Advanced Matrix Theory and Linear Algebra for Engineers

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Lecture No. # 24

Inner product and Orthogonality - Part 3

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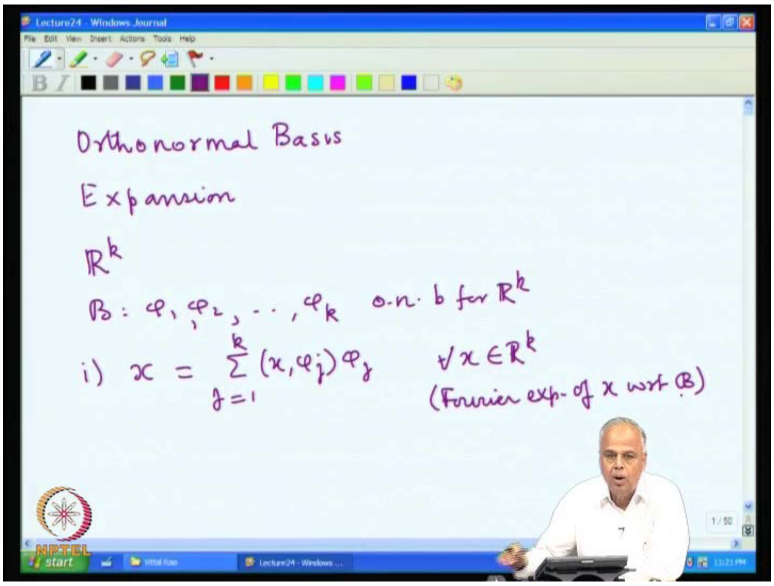

In the last lecture, we introduced the notion of an orthonormal basis. And we looked at the expansion in terms of an orthonormal basis. We found that, if we start with R k and we have any orthonormal basis phi 1, phi 2, phi k an orthonormal basis for R k then, we have the following result. One, any vector x in R k has this expansion x comma phi j phi j, where you recall that x comma phi j denotes the inner product which is same as phi j transpose x and this is true for every x in R k. And this is called the Fourier expansion of x with respect to the orthonormal basis B.

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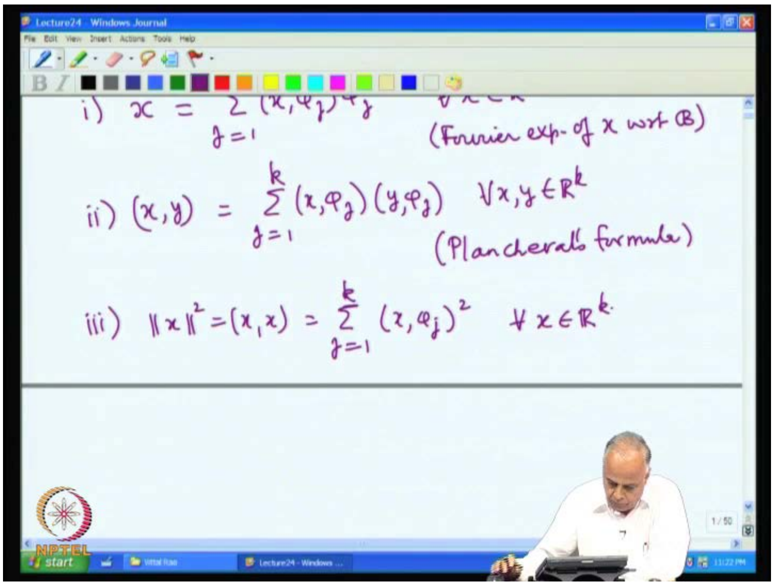

Then we found that, the inner product x comma y is summation j equal to 1 to k x comma phi j y comma phi j for every x y in R k, and this was called the Plancheral's formula. Then we had for the length, the identity that (( )) x square which is x comma x which you put y equal to x above, we get j equal to 1 to k x phi j square for every x in R k.

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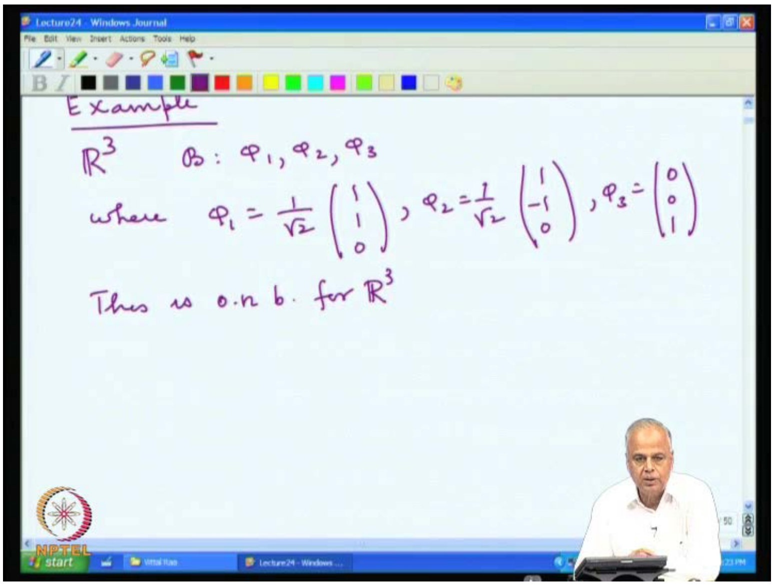

Now, let us look at an example to illustrate all these facts. Let us, take R 3 and let us take B to be the basis phi 1, phi 2, phi 3, where phi 1 is the vector 1 by square root of 2 into 1

1 0, phi 2 is the vector 1 by square root of 2 into 1 minus 1 0, and phi 3 is the vector 0 0 1. We have seen that, this is an orthonormal basis for B. So, this is orthonormal basis for R 3. So, let us look at this Fourier expansion and Plancheral's formula and Parseval's identity.

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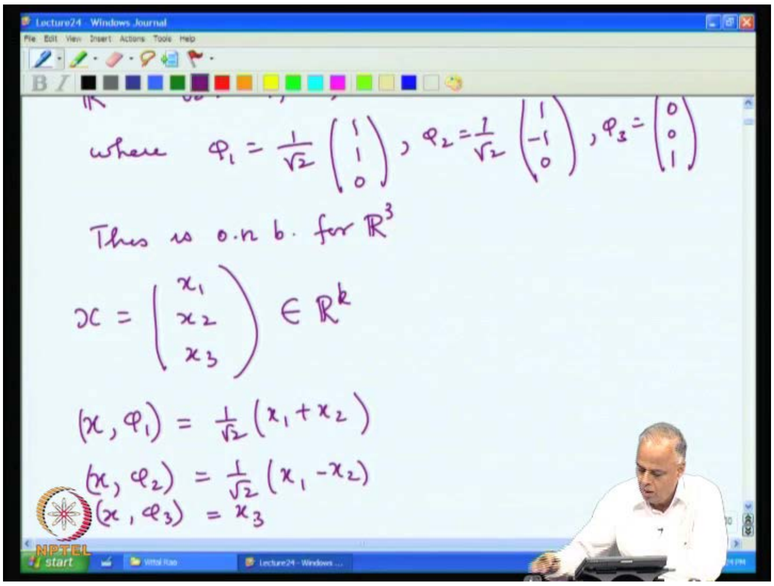

Now, we have take any vector x equal to x 1 x 2 x 3 in R k. So, look at any vector x in R k, we have x phi 1 which is the dot product of x with phi 1, which is 1 by root 2 into x 1 plus x 2 and since the third factor the co efficient here is 0, there is no contribution. Similarly, x phi 2 is 1 by square root of 2 into x 1 minus x 2. And x phi 3 is x 3. So, we have these three Fourier coefficients.

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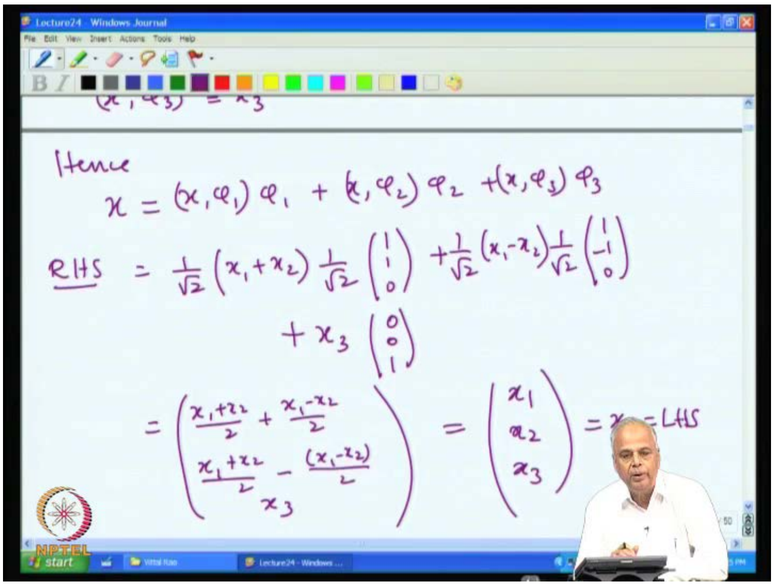

Hence, x must be equal to x phi 1 phi 1 plus x phi 2 phi 2 plus x phi 3 phi 3, let us check whether this is true. The right hand side is, x phi 1 which is 1 by square root of 2 into x 1 plus x 2 which we obtained here into phi 1, phi 1 was 1 by square root of 2 into 1 1 0 plus x phi 2 which we have here, if you substitute that, we get 1 by root 2 into x 1 minus x 2 into phi 2 is 1 by root 2 into 1 minus 1 0 plus x 3, x phi 3 is x 3 times phi 3 which is 0 0 1.

If we now simplify this, this is nothing but, x 1 plus x 2 by 2 plus x 1 minus x 2 by 2, if you look at the first components all along and then, x 1 plus x 2 by 2 minus x 1 minus x 2 by 2 and x 3 which is exactly equal to x 1 x 2 x 3, which is x the left hand side. So, this verifies the Fourier expansion for this vector.

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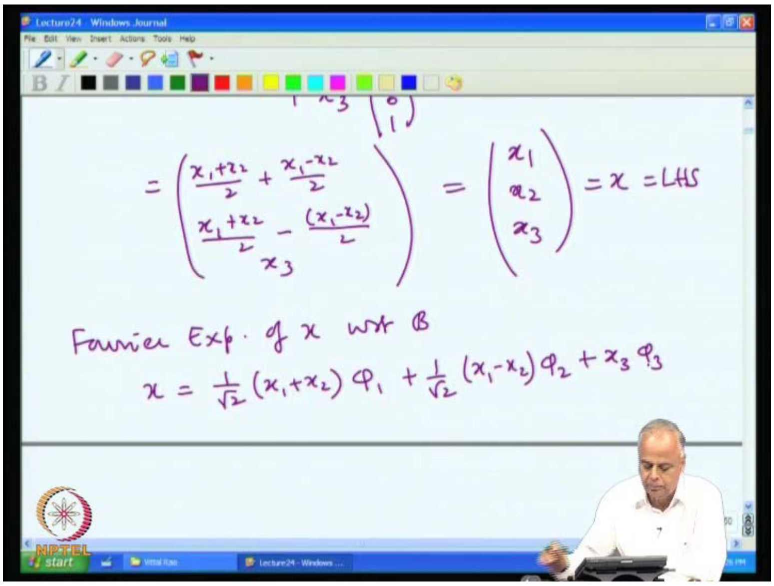

So, the Fourier expansion of x of x with respect to B is therefore, x is equal to x x comma phi 1, which is 1 by root 2 into x 1 plus x 2 into phi 1 plus 1 by root 2 into x 1 minus x 2 into phi 2 plus x 3 into phi 3. So, this verifies the Fourier expansion.

