Advanced Matrix Theory And Linear Algebra For Engineers

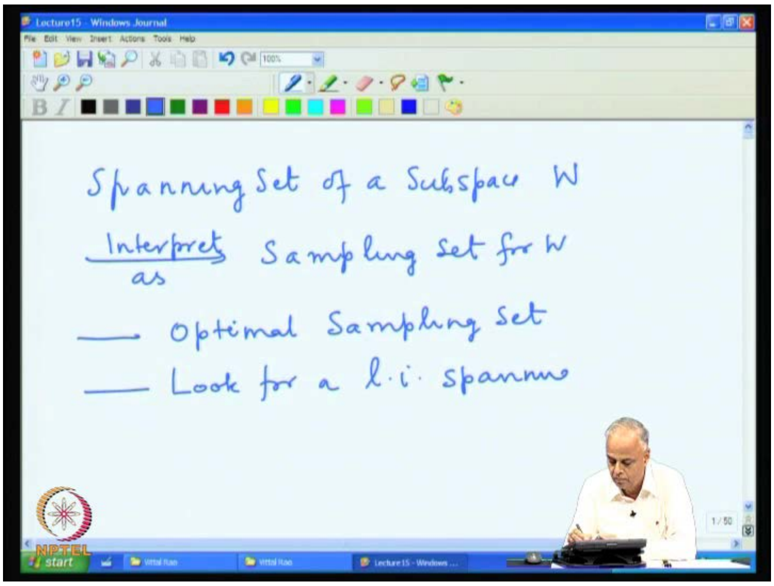
Prof. Vittal Rao

Centre of Electronics Design And Technology

Indian Institute of Science, Bangalore

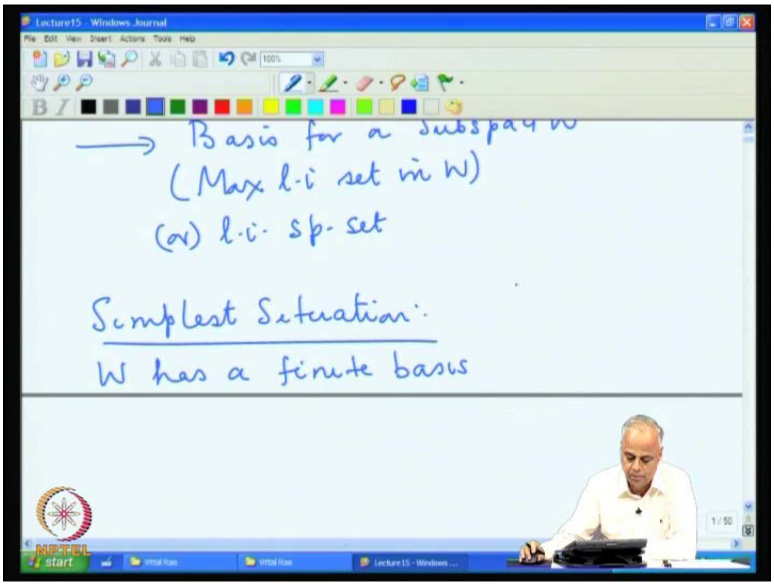
Lecture No. # 15

(Refer Slide Time: 00:22)



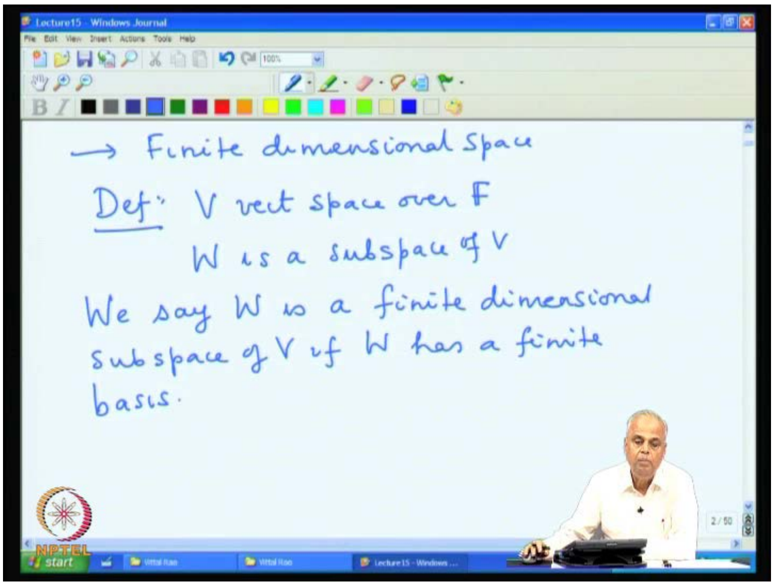
In the last lecture we found that we can interpret spanning set of a sub space we could interpret this as a sampling set for the sub space is W. We can interpret spanning set the sampling set for W view from this point of you we would like do to have an optimal sampling set. This means we must remove redundancy from sampling and this met we should remove linear dependent and look for a linearly independent spanning set.

