Advanced Matrix Theory and Linear Algebra for Engineers

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Lecture No. # 23

Inner Product and Orthogonality- Part 2

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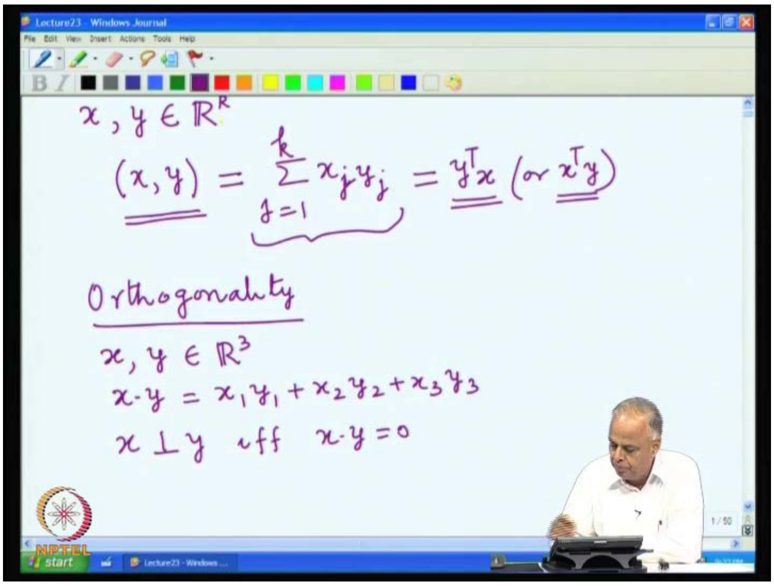

In the last lecture we introduce the notion of an inner product and we are looking at the geometry induced by this notion of inner product. How did we define the inner product suppose x and y are 2 vectors in r k. we defined and denoted the inner product but, x comma y and we defined it to be the sum of the product of the components which is just the generalization of the notion of the dot product. We had when we dealing with vector calculus into an 3 dimensions this can also be written as y transpose x or x transpose y all of them mean the same thing. So from now on when we say inner product we would be either denoting it by x comma y or we denoting by y transpose x or we shall denote it by x transpose y and all of them mean the same thing. The sum of the products of the components then we found some important properties of the inner product. Which was that the inner product was distributed and the inner product of a vector with itself. Gave the length square and that is 0 only when the vector is 0 and the most important thing that the inner product induces and which we shall be focussing is the notion of orthogonality.

What do we mean by this suppose we take 2 vectors in r 3 for example, then we had the dot product in our earlier calculus courses, vector calculus courses. Which we defined as x 2 y 2 plus x 3 y 3 and we found in the normal Euclidean geometry the x and y are perpendicular to each other if and only if this dot product 0.

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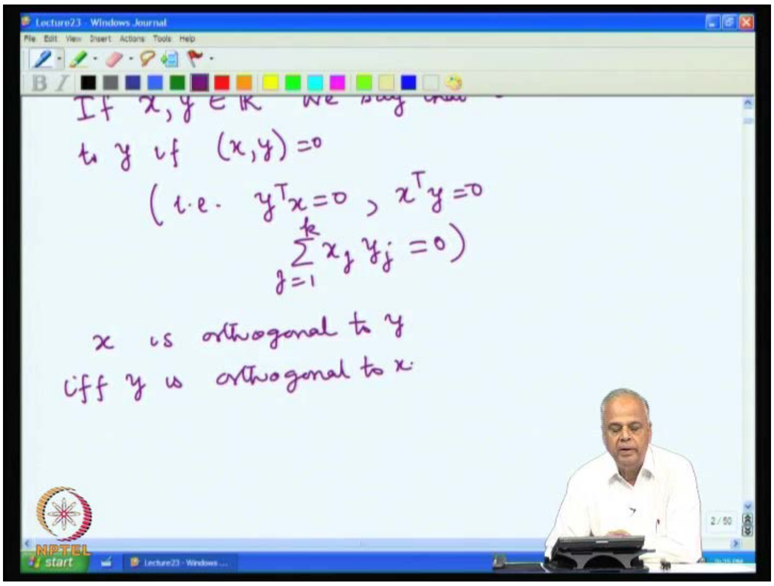

And we now generalize this notion through the r k using this inner product if x and y belong to r k we say that x is orthogonal or perpendicular to orthogonal to y. If x comma y is equal to 0 that is the dot the inner product what does that mean that is y transpose x is 0 or we can write this as x transpose y is 0 or if we expand this we can write this as x j y j equal to 0 so when we say x and y.

Are orthogonal we need all these things the notation is x comma y equal to 0 and that means that the y transpose x or x transpose y are sum of the products of the component is equal to 0 from the definition. The symmetry tells us that x is orthogonal to y if and only if y is orthogonal to x so from now on instead of a x is orthogonal to y and y is orthogonal to x we will just say x and y are orthogonal to each other x and y are orthogonal to each other. So the orthogonality comes from the fact that the inner product is 0 the inner product induces the notion of orthogonality and orthogonal is geometric notion so the inner product induces the idea of orthogonality.

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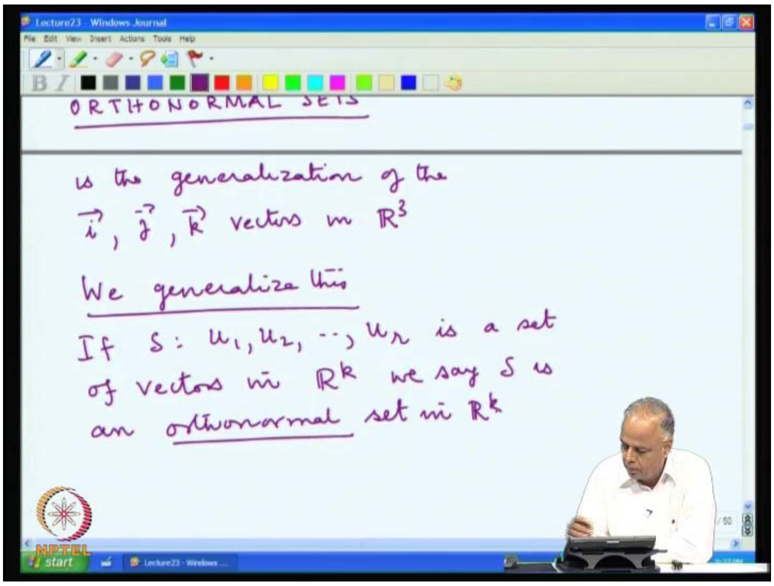

We next began looking at the notion of orthonormal sets this is the this notion is the generalization it is the generalization of the i j k vectors which we would have seen in vector algebra in r 3. This is the generalization of these 3 vectors what do these 3 vectors have special each vector is orthogonal to the other vector i dot j is 0 i dot k is 0 k dot i is 0 so they are mutually orthogonal vectors and each vector has length 1. So this is a collection of vectors which have the special property that any 2 of them is orthogonal to each other and each vector has length 1. Now we generalize this idea in r k because we have the notion of orthogonality induce by the inner product. We have the notion of length which comes out as the inner product vector with itself and therefore, we can generalize this whole notion of orthonormality to the case of r k.