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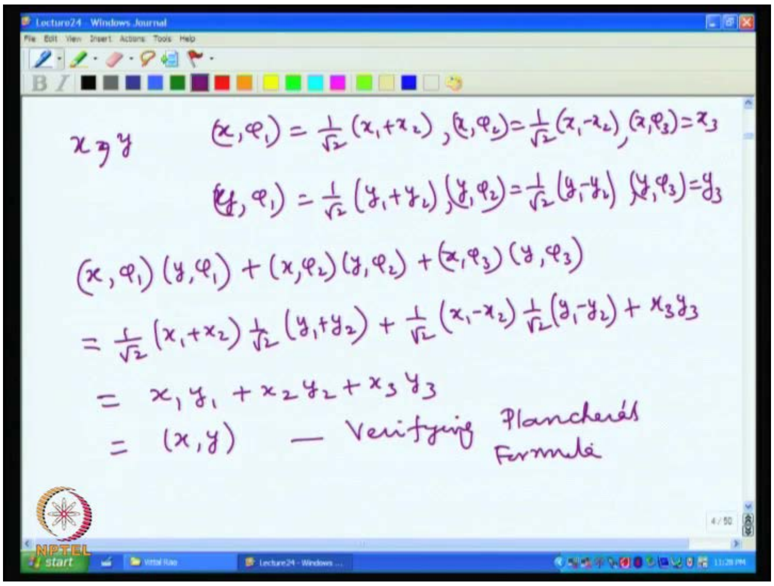

Now, let us look at a vector x and a vector y, then x phi 1 is 1 by root 2 into x 1 plus x 2 as we found above, x phi 2 is 1 by root 2 into x 1 minus x 2 and x phi 3 is x 3, this is what we found. Similarly we have, y phi 1 is 1 by root 2 into y 1 plus y 2, y phi 2 is 1 by root 2 into y 1 minus y 2, and y phi 3 is y 3.

So, if we take the product of the corresponding Fourier coefficients we get x phi 1 into y phi 1 plus x phi 2 into y phi 2 plus x phi 3 into y phi 3, which is equal to x phi 1 is 1 by root 2 into x 1 plus x 2 into y phi 1 is 1 by root 2 into y 1 minus y 2 plus x phi 2 is 1 by root 2 y 1 plus y 2 x 1 minus x 2 into 1 by root 2 into y 1 minus y 2 plus x 3 y 3. Now, if we simplify this, this is nothing but, x 1 plus x 1 y 1 plus x 2 y 2 plus x 3 y 3, which was the inner product of x and y.

So, verifying the Plancheral's formula. What does the Plancheral's formula say? That, the product of the corresponding Fourier coefficients one (Refer Slide Time: 08:06), the product of the second Fourier coefficient, the product of the third Fourier coefficients, then they are all added up we must get the inner product and that is what we get here.

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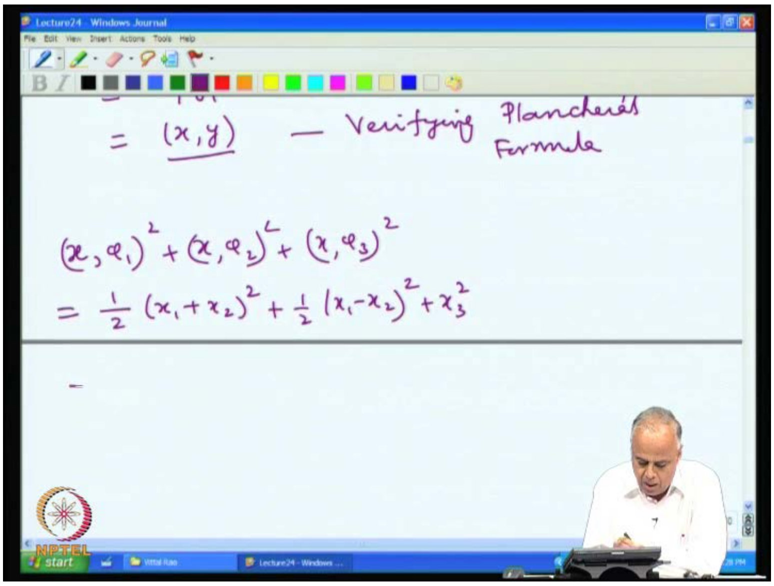

And finally, we have x phi 1 squared plus x phi 2 squared plus x phi 3 squared is equal to x phi 1 was 1 by root 2 into x 1 plus x 2, so this is x 1 plus x 2 square plus 1 by 2 into x 1 minus x 2 square plus x 3 square.

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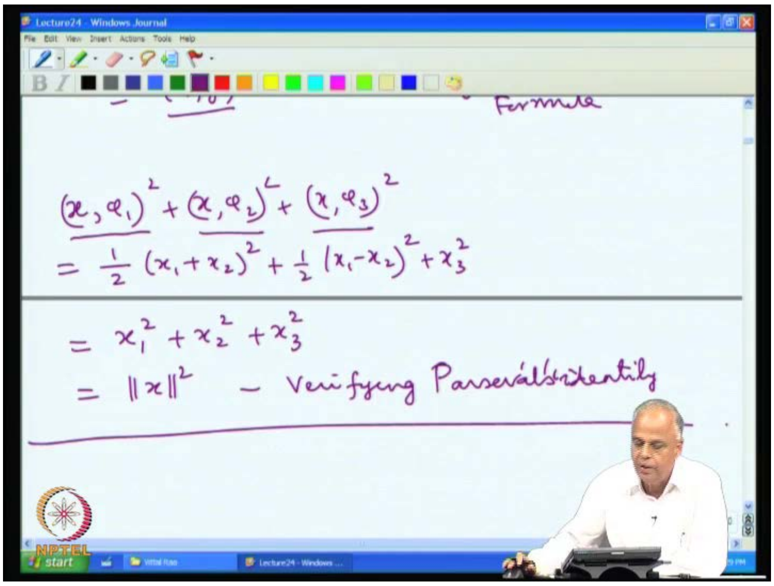

And if you simplify this, this is just the x 1 square plus x 2 square plus x 3 square, which is the length of x square. So, this verifies the fact, the sum of the squares of the Fourier coefficient is the sum of the the square of the first one, square of the second one, the square of the third one. If you add the sum of the squares of the Fourier coefficients, we get the length square and that is verifying the poison's formula sorry this is called the Parseval's identity verifying Parseval's identity.

So, that is we have the expansion in terms of an orthonormal basis giving rise to easy ways of computing the inner product, the (( )) in terms of the Fourier coefficients as we would have done normally with the standard harded orthonormal basis. Then at the end of the last lecture, we raise the following question.

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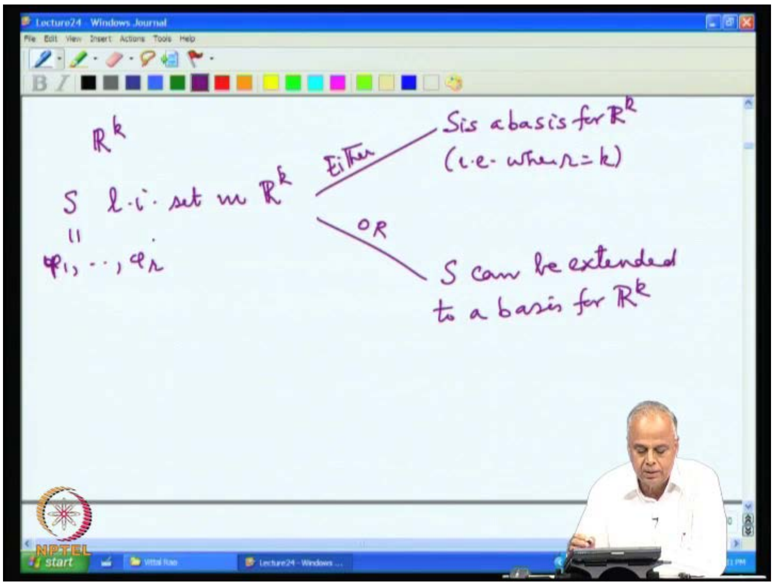

Supposing I have R k and I have S a linearly independent set in R k, then either S is a basis for R k. And when can this happens, that is when let us say s is the set u 1 or phi 1, phi 2, phi r, this will happen when R equal to k. Or S can be extended to a basis for R we have seen this before that, any linearly independent set, if it is not a basis, then we we can (( )) k minus R vector set to get a basis. Now suppose, if it is a basis then which already a linearly independent set and we we have no problem, if it is not, we have to (( )) k minus R.

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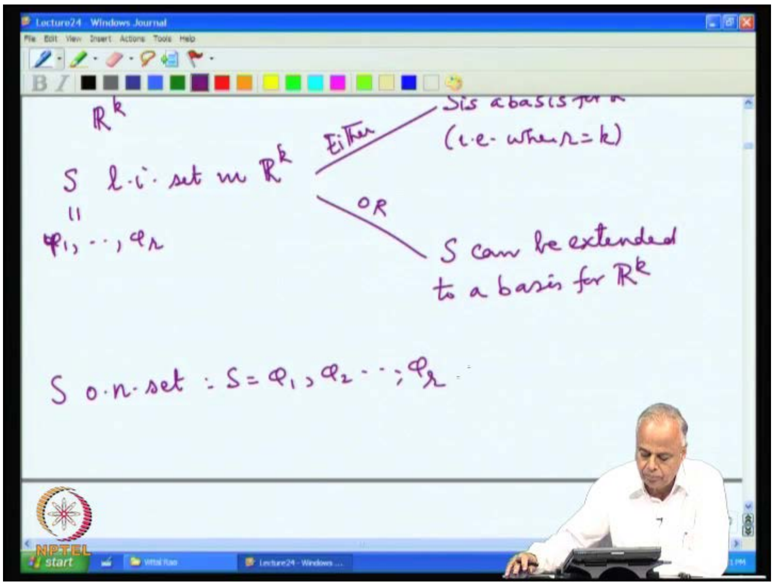

Now, suppose we start with S an orthonormal set, say S is phi 1, phi 2, phi k phi r.

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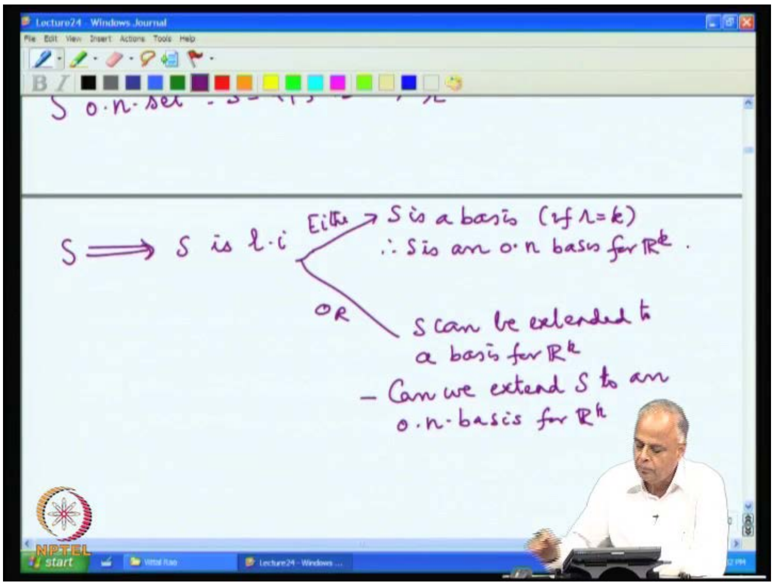

Then again because is orthonormal that says, S is linearly independent. Now, if S is linearly independent as we observed before, either S is a basis that is if r equal to k, but if it is a basis and since it is already a orthonormal set therefore, S is a orthonormal basis. If it were not a basis, S can be extended to a basis.

