Advanced Matrix Theory and Linear Algebra for Engineers
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Lecture No. # 19
Linear Transformations - Part 3

In the last lecture, we saw several examples of linear transformations. We shall now continue to study the structure of linear transformations, the answers too many of the questions that we raised lies in the study of the structure of a linear transformations.

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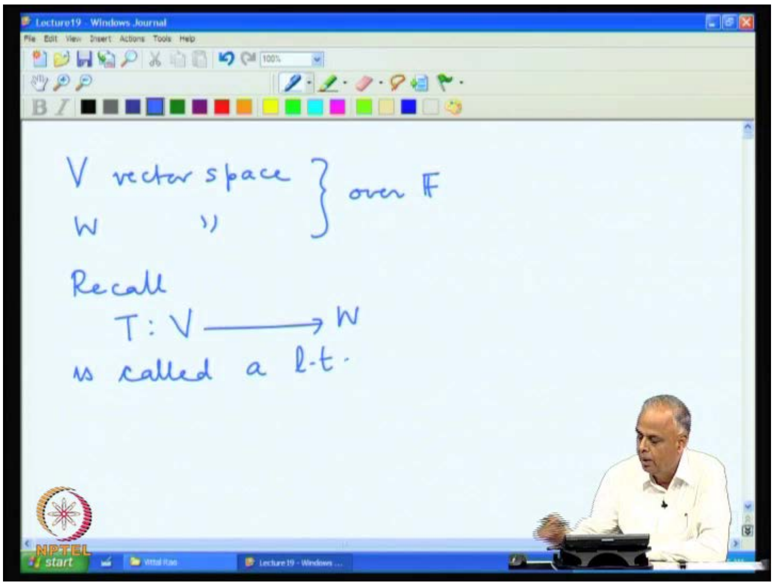

Let us now consider a vector space V and a vector space W, both are vector spaces over a field F and recall that a transformation from V to W is called a Linear Transformation, if it preserve the basic algebraic operation in the vector spaces.

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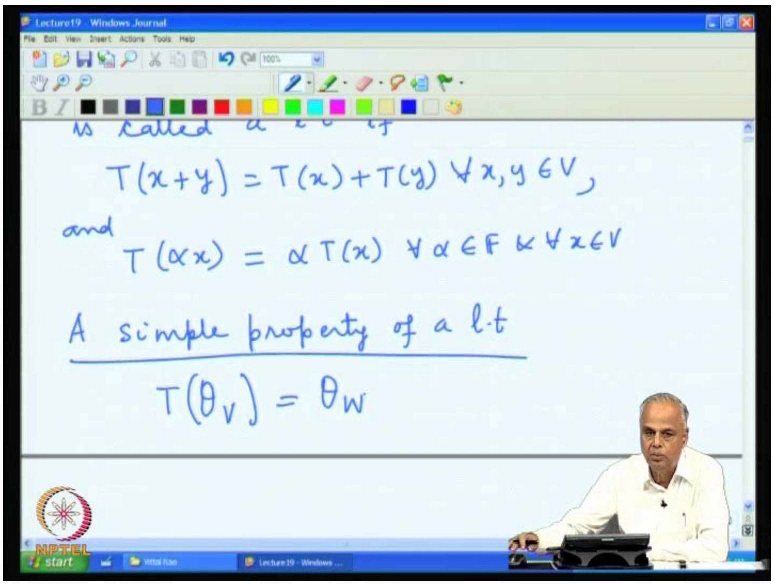

If it preserves addition T of x plus y is T of x plus T of y for every x y in V and it preserves scalar multiplication, T of alpha x is alpha T of x for every alpha in F and for every x in V. Such a Linear Transformations is going to hold the key for our studies on various questions that we raised and important property that we have observed a very simple, but important property. So, let us note that a simple property of a Linear Transformations which we saw last time was that if we take theta V, the 0 vector in the V space T always maps it to the 0 vectors in the W space.

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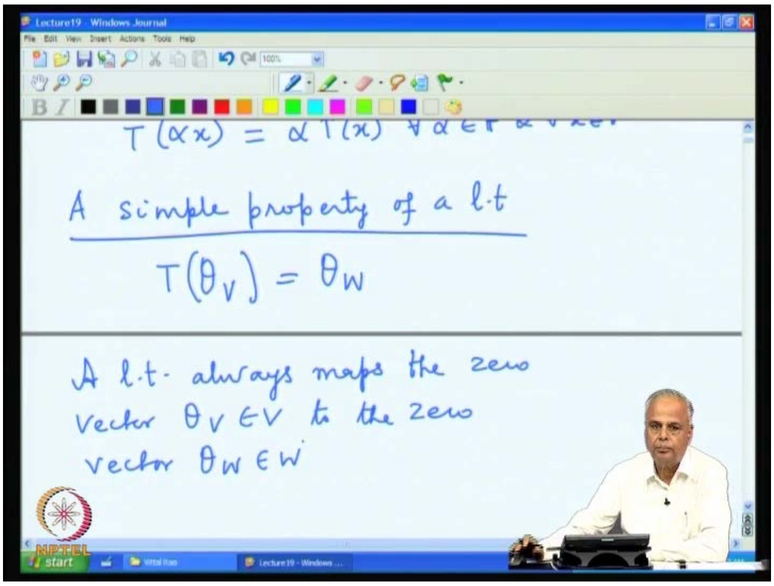

So, a Linear Transformation always maps the zero vector theta V in V to the zero vector theta W in W.

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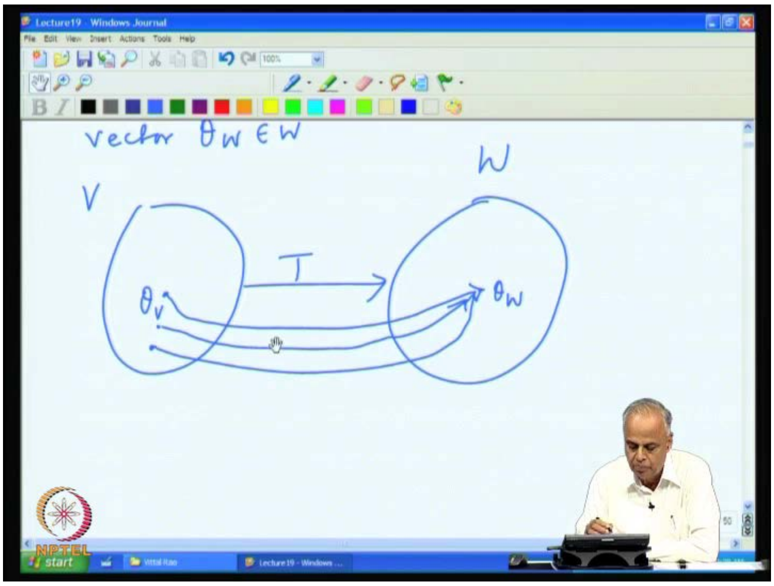

So, what does that mean? We have the vector space V and vector space W and T is the transformation that is taken V vectors into W vectors. What the above property says is, theta V is a vector in V and theta W is a vector in W, theta V is the 0 vector in V, theta W is the 0 vector in W, T pulls along this 0 and then maps into that 0 this is a typical property of Linear Transformations. Now, it may so happen that some other vector in V may also get pulled to the 0 vector, there may be another vector which get pulled to 0 vectors. In other words, there may be a lot of vectors in V which are all going to get focused towards theta W, so all of them are going to be focused towards theta W by this lens T. So, we collect these vectors which are going to be focused to theta W.

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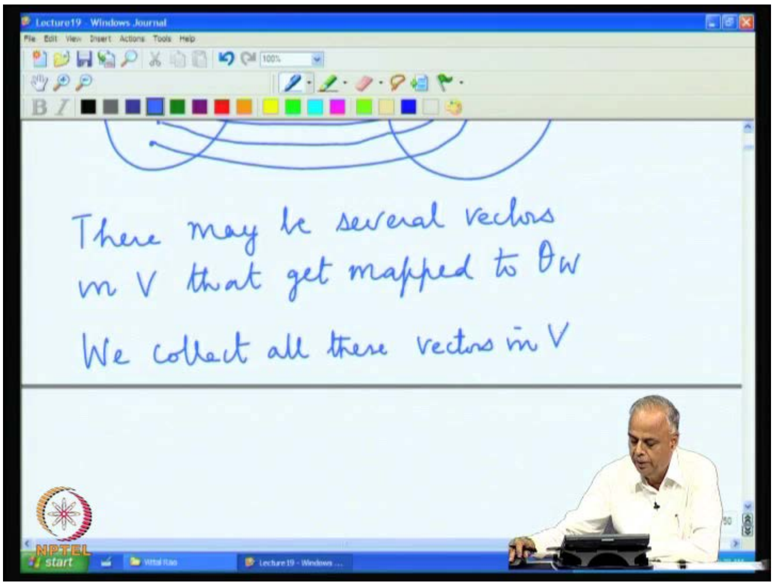

So, there may be several vectors in V that get mapped to the 0 vector in W, in addition to the theta V which we already know gets mapped to the theta W. We collect all these vectors in V.

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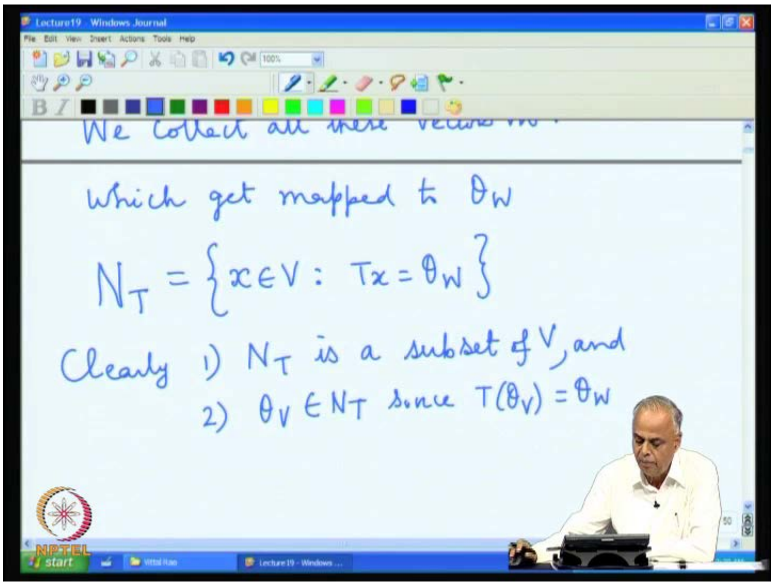

Which get mapped to 0 the 0 vector under the map T, so we denote this collection by N T. So, N T is the collection of all the vectors in V such that, they get mapped to the 0 vector. Now clearly, the first thing we observe is that N T is a collection of vectors from V having certain specific property.

Therefore, apriority they are all vectors in V therefore, N T is a subset of V (no audio from 05:16 to 05:22) and the second trivial thing that we know is that the 0 vector of V is certainly in N T because the 0 vector get mapped to the 0 vector. Two, theta V belongs to N T, since T of theta V equal to theta W.

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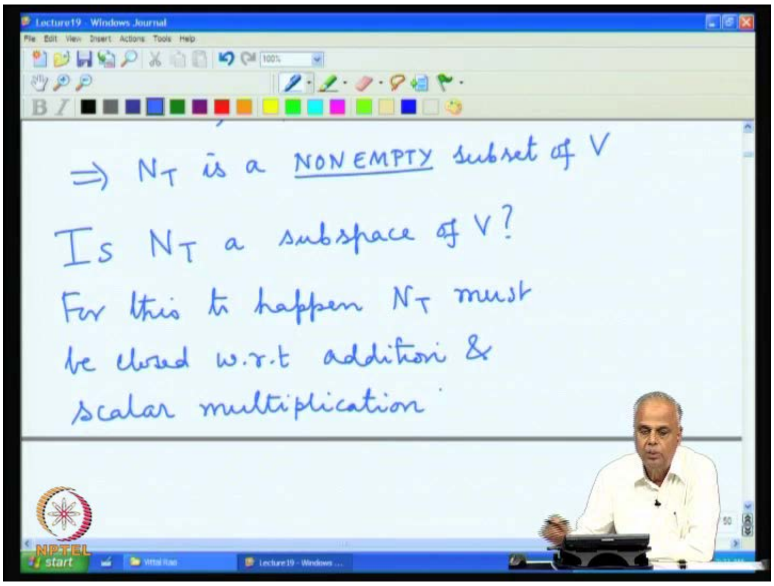

And therefore, N T is a non empty subset of V because at least one vector namely theta V which belongs to N T. Now whenever we have a non empty subset of a vector space, the natural question that we ask is whether that is a sub space, so is N T a subspace of V? This always whenever there is a non empty subspace of a vector subset of vector space, we are always interested in knowing whether it is a subspace. In order to make sure that NT is a subspace, we must make sure that N T is closed with respect to the two basic operation of the vector space. So, for this to happen N T must be closed with respect to addition and scalar multiplication that is the major requirement for any subset to get qualification qualified as a subspace. So, let us verify whether N T is closed under these two operations.