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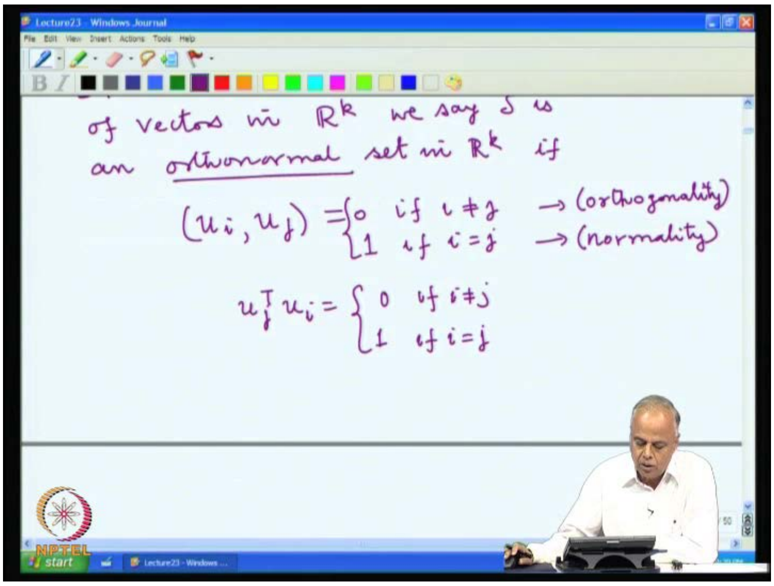

We generalize this if s is u 1 u 2 u r is a set of vectors in r k we say s is an orthonormal set it is orthonormal. Set in r k if any 2 of the vectors must be orthogonal to each other u i and u j must be equal to 0 if i is not equal to j that says if you.

Take any 2 different vectors from the set the dot product is 0 the inner product is 0. Which means they are orthogonal what do we mean by the length is 1. If we take i equal to j then i get u i comma u. It hast give the length of u i square but we want that to be 1 so that means this must be equal to 1 if i equal to j. So the first condition is the orthogonality condition the fact that any 2 vectors are orthogonality the second 1 is the normalization condition normality. Condition each vector has been normalized to have length 1 and that is why we call the set as orthonormal set or if you want to write it in terms of the transpose notation this means u j transpose u i is equal to 0 if in not equal to j 1 if i equal to j. So the vectors are orthonormal if any pair of them is orthogonal to each other and every vector has length 1.

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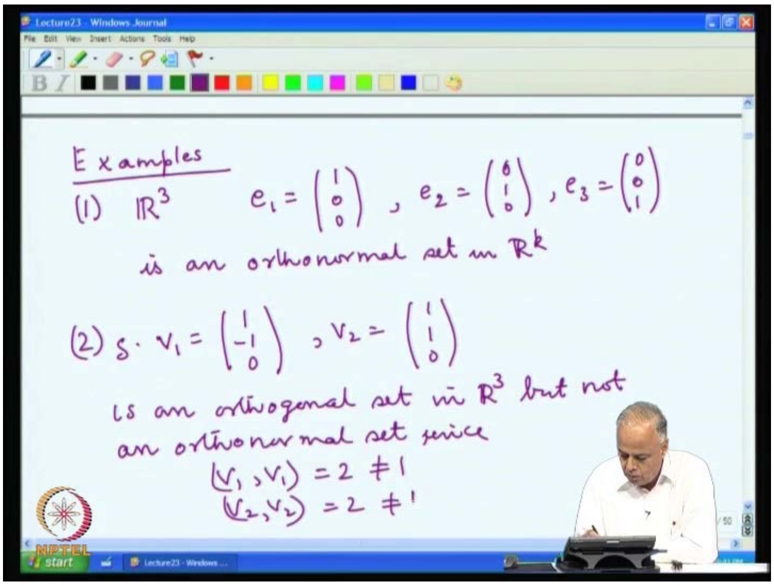

Let us look at some simple examples since we have seen that this notion of orthonormality itself is a generalization of the i j k vectors the i j k vectors come out have the first natural example of orthonormal sets. So we have r 3 in r 3 now we will in the in a vector space notation. We will denote the i vectors u 1 1 0 this is the j vector 0 1 0 and e 3 0 0 1 is an orthonormal set in r k notice that the vector v 1 equal to 1 minus 1 0 v two. Equal to 1 1 0 is an orthogonal set because a the 2 vectors are orthogonal to each other is an orthogonal set in r 3 but, it is not an orthonormal set in the normalization of length being 1 is not satisfied. The length of 1 is v 1 is square root of 2 and the length of v 2 is also square root of 2 but not and orthonormal set since v 1 v 1 is equal to 2 not equal to 1 v 2 v 2 is 2 not equal to 1. Now if we take the new vectors that we are going to form w 1 and w 2 which are obtained by normalizing v 1 v 2. What is meant by normalizing v 1 v 2 they do not have length 1. Now we divide by the length then we get a unit vector so we take v 1 and divide by its length which is square root of 2 and we take v 2 and divide by its length.

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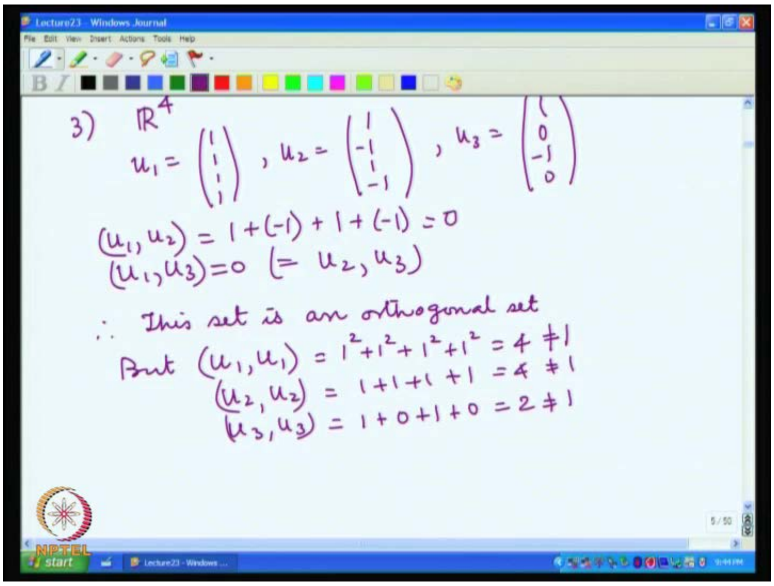

Now this set is an orthonormal set in r 3 because now the w 1 w 2 are orthogonal to each other and each 1 of them has length 1 let us take r 4 again in r 4 look at these vectors u 1 1 1 1 u 2 is equal to 1 1 minus 1 1 minus 1 u 3 equal to 1 0 minus 1 0. If you now look at u 1 comma u 2 the inner product of u 1 and u 2 which is simply the sum of the product of the components it is 1 into 1 which is 1 plus 1 into minus 1 which is minus 1 plus 1 into 1 plus we have the 1 into minus 1 which gives me 0 and therefore, we have that u 1 is orthogonal to u 2 similarly, u 1 is orthogonal to u 3 and u 2 is orthogonal to u 3. So this u 1 u 2 u 3 are orthogonal to each other. Therefore, the set is an orthogonal set the set is an orthogonal set but, we have the length of u 1 square which is the u 1 comma u 1 is 1 square plus 1 square plus 1 square plus 1 square. The sum of the squares of the components which is 4 but, we want it to be 1 for normality. So it is not 1 similarly, u 2 comma u 2 is again 1 plus 1 plus 1 sum of the squares of the component that is not 1 u 3 comma u 3 the dot product or inner product of u 3 with itself is 1 square plus 0 square plus minus 1 square plus 0 square. Which is 2 which is not 1 so n1 of these vectors have length 1 but, they are orthogonal to each other. So this is not hence this is not an orthonormal. Set this is not an orthonormal set as before since we already have orthogonality we can force now.