(Refer Slide Time: 00:22)



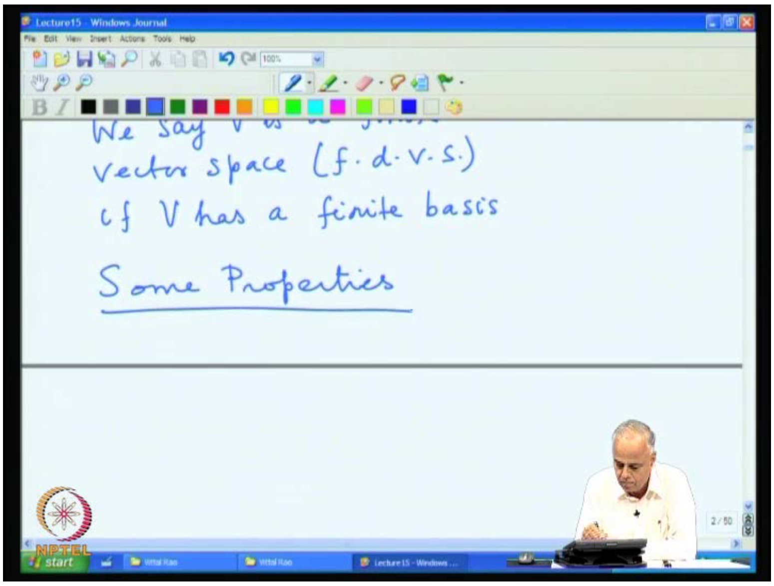
This let us to the motion of a basis for a sub space as a maximum a linearly independent set in W. We also look at it as the linearly independent spanning set both were equivalent the basis maximum linearly independent set as could be interpret as a linearly independent set it spans the space now from the point of the sampling the ideal situation are the simplest situation is went W has a finite basis that our sampling set is a finite sampling set.

(Refer Slide Time: 02:18)



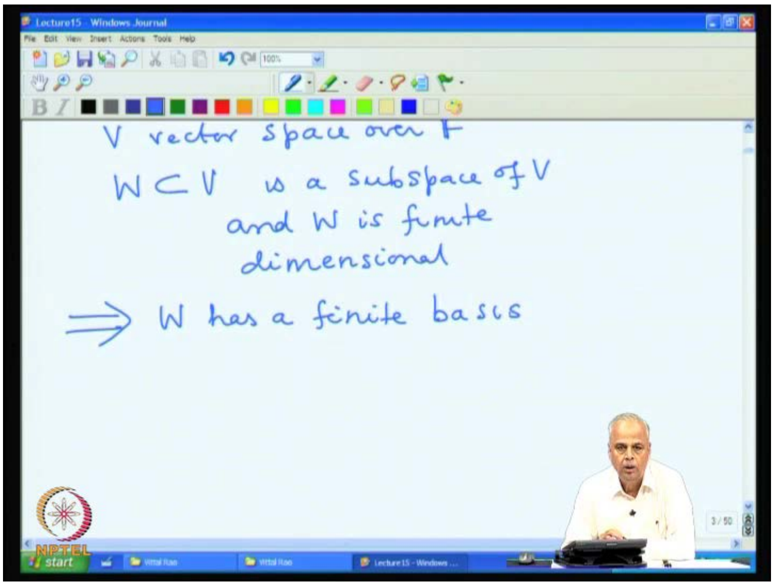
So this let us to the notion of finite dimensional space let us recall the definition suppose V is the vector space over a field F and W is a sub space of V and then we say W is a finite dimensional sub space of V if W is the finite basis W has the finite basis in particular in V as we itself.