The question arises the extended basis may not be orthonormal and therefore, we ask can we extend S to a orthonormal basis. So, can we extend S to an orthonormal basis for R k, whenever I said basis I always (( )) R k, for R k. Now, we shall investigate this question.

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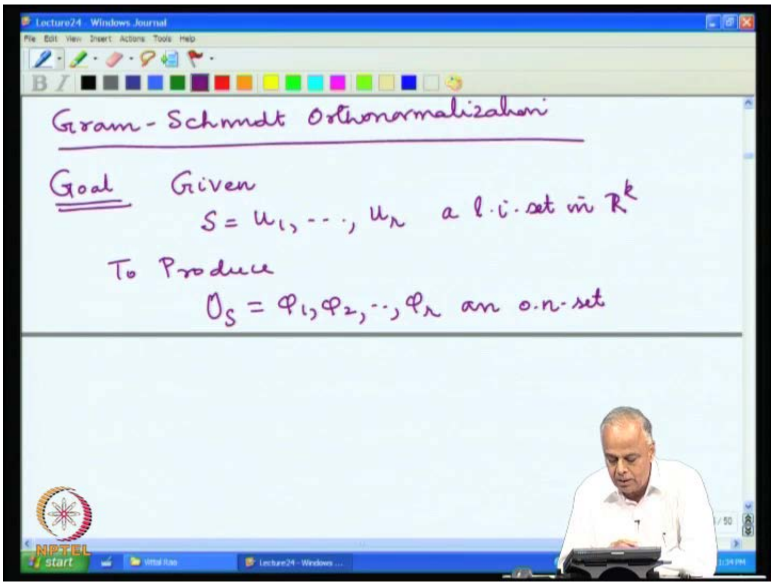

And as I mentioned, the main technique involved in this is what is known as the Gram-Schmidt orthonormalization. Before we describe this procedure let us say, what does this Gram-Schmidt do, what is the goal of this process? The goal of this process is the following, we are given u r a linearly independent set in R k we are given, we are start with a linearly independent set in R k. So, this is the starting ingredient.

Now, we must do something with this then we want to produce O S, O stands for orthonormal, S stands for that you start with this set S. So, O S which we call as phi 1, phi 2, phi r an orthonormal set of the same size as S, S had R vectors, we want O S also to have R vectors, but that is not a big deal.

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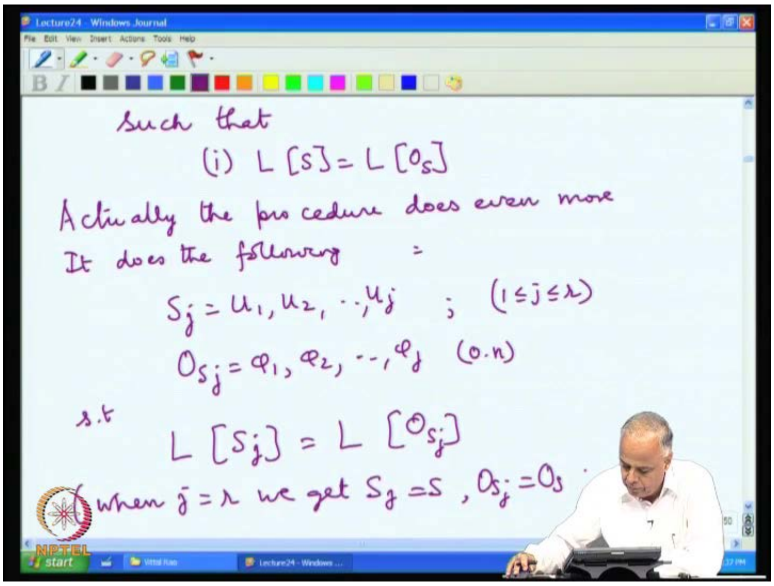

You wanted to be such that, we want to construct this orthonormal set in such a way that, certain things happen. One obviously, we would like that whatever subspace that S spans O S must also span the same subspace such that 1, L S equal to L O S, this is our main aim. Whatever subspace S spans, the O S must also span the same, what happens then? Then O S becomes an orthonormal set in S, it spans S and therefore, it will become a basis for O S. So, this O S orthonormal set in R k.

So, we would like to construct in such a way that, the subspace span by O S is the same as the subspace span by S and hence, O S will become an orthonormal basis for L S. Actually the procedure does even more, it does the following. It constructs this vectors phi 1, phi 2, phi 3, phi r in a recursive manner such that, at each stage for example, at the first stage will construct phi 1, and phi 1 will generate the same subspaces u 1 then at the second stage, we will got phi 2. And the phi 1 and phi 2 will generate the same subspaces u 1 and u 2 it will go on step wise and at the end L S will be L O S.

So, it does the following, L of let us denote S j O S let us use a certain convenient notation before we state this. Let us, take S j to be u 1, u 2, u j we had r vectors u 1, u 2, u r out of these we are selecting the first j vectors. So, the j has to be something between 1 and r.

Then correspondingly, we will construct O S j, which is phi 1, phi 2, phi j that is the first j vectors that we construct orthonormal such that, the space span by S j is the same as the space span by O S j. So, that at the end when j equal to r, S j will become S, O S j will become O S and L S will be equal to O S. So, when j equal to r we get S j equal to S, O S j is equal to O S and L S equal to L O S. So, this is the goal of this Gram-Schmidt process.

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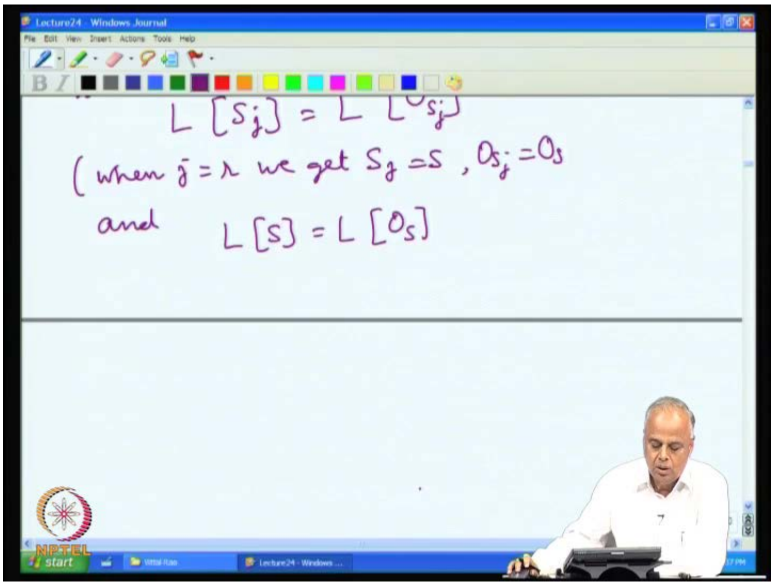

At each stage, it produces orthonormal set, which sweeps the vector space or the subspace span by all those vectors in the (( )) up to that stage. So, let us now describe the procedure.

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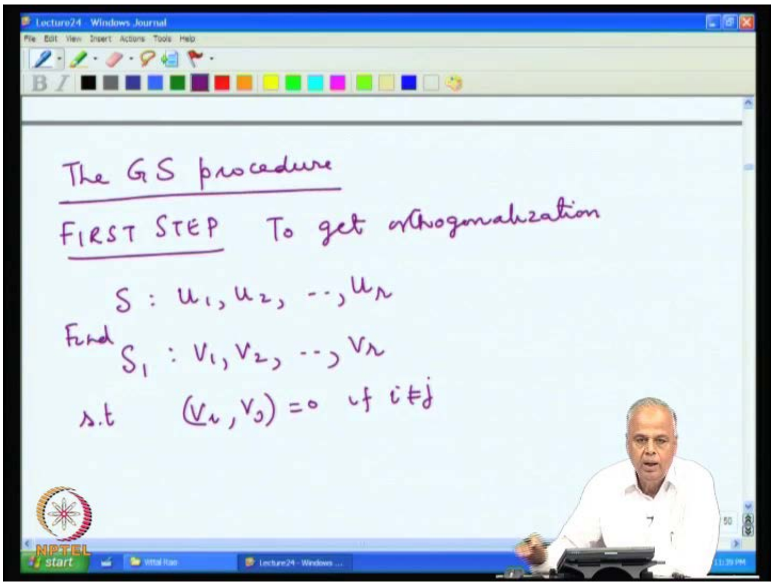

So, the G S procedure the Gram-Schmidt we will denote it by G S. The Gram-Schmidt orthonormalization procedure goes as follows. As we said, our aim is to eventually get an orthonormal set. So, there are two things involve, one is we have to get orthogonalization and then, we have to get a normalization.

The first stage is to get the orthogonalization done, so the first stage is to get orthogonalization and then, once we get orthogonalization; the next step will be to do normalization by just dividing by the length. So, we have the set S u 1, u 2, u r our job is we now find v 1, v 2, v r such that, v i comma v j is equal to 0, if i not equal to j that is, they are orthogonal to each other. That is what is meant by saying getting the orthogonalization.

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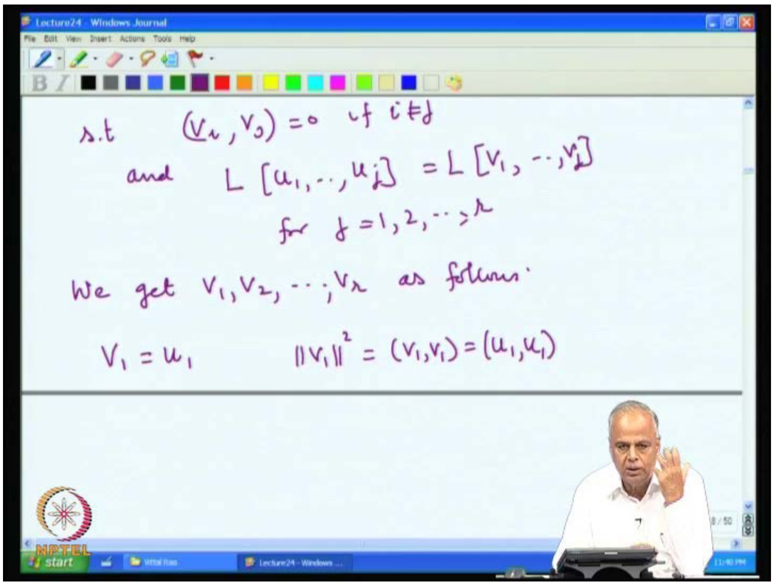

And the space span by u 1, u 2, u j the first j vectors is the same at the space span by v 1, v 2, v j and this is true for j equal to 1, 2, up to r. So, now we are not worried about normalization, we have only worried about orthogonalization. So, at the stage when j equal to r when the process ends, we would have got an orthogonal basis for this v 1, v 2, v r will form an orthogonal basis for L S.