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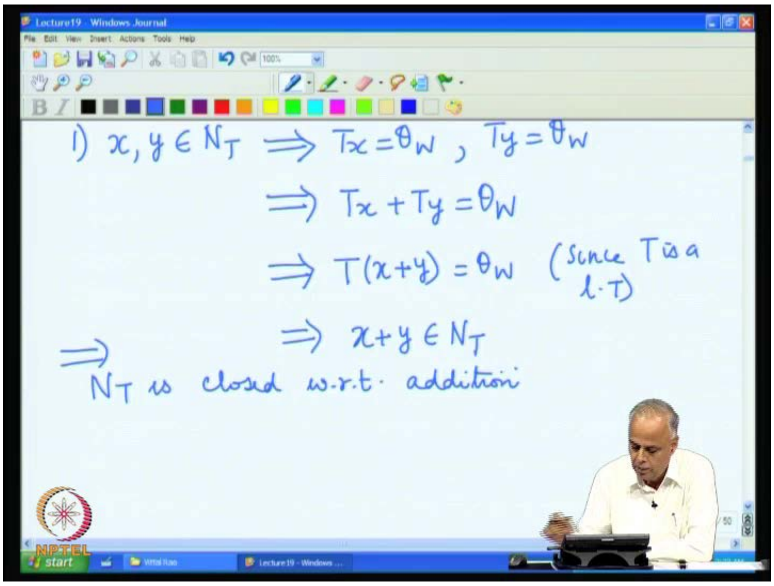

So, let us first check with addition suppose we have two vectors in N T, we want to know whether there sum will also be in N T. Now all we know at present is that x y are in N T, but N T is the collection of all those vectors which are mapped to 0 vector and therefore, T x must be 0 vector that means x must be mapped to the 0 vectors and T y must be the 0 vector because x and y are in N T and therefore, the map to the 0 vector. That says I can add the two and I will get equal to theta W plus theta W, the 0 vector plus the 0 vector is the 0 vector itself.

Now, we know that T is a Linear Transformation therefore, T preserves addition and hence T of x plus y is the same as T of x plus y; T x plus T y is the same as T of x plus y, that says this is because T is a Linear Transformation, we know T preserves addition. That says the vector x plus y is also getting mapped to the 0 vector and hence, x plus y also belongs to N T. So, thus x and y belong to N T mean x plus y belong to N T hence, N T the whole thing implies N T is closed with respect to addition. The next thing we have to verify is whether N T is closed with respect to scalar multiplication.

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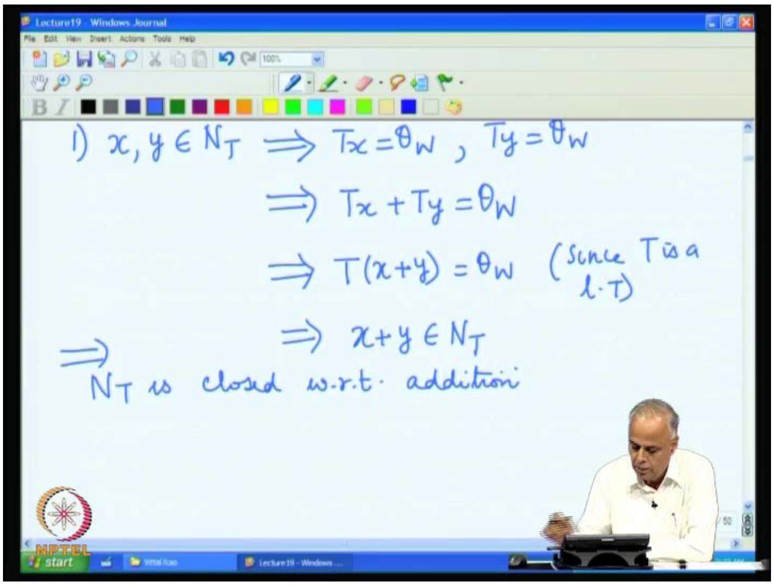

So, let us take a vector x in N T what does that mean? We want to verify that it is closed with respect to scalar multiplication, that means when we multiply x by any scalar, the resulting vector must also be in N T. Now first of all, we given x is in N T this means T carries x to the 0 vector, that is the qualification for being in N T, and if that is so for any scalar if I multiply both side by alpha, I get alpha theta W which is theta W, when the 0 vector is multiplied by any scalar we get the 0 vector. Now since T is linear, T of alpha x is same as alpha of T x, because T preserves scalar multiplication, so alpha T x is same as T of alpha x, since T is a Linear Transformations. So, that says vector alpha x is going to carry over to the 0 vector, and hence alpha x qualifies to be in N T.

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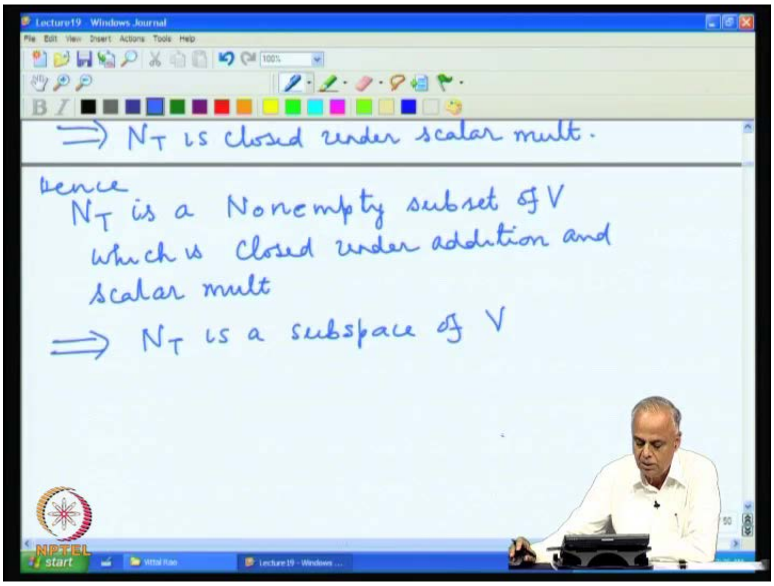

So, this says N T is closed under scalar multiplication. Therefore, N T so what are the various properties we observed? N T is… So, hence N T is first of all, a non empty subset of V which is closed under addition and scalar multiplication, and that makes it a subspace. So, N T is a subspace of V.

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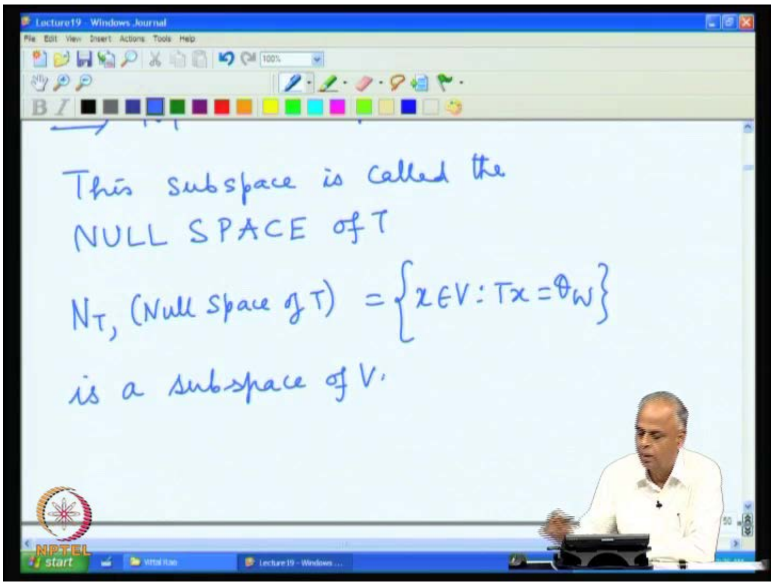

This subspace is called the Null space vectors which get nullified Null space of T, so we have N T is called Null space of T is the set of all vectors in V such that, T x equal to theta W and this is subspace of V. (no audio from 12:05 to 12:12)

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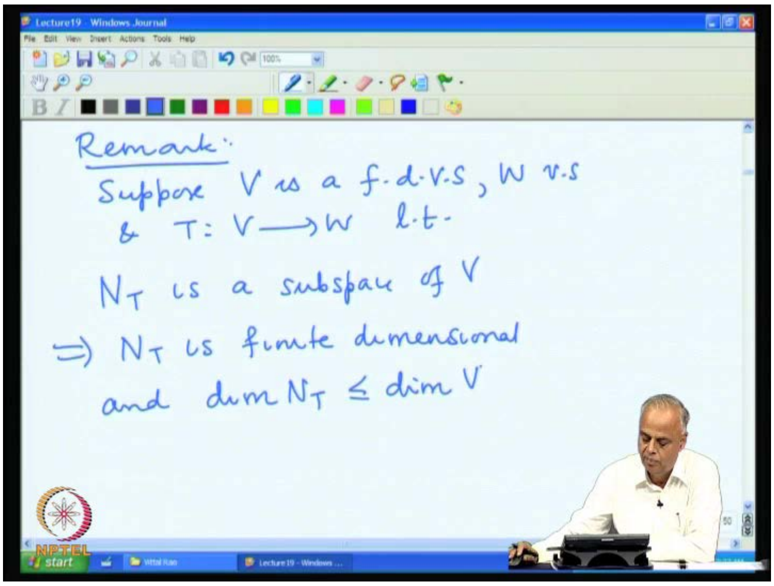

So, make one simple observation suppose, V is a finite dimensional vector space and T maps V to W and W is a vector space, we do not know whether it is finite or infinite dimensional space. So, it is some vector space and as a Linear Transformations; T mapping V to W is a Linear Transformations. Now, N T is a subspace of V and V is a finite dimensional space, so we observed that any subspace of a finite dimensional vector space must be finite dimensional and it is dimension should be less than or equal to the dimension of the full space and hence, we get N T is finite dimensional and dimension of N T is less than or equal to dimension of V because it is a part of V. This dimension is called the nullity of T.

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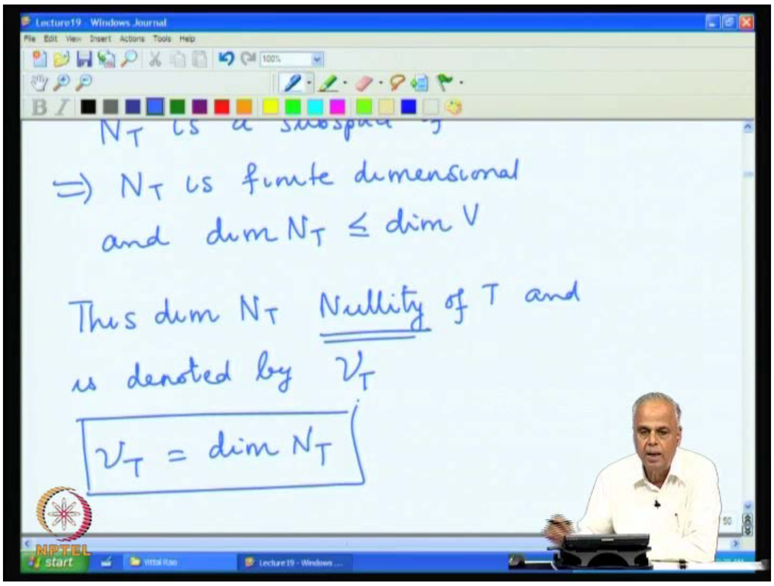

And is denoted by the nu of T, so what do we have? nu of T is dimension of N T where N T is the Null space, so the nullity of Linear Transformations is just the dimension of the Null space of T. So, now we have collected all the vectors which are focused to 0 and then we have studied them and we find that they form a subspace and that subspace is called the null space and it is dimension is called the nullity.

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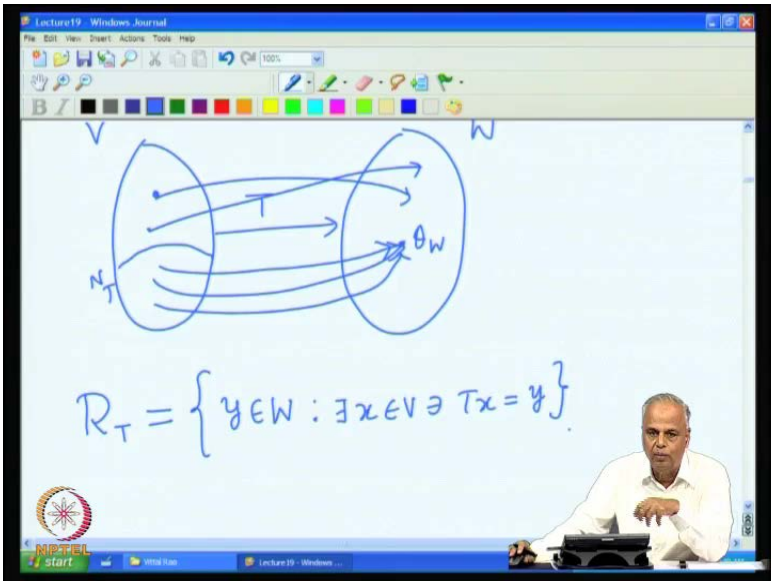

Now let us look at again the transformation we have V, we have W and T transforms V vectors to W, now we have say this is N T, that is all the vectors here are focused to W;

All the vectors in this portion are going to be focused to W. The 0 vector in W now, if I take any vector that is not in N T, it will now get focus to elsewhere. This one may be focused somewhere. So, now therefore, we see theta W is one focal point and there may be other focal points, that mean the other points in W where the image of a vector at V may come and form, we collect all these focal points.