(Refer Slide Time: 02:18)



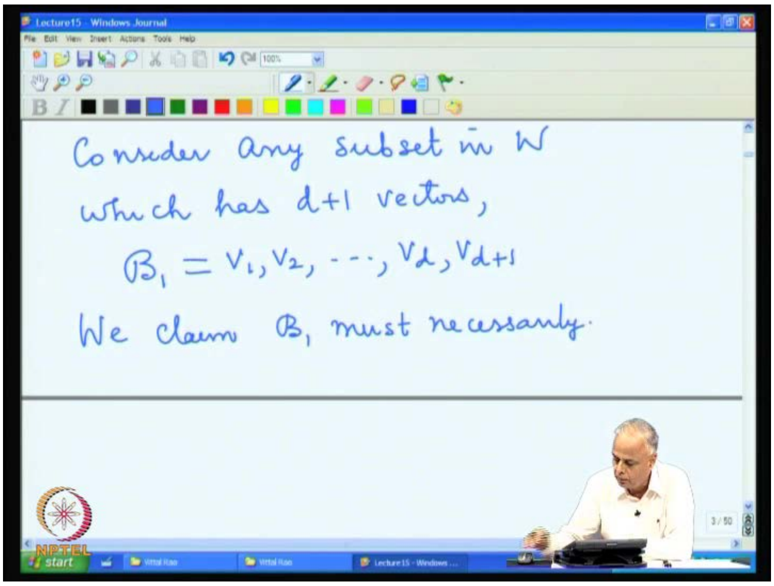
We say V is finite dimensional vector place a finite dimensional vector place. Which from known an in short will right it as will as f d V S finite dimensional vector place if V has finite basis will now look at some properties of finite dimensional spaces.

(Refer Slide Time: 04:11)



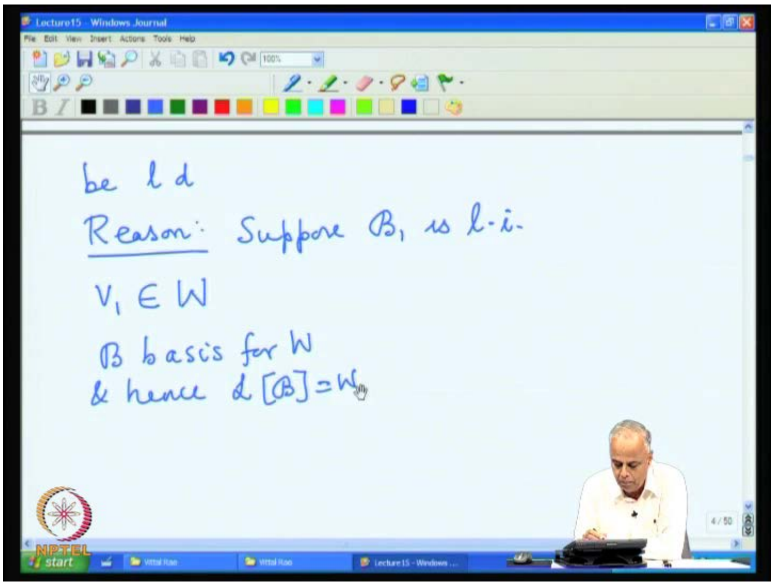
Suppose then start V vector space over F and W contain in V is a sub space of V and W is the finite dimension. We look at a sub space which is finite dimensional and we look at some of the properties of such a sub space now what do we mean by saying that W is the finite dimensional sub space by definition this means that W is must have a finite basis this implies W has the finite basis.