The way to get this v 1, v 2, v r is what we will describe now. So, we get v 1, v 2, v r as follows. We first define v 1 to be just u 1 that is the first term we start with the first given vector, then we find the length of v 1 square which is v 1 comma v 1, which is the same as u 1 comma u 1. So, since v 1 is u 1 is given to us, we know v 1.

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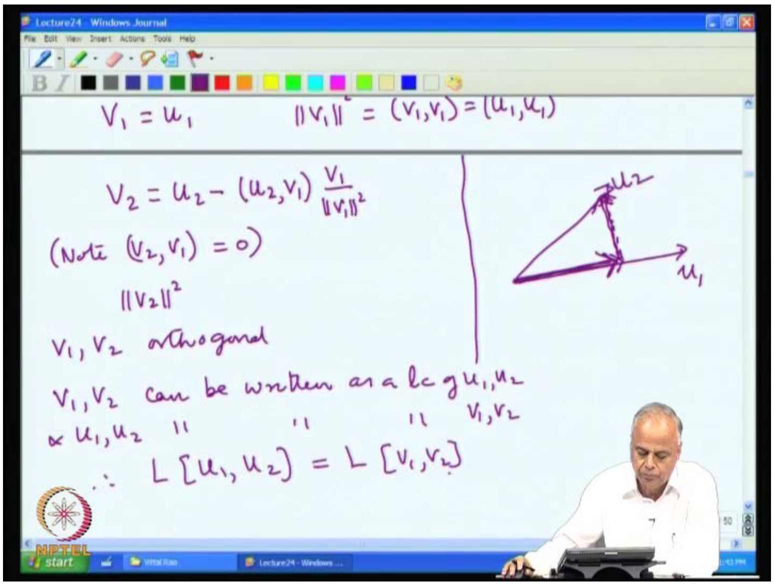

The next step is to given to go to v 2, the v 2 we now use the u 2 information, if you look at a bit geometrically we have u 1 here and u 2 here (Refer Slide Time: 20:41). Now, the u 2 may be sort of as having two pieces of information, one which is along the direction of u 1, the other one perpendicular to that. So, the u 2 is made up of this vector and this vector.

And since, we already have the u 1 direction under control with our v 1, we have to only worry about producing this orthogonal direction u 2 for that, from v 1 we must subtract this projection and that is obtained as follows. You start with u 2 from that, the subtraction of this projection is given by you take the inner product of u 2 with v 1 and divide it by v 1 square that unit vector in that direction.

Now see, what happens, we can divide by v 1 we must be very careful that the denominator is not 0. So, the denominator is not 0 requires that, v 1 is not the zero vector. Now v 1, if v 1 at the zero vector that would mean that, the u 1 is the zero vector, but u 1 cannot be zero vector, because we are assuming u 1, u 2, u r are linearly independent. So, this is perfectly well defined.

And the note now, if you take the inner product of v 2 and v 1 it is 0, because then that was therefore, that v 2 and v 1 are orthogonal to each other. Now, having obtained v 2 we find the length of v 2 square, now we have got v 1 and 2 are orthogonal and we observe that anything that can be written as u in terms of u 1 and u 2 as a linear combination can also be written as a linear combination of v 1 and v 2, because u 1 can be written in terms of v 1 and u 2 can be written in terms of v 1 and v 2.

So, we have v 1, v 2 can be written as a linear combination of u 1, u 2. And u 1, u 2 can be written as a linear combination of v 1, v 2. Therefore, L u 1 u 2 the space span by u 1, u 2 is the same as space span by v 1, v 2. Now therefore, once we know v 1, we have the definition for v 2. Now, we define recursively.

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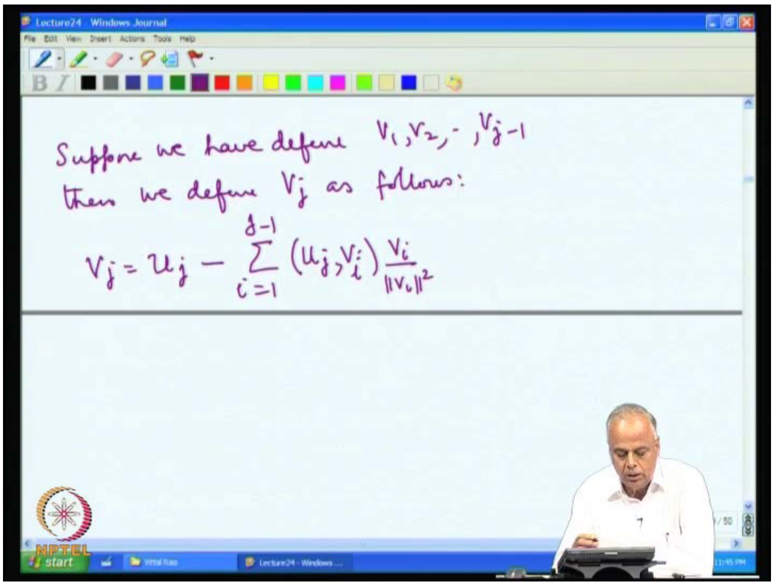

Suppose, we have define v 1, v 2, v j minus 1 then we define v j as follows; v j for v 1 we started with u 1, for v 2 we started with u 2, for v j you start with u j and then, you have to subtract all the information along the projections of the previous directions.

So, that is obtained by taking i equal to 1 to j minus 1 all the previous stages, take the vector u j look at this projection along v j sorry v i the previous direction (Refer Slide Time: 24:38), and subtract it, then look at the unit vector in that direction. This is how v j is defined for once we know all the previous (( )). Again you know that this will not be 0 and you get the orthonormal orthogonal vectors.

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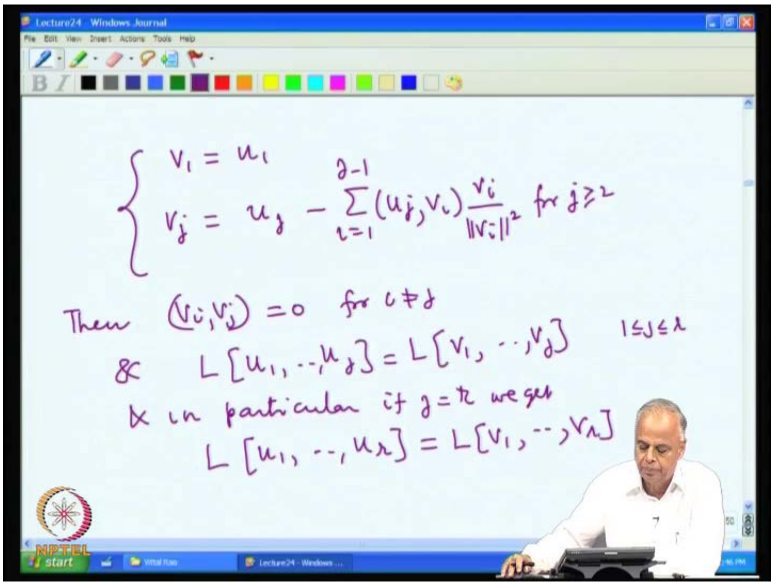

So, the the general procedure therefore is v 1 is defined as u 1 for j greater than or equal to 2, it is defined as i equal to 1 to j minus 1 u j v i v i by length v i square for j greater than or equal to 2. So, once we know v 1 we know v 2; and once we know v 1, v 2 we know v 1, v 2, v 3; and once we know v 1, v 2, v 3 we know v 4 and we go on recursively like that. And at the r th stage we get, then v i v j is equal to 0 for i not equal to j and the subspace span by u 1, u 2, u j is equal to the subspace span by v 1, v 2, v j and this is true for any j between l and r.

And in particular, if j equal to k j equal to R we get the subspace span by the set u 1, u 2, u r that is the original set given to us is the same as the subspace span by v 1, v r. So, therefore, first we have done the orthogonalization process, we were given linearly independent vectors u 1, u 2, u r. Recursively, step by step we are produce the sequence v 1, v 2, v r of vectors which are orthogonal to each other and in such a way, this the same space at each stage has work by these corresponding use.

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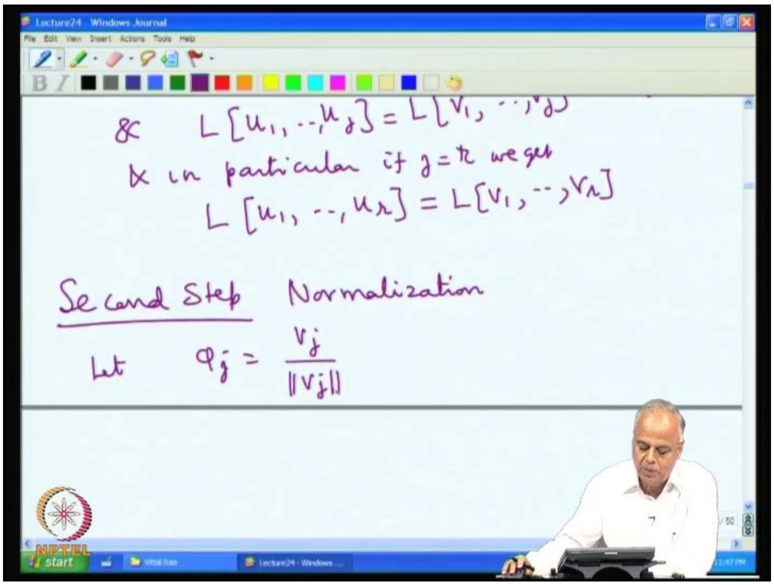

Then the second step in the process is the normalization. We want all these vectors to be length be let phi j to be v j divided by the length of v j.

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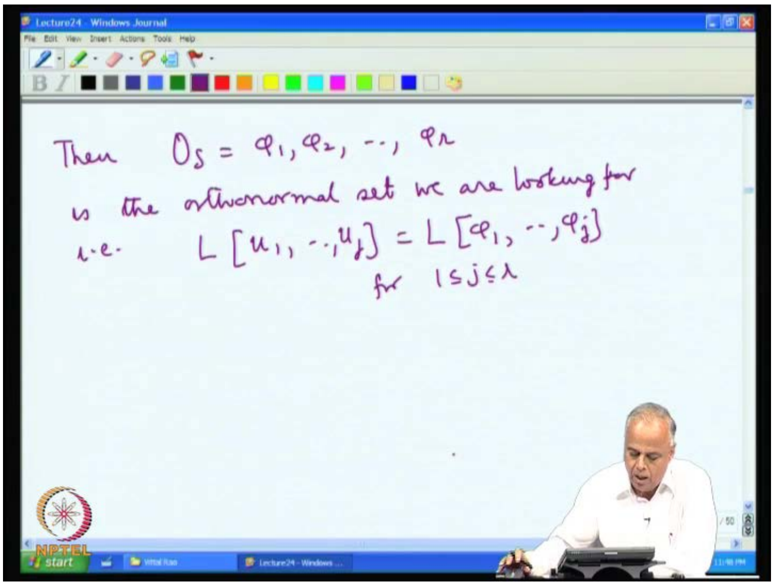

Then phi j is become then O S equal to phi 1, phi 2, phi r is the orthonormal set we are looking for. What do we, what are we looking for? We want them to the orthonormal set such that, that is L of u 1, u 2, u j is equal to L of phi 1, phi 2, phi j for 1 less than or equal to j less than or equal to r.