So, we now look at the set R T which is the collection of all these focal points; these focal points will be in W; if a points in W we are trying to see where they come and fall. These are points in W such that, somebody comes and falls there is unique comes here under T, that is there exist a x in V such that the image of x under T is y. So, y is the focal point for x then he is take into R T, if y is not the focal point for anybody, it is not going to be in R T. So, R T is the collection of all such y for which there is a pre image x such that T x equal to y, we also say that why the R T is the collection of all the values taken by the function T? T of x is the value of that function at the point x, so y is the value of the function at the point x. We are collecting all the possible values that the function T takes.

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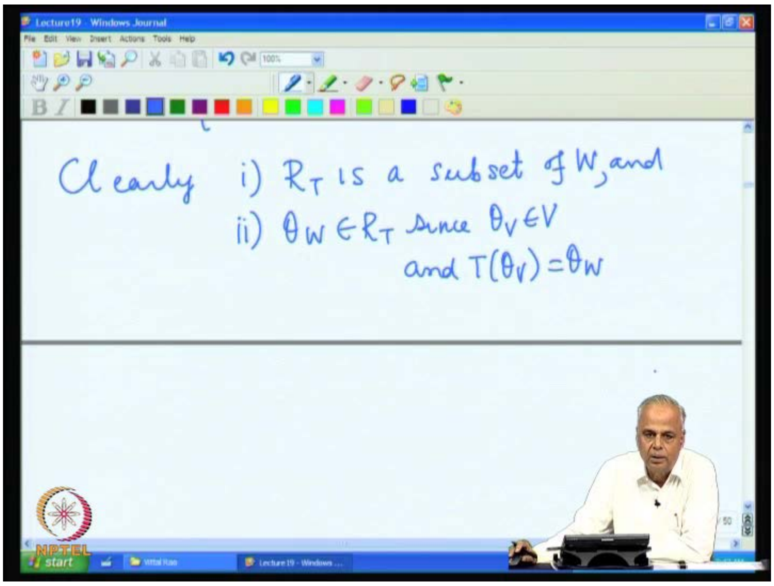

Now clearly, since this is a collection of vectors in W with certain specific property it is the subset of W. R T is a subset of W and we have already seen, that theta V goes to theta W therefore, the theta W is the one of the values taken by T. So, theta W belongs to

R T since, theta V belongs to V and value of T at theta V is precisely theta W. So, there is at least one point in R T therefore, R T is a non empty subset of W.

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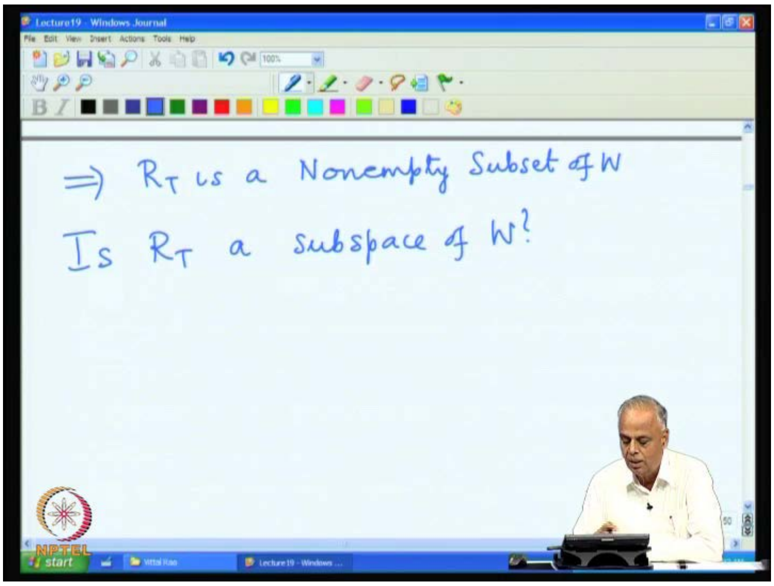

So, that implies R T is a non empty subset of W and as we look in the case of N T, whenever we have a non empty subset vector space; we always interested in knowing whether it is a subspace. So, is R T a subspace of W? It can be a subspace of W because it is a subset of W. Once again, in order to check whether it is a subspace of W or not, there are two properties that we have to check, whether it is closed under addition and whether it is closed under scalar multiplication.

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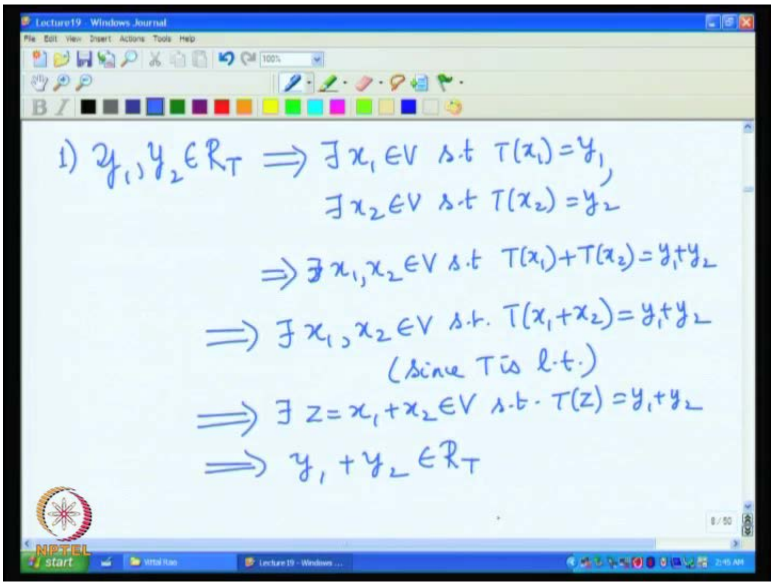

So, let us check whether it is closed under addition, so suppose I take y 1 and y 2 in R T we want to know, whether y 1 plus y 2 is also in RT. Now first of all, what does it mean to say that y 1 is in R T? y 1 is in R T means it is a focal point, which means? There is a pre image x 1 in V such that, T of x 1 will be focused on y 1, T of x 1 is y 1. Similarly, y 2 is in R T means there exist a x 2 in V such that, T of x 2 is y 2, which implies there exists x 1 x 2 in V such that, T of x 1 plus T of x 2 is y 1 plus y 2. Now, since T is linear T of x 1 plus T of x 2 is T of x 1 plus x 2. So, this means, there exist x 1 x 2 in V such that, T of x 1 plus x 2 is equal to y 1 plus y 2 since, T is linear, if T is Linear Transformation, T of x 1 plus x 2 is same as T of x 1 plus T of x 2.

Now, if you call x 1 plus x 2 as z then, since x 1 is in V; x 2 is in V; x 1 plus x 2 will also be in V because V is a vector space. So, there exists z which is equal to x 1 plus x 2 in V such that, T of z is y 1 plus y 2. This means y 1 plus y 2 is also a focal point with z being focused at y 1 and y 2, that is the value of T at the vector z is y 1 plus y 2, so y 1 plus y 2 is a value taken by T therefore, y 1 plus y 2 also belongs to R.

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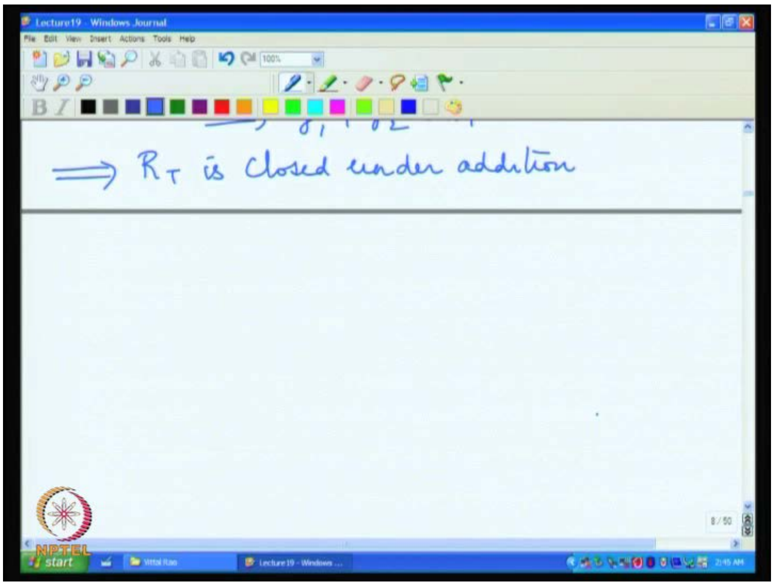

So, y 1 and y 2 are in R T; y 1 plus y 2 is also in R T which means R T is closed under addition. Other thing that we have to check whether R T is a subspace or not, is to see whether R T is closed under scalar multiplication.

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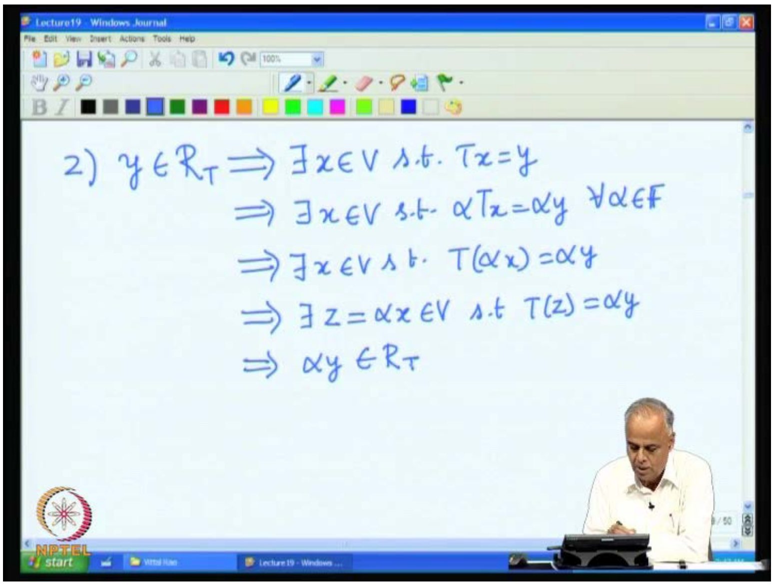

So, take a vector y in R T we want to know whether alpha y will also be in R T, if y is in R T that means y is the value taken T at a point x, that means there exist a x in V such that T x equal to V, T x equal to y. If T x equal to y; if we multiply by alpha, alpha T x will be equal to alpha y for every alpha in F. Once again, since T is linear alpha of T x will be T of alpha x, so there exist a vector x in V such that, T of alpha x is equal to alpha y. Now, if we call alpha x is z so since, x is in V alpha is a scalar and V is a vector space therefore, it is closed under scalar multiplication, so z equal to alpha x will belong to V such that, T of z is alpha y. That means, alpha y is the value of T at the vector z or it is a focal point for the vector z and hence, alpha y must also belong to R T.

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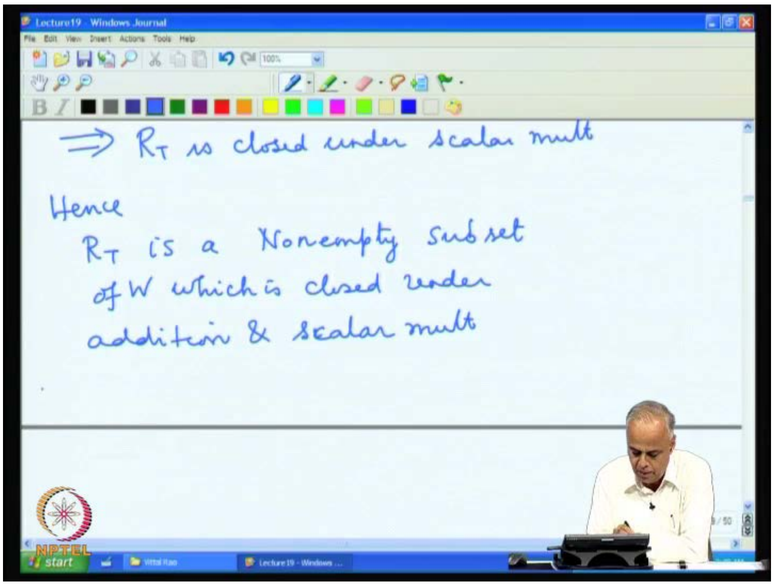

So, that says R T is closed under scalar multiplication, thus we have R T is a non empty subset of W which is closed under addition and scalar multiplication.