Normality by dividing each 1 of these vectors by it is length. Suppose we now take the set s 1 consisting of this vectors v 1 which is obtained from we have to obtain v 1 from u 1 by dividing you have to take this vector u 1 and divide it by its length. We get 1 by 2 into 1 1 1 1 similarly, we divide u 2 by its length and we get u 3 by its length and then we get this vectors. Now this is an orthonormal set in r 4 so thus the orthonormality we need 2 requirements for a set of vectors to be orthonormal. The name suggests is ortho and normal the word ortho refers to the fact any 2 vectors are mutually orthogonal to each other the word normal refers to the fact. That the vectors have been normalized to have length 1. So we now have this notion of orthonormality in a inner product space particularly r k with this inner product.

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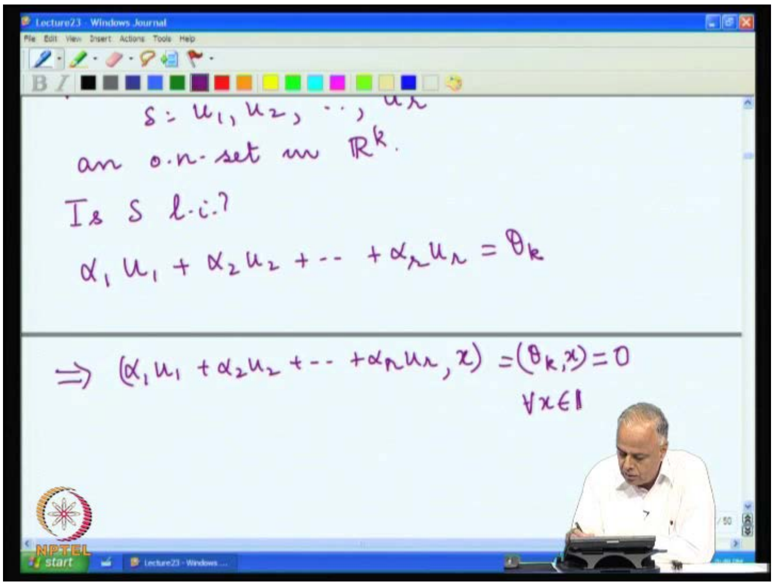

Now we are going to look at a very important property of inner product. Orthonormal sets an important property of i will write o n sets for orthonormal sets. So we now look at a very important property of orthonormal set so suppose we have s u 1 u 2 u r an orthonormal set in r k. Whenever we have a set of vectors in r k the first thing we investigate is whether the set is linearly independent or not. Whenever we get a set of vectors we always first look at the fact whether it is linearly independent or linearly dependent. If is linearly dependent now lot of redundant information we would like to throw it out. So first we check whether this set is linearly independent is s linearly independent for this. We must check whether a linear combination of this u 1 u 2 are when it is the 0 vector thus forces all the co efficient to be 0. So we start with a linear combination of these vectors and suppose it is equal to the 0 vector. We want to investigate whether that will force all the co efficient to 0. If it forces all the co efficient to be 0 then we are linearly independent but, if you have non 0 co efficient we give 0 vector then we have linear dependent now this implies if we take any vector and take the inner product with the sum take any vector x and inner product with this that is the same as theta k comma x. Because the sum is equal to theta k but the inner product of the 0 vector is always 0 with any vector. So this implies the inner product of the sum with any x is equal to 0 for every x in r k .

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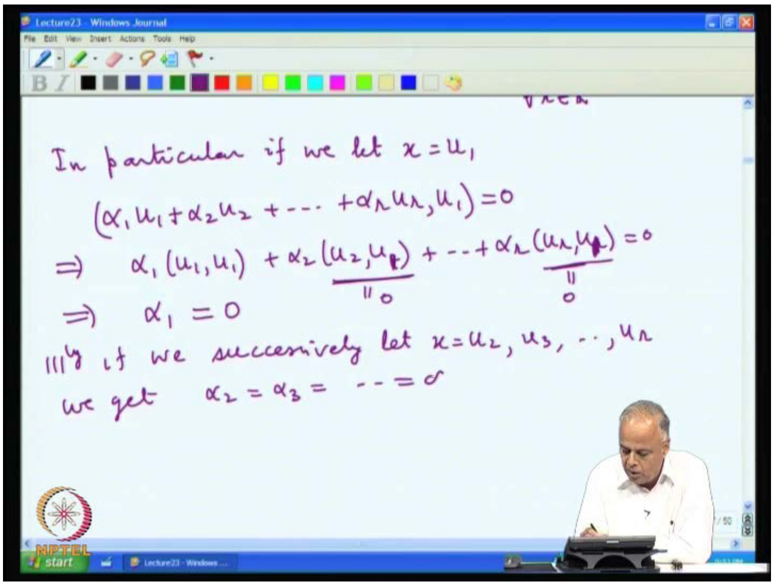

Now if I particularly take in particular if we let x equal to u 1 what do we get we get therefore, alpha 1 u 1 plus alpha 2 u 2 plus alpha r u r u 1 must be equal to 0. Because this inner product is 0 whatever x i take in particular I have taken x is equal to u 1. Now the inner product of a sum as the sum of the inner product which property we have seen last time and constants can be pulled out of the inner product. So this whole thing implies alpha 1 u 1 u 1 plus alpha 2 u 2 u 2 and so on alpha r u r u r is equal to 0 u 2 u 1 u r u r because you are taking the inner product with u 1. Now alpha 1 u 1 u 1 u 1 is 1 why is u 1 u 1 1 because we are given that u 1 u 2 u r is an orthonormal set. When you have a orthonormal set in that set every vector has length 1 u 1 is the member of that orthonormal set and hence it must have length 1. Therefore, u 1 u 1 is 1 so the first term becomes 1 now u 2 u 1 the second term u 2 u 1 is 0 because u 2 and u 1 are members of this. Orthonormal set and any 2 vectors in the orthonormal set are orthogonal to each other and therefore, their inner product is 0 and hence u 2 u 1 is 0. Similarly, u 3 u 1 is 0 u r u 1 is 0 because all these vectors are in that set and any 2 vectors are orthogonal. So we simply get alpha 1 equal to 0 similarly, if we successively take let x equal to u 2 next then u 3 and so on when i let x equal to u 1 x alpha 1 is 0 if i let x equal to u 2 in this place and take this i get alpha 2 equal to 0 and so on we get alpha 2 equal to alpha 3 equal to alpha r equal to 0.