(Refer Slide Time: 04:11)



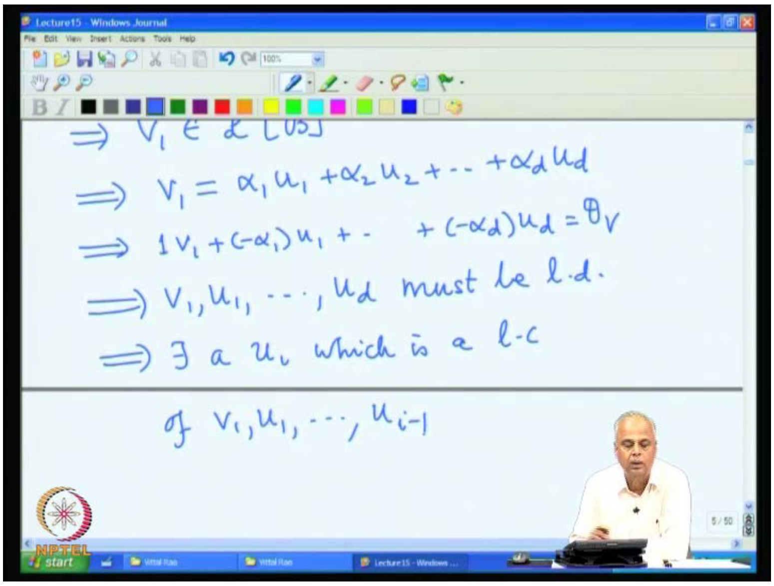
So let us look at finite basis say it is a finite basis it must have a finite number of vectors let has been to be vector u 1 u 2 and so an u d. W has a finite basis which consists of d vector we no it has finite basis because no it is a finite dimensional sub spaces it must have a finite basis suppose we have finite basis and it has been vector we call the u 1 u 2 u d now consider any sub set in W which has d plus 1 vector let call it us B 1 to be V 1 V 2 and V d V d plus 1. We have sub space which is finite dimensional and basis consists of V vectors now we consider another set which has d plus 1 vector one more than the eye of the basis we claim B 1 must necessary linear independent.

(Refer Slide Time: 06:42)



b 1 must necessarily linearly independent what the reason suppose not suppose V 1 is not linear independent what does mean suppose B 1 is linearly independent suppose B 1 linearly independent. We will arrive at the quant addition and there by showing but B 1 must be linearly independent suppose B 1 is linearly independent now look at V 1 V 1 belongs to W and we had B basis for B basis for W which means it has two property 1 it must be linearly independent and two it must span W. We use the spanning part of it and hence the space span by must be equal to W we have V 1 is W but, W is span by V so V 1 is W w is span by B therefore; V 1 being in W must in L B.

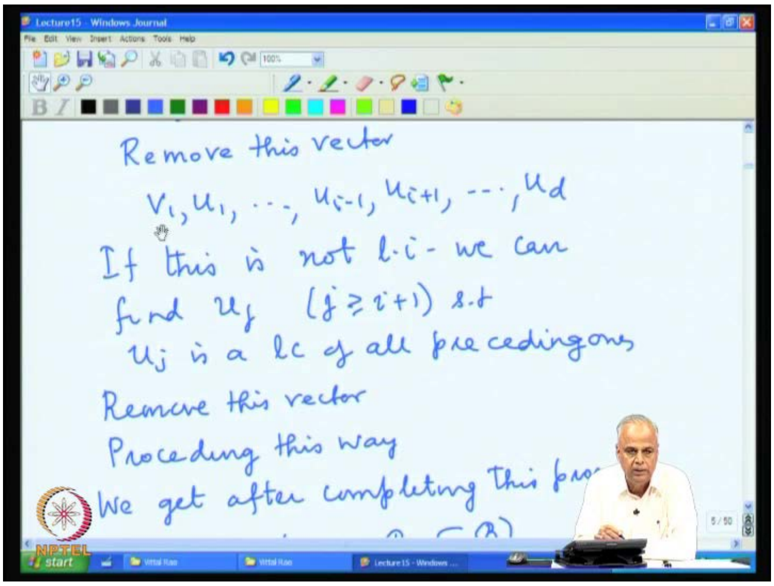
(Refer Slide Time: 06:42)



So that says V 1 is L B what is mean by V 1 is L B that means V 1 can be obtain as a linear combination of the B vectors alpha 2 u 2 plus alpha d u d fact that V 1 is L B being V 1 is equal to alpha 1 u 1 plus alpha 2 u 2 plus alpha d u d this implies but 1 times V 1 plus minus alpha 1 u 1 plus etcetera plus