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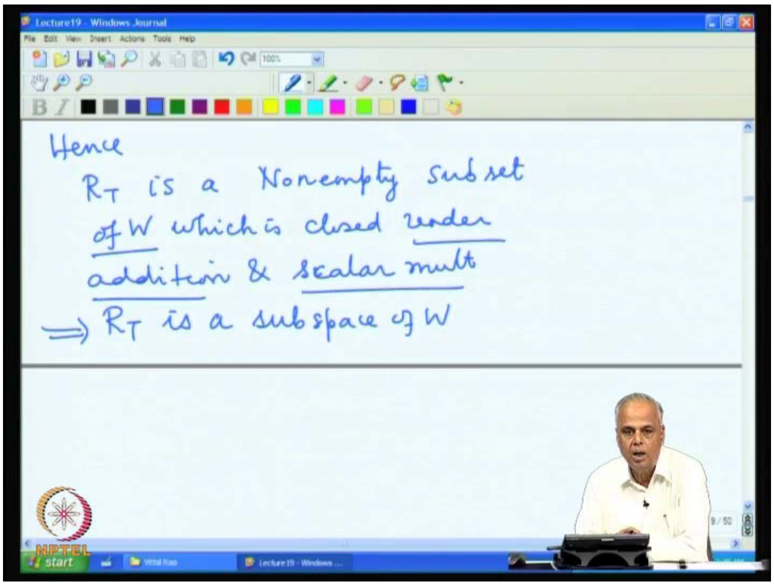

Which means RT is a subspace of W; R T is a non empty subset of W which is closed under addition and scalar multiplication and hence, R T is a subspace of W.

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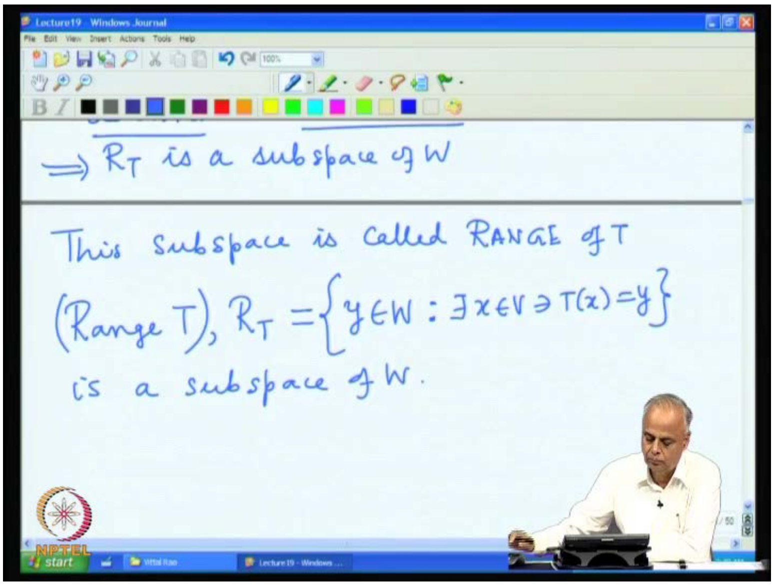

This subspace is called range of T, so the range of T which is denoted by R T is equal to all those y's in W such that, there exist a x in V whose image under T is y and this is a subspace of W.

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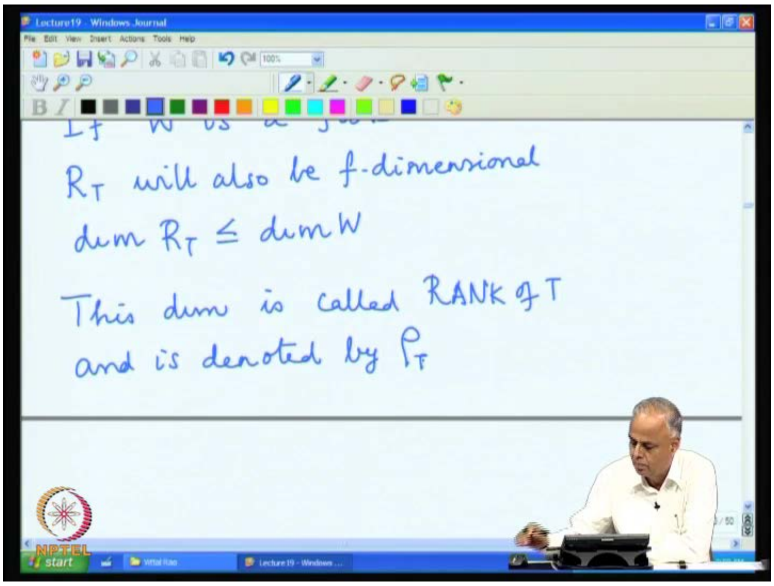

If W is a finite dimensional vector space then, R T being a sub space of the finite dimensional subspace W will also be finite dimensional and since, is a subspace of W;

dimension of W will be less than or equal to dimension of R T will be less than or equal to dimension of W. This dimension is called rank of T and is denoted by rho T.

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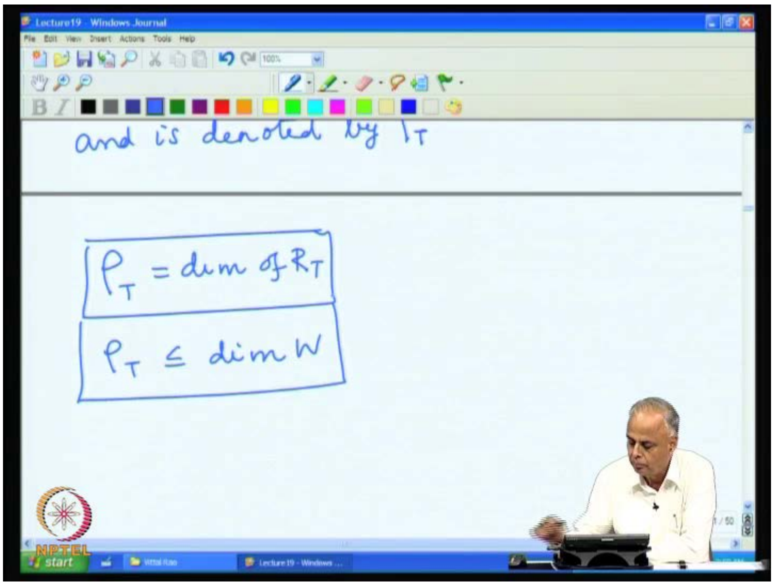

So, rho T is just the dimension of range of T by dimension and rho of T will be less than or equal to dimension of W.

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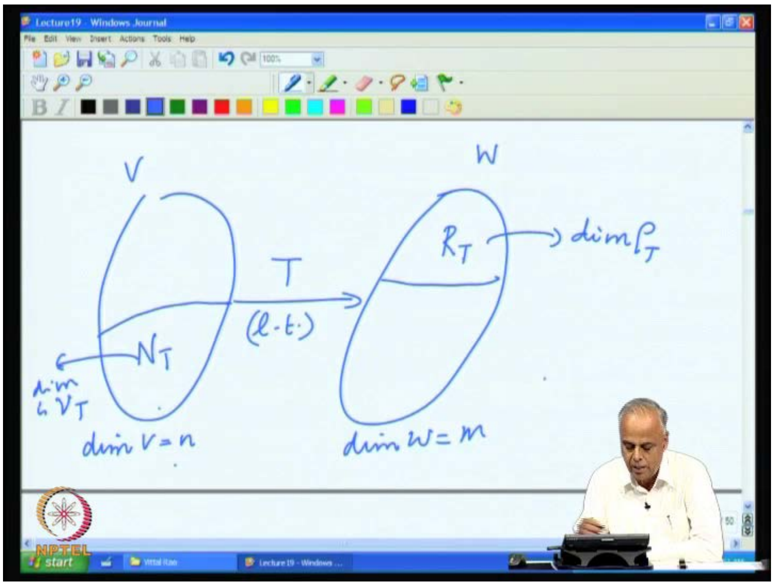

So, suppose now we have V finite dimensional, say dimension of V is n and say W finite dimensional, dimension of W equal to m and T is a Linear Transformations. So, I have two finite dimensional spaces and I have a Linear Transformation T from V to W. Now, we have seen one subspace of V which is connected with T namely the N T and we have seen, one subspace of W which is connected with T namely range of T and this N of T is a finite dimensional subspace of V and it is dimension is called nullity and that is denoted by nu T. So Dimension, it is Dimension is nu T and here, the dimension is rho T. At this dimension is smaller than or equal to n and this dimension is smaller than or equal to m.

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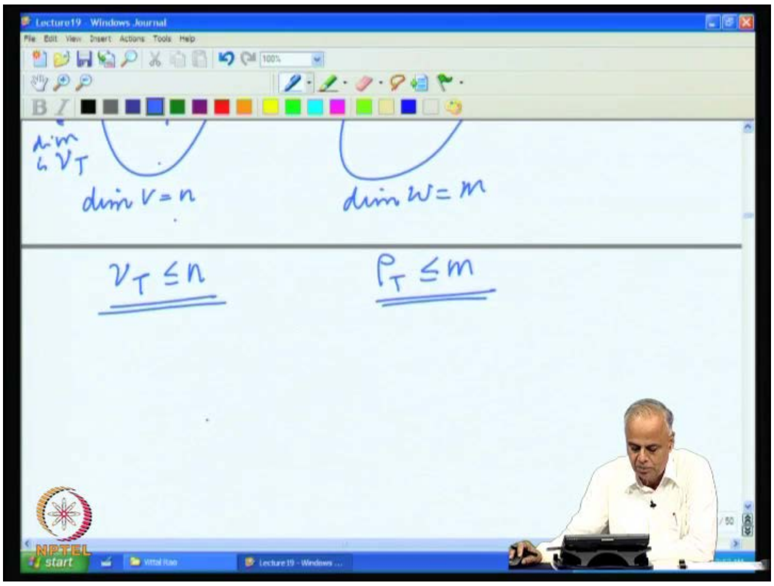

So, nu of T is less than or equal to n and rho of T is less than or equal to m, so we have two important subspaces associated with the Linear Transformationship. As we go along, we will a see lot of subspaces that are connected with a Linear Transformation and these subspaces come into play in the analysis of the structure of a Linear Transformations.

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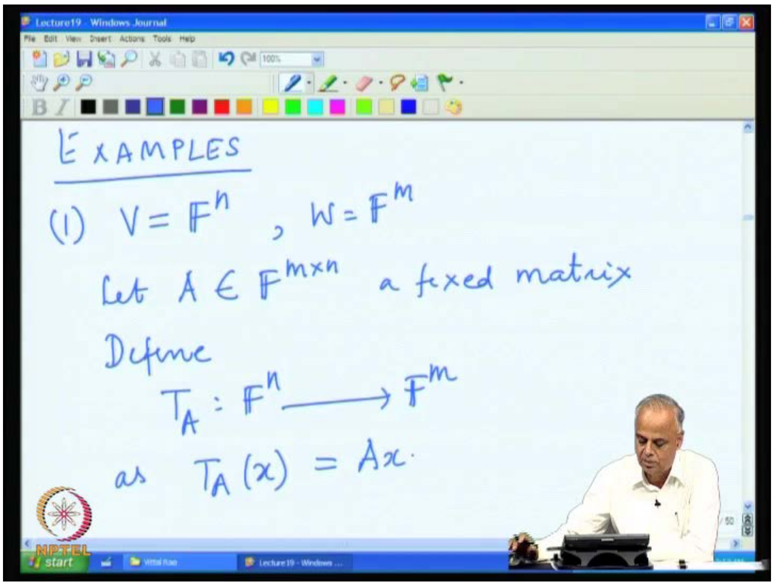

Let us look at some examples, (no audio from 27:55 to 28:04) to consider V to be F n where F is a field, W to be F m where F is a field. So, from n component vectors or n column vector with n entries to the column vectors with m entries. Now we have a transformation, how did we define a transformation from V to W? And the last lecture, we saw any m by n matrix will generated transformation a Linear Transformations from F n to F m. So, let us now consider a fixed matrix in to a fixed matrix, so consider a fixed n by m matrix with entries in m then, we defined a Linear Transformation T A from F n to F m as T A takes any vector x to the vector A times x and since, A is m by n and x is m by one, the result will be m by one and therefore, it will be F m. We have already seen in the last lecture, that this is a Linear Transformations from F n to F m.