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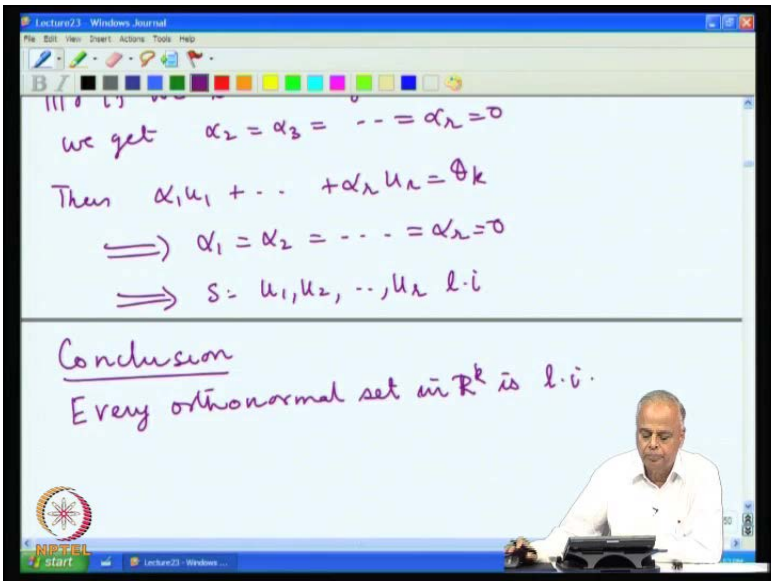

What we have this alpha 1 u 1 plus alpha r u r equal to theta k implies alpha 1 equal to alpha 2 equal to alpha r equal to 0 this means that the set s of vectors. The set of vectors u 1 u 2 r is linearly independent which implies the set of vectors u 1 u 2 u r linearly independent. So what we have shown is we start with any linearly any orthonormal set it is automatically forced to be linearly independent conclusion every orthonormal set is linearly independent. So conclusion every orthonormal set in r k is linearly independent that is a very important property of orthonormal. But now let us look at what does mean an orthonormal set is automatically linearly independent and the moment you have a linearly independent. Set you wonder whether it is a basis for that to be a basis it must also span the space. So an orthonormal set will become a basis since it is already linearly independent the only requirement that will be force will be further required will be that its spansely space.

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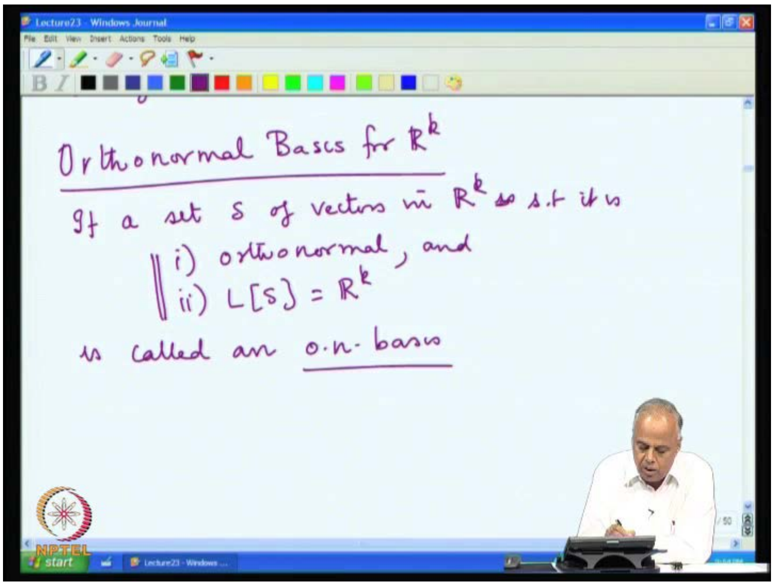

This links us to the notion of an orthonormal basis for r k. So if a set s of vectors in r k is 1 orthonormal remember that when we said we want a basis we want linear independence and we want spanning. Now linear independence it can be now replace the orthonormal because orthonormal automatically implies linearly independent. So we want orthonormal and its spans l s is r k the span of the set s is r k. Such a basis is called vectors in r 4 such that it is orthonormal is called an orthonormal. Basis you put it this way so a set is an orthonormal basis if it is orthonormal and it is a spanning set so these are the 2 things required for a set to be an orthonormal basis.

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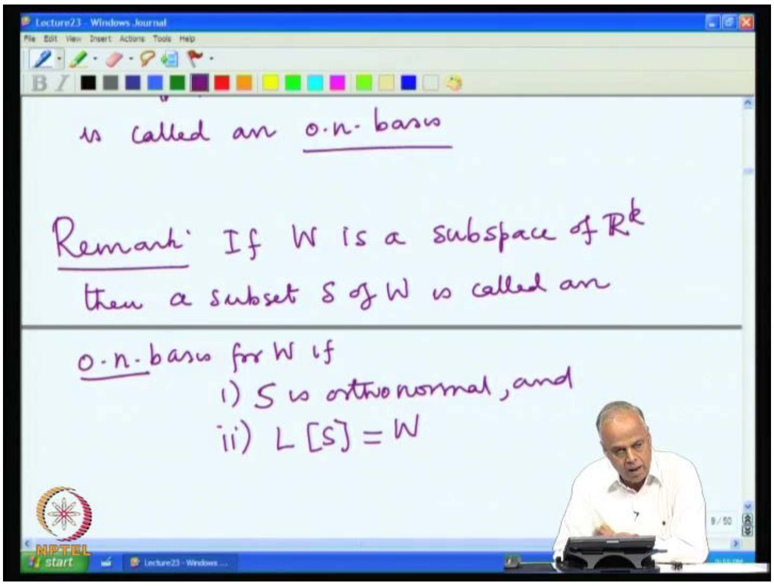

Remark similarly, if w is a sub space of r k then a subset s of w is called an orthonormal basis for w if 1. We want orthonormal so s is s must be orthonormal so we have s is orthonormal and we want it to span that means l s. It must span what now it must we are looking for a basis for w and therefore, it must span w so a orthonormal set in w which also spans w is called an orthonormal basis for w.

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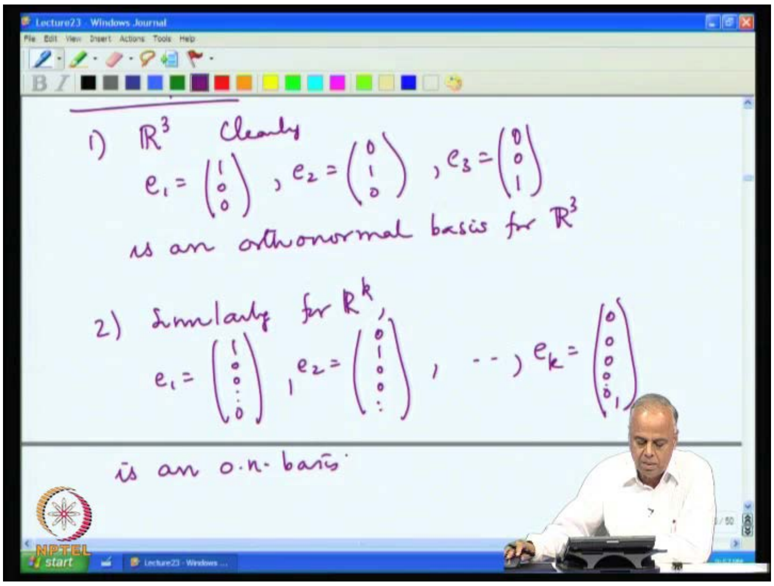

So let us look at 1 of the simple example. Let us take r 3 clearly e 1 equal to 1 0 0 e 2 equal to 0 1 0 e 3 equal to 0 0 1 is an orthonormal basis or r 3. Why is it we know clearly that these are all orthogonal to each other. Because the dot product of any 1 of them is 0 with the other and then each 1 of them has length 1 and any vector x 1 x 2 x 3 or r 3. Obviously x 1 times e 1 plus x 2 times e 2 plus x 3 times e 3. So the spans r 3 so this is linearly this is orthonormal and spans and therefore, it is a basis so this is simplest example similarly, for r k e 1 equal to 1 0 0 0 with k. Components e 2 has second component 1 all other 0 and if go on like that e k has the last component 1 the k th component all other 0 is an orthonormal basis these are very simple examples.