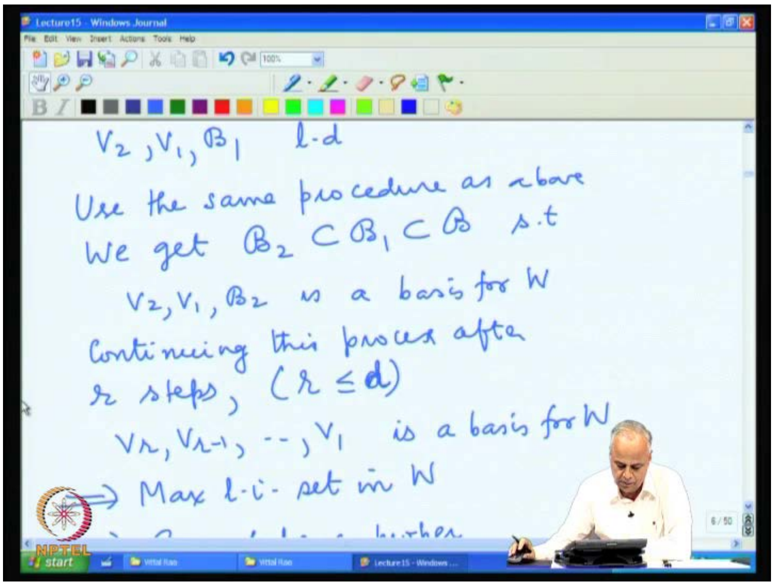
Minus alpha d u d is the 0 vectors and therefore, we have here a linear combination of V 1 vector and u 1 u 2 u d V giving rise to the 0 vectors at the same time at least one of the component 1 of the coefficient the linear combination is not 0 here is the non 0 linear combination coefficient namely one namely the coefficient of V 1 that says V 1 u 1 u d must be linearly independent. We have the linear combination in which not all the coefficient of the 0 and hence this vector linearly independent. We have seen that we have finite set of linear independent vectors then we scan then from the left we will hit the first vector which is linear combination of the previous vector what we this we scan from the left V 1 cannot be linear combinational previous as well as therefore, there Is the vector between u 1 and u d as we know first hid the vector which is linear combination of the previous follows there excites a u i which is the linear combination of V 1 u 1 u 2 u i minus 1 is the linear combination of the previous follows.

(Refer Slide Time: 10:42)



We not that vector of remove this vector what do we get V 1 u 1 u i minus 1 u i plus 1 etcetera u d this set may or may not be linearly independent suppose this is further linear dependent then we can do can do the scanning again and then make a nock out the vector which is linear combination of the preceding one now that cannot be V 1 that cannot be u i minus 1 because we have seen this are not linear combination of previous follows now could be u i plus 1 or u i plus 2 and an we nock of the vector u j which is j is bigger than i therefore, if this is not linearly independent we can find u j where j is greater than or equal to i plus 1 such that u j is a linear combination of all preceding 1s then we say preceding 1s you known preceding 1 this set the u i already having in locked out then lock out this vector remove this vector and then continue this process preceding this way finally, we should reach the end of the sure because locked out u i and something beyond that something beyond the in finite number of vector this process will end at a certain stage therefore, we get after completing this process V 1 and will be certain number of u i that would have been remove from what is left he is a sub set what is left is the sub set of B 1 we write this like this where B 1 is the sub set of B. Which is linearly independent therefore all the un necessary vector has been removed no more to be removed and whatever could be span by V 1 u 1 u 2 u d could also spent by V 1 and B 1 V because only redundancy vector would remove and what V 1 u 1 u 2 u d can span is all of W because u 1 u 2 u d already spans W therefore, which is linearly independent V 1 B 1 linearly independent and spans W.