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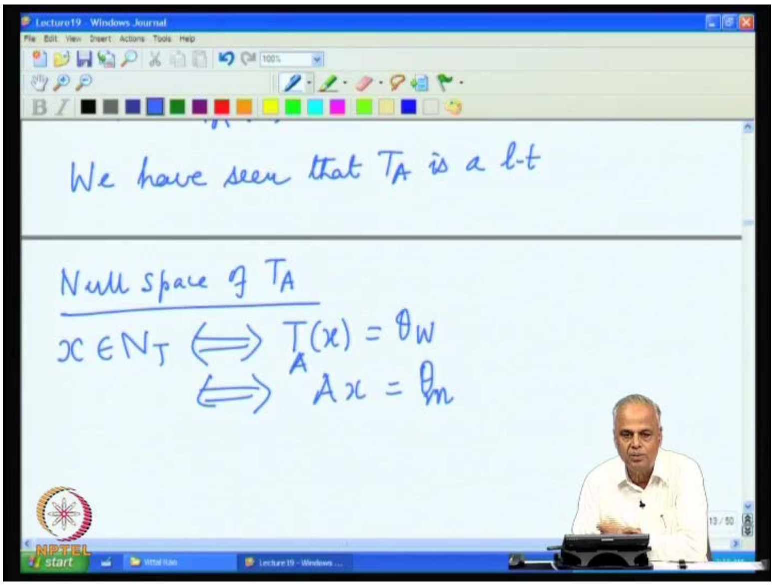

So, we have seen that T A is a Linear Transformation from F n to F m, now once we have a Linear Transformation we want to what is this Null space? What is it is dimension? What is the range space? And what is the dimension? So, let us look at the Null space of T A, so in order to find Null space of T A we want to find all those vectors which get mapped to the 0 vector. So, x in V belongs to the Null space of T if and only if somebody gets qualified to be in n T; if and only if it is get carried to the 0 vector; so if and only if T x is theta W, but then by definition I shall put T A the definition of T A is that, T A of x is A times x. So, the image of any vector is obtained by pre multiplying it by the matrix A. So therefore, T A of x is A of x is equal to theta W; what is theta W? This is theta m because W is F m. So, this is simply the homogenous system of equation A x equal to theta m and the solutions of this is what we know as the Null space of the matrix A.

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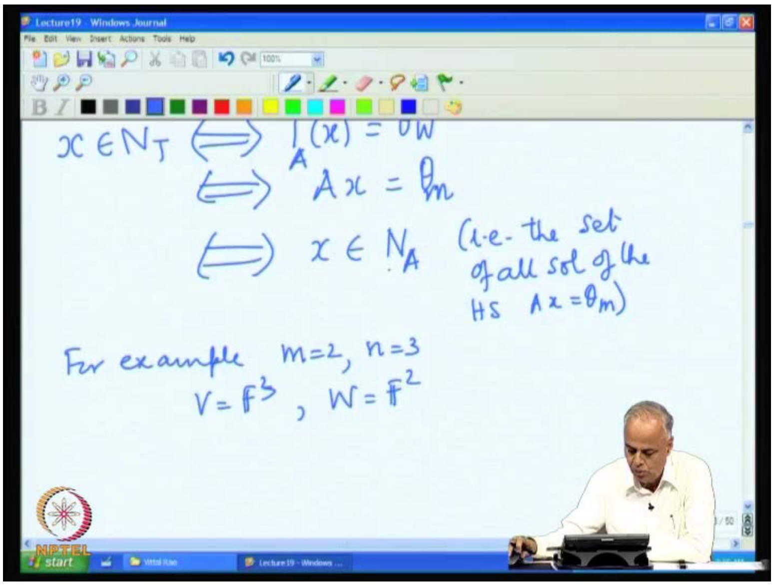

So, if and only if, x belongs to the Null space of matrix A that is the set of all solution of homogeneous system A x equal to theta, so the Null space of T is the same as the Null space of A. For example, if we take m equal to 2, n equal to 3 and consider F 3 to be our V and F 2 to be our W. We have to look for a matrix in m by n, F m by n remember we want to, when we go from n component to m component, we need a matrix which is m by n in this case, we have m is 2 and n is 3.

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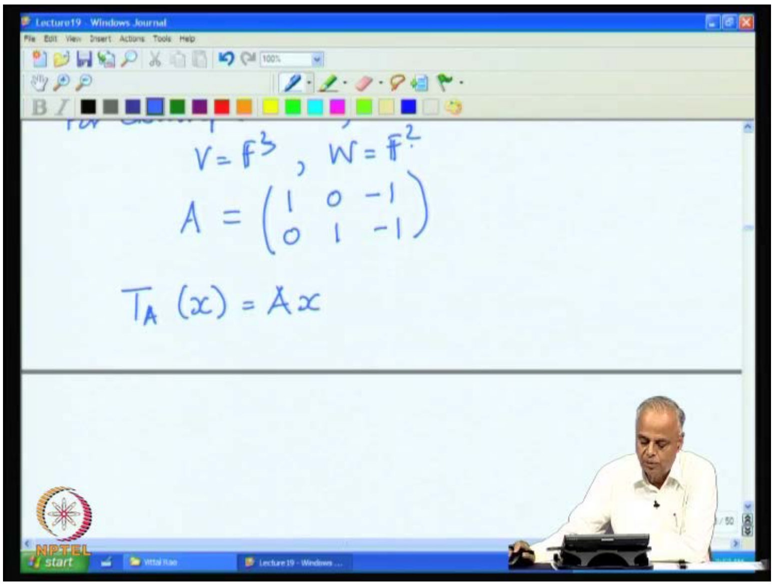

So, we have to take a matrix which is 2 by 3, so let us say 1 0 minus 1 0 1 minus 1 then, the transformation T A is T A of x, now x is in F 3 will be A times x. Since x is in F 3, A is in 2 by 3 the result will be in 2 by 1 and therefore, it will be in F 2.

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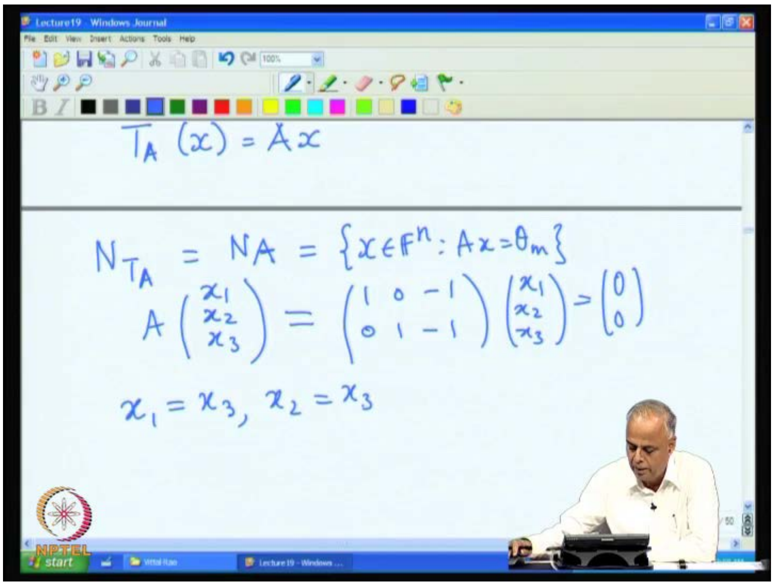

Now, what is the Null space of T A? It must be equal to the Null space of A, now what is Null space of A? The set of all x in F m such that, A x equal to theta m. Now, what is A x? A is this matrix I do not know what x is? x must be in F 3, so it must have three components which means 1 0 minus 1 0 1 minus 1 into x 1, x 2, x 3 is equal to theta 2. Now, the matrix A is already in rho reduced echelon form and therefore, we can write down the solution by inspection by eliminating two of the pivotal variables; there are two pivotal variable x 1 and x 2 and the non pivotal variable is x 3 and we can eliminate x 1 and x 2 in terms of x 3, we get x 1 equal to x 3, x 2 equal to x 3.

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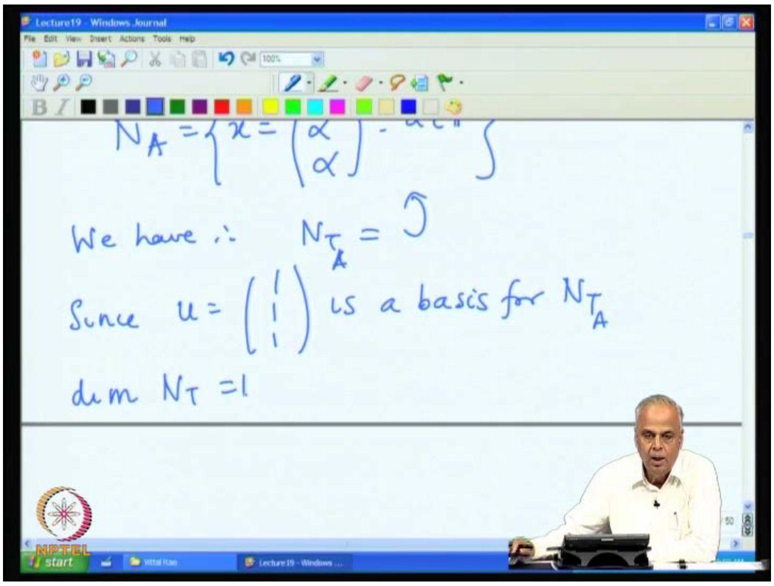

And therefore, the Null space of A consists of all vectors for which x three can be chosen arbitrarily the non pivotal variable. Once, you choose the non pivotal variable, the pivotal variable have to be chosen to be equal to them and alpha can be chosen arbitrarily. And we have therefore, N T also equal to this because N T is equal to N A. Now, what is the dimension of N T? Since, the vector 1 1 1 is a basis for N T now, N T is same as N A, I should write N T A and since the basis has exactly one vector and the number of vectors in basis is called dimension, we get dimension of N T is 1 and this dimension is called nullity.

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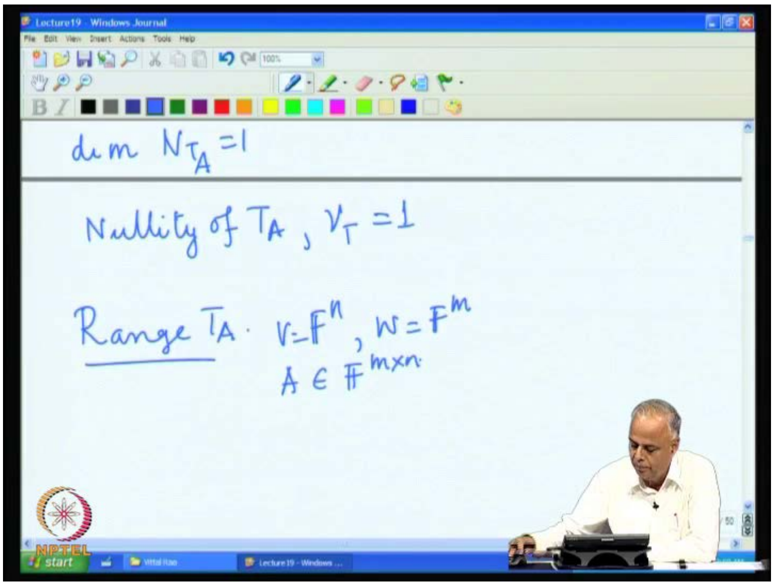

Therefore, the nullity of T A which is nu T is 1, now let us find the range of T A for the same transformation. First, let us look at the general matrix and then look at these examples, so once again we look at F n to be V, W to be F m and we take a fixed matrix in m n.

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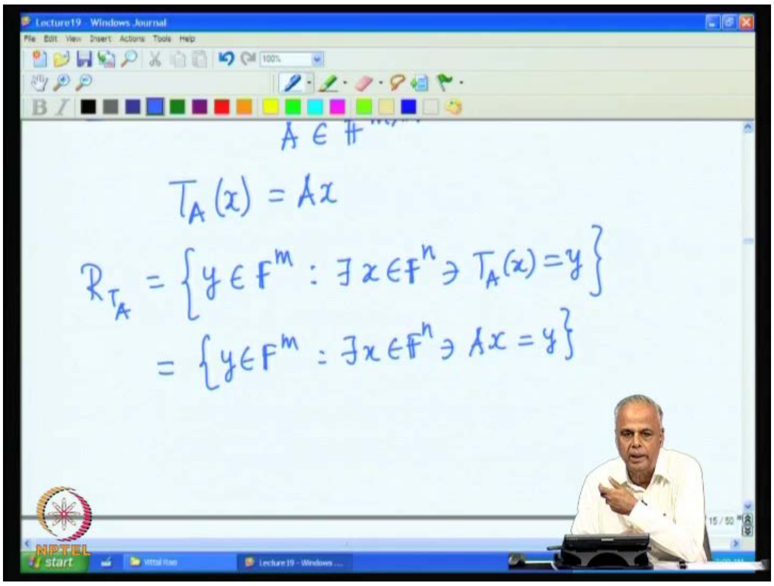

And we look at the Linear Transformations which takes the vector x to A of x, now we want to find the range of T A what does this range of T A mean? To want to know the vector which are all focused by the vectors in x. So, we want to look at the range of T A, the set of all y in F m that is the W such that, there exist a x in V, V in this case F m such that T A x is equal to y. This means we are looking at there exist a x in F m such that, but now again T A of x is multiplying the vector x by the matrix A, but this means we are looking at all those y for which the non-homogenous system has a solution.