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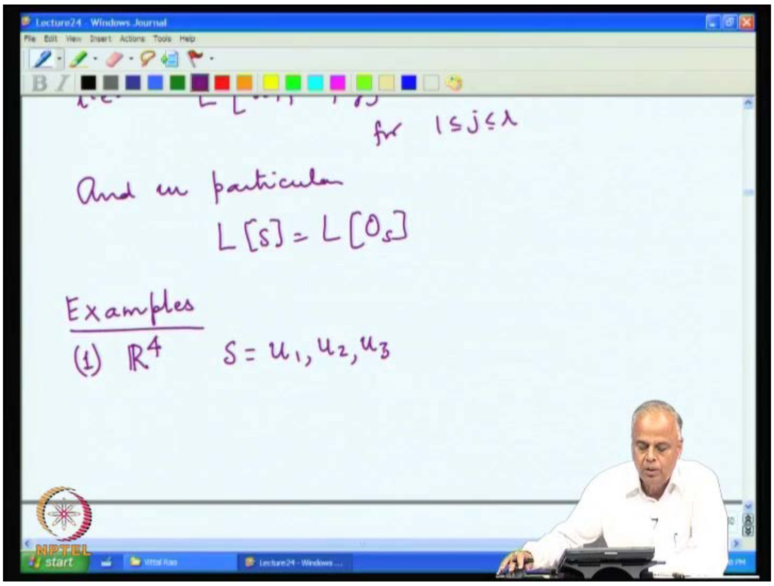

And in particular, L of S is equal to L of O S, so O S becomes an orthonormal basis for the subspace span (( )). This process is called the Gram-Schmidt orthonormalization process. Let us, look at one or two examples. Let us, take R 4 and let us take S to be consisting of this three vectors u 1, u 2, u 3.

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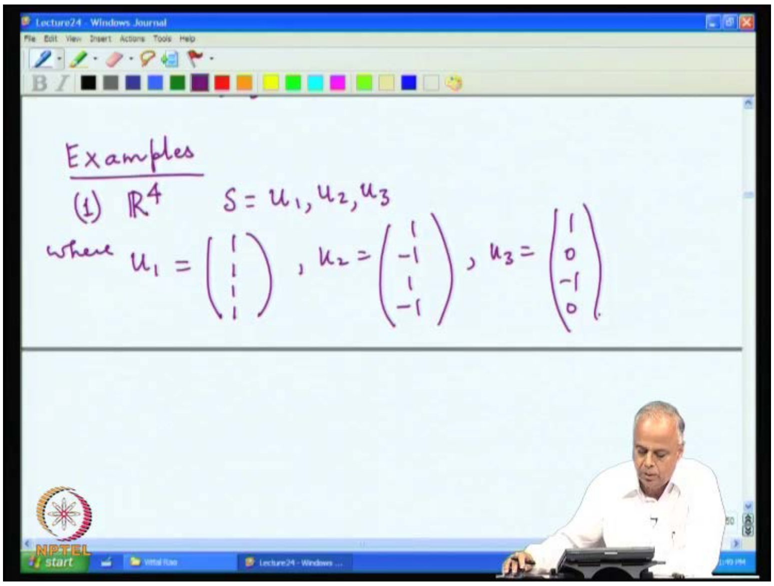

Then, where will define u 1 to be 1 1 1 1, u 2 to be 1 minus 1 1 minus 1, u 3 to be 1 0 minus 1 0.

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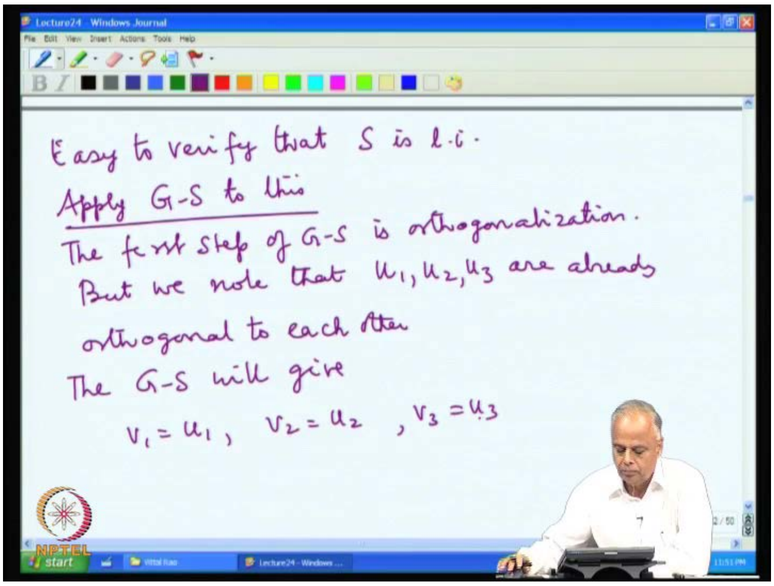

It is easy to verify will leave the as an exercise. It is easy to verify that, S is linearly independent, since it is a linearly independent set, now we are going to apply G S state, the Gram-Schmidt operation to this. Now, what is the first stage of the Gram-Schmidt, it is orthogonalization, now if you notice that u 1, u 2, u 3 are orthogonal. So, the first step of G S is orthogonalization.

But, we note that u 1, u 2, u 3 are orthogonal are already, orthogonal to each other, why is it so. If you take the dot product of u 1 with u 2, we get 1 minus 1 plus 1 minus 1 which is 0. Similarly, dot product u 1 with u 3 is 0, and the dot product of u 2 with u 3 is 0, so pair wise they are all orthogonal to each other.

So, the Gram-Schmidt process we will simply produce B 1 equal to u 1, v 2 equal to v 2, v 3 equal to u 3, because they are already orthogonal set. So, the G S will give v 1 equal to u 1, v 2 equal to u 2, v 3 equal to u 3 check, actually it is work carrying out the Gram-Schmidt expressions explicitly and see that, you get v 1 equal to u 1, v 2 equal to u 2, v 3 equal to u 3.

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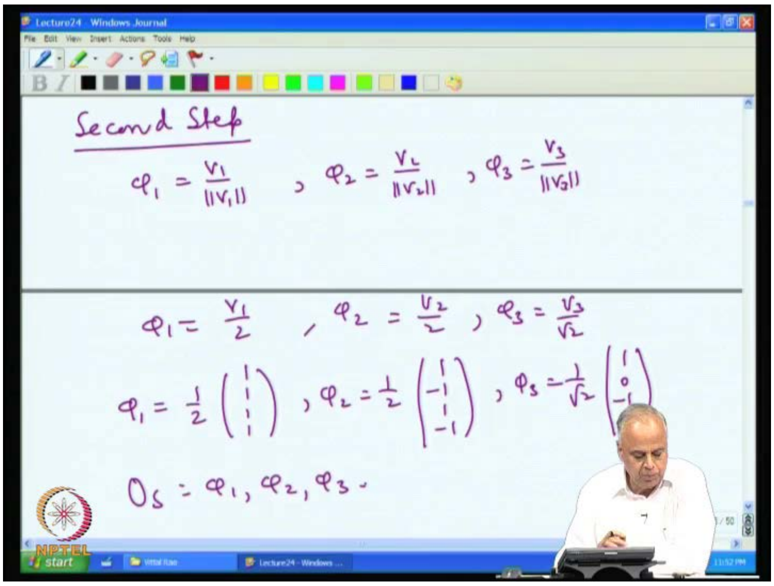

So, the second step is the only thing that has to be done in this case, because they are already orthogonal. Second step involves dividing by the length of the vector. So, phi 1 is v 1 by length of v 1, phi 2 is v 2 by length of v 2 and phi 3 is v 3 by length of v 3. Now, the length of v 1 is, length of v 1 square is 4, so the length of u 1 is 2; length of u 2 square is 4, so the length of u 2 is 2; length of u 3 square is 2, so the length of u 3 is 2.

So, this is going to be equal to v 1 by 2, phi 2 is v 2 by 2 phi 3 is v 3 by root 2. And therefore, we get phi 1 as 1 by 2 into 1 1 1 1, phi 2 is 1 by 2 into v 2 is 1 minus 1 1 minus 1 and phi 3 is 1 by root 2 v 3 is 1 0 minus 1 0. So, we have got this O S, the O S required in this case is phi 1, phi 2, phi 3. In this case, the Gram-Schmidt process was simple, because the vectors were already orthogonal to each other.

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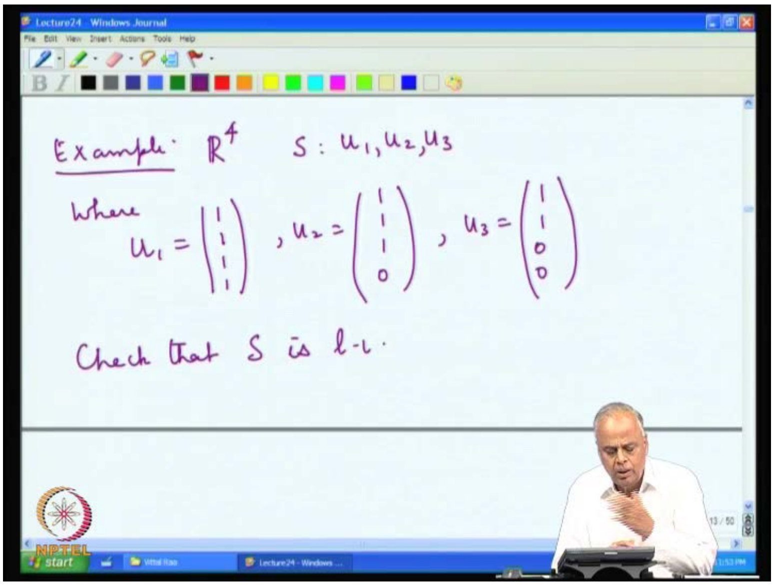

Let us look at another example. Let us take again R 4 and let us take the set S to be u 1, u 2, u 3, where u 1 is 1 1 1 1, u 2 is 1 1 1 0, u 3 is 1 1 0 0. Now we see that, these vectors are not orthogonal and therefore, the Gram-Schmidt process will change to produce v 1, v 2, v 3. Check that, S is linearly independent, this easy to check that S is linearly independent, you take alpha 1 u 1 plus alpha 2 u 2 plus alpha 3 u 3 equal to zero vector and so all the coefficients must be 0.

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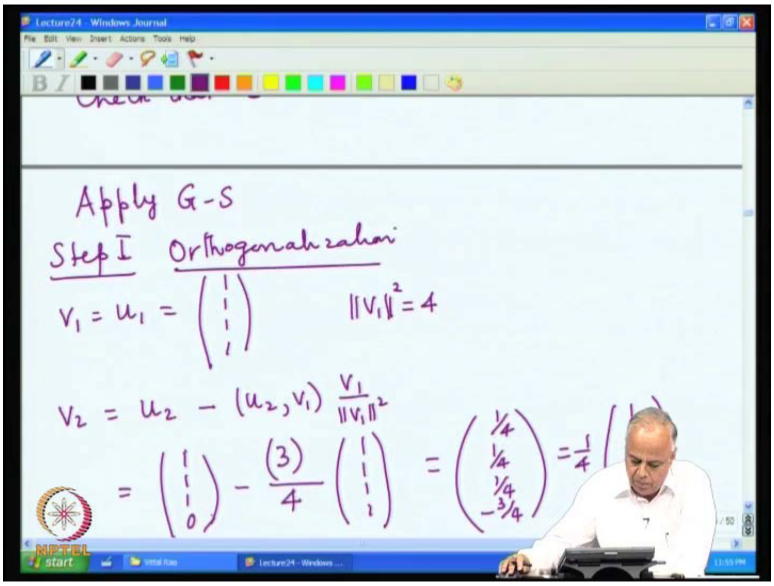

So, we will assume that this checking has been done, since it is a linearly independent vector set, we can apply the Gram-Schmidt rule. So, what is the step one, in the step one is the orthogonalization process, we are going to look at v 1, v 2, v 3 we first are going to define v 1 as u 1 which is 1 1 1 1 in the calculation we are going to use the norm, so we will take the length of v 1 square is 4.

