthonormal basis here and the orthonormal basis here and notice that because the dimension of the range of A transpose equal to dimension of range of A or the rank of A transpose is the rank of A, RA transpose and RA have the same dimension which we will call as rho. Since, rho A equal to rho A transpose we will not put the suffix A and A transpose we will simply call it as rho the rank of the matrix and the dimension here is also rho the dimension here is the nullity of A and the dimension here is the nullity of A transpose.

So therefore, the main analysis of the matrix will boil down to finding suitable orthonormal basis for these four pieces of subspaces two subspaces on the Rn side and two subspaces on the Rm side and it is this analysis of finding a suitable basis, a suitable orthonormal basis which will occupy our attention for almost the rest of the course. therefore, we have to understand what is meant by a suitable basis, how to choose this orthonormal suitable basis, how you put all this information together to get the answers to all the questions that we raised about a matrix. Now, before we do that let us also observe one fact.

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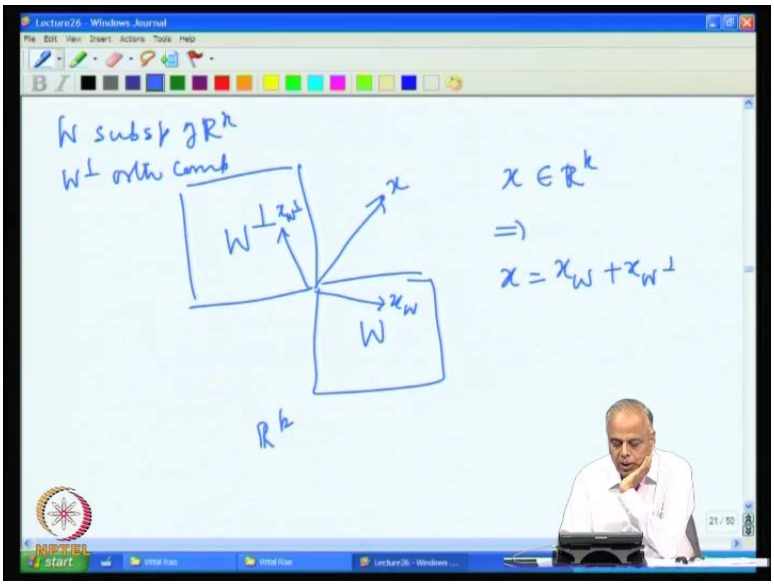

Suppose, I have a Rk and I have a subspace W and I have a orthogonal complement W perp, we said that if you take a vector x in the space it can be split into two parts xW and xW perp. Therefore, x belong to Rk so, we have W subspace of Rk and w perp orthogonal complement then x belong to Rk implies x can be written as xW plus xW perp.

Now what is this role of this xW? Suppose now, we have this vector x and we do not have any information of W perp and therefore, we would like to get as much information about x from W alone.

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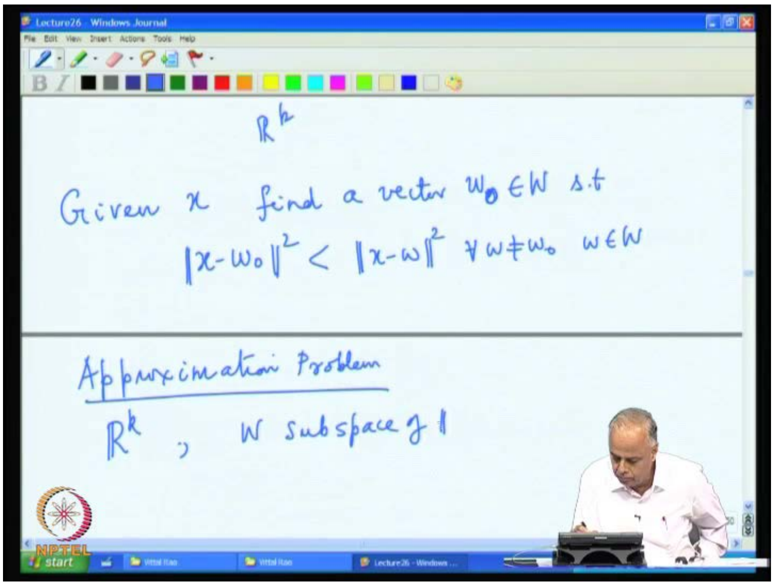

What this means is given x find a vector which we will call as W naught in W such that if you now look at the suppose, I take W naught as the approximation for x then the error is x minus W naught and the quantification of the error squared is the length squared so this is the error by taking W naught as the approximation of x.

Now, I want to find a W naught such that that is the least err that is if you take any other W in W the error must be more x minus W naughts must be giving the least error. So, in other words can we approximate x in W with the least error so, this is called the approximation problem. So, approximation problem is the following Rk and then we have W subspace of Rk.