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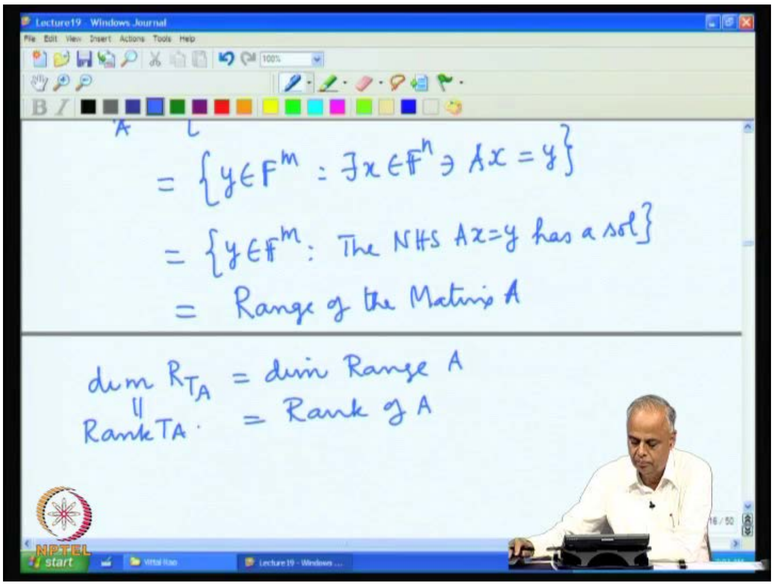

So, this is the set of all y in F m such that, the non homogenous system A x is equal to y has a solution and this is what we call as a range of the matrix A and the dimension of this is called the rank of A. So, the dimension of R T A is same as the dimension of range A is called the rank of the A and it is also because the dimension of the range of T A which is the rank of T A.

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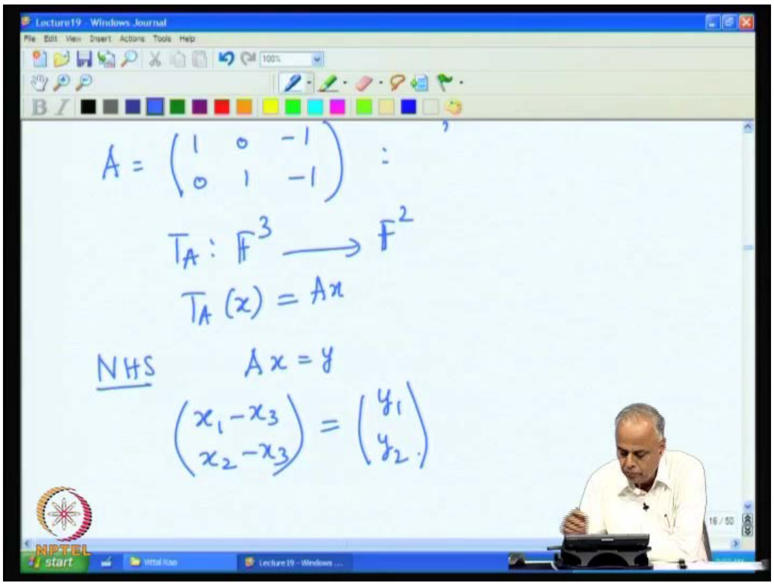

Then let us go back to that example, 1 0 minus 1 0 1 minus 1 mapping F then, we have T A mapping F 3 to F 2 where T A of x is A x. We have already found the Null space of this matrix; we shall now find the range of this matrix. So, now we want to know for what wise A x equal to y has the solution, so look at the non homogenous system A x equal to y that means A x is x 1 minus x 3, x 2 minus x 3 that is what A x is, if we take a vector x 1, x 2, x 3 and pre multiplied by the matrix A, you get this and we want this to be equal to be y 1 and y 2; we want to know for what y 1 and y 2 will the system have a solution.

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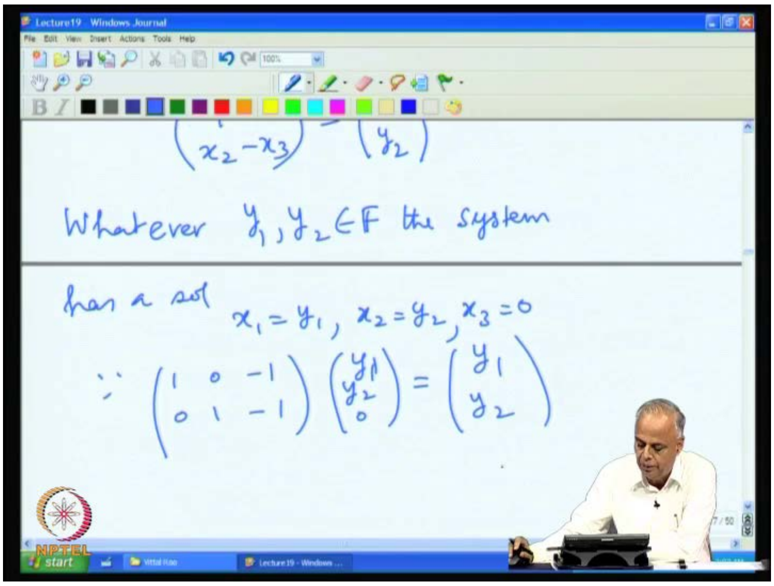

We see that whatever y 1, y 2 we chose in F, the system has a solution x 1 equal to y 1, x 2 equal to y 2, x 3 equal to 0 because a into x 1, x 2, 0, y 1 y 2 0 is precisely equal to y1 and y2 and therefore, given any y 1 by 2.

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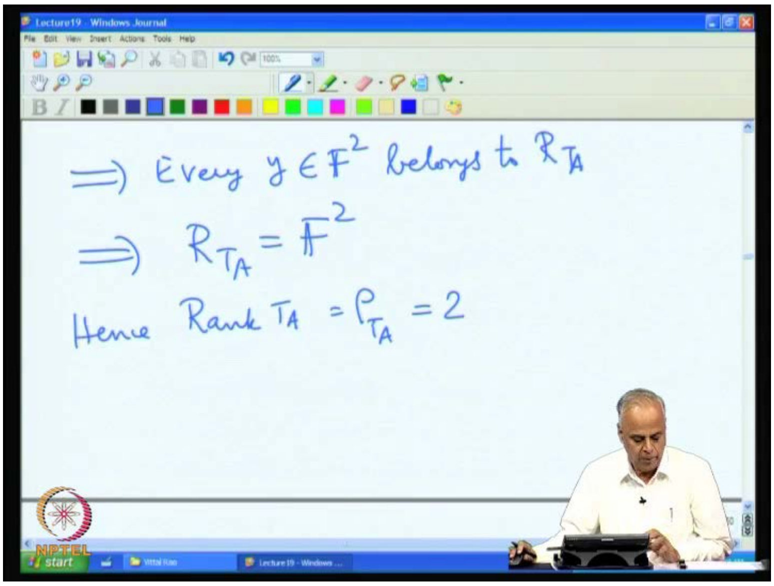

Were able to construct a solution therefore, every y in F 2 belongs to R T and therefore, R T is all of F 2 and hence R T A therefore, rank of T A which is rho T A is 2. So, thus we have very simple example of Linear Transformations from F 2 to F 3 to F 2 for which we found the Null space and the range.

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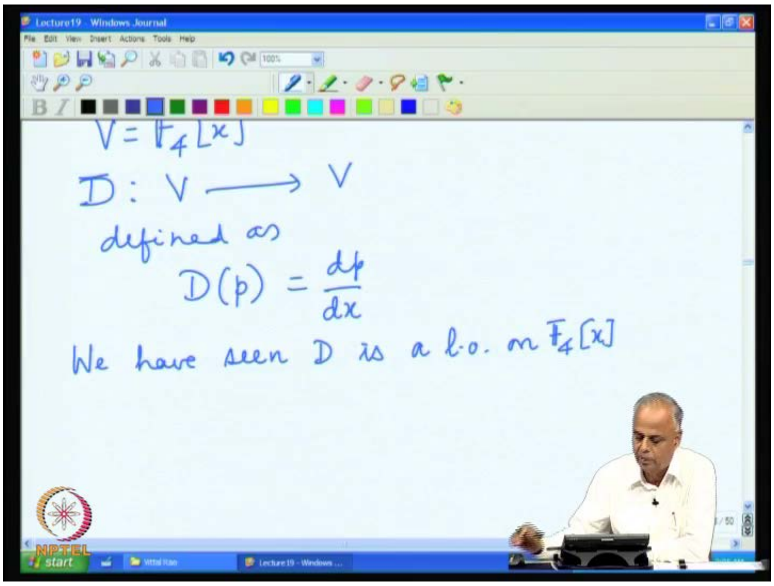

Let us look at another example; let us consider V to be the collection of all polynomials of degree less than or equal to 4 with coefficients from F. We now look at the linear operator that is a Linear Transformations from V to V, defined as D of any p is dp by dx.

The differentiation operator, we have seen in the last lecture that this is a Linear Transformation or a linear operator. We have seen D is a linear operator on F 4 (x), our V now is F 4 (x). So, let us find the Null space and the range of this operator.

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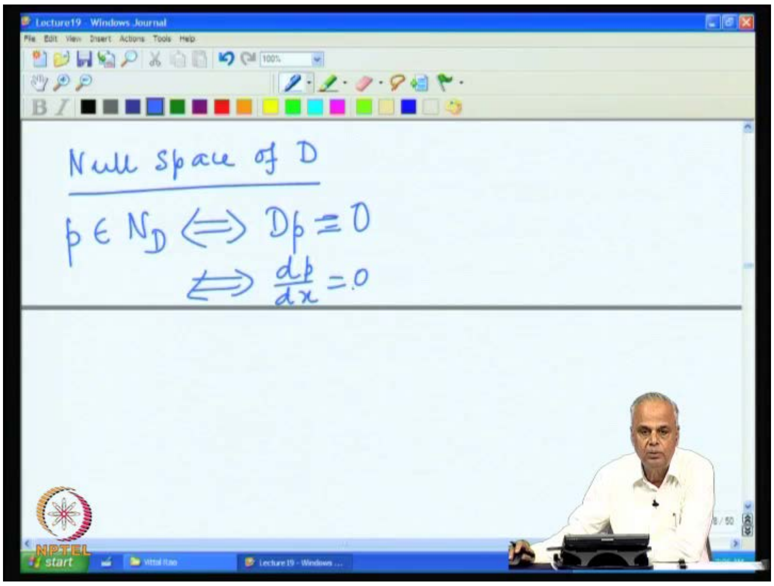

So, let us find Null space of D, so we want to find those vectors in V which get mapped to the 0 vector under the transformation D. Now vectors are all polynomials because our vector space is the space of all polynomials, so we want to find all those polynomials which when differentiated gives me the 0 polynomial go to zero vector. Zero vector is the 0 polynomial so x belongs to the Null space of D; if and only if dx is the 0 polynomial, that is if and only if let us use the notation p because we have polynomials, so dp equal to 0. Now dp by definition, dp by dx that must be equal to 0 that is how the transformation D is defined; the transformation d is the differentiation transformation.

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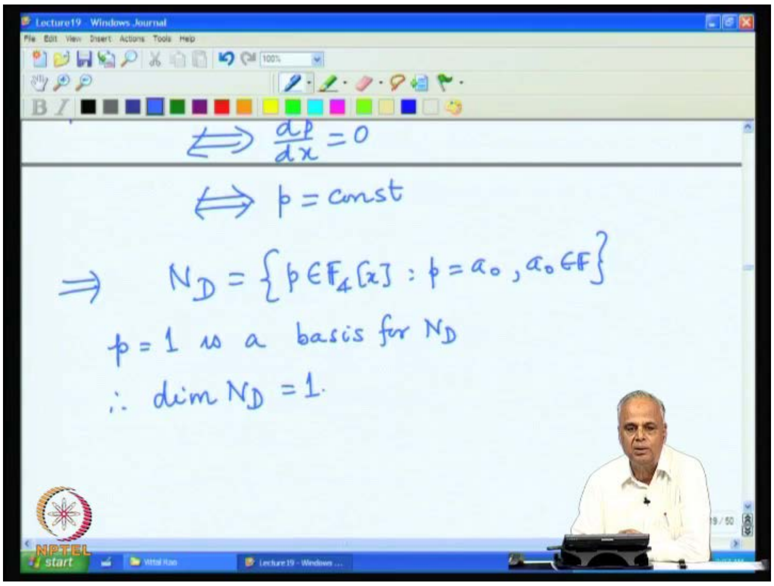

So, dp by dx is equal to 0 the derivative 0 if and only if, p is a constant. So therefore, only the constant polynomials qualify to be in the Null space of D. So, thus we get the Null space of D is set of all polynomials in the vector space which are of the form p equal to some constant a naught; a naught belongs to F, they are all constant polynomials.

Now what is a basis for this space? Well the constant polynomial one is a basis for this space, so p equal to 1 is a basis for N D because everybody else is a multiple linear combination a naught times p will get all the vectors in N D and therefore, dimension of N D is equal to 1 because we have a basis consisting of one vector and since, dimension of the Null space is 1; the nullity of D is 1.