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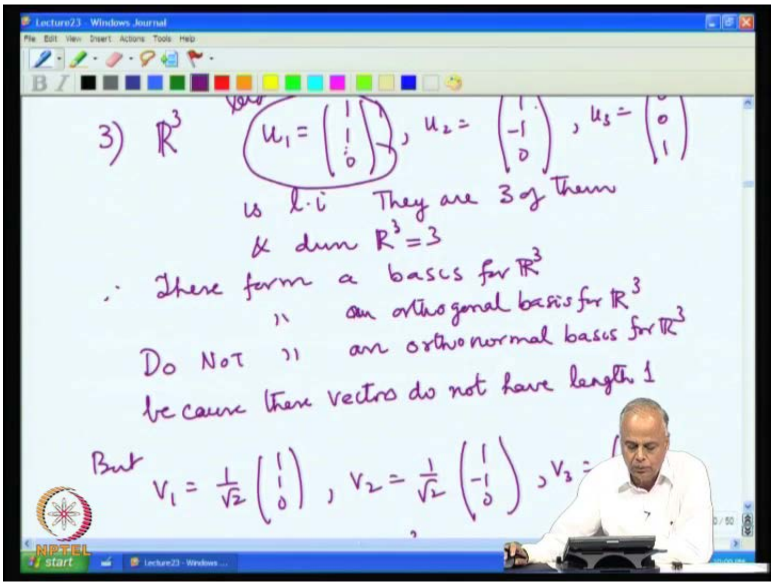

Let us look at another example let us take r 3 let us take the vectors u 1 equal to 1 1 0 u 2 is equal to 1 minus one. 0 u 3 is equal to 0 0 1 this is a basis is easy to verify that this is linearly independent. This is there are 3 vectors for r 3 for something to form a basis any 3 linearly independent vectors in r 3 will form a basis for r 3 there are 3 linearly independent vectors. So they are 3 of them and dimension of r 3 is 3 therefore, these form a basis first thing, we know it is that these form a basis for r 3 now every vector here is orthogonal to each other. Because the dot product of u 1 and u 2 is 1 into 1 plus minus 1 into 1 where 0 into 0 which is 1 minus 1 which is 0. Similarly, u 2 and u 3 are orthogonal and u 3 and u 1 are orthogonal. So these vectors are orthogonal so if they form an orthogonal basis for r 3 they form an orthogonal basis for r 3. However they do not form an orthonormal basis because the normality condition is not satisfied these vectors do not have length 1. Because these vectors do not have length 1 therefore, since we already have orthogonality in order to get normality. All we have to do is divide each 1 of these vectors by length 1. When we say that they do not have length 1 u 3 has length 1 but, u 1 and u 2 do not have length 1. Even if 1 vector fails to have length 1 we will use the normality condition. Therefore, if we now define v 1 to be 1 by root 2 1 1 0 this is obtained by dividing u 1 this vector u 1 by its length what is its length is 1 root 2. Therefore, we divide by the length root 2. Similarly, we divide v 2 by its length and v 3 does not require any division because it has already has length 1 this is an orthonormal basis for r 3. Let therefore, even though this these original vectors u 1 u 2 u 3 did not form a basis we have got a new basis by dividing them by the length. Because these vectors were already orthogonal we needed to do only normality.

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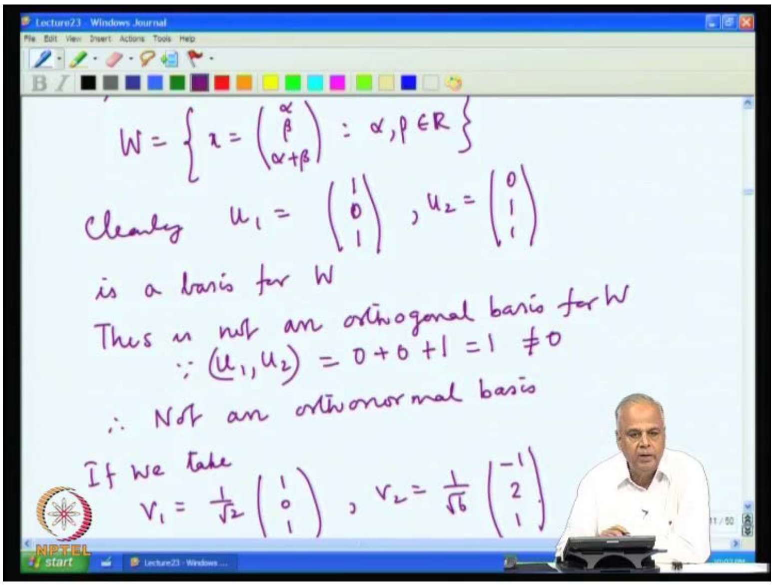

Let us now look at r 3 in r 3 consider the subspace w. It consists of all those vectors which are of this form alpha beta alpha plus beta alpha beta real numbers. What it means is all those vectors for which the third component is the sum of the first 2 components. All the third component is alpha plus beta which is the sum of the first 2 components alpha and beta clearly u 1 equal to 1 1 1 0 1 u 2 equal to 0 1 1 is a basis for w. Because these 2 vectors belong to w they are linearly independent and every vector an w is a linear combination of these vectors. So all the conditions required for basis is satisfied however this is not an orthogonal basis because the 2 vectors are not orthogonal to each other. Because u 1 u 2 the inner product is 1 into 0 0 plus 0 into 1.

The product of the second components is 0 into 1 is 0 but the product of the third component is 1 into 1. So the u 1 u 2 is 1 not equal to 0 and therefore, the vectors are not orthogonal and therefore, it does not form an orthogonal basis and therefore, not even orthonormal basis it is not even normal. So it does not have either orthogonality property or the normality property however if we take v 1 equal to 1 by root 2 1 0 1 and v 2 is equal to 1 by root six minus 1 2 1 we see that v 1 belongs to w. Because v 1 be simply the multiple of the vector u 1 1 by root 2 multiple of the vector u 1. We have here so v 1 is just the 1 by root 2 multiple of u 1 and since u 1 is in w any multiple in w v 2 is a vector in w why first of all. If we look at minus 1 2 1 the third component is the sum of the first and the second and therefore, this part belongs to w and any multiple of that will belong to w and. Therefore v 1 v 2 belong to w that is the first thing that we observe secondly v 1 are v 2 are orthogonal to each other because 1 minus 1 plus 1 plus 1 is 0 so the dot product is 0 thirdly we observe that the length of v 1 is 1 and the length of v 2 is 1