In the second, we will have to calculate v 2 we calculate the v 2 we start with u 2 and from that, we subtract the v 1 information, the v 1 information in u 2 is given by u 2 comma v 1 into v 1 by norm v 1 square.

Let us substitute u 2 was 1 minus 1 1 minus 1 you can recall that, u 2 I am sorry u 2 has 1 1 1 0 let us substitute the correct values, u 2 has 1 1 1 0 minus u 2 comma v 1 this vector u 2 must be inner product with v 1, we get 1 into 1 plus 1 into 1 plus 1 into 1 plus 0 into 1, so that is 3 then divided by norm v 2 square which is 4 into v 1, which is 1 1 1 1. If we now simplify that, that is just 1 by 4, 1 by 4, 1 by 4, minus 3 by 4 or we can write it as, 1 by 4 into 1 1 1 minus 3, so that is my v 2.

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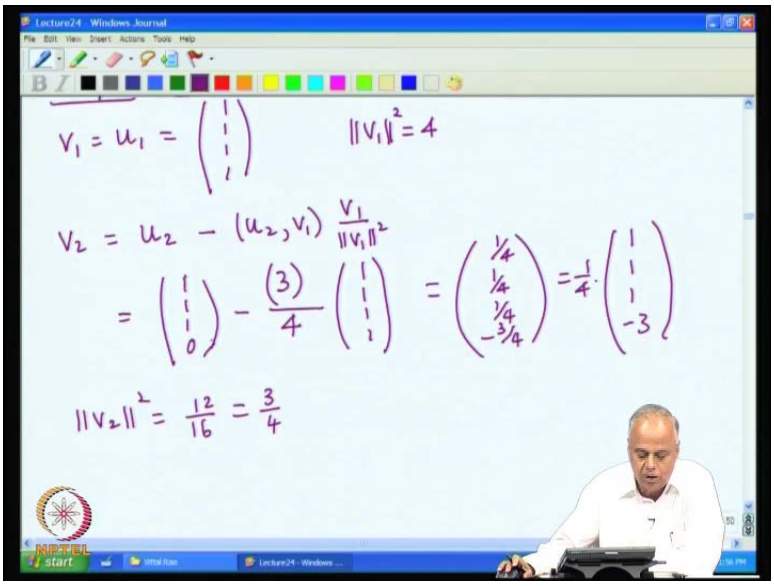

And since I would meet the length calculations I will write length of v 2 square is 1 by 16 the component square it will be 12 by 16, which is 3 by 4, (( )) square plus 1 square plus 1 square plus 3 square that will give me a 12, there is a denominator square which is 4 square 16. So, I will get 12 by 16 which is 3 by 4.

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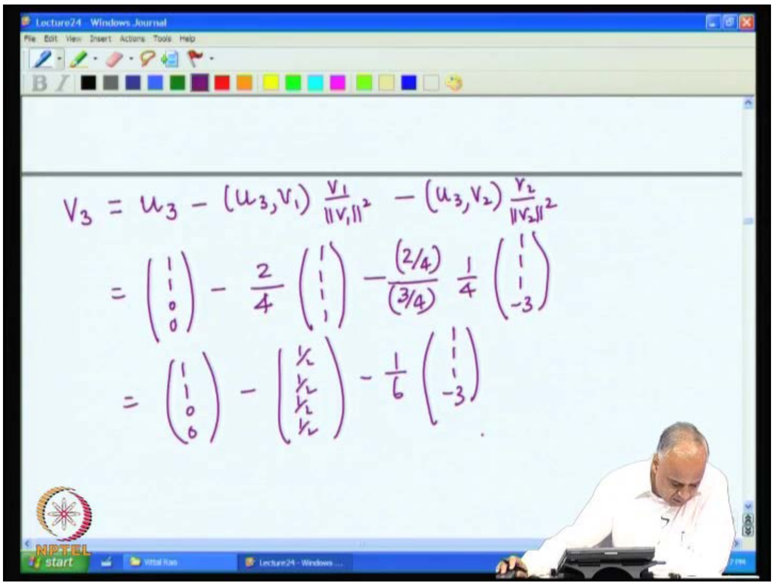

The next one is to calculate v 3. How do we calculate v 3 from u 3, we may subtract v 1 information, and also v 2 information that is the projections basically. Now let us say, what is u 3 was 1 1 0 0 and the dot product of u 3 with v 1 is 1 into 1 plus 1 into 1 the remaining components of u 3 are 0, so that is just 2 divided by the length of v 1 square which was 4 into v 1 minus u 3 v 2, v 2 is this vector and u 3 is 1 1.

So, it will be 1 plus 1 2 by 4, because each component is there 1 by 4. So, it will be 2 by 4 divided by norm v 2 square, which is 3 by 4 we have here into v 2 which is 1 by 4 into 1 1 1 minus 3. So, 1 1 0 0 minus half, half, half and half minus there is a (( )) becomes 1 by 6 into 1 1 1 minus 3.

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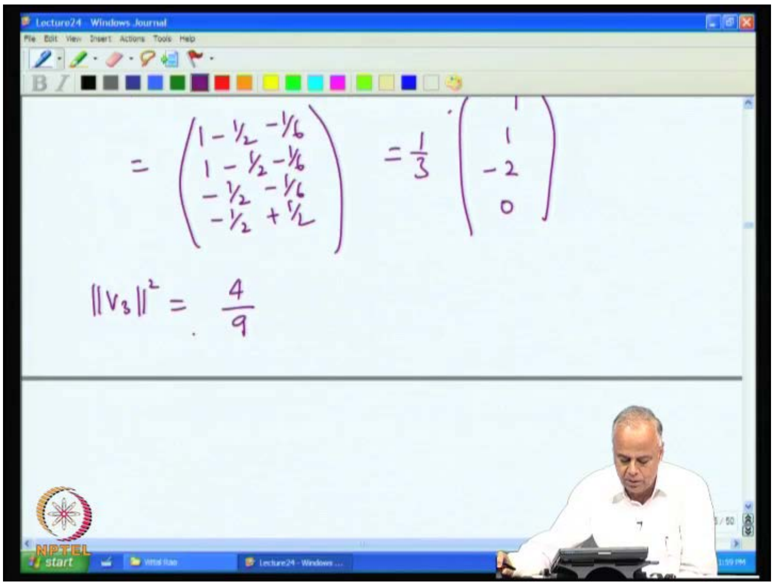

When we simplify this, we eventually can write this as 1 minus half minus 1 6 1 minus half minus 1 6 and then minus half minus 1 6 and then minus half plus 2 minus half 12 plus half and that simplifies to 1 by 3 into 1 1 minus 2 0, so that is what v 3 is. So, having got v 3 will calculate norm v 3 square, which is 1 plus 1 2 6 by 9, norm v 3 square.

So, now the process stops here orthogonalization process, because we started with three vectors u 1, u 2 and u 3. So, we have to produce v 1, v 2 and v 3. So, we have produced v 1, we have produced v 2 and we have produced v 3. So, that the orthogonalization process.

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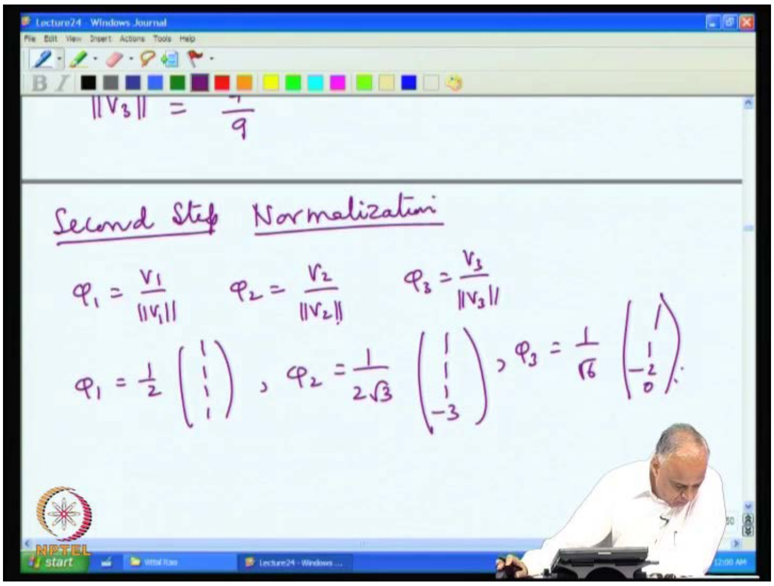

And the second step is the normalization process. In the normalization process, all we have to do is divide v 1 by its length, v 2 by its length, v 3 by its length. So, we define phi 1 as v 1 by norm v 1, phi 2 as v 2 by norm v 2 and phi 3 by v 3 by norm v 3, when we do that we get phi 1 to be 1 by 2 into 1 1 1 1.

We have all this information above, we have calculated v 1, we have calculated norm v 1, we have calculated v 2, norm v 2, v 3 and norm v 3. If we substitute all that, we get phi 2 to be 1 1 1 minus 3 and phi 3 to be 1 by root 6 into 1 1 minus 2 0. So, this is the Gram-Schmidt orthonormalization process with the given vectors.

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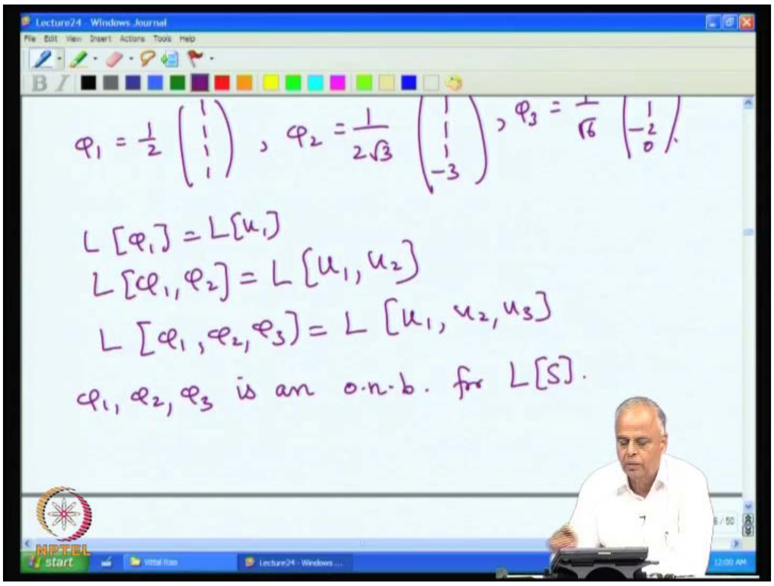

Then, we will have L of phi 1 will be equal to L of u 1, L of phi 2 the space span by I am sorry the space span by phi 1 and phi 2 will be the same as the space span by u 1 and u 2 (Refer Slide Time: 40:20). And finally, L of phi 1, phi 2, and phi 3 will be the same as the space span by u 1, u 2, u 3; and therefore, phi 1, phi 2, phi 3 is an orthonormal basis for L of S.