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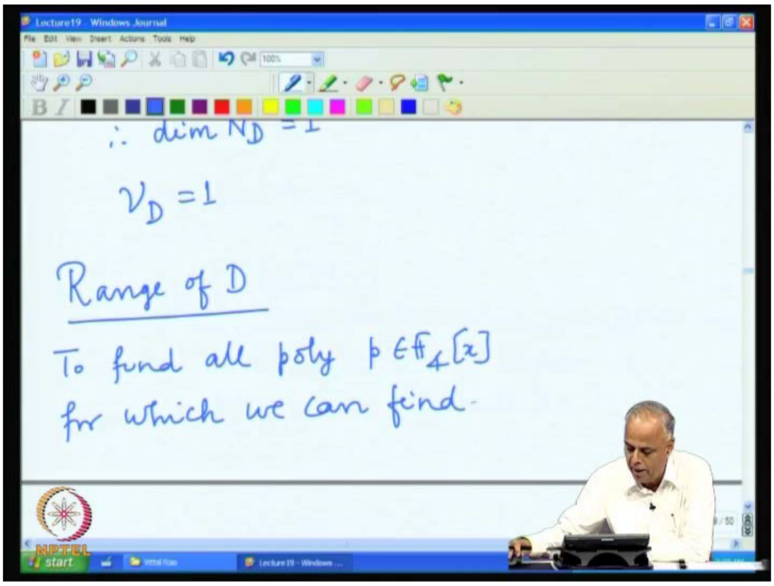

So, thus we have found the Null space of this differentiation operator F 4 (x) and the nullity. Now let us look at the range of d, so to find the range of D we want to find all polynomials p in F 4 (x) remember now, we are looking at d as a linear operator on V. So, even the W space is now V; so the W space is also V four x, so we are looking at all those p in F 4 x for which we can find a pre image what does that mean?

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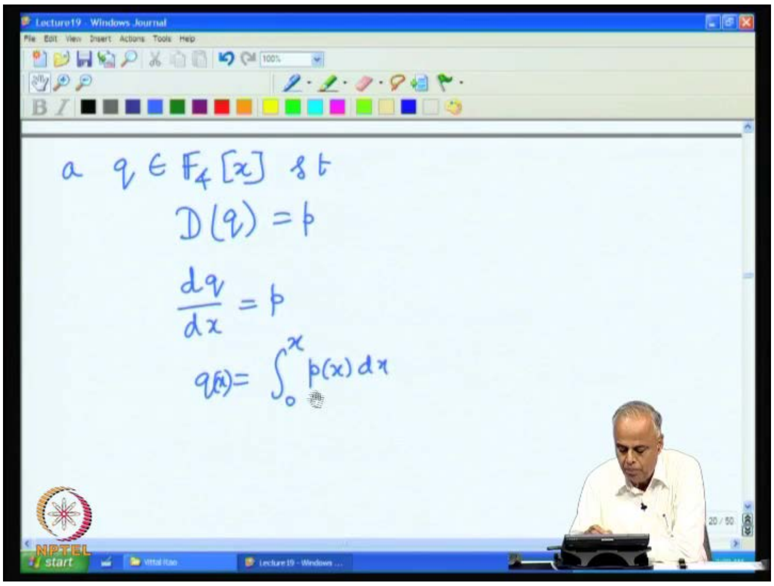

We can find a q which is also in that same space because they are linear operator such that, the image of q is equal to p. So, we want to find all those p's which can be obtained as the image of q in F 4 which means we want dq dx must be equal to p. So, we would like to find for those p's for which dq dx will be equal to T then, q must be equal to integral 0 to x p x dx, but then if you take any polynomial of degree four, the integral will become a polynomial of degree five and therefore, in ordered that q belongs to; we want q to belong to F 4 (x), so in order that q is a polynomial of degree less than or equal to 4, p must be a polynomial of degree less than or equal to three. So, only polynomials of degree less than or equal to three have a pre image.

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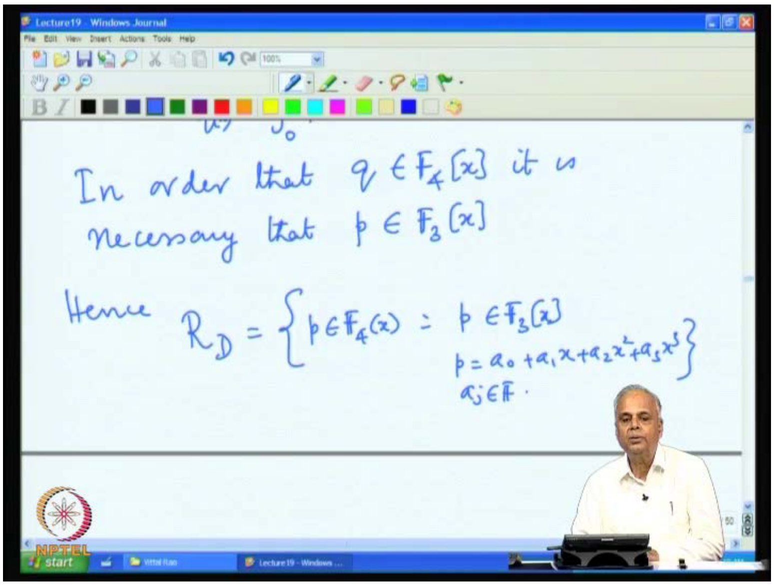

In order that q belongs to F 4 (x), it is necessary that p belongs to F 3 (x); it is a polynomial of degree less than or equal to three. Hence, the range of D is set of all polynomials in F 4 (x) such that p belongs to p d of 3 x. They are of form p is equal to a naught plus a 1 x plus a 2 x square plus a 3 x cube where all the a j's are in F. So, R D consist of all polynomials of degree less than or equal to 3.

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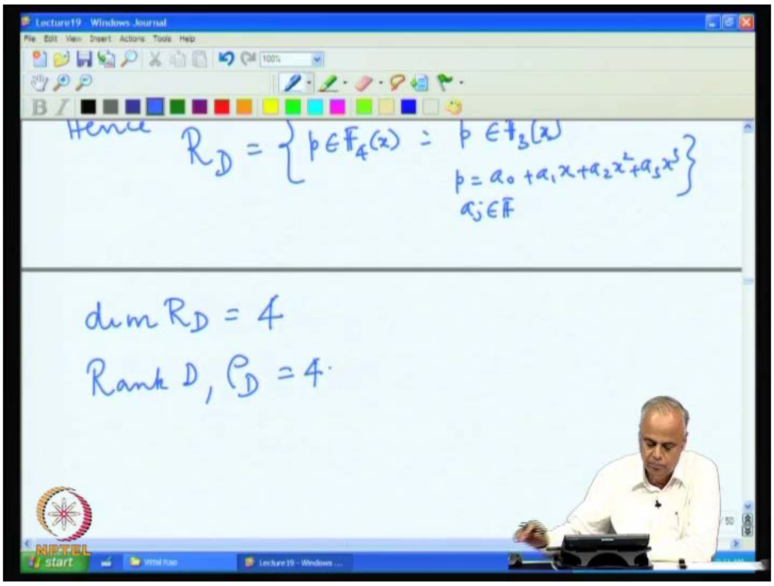

Now what is the dimension R D, we have seen that one x, x square x cube form a basis and therefore, the dimension is 4 and therefore, rank D which is rho D is equal to 4.

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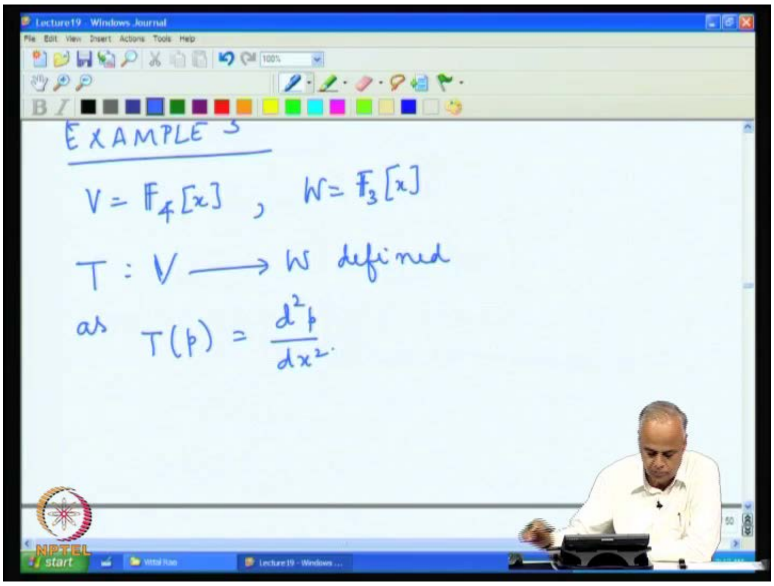

We will see one more example, similar to the one above. Take V equal to F 4 (x) and then say W equal to F 3 (x) and then consider the transformation D, let us call it as T because D we use for differentiation, T mapping V to W defined as T of p is the second derivative of p. Now again in last lecture, we verified that this is a Linear Transformations from V to W.

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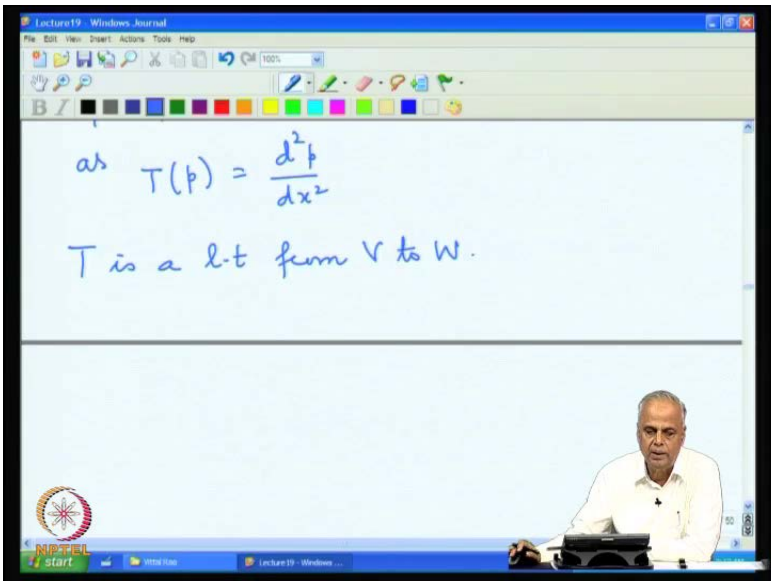

So, T is a Linear Transformation from V to W, the once we have Linear Transformation we want to find is Null space and the range.

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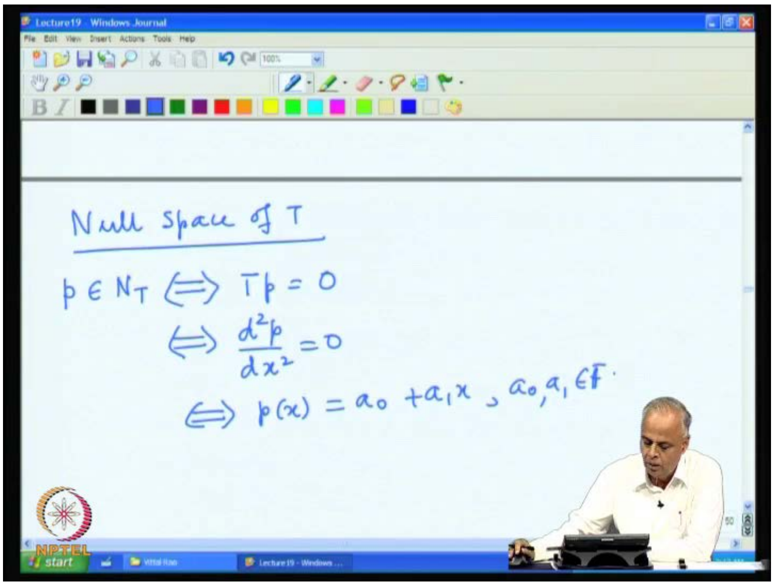

Now let us find the Null space of T; now the Null space of T is all those vectors which get focused to the 0 vector. Vectors here are all polynomials so p belongs to the Null space of T, if and only if it gets focused to the 0 polynomial or its value under T is the 0 polynomial. If and only if T is defined as d squared, so it is d square p by dx square is 0 this means, p x is a linear polynomial a naught plus a 1 x where a naught and a 1 belong to F. Therefore, only linear polynomials qualify to be in the Null space of T.