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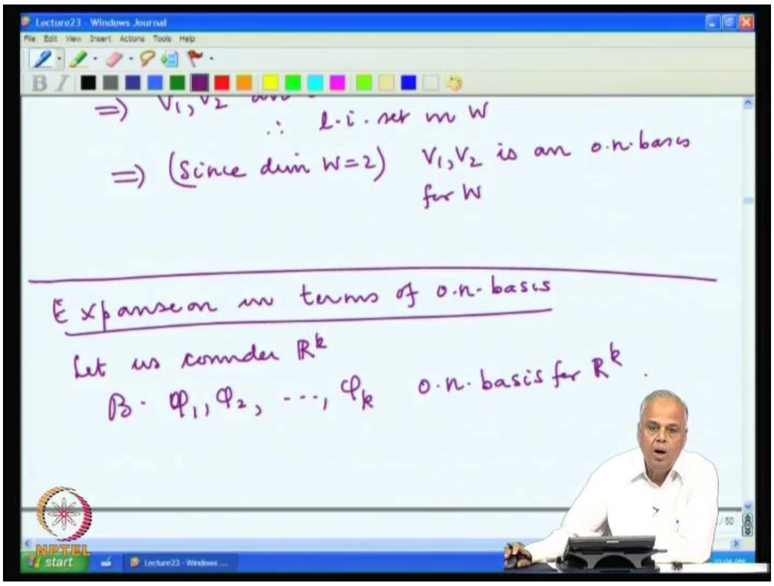

And therefore, v 1 v 2 is an orthonormal set in w and therefore, linearly independent set in w. Since u 1 u 2 is a basis for w the dimension of we have seen that the u 1 u 2 be the basis for w. So dimension of w is 2 any 2 linearly independent vectors in w will form a basis which says since dimension w equal to 2. This implies that v 1 v 2 is an orthonormal basis for w. So we have a sub space here for which we have a orthonormal we have found the orthonormal basis. Now we shall look at what is effect of this orthonormal basis we have seen that whenever we have a basis every vector in that space can be expanded as a linear combination of the vectors.

In that basis and therefore, in particular if we have an orthonormal basis then every vector in that space can be expanded as a linear combination of this orthonormal basis. Let us look at this expansion so while call this the expansion in terms of orthonormal basis. So first let us look at r k it is considered r k and let us say b let us call it phi 1 phi 2 phi k any. Basis must contain exactly k vectors so an orthonormal basis for k or r k so suppose we have a orthonormal basis for r k .

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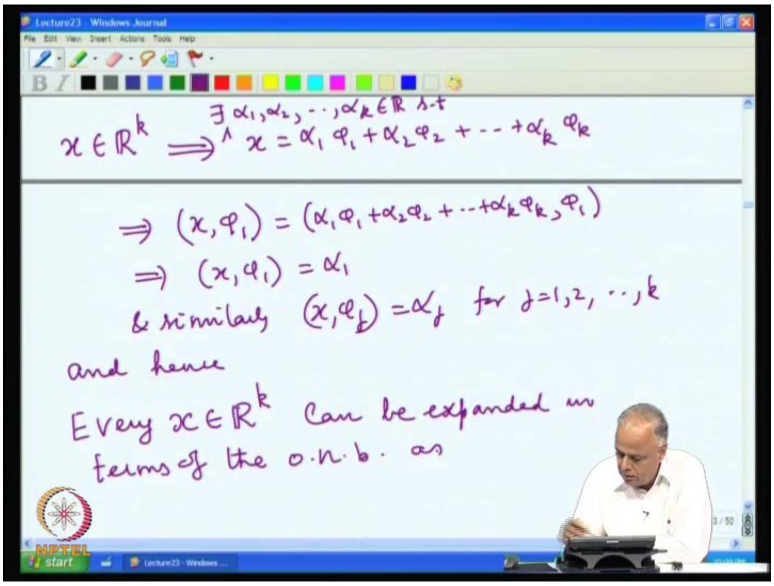

Then any vector x in r k we can expand it as x is equal to x 1 phi 1 plus x 2 phi 2 plus x k phi k. What we do not what this x 1 x 2 x k are so let us call them as alpha 1 alpha 2 alpha k at the moment. We do not know what they are so given a vector x in r k all we can say is there exists numbers alpha 1 alpha 2 alpha k. All of them are real such that x can be written as a linear combination the phi 1 phi 2 phi k if this alpha is are the coefficients. Now suppose now I take the inner product x phi 1 the dot product x phi 1 that is the same as alpha 1 phi 1 plus alpha 2 phi 2 plus alpha k phi k comma phi 1. Because x is the sum again as before if we take the dot product phi 1 phi 1 will give 1 phi 2 phi 1 will give 0 phi 3 phi 1 will give 0 phi k phi 1 will give 0. Because phi 1 phi 2 phi k is an orthonormal set. So that says x phi 1 is equal to alpha 1 so this where the co efficient of phi 1 in expansion of x is precisely the dot product or the inner product of x with phi 1 similarly, the co efficient of phi 2 will be the dot product of x with phi 2.

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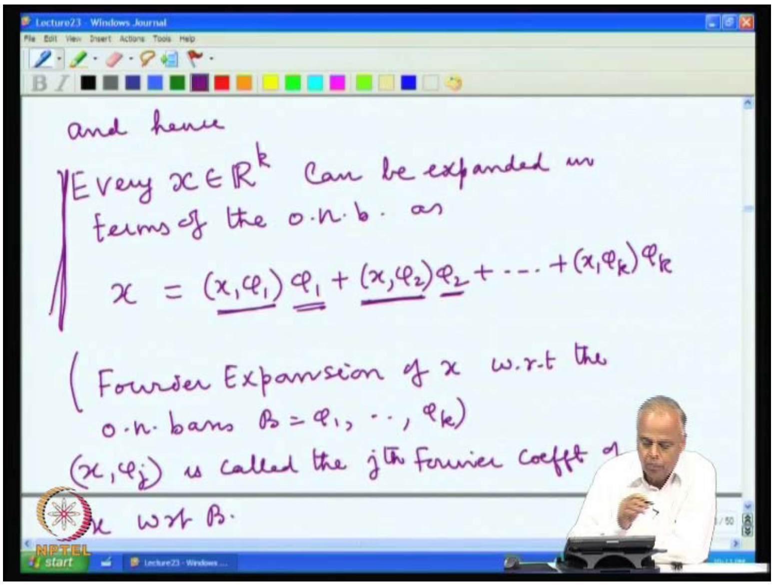

And similarly, x phi j will be equal to alpha j for j equal to 1 2 up to k and hence every x in r k can be expanded in terms of the orthonormal basis. As x is equal to x comma phi 1 phi 1 plus x comma phi 2 phi 2 plus x comma phi k phi 1. So we know precisely how to find the co efficient what is that is much easier to find the co efficient in the expansion with respect to the orthonormal basis. Because to find the co efficient with respect to phi 1 we need to know only the relationship between x and phi 1. Namely the dot product of x 1 phi 1 we find the co efficient corresponding to phi 2. We need to know only the relationship between x and phi 2. Namely the dot product and so x with and so on therefore, in these cases it is much easier to find this orthonormal expansion in terms of orthonormal basis.

In general situation when we deal with vector spaces abstract and abstract inner product. Which we generalize the dot product such expansions are reflective as in Fourier expansion. We generalize Fourier expansion of x with respect to the ortho normal basis b and the x phi j is the co efficient the co ordinate or the component of x with respect to this order basis b is called the j th Fourier co efficient of x with respect to this order basis b. We have an order here phi 1 phi 2 phi k so let us treat this as an order orthonormal basis. Therefore, the first conclusion is that every vector can be expanded in this form. So this is the first important conclusion once we have an orthonormal basis every vector can be expanded in a Fourier expansion with respect to this orthonormal basis.