(Refer Slide Time: 14:07)

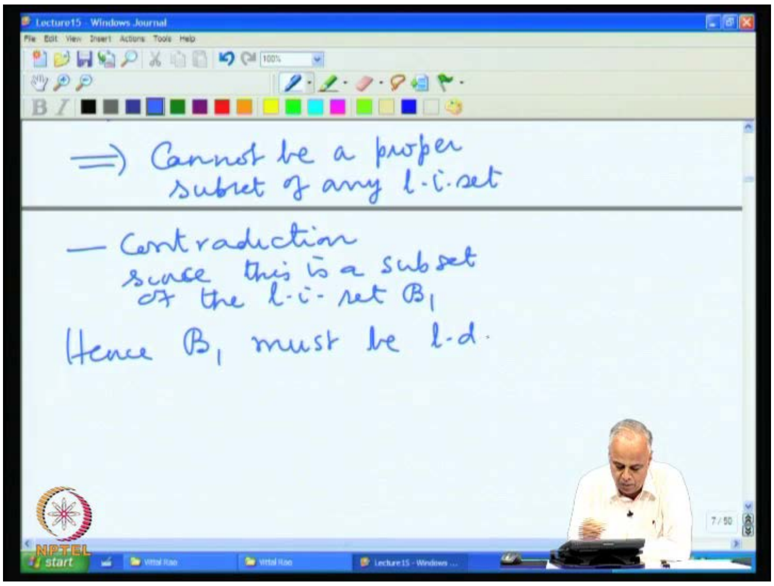


Therefore, this V 1 this vector V 1 and all this remaining and removed vectors of B 1 forms a basis for W. Now what we this what is V 1 done V 1 has L board out at least 1 vector of V because remember in the scanning process of we would at least through 1 vector out of the V vectors V 1 has L board at least 1 vector out the now, we push in V 2 and look at this linearly dependent again because V 1 B 1 is already a basis and V 2 in that space and that fore V 2 return the linear combination of the this follows, this is linear independent . We can use this same procedure as the about and again we should locking out vectors, we can lock out V 2 because the linear combination of the previous one we can lock out V 1 because it is not the linear combination of the V 2. The reason is V 1 V 2 V d plus 1 the linearly independent we assume that assume that V vectors are linearly independent , V 1 cannot be is linear combination V 2 so again the locking out the vector will take place only among from the B vectors using the same procedure. Now get sub set B 2 or B 1 which is sub set of B set that V 2 V 1 B 2 is a basis for W, B 2 would have obtain from B by locking out at least one vector in the first vector V 1 L boarding and 1 vector V 2 L boarding at least 2 vector be would have been locked out to get B 2.

Now continue this process after r steps we would have L board in V r V r minus 1 V 1 and locked out all the vectors in c. If you lock out all the vectors in B at each step at least 1 so r can be at most n at most d after d steps. All the u vectors would have been knocked out we get after r steps r is less than r equal to d V r is a basis for W since it is basis for

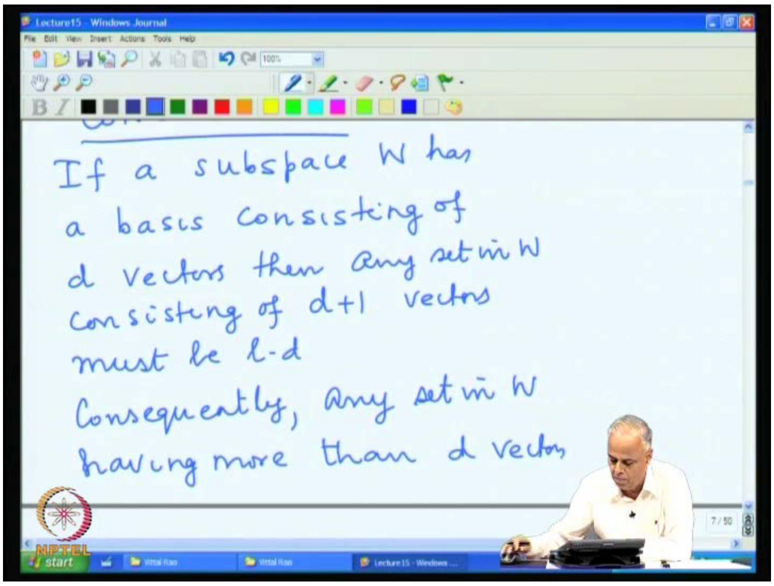
W it must be a maximum linearly independent set in W because is a basis maximum linearly independent set.