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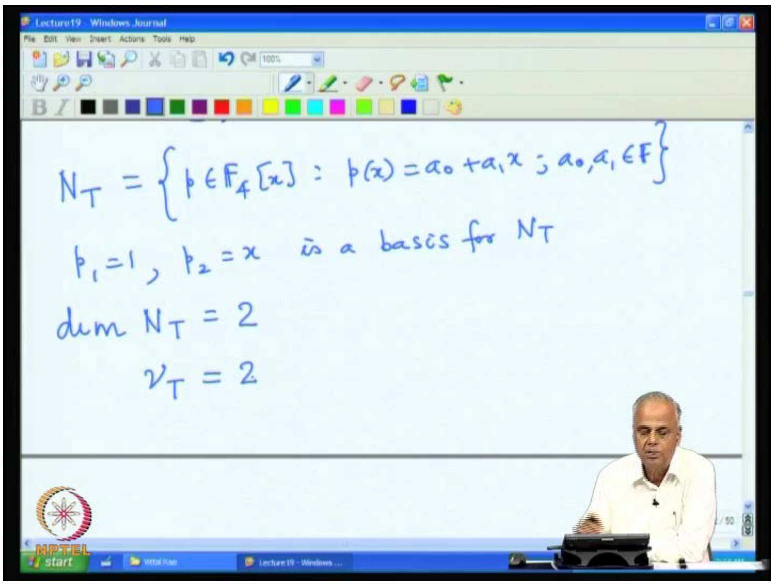

So therefore, we get Null space of T consist of all those polynomial in F 4 (x) which are of the form p x equal to a naught plus a 1 x; a naught a 1 in F. Now clearly p 1 equal to 1, p 2 equal to x is a basis for N T because as we seen here, every other polynomial is a linear combination of polynomial one and the polynomial x and therefore, they form a spanning set and obviously, linearly independent and therefore, they form a basis. So, the dimension of N T since the basis consists of two vectors now, the dimension of N T is 2, so the nullity of T it is dimension is 2 which implies nullity of T; which is defined to be the dimension of N T is 2.

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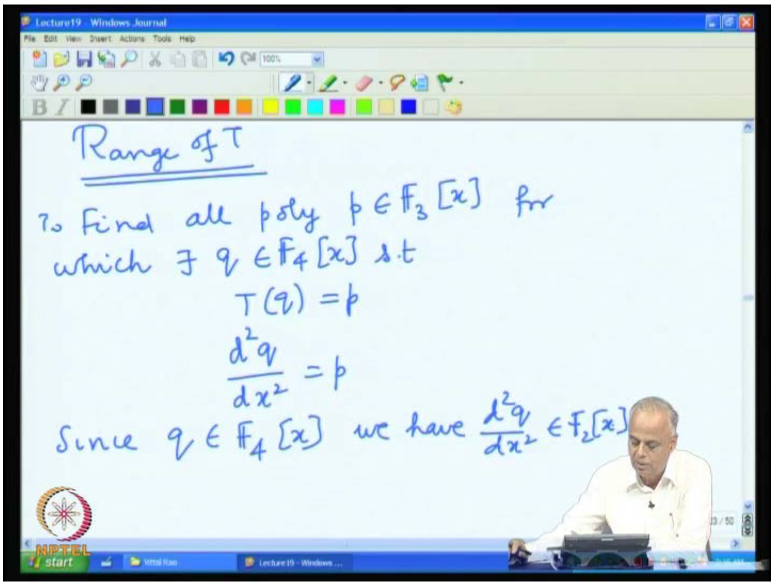

Let us now look at the range; the range of T we want to find all polynomials p in now, the W space is F 3 (x); our W space in this case is F 3 (x). So, we want to find all those polynomials in F 3 (x) that means polynomials of degree less than or equal to 3. We want to find all those polynomials for which there is a pre image; for which there exist a q pre image must be from V, V is F 4 (x) in our case therefore, q belonging to F 4 (x) such that T (q) equal to p. If you want T (q) to be equal to p, since T is defined as d square q by dx squared, we want to find those p's in F 3 (x) for which we can find q in F 4 (x) such that d square q is equal to dx squared.

Now, if q has to be in F 4 (x) is a polynomial of degree less than or equal to 4, and so when we differentiate it twice, it will lose two degrees. Every derivative reduces the power by one in the polynomial; degree by one in the polynomial. So, if we take any polynomial in F 4 (x) and differentiate it twice on the left hand side, we will get only polynomials of degree two or less and therefore, p has to be a polynomial of degree two or less. Since, q belongs to F 4 (x) we have d square q dx square belongs to F 2 (x), it has to be polynomial of degree less than or equal to 2.

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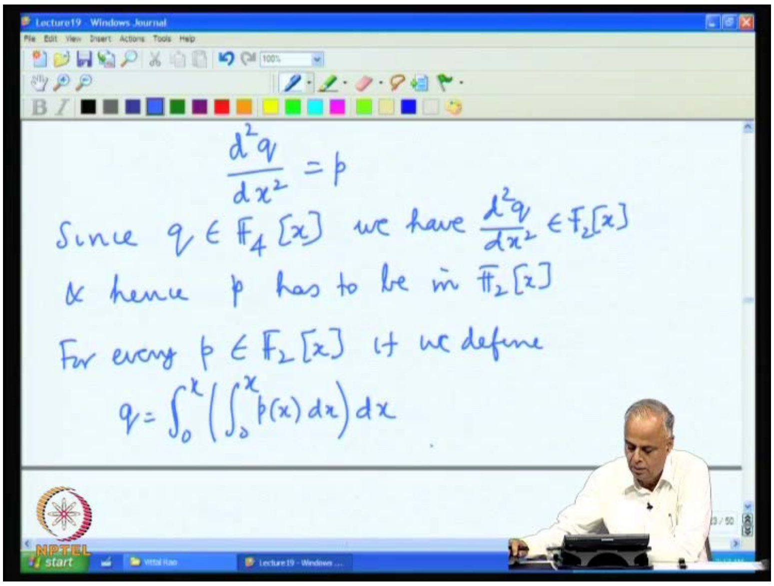

And hence, p has to be in F 2 (x), so we know that if at all there is going to be a solution for this you mean better start with p which is in F 2 x, but then for every p in F 2 (x). If we define q to be integral 0 to x integral 0 to x p x dx dx then, since p is a polynomial of degree less than or equal to degree two, when I integrate it will be polynomial of degree less than or equal to three and if I integrate further, I will get a polynomial of degree less than or equal to four.

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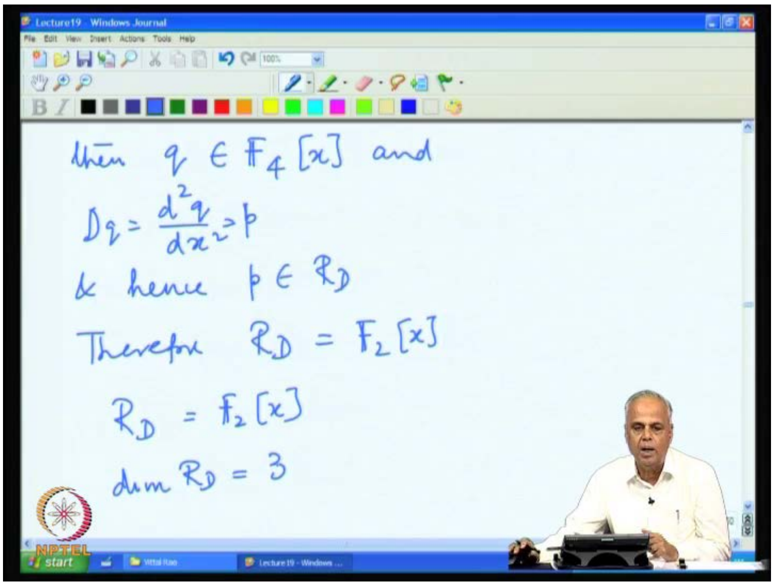

So, q will be in F 4 (x) and if we differentiate that is D of q will be precisely p and hence, p will belong to range of D. Therefore, range of D is precisely F 2 (x), what we have shown is take any vector in F 2 (x) it is in the range and previously we showed, that if it has to be in range, it has to be in F 2 (x) and therefore, F 2 (x) is precisely the range of D. So, the range of D is F 2 (x) and therefore, the dimension of R D is dimension of F 2, which is three because one x x square form basis for all polynomials, whose degree is less than or equal to 3 or less than or equal to 2.

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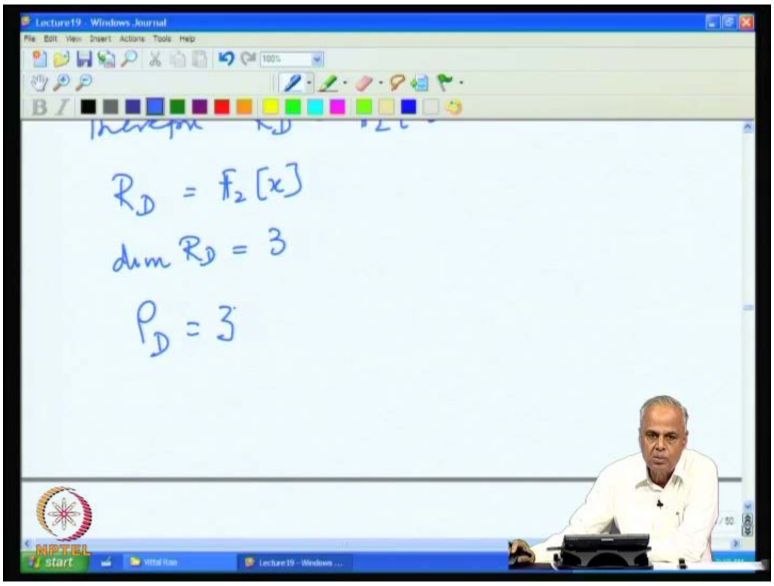

And therefore, the rank of D is 3. So, these are some simple example of looking at the range and the Null space.

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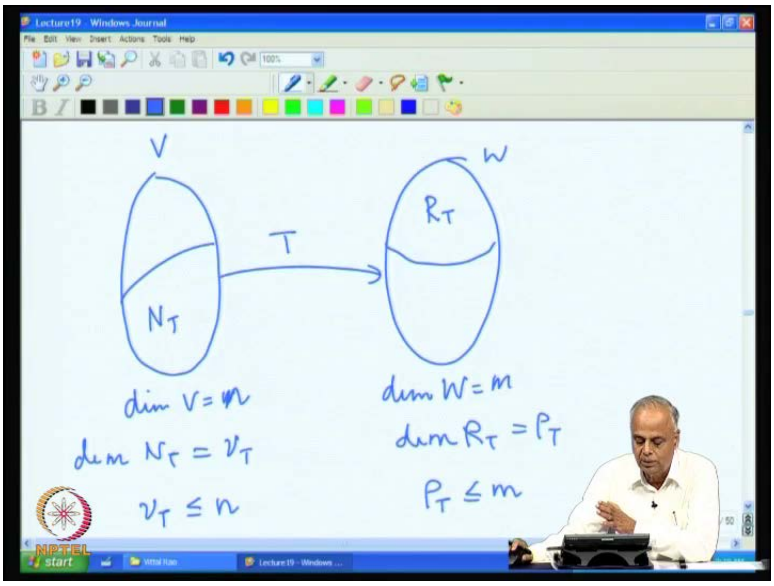

Now what we have so far is; here is the vector space V; here is the vector space W and the dimension of V is equal to n, say dimension of W equal to m and there is a Linear Transformation T and then, a part of this is what is known as the Null space of T and the part of this is what is known as the range of T and the dimension of n of T is what is known as nu of T and the dimension of rho of R of T is what is known as rho of T and since, Null space of T is a part of V and we have nu T is less than or equal to n similarly, we have rho T is less than or equal to m.

Now, we have one subspace on V which comes from T; we have one subspace on W which comes from T. Is there a connection between these two? And there is a connection between the dimensions and that is what is known as the rank nullity theorem. We shall first take this and we have look a proof of this in the next lecture. Now, what is the statement? Let us look at the three examples we had, in each of this examples if you see the first example, let us even take the last example. We have the n is 5 in this case because F 4 m is 3, so n is 5, nullity was 2 and rank was 3.

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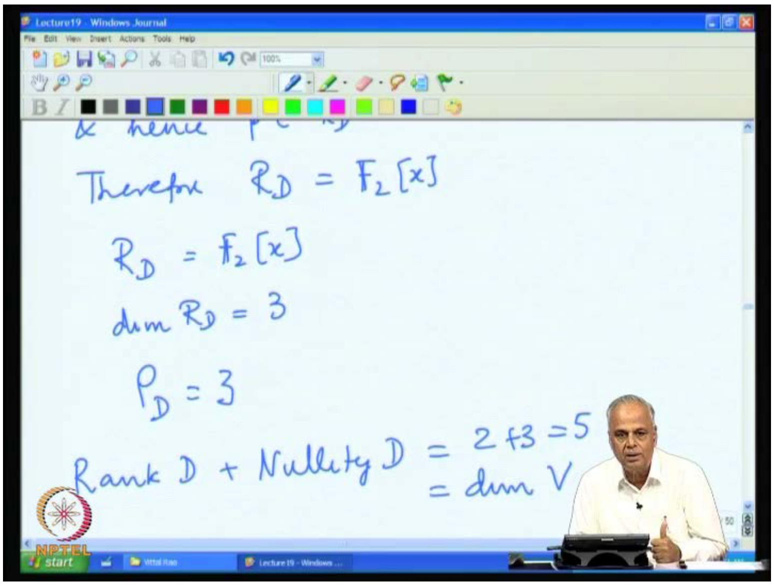

And the rank plus nullity came out to be in this example, we got rank D plus nullity D was equal to 2 plus 3 which is 5; which is equal to the dimension of V. And now this is not an accident, and the fact this is not an accident and this is always true for Linear Transformations is known as the rank nullity theorem which we will look at in the next class.