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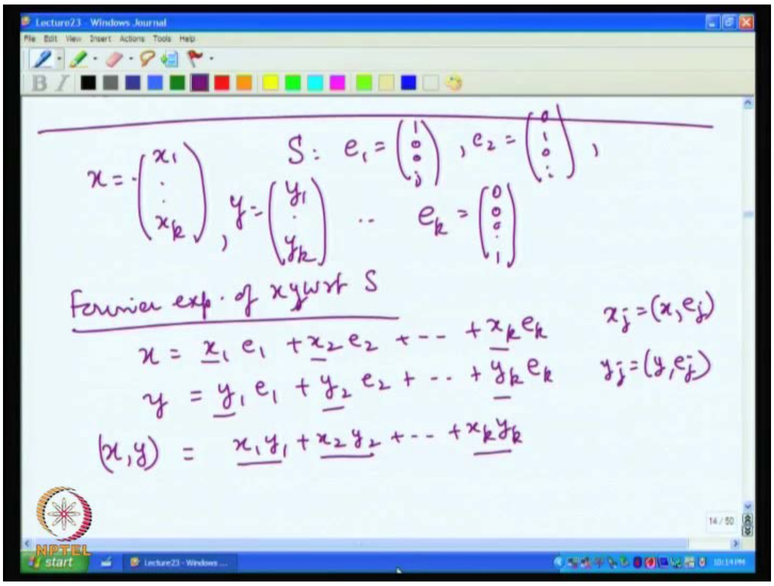

Something interesting happens suppose we had x and we had the first. The standard basis we call this the standard basis 1 0 0 0 e 2 as 0 1 0 0 and so on and finally, e k has 0 0 0. This is the standard basis and if I have x which is x 1 x 2 x k then if I expand with respect to this standard or the Fourier expansion of x with respect to this basis is nothing but, x is equal to x 1 e 1 plus x 2 e 2 plus x k e k. Because x comma e 1 the dot product of x and e 1 picks up only the first co efficient the dot product of x and e 2 picks up the second co efficient and therefore, we have the Fourier expansion of x with respect to has as this and similarly, if i take a vector y which is y 1 y 2 y k then the Fourier expansion of y with respect to this will be y 1 e 1 plus y 2 e 2 plus y k e k where x j is actually.

Equal to x e j and y j is equal to y e j now what is the dot product of x and y or the inner product of x and y it is x 1 y 1 plus x 2 y 2 plus x k y k which is the sum of the products of the Fourier co efficients with respect to this basis s remember this x 1 x 2 x k and y 1 y 2 y k these are all the fourier co efficient so this is the product of the first 2 fourier co efficients of x and y this is the product of the fourier co efficient of x second fourier co efficients. This is the product of the k th fourier co efficient so the inner product of the dot product is the sum of the products of the first k fourier the k th fourier co efficient

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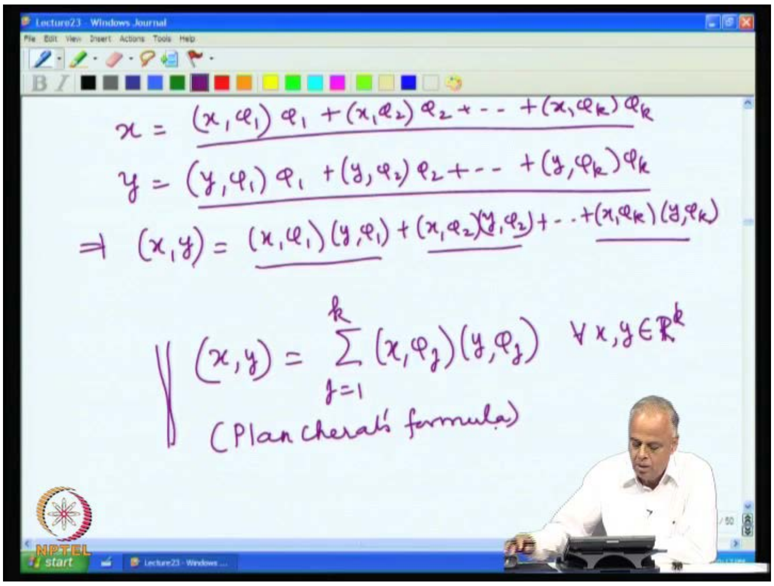

Now let us look at the fourier expansion in terms of our order orthonormal basis general orthonormal basis phi 1 phi 2 phi k. Then we have x is equal to x phi 1 phi 1 plus x phi 2 phi 2 and so on x phi k this is what the fourier expansion we obtain that the fourier expansion of any vector. The co efficients are simply be dot product of x with vectors similarly, y is y phi 1 phi 1 plus y phi 2 phi 2 and so on y phi k. Now I take the dot product of x and y. I will have to take the dot product of this sum with respect to this sum. Now when we take the dot products the cross terms go away because phi i comma phi j will be 0. If i is not equal to j and the direct the direct terms phi 1 comma phi 1 will give you 1 so this will simply be x phi 1 y phi 1 plus x phi 2 y phi 2 and so on x phi k y phi k.

Which simply says again we get the inner product of x and y are the product of corresponding fourier co efficients. So whether you choose this standard ordered basis you namely u 1 u 2 u k or whether you choose any arbitrary orthonormal basis the inner product is always the sum of the product of the corresponding fourier co efficients. Therefore, the next important property is the x comma y is equal to summation j equal to 1 to k x phi j y phi j for every x y in r k. This is referred to as the plancherals formula once again this we put y equal to x. In the above we get x equal to x which is the length of x square the sum of j equal to 1 to k x phi j whole square now again this to same whatever ordered basis.

Whatever ordered orthonormal basis you choose the length square is alpha always the sum of the corresponding fourier co efficient square in the standard ordered basis. We take this vector as x 1 x 2 x k the length is simply x 1 square plus x 2 square plus x k square. If you take an arbitrary ordered orthonormal basis then the length of x square is the sum of the squares of the corresponding fourier co efficients this is true for every x is r k and this is called the parsevals identity this is called the parsevals identity. Now we have given an orthonormal basis.

We can expand every vector in terms of this orthonormal basis the co efficient are called the fourier co efficients and whenever you want to take the inner product of 2 vectors. You have to simply take the sum of the products of the corresponding fourier co efficients. Whenever you want to find the length square you have to only find the sum of the square of the fourier co efficients of that vector with respect to this orthonormal. Whatever orthonormal basis we choose this is this identities hold these are very important facts.

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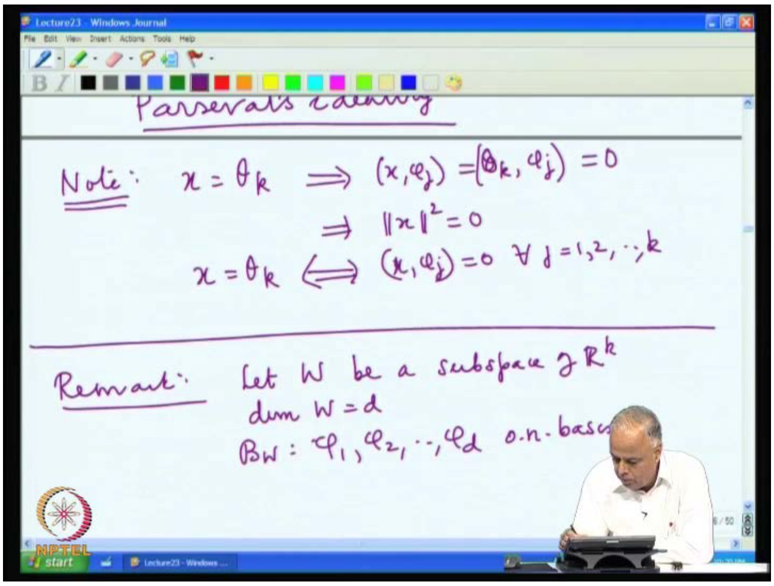

Note that if the vector x is theta k then all this fourier co efficients must be 0. Because x phi j is equal to 0 first of all it is equal to theta k phi j that the inner product of this 0 vector with anything is 0. So all the fourier co efficients are 0 and therefore, we get now x square is 0 which is what we want look like the length vector is 0. So we have whenever x is the 0 vector that says the fourier co efficients are all 0 for every j equal to 1 2 k conversely if x phi j is 0 for all j then.