(Refer Slide Time: 17:33)



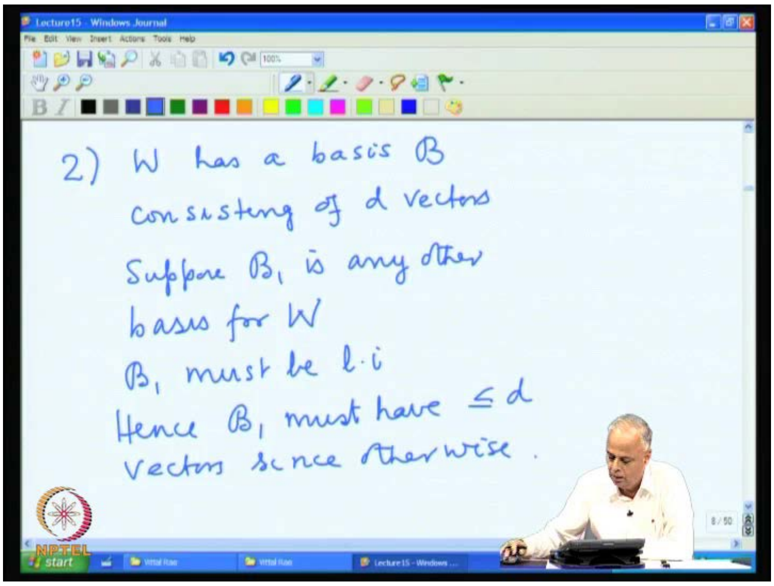
And therefore, it cannot be a proper sub set of any linearly independent set this is the contradiction because r is less than d and V 1 consist of V 1 V 2 V d plus 1 and V 1 must assume to the linearly independent. Because are since this is sub set of the linearly independent set B 1 let us go on the argument one more time we have started with a finite dimensional sub space and therefore, this finite dimensional sub space the assuming must have finite basis and assuming the basis which has B vector and then wanted to claim but any d plus 1 vector will from linearly independent set to prove that we assume the contradiction suppose it is linearly independent repeat where the linearly independent then step by step the i board in 1 vector of the other we said in to the basis locking out 1 vector out of it at end of the r steps we get V r V r minus 1 V 1 and V on locked out all the vectors of the B. Since we lock out one vector in each step at most end d steps r less than or equal to d at most they may be would locked out two less than the steps locked out all the vectors at most these steps we would got it basis consisting at most V of the V vectors and it is the basis because maximum linear the independent set it cannot be sub set proper of any linear independent set but, it is proper linear proper set of the linear independent set B 1 which gives a contradiction therefore the our starting association that B 1 is linearly independent must be falls hence B 1 must be linearly independent.

(Refer Slide Time: 20:19)



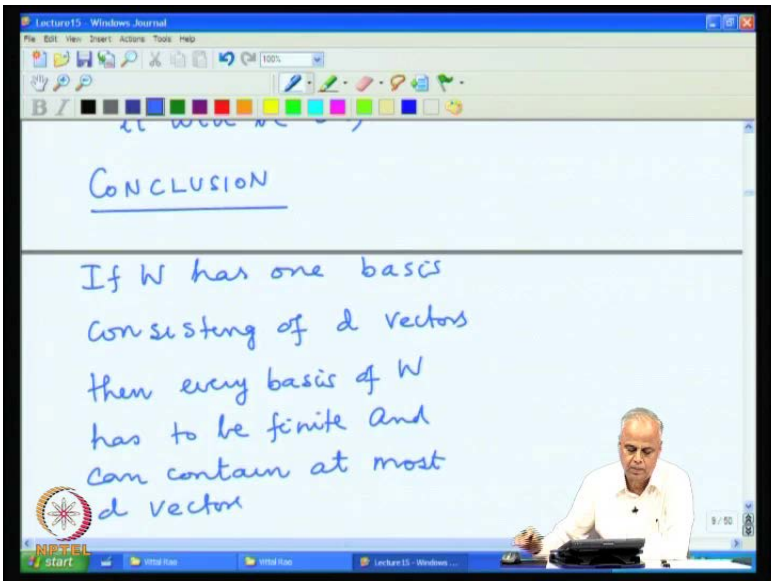
So what is the conclusion that conclusion is that if a sub space W has a basis consisting of d vector then any set consists in W consisting of d plus 1 vectors must be linearly dependent anything more than d vectors is force to the is linearly dependent d plus consequently any set in W having more than d vectors must be linear dependent must be linearly independent .If you have one basis which as d vectors than anything size bigger than this must be linearly dependent what is the important property of finite dimensional spaces that we look at.