So thus, given any linearly independent set by using the Gram-Schmidt process we can extract an orthonormal basis for the subspace span by these given set of vectors, we will be using this repeatedly.

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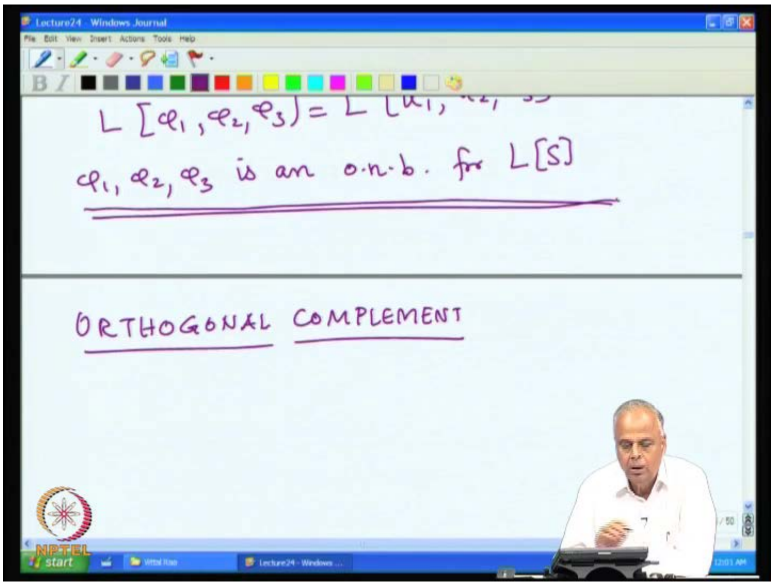

Before, we actually put this thing into action we will now look at an important concept of orthogonal complement of a subspace then, we will see how this Gram-Schmidt comes into the picture. Before that, let us make one small (( )) what does what does this mean to us, this means to us that whenever you produce a linearly independent set whenever you have a linearly independent set, you can always convert that into an orthonormal set producing the same space.

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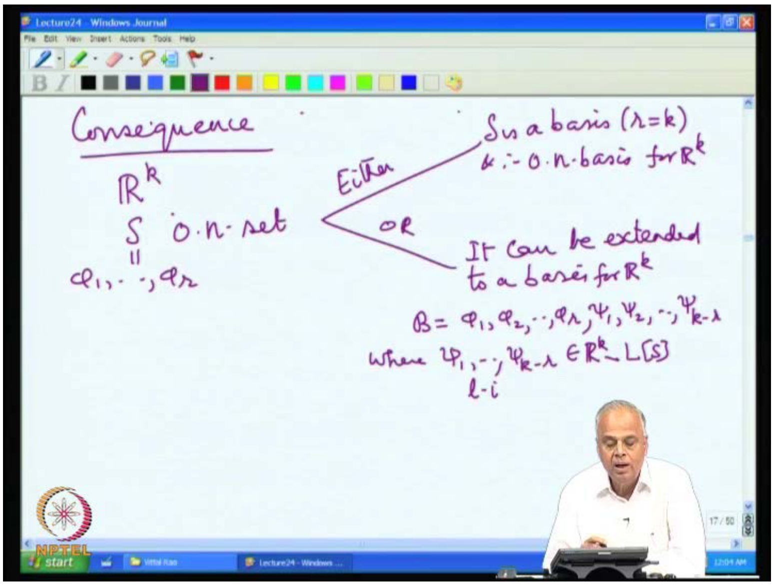

So, what is a consequence of this, let us immediately look at a consequence of this. Now suppose, I have R k and I have a linearly I have a say even it is an orthonormal set (( )) we have an orthonormal set; now either S is our basis when does that happen? If S is phi 1, phi 2, phi r this will happen when r equal to k.

So, suppose have an orthonormal set, if r equal to k then automatically it is a basis, because there are k vectors and they are linearly independent, because every orthonormal set is linearly independent; and therefore, since it is a basis and it is already orthonormal it is an orthonormal basis for R k.

So, given an orthonormal set, either it is a basis or it can be extended to a basis (( )) that we can extend it to a basis, because it is a linearly independent set, any orthonormal set is a linearly independent set and any linearly independent set can be extended to a basis. So, it can be extended to a basis, let us call this as phi 1, phi 2, and phi r.

How can we extend it to a basis for R k? We can extend it to a basis for R k by appending an (( )) number of vectors. How do we append? We already have r vectors, the dimension is k. So, we have to append k minus r vectors, let us call them as psi 1, psi 2, psi k minus r, where psi 1, psi 2, psi k minus r belong to R k minus L S and linearly independent. So, we have extended the given orthonormal set to basis for R k, but this basis may not be an orthonormal basis. What do we do next we apply G S to B the Gram-Schmidt.

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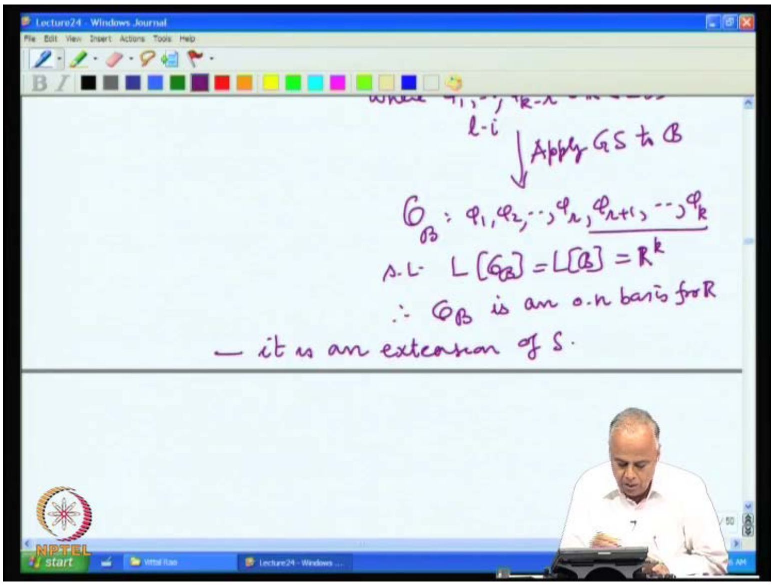

We can apply the Gram-Schmidt to B, because these linearly independent we usually apply the Gram-Schmidt operation to a linearly independent set, since B is a basis for R k, it is a linearly independent set. So, we can apply the Gram-Schmidt operation to B, when we do the Gram-Schmidt operation we go on doing orthogonalization and normalization, but the first r vectors are already orthogonal and normal. So, the Gram-Schmidt operation will do nothing to them, and (( )) the later stages it will come into action to orthogonalize and normalize the psi vectors. So, it will orthonormalize and will give as a O B will retain this phi 1, phi 2, phi r, because they are already orthonormal.

And now, give us phi r plus 1 phi k such that, L O B is the same as L B that is what the Gram-Schmidt does, but L B is R k because, B is a basis for R k. And therefore, O B spans R k, it is orthonormal. Any orthonormal set which spans is an orthonormal basis, so O B is an orthonormal basis for R k. And you observe now that, that is an extension of phi 1, phi 2, phi r it is an extension of S, the given the set we started with was this, we have now extended by adding this, so it is an extension of S.

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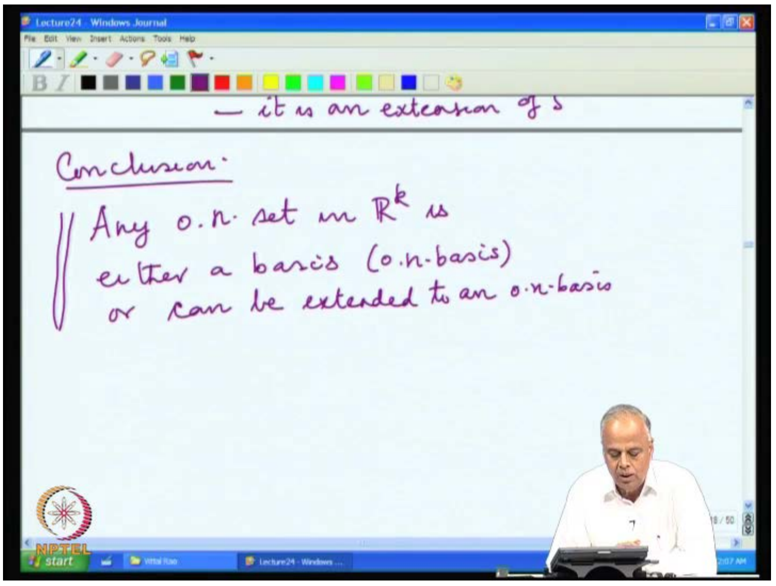

So, conclusion any orthonormal set in R k is either a basis in fact, (( )) once it is a basis, it is an orthonormal basis, either an orthonormal basis or can be extended to an orthonormal basis. This is the question we raised earlier that, whether any orthonormal set can be extended to an orthonormal basis, we have answered that question in the affirmative now that any orthonormal set can be extended to an orthonormal basis.

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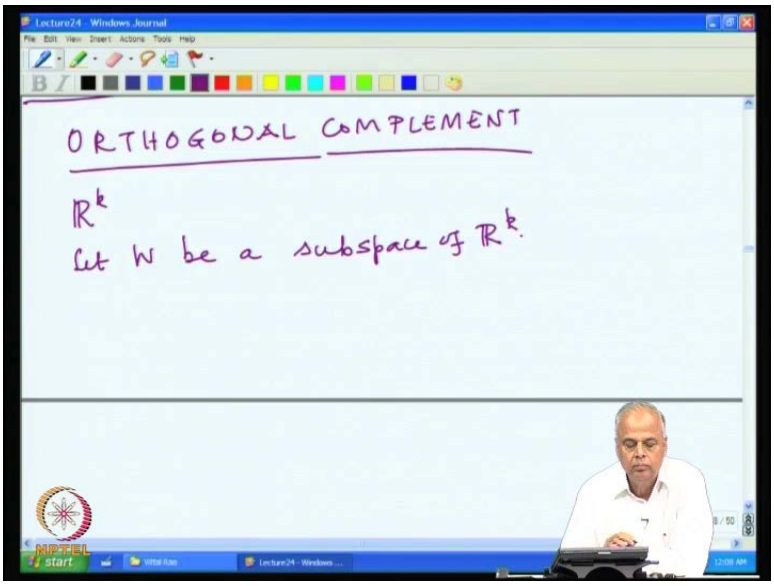

We now introduce a important concept called orthogonal compliment and we will see, how we use this idea of extending an orthonormal set to an orthonormal basis. This idea of orthogonal compliment is essentially to extend the Pythagorean geometry in two dimensions to this general set of R k, so we get many results analogous to the Pythagoras theorem. The Pythagoras theorem involves orthogonality and therefore, once we have this orthogonality concept, we can try to look at this Pythagorean property. So, we have R k and let W be a sub space of R k.