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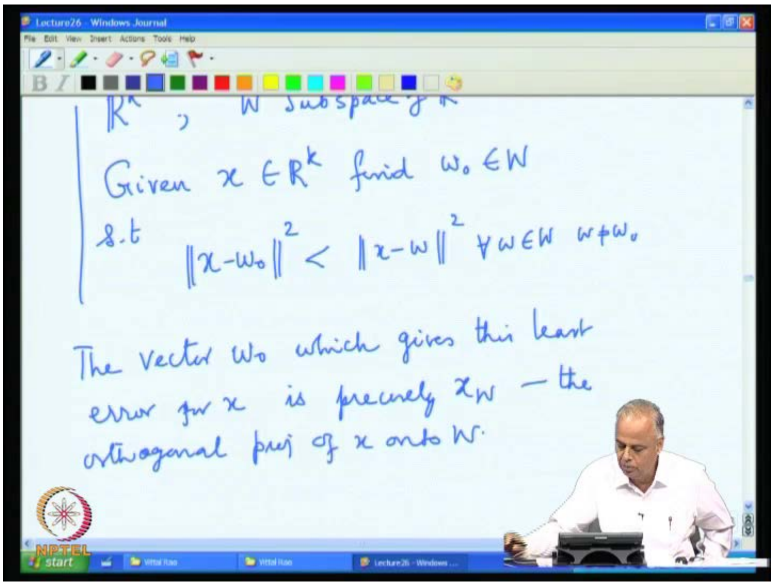

Then given x in Rk find W naught in W such that if we take W naught as the approximation for x the error length of the error squared is less than the error for any other vector in W. This is called the actually we will call it the best approximation problem because any other vector can do only a worst job because the error will be more here we are trying to minimize the error.

Now, we will see that the vector x naught we do not even know that whether there is a solution for this problem .We shall see in the next lecture that the vector W naught which gives this least error for x is precisely xW the orthogonal projection the orthogonal projection of x onto W.

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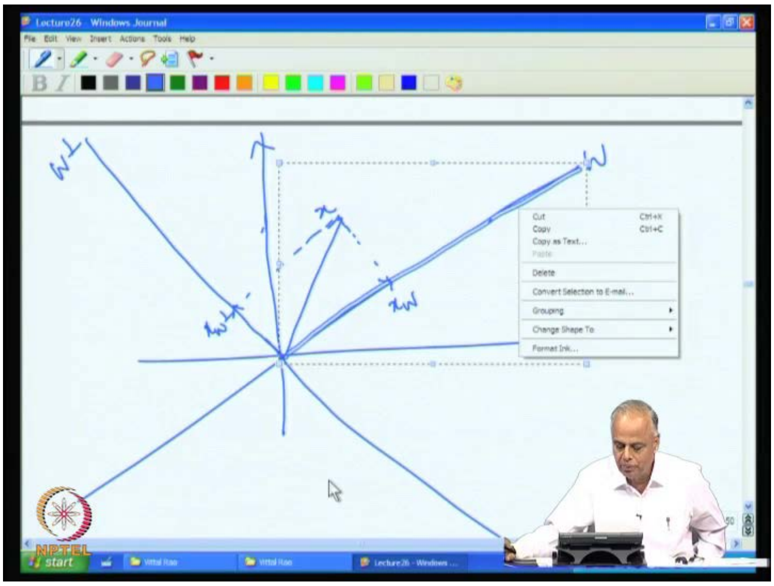

You can visualize this as follows, suppose,we look at two dimensions and we have a subspace a subspace in two dimensions is a line through the origin and then you take any vector x. Now, what is orthogonal complement of W it is that perpendicular line so that is W perp now, take any vector x what is it splitting this is the orthogonal projection that is drop the perpendiculars this vector is xW and this vector is xW perp. If you see if you take any vector on the line wW its distance from x will be more than the perpendicular distance this geometric fact is what we are trying to prove in the theorem.

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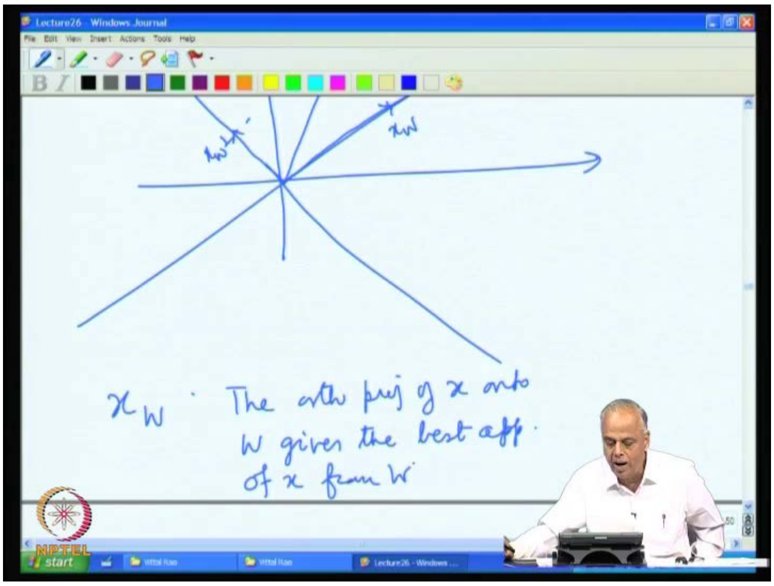

So, we find that the orthogonal projection xW perp xW the orthogonal projection of x onto W gives the best approximation of x from W so, if this simple geometric fact is what we are going to approximate and now this fact will be used eventually when we are analyzing this matrices. Recall that when we had a vector b for which we could not find a solution that will be because it will lie outside the range of A then Ax equal to b will not have a solution then, what we will do is we find the closest vector to be in RA and that closest vector being in the range will have a solution and that will generate the notion of the so, called least square solutions, so these are some of the ideas that will be utilized in analyzing the matrix problem.

In the next lecture we would look at in detail, and get exactly this idea of this best approximation and show that these best approximations are given by this orthogonal projection. Then our main problem will be focussed about finding the suitable basis for the four subspaces. We should analyze what sort of basis will make our work of answering the questions regarding the matrix, the fundamental questions that work must become easier by the choice of our basis, we shall work a strategy towards that direction.