The fourier expansion tells you all the co efficients are 0 and therefore, the vector must be 0 so a vector is the 0 vector. If and only if it is orthogonal to all the basis vectors this is 1 criterian for phi 1 phi 2 phi j to be a basis. We look at it later remark we can do the same thing in a sub space also, let w be a sub space of r k dimension of w is d b w is w 1 let me call it again the same phi notation for a orthonormal basis phi 1 phi 2 phi d orthonormal basis for w.

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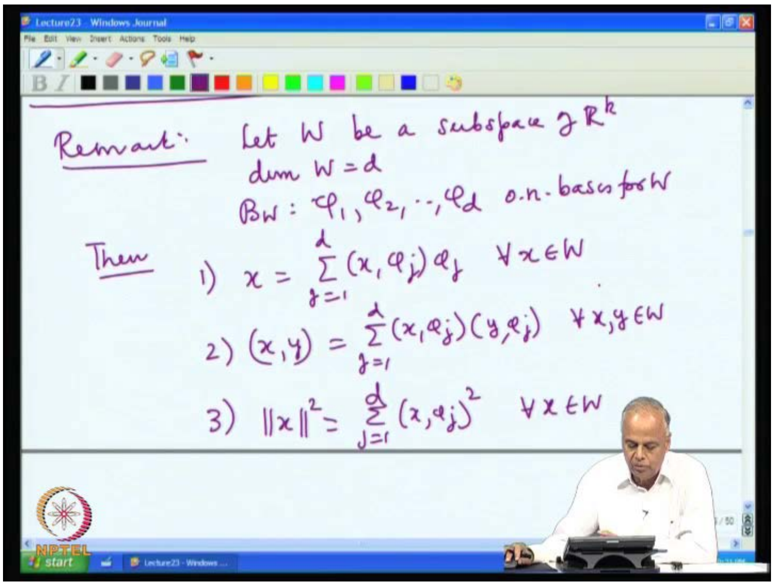

Then we can restrict our fourier expansion within w x is equal to summation j equal to 1 to d x phi j phi j for every x in w. So every vector x in w can be expanded in the fourier expansion x comma y is equal to summation j equal to 1 to d x phi j y phi j for every x y in w and finally, this norm condition norm x square is equal to summation j equal to 1 to d x phi j square for every x in w. So if you take w equal to r k we get all the results we had before but, we can also restrict our self sub space and we get the corresponding.

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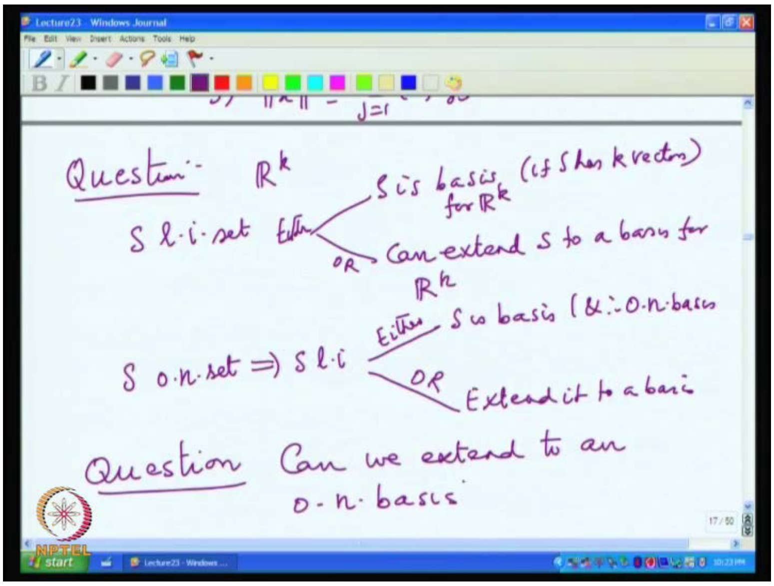

Now we raise a question we have r k, we have a linearly independent set ,we saw that either s is basis this will happen if s has k vectors. So if s has k vectors because if we have k vectors and any k linearly independent vectors will form a basis. So if s has k vectors s is already a basis or can extend s to a basis for r k. This is a basis it is already a basis for r k or it can be extended to a basis for r k. Now if you start with s an orthonormal set we have seen that s is linearly independent and therefore, any linearly independent. Set is either s is basis and this will happen if s have a k vectors is observed now if it is a basis it is already orthonormal and therefore, orthonormal basis but now because is a r if it does not a k vectors. We can extend it to a basis but we do not know whether what we have extended to is an orthonormal basis. So question is can we extend to an orthonormal basis.

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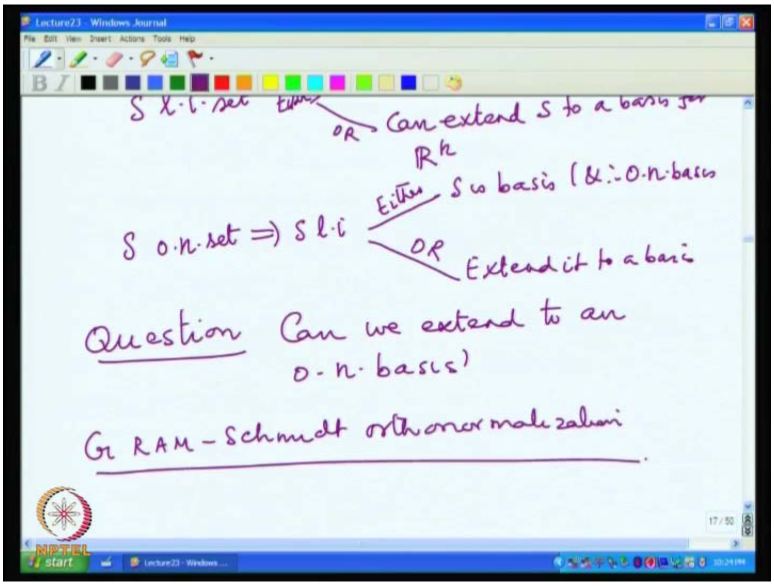

So given any orthonormal set it is either a basis an orthonormal basis or it is a linearly independent set. Which can be extended to a basis the question is can we extend it to an orthonormal basis in other words is every orthonormal set a basis r can be converted to an orthonormal basis. Now we shall investigate this question the main ingredient that is required for this is what is known as the gram schmdt. Ortho normalization the main idea of the gram schmdt ortho normalization is the given any linearly independent set we convert it to an orthonormal. Set in such a way the space that we span are not lost we shall look at the details of this conversion in the next lecture.