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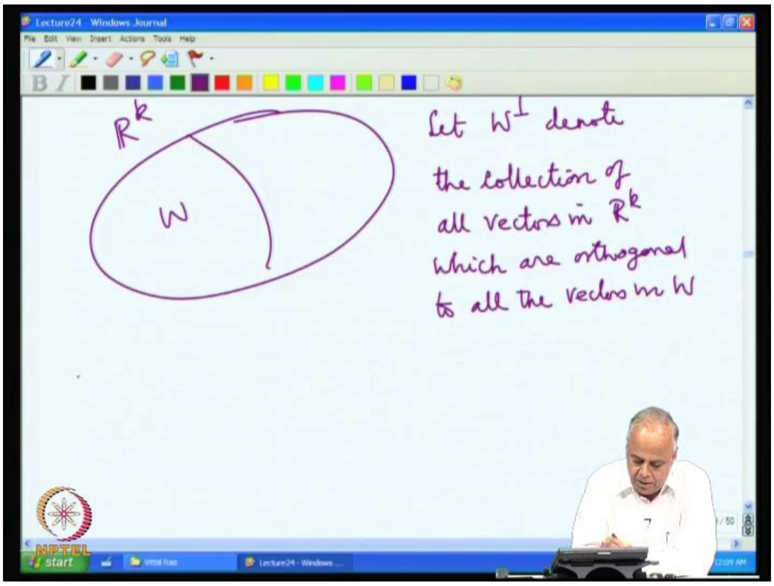

So, we have this big space R k and in that big space, we have taken a sub space W. Now, what we do is, we look at all those vectors which are perp, this is the W, we look at all those vectors which are perpendicular to W. So, we denote that by W, so let W perp denote the collection of all vectors in R k, which are orthogonal to all the vectors in W.

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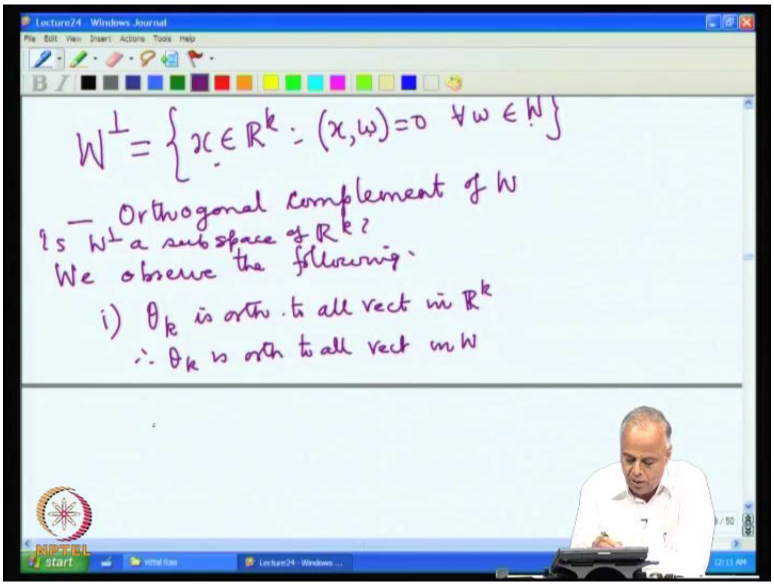

So, let us write this in a mathematical notation. So, W perp consists of all those vectors, the collection of all vectors in R k which are such that, they must be orthogonal, so x W must be equal to 0 for what, for all those w which belong to W. So, it is the collection of all vectors, which are orthogonal to those w which belong to W. So, this is called orthogonal compliment of W this is called the orthogonal complement of W.

Now, first of all is there any vector at all that is orthogonal to all the vectors in W, because we want to make sure, there is something in W perp or not. Now, clearly the zero vectors is orthogonal to all the vectors and hence in particular, it is orthogonal to all the vectors in W and hence it belongs to W perp.

So, we observe the following. one of course, W perp consists of vectors in R k, so it is a subset of R k; whenever you have a subset of R k we wonder whether it is subspace and that is what we are going to investigate, so is W perp a sub space of R k. Now, you may recall that in order to check whether something is a sub space we must make sure it is non empty, we must make sure is closed under addition; we must make sure it is closed under scalar multiplication. So, first thing is theta k is orthogonal to all vectors in W in R k therefore, theta k is orthogonal to all vectors in W, because W is a part of R k.

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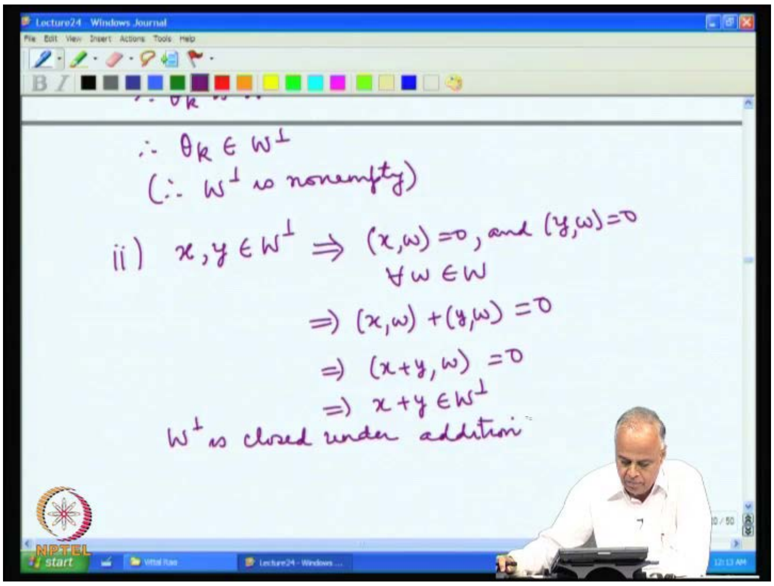

Therefore, theta k belongs to W perp and therefore, W perp is non empty. So, there is something in W perp (( )) zero vectors belongs to W perp. So, we have a non empty set, next we have to check whether it is closed under addition. So, suppose x and y belong to W perp. What does that mean, x belongs to W perp means x is orthogonal to all the vectors in W and y belongs to W perp means y is orthogonal to all the vectors in W. Therefore, x comma W is 0, the inner product of x with W is 0 and the inner product of y with W is equal to 0.

Now therefore, if I add (( )), I get 0, so x w plus y w is equal to 0, but the inner product is (( )). So, this is the same as x plus y and w, because x plus y w is x comma w plus y comma w and that is 0. This means x plus y is orthogonal to W and therefore, x plus y belongs to W, thus W perp is closed under addition, it is closed under addition.

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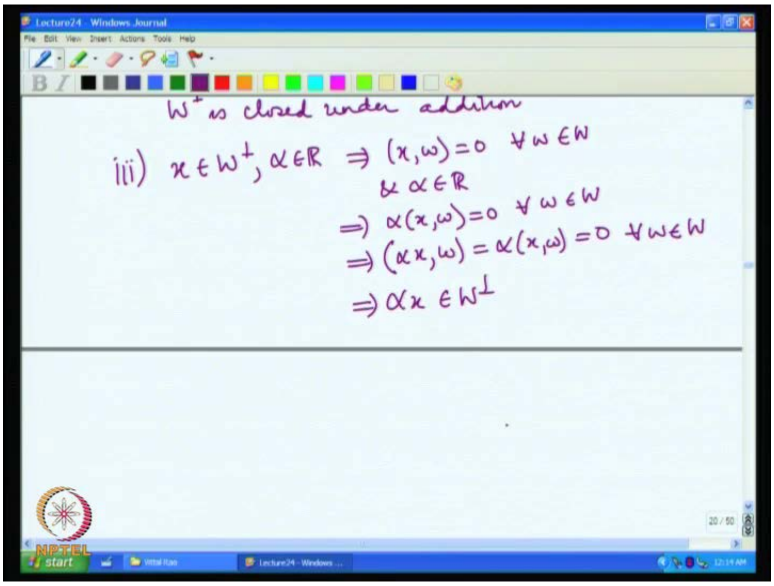

And we have to look at closer under scalar multiplication, x belongs to W perp alpha is any scalar; now the x belongs to W perp means x w equal to 0 for every w in W and we are given alpha is R. Now, if I multiply 0 by alpha I am going to get 0, so alpha x w is equal to 0 for every W, but the inner product the scalar constants are the numbers can be pulled and out of the inner product. So, it can be written as alpha x w is the same as the alpha x w and alpha x w 0 from previous (( )). So, that says alpha x is orthogonal to for every w in W and therefore, alpha x belongs to W perp.

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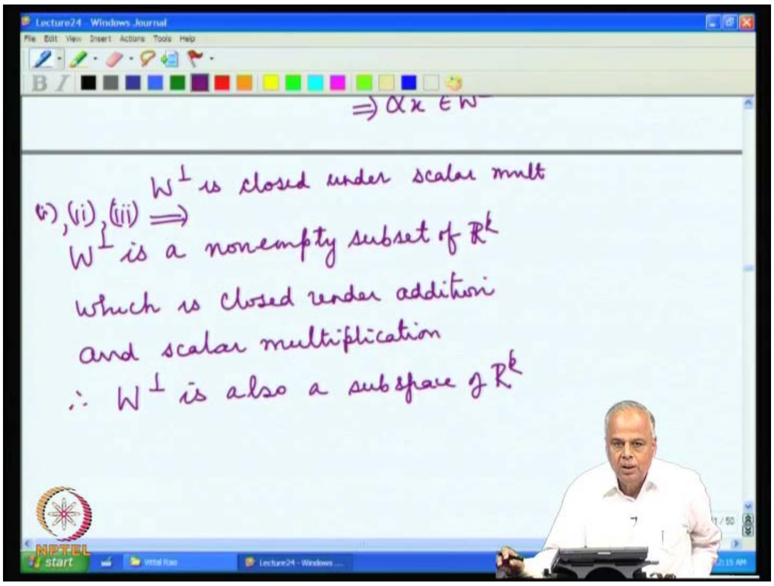

Therefore, W perp is closed under scalar multiplication. So, W perp from 1 to 3 what do we get, first one was non empty, second one was closed under addition, third one was closed under scalar multiplication. So, W perp is a non empty subset of R k, which is closed under addition and scalar multiplication and therefore, W perp is also a subspace of R k. So, we are started with a subspace of R k and we have constructed another subspace which is orthogonal subspace all the vectors there are orthogonal to this.

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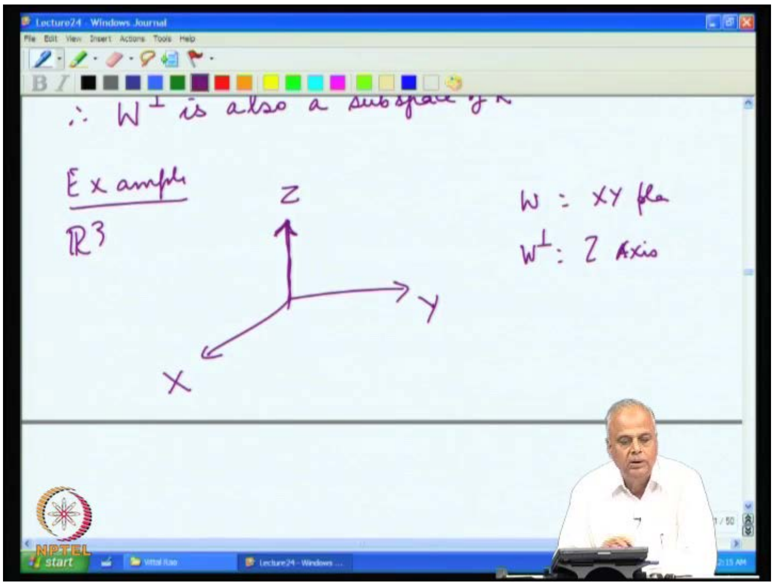

Let us look at geometrically a simple example of this. Let us, take R 3 this is what we normally call as an x y z axis, let us say W is the x y plane. Now, what are the vectors which are perpendicular to all the vectors in W then we get the z axis. So, the W perp is the z axis in this case.