(Refer Slide Time: 22:05)



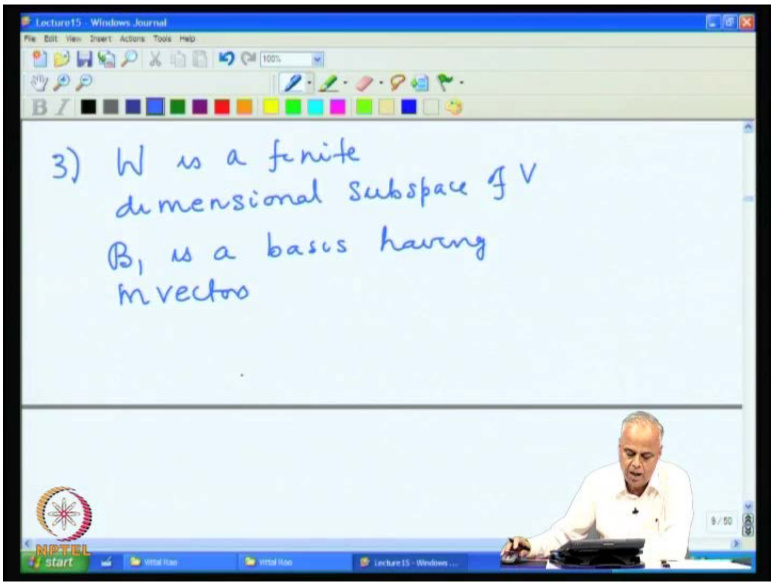
Now let us look at the next important property suppose W has a basis consisting of d vectors W is the finite dimensional space it has it is finite basis and suppose it has the finite basis having exactly d vectors let us call this basis B suppose B 1 is say other basis for W suppose B 1 is any other basis for W then B 1 must be linearly independent because, any basis must be linearly independent said how ever since B already has the basis consisting of d vector you have seen just now but a movement there is basis consisting of d vectors any set consisting of d plus 1 or more vector must be linearly dependent therefore, the B 1 has more than B vector it will linearly become dependent but since B 1 is basis B 1 must be linearly independent hence B 1 must have less than or equal to d vectors since otherwise it will be linearly dependent and therefore, it say the movement one finite basis all other basis has finite can at most that many vectors.

(Refer Slide Time: 24:12)



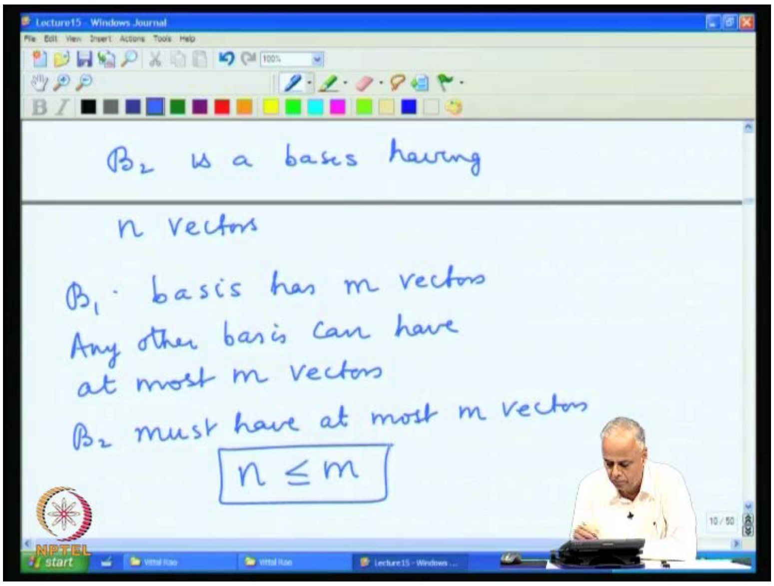
So what is the conclusion the conclusion is if W has one basis consisting of d vectors then every basis of W has to be finite and can contain at most d vectors the movement one finite basis all are other basis force to be finite and nothing can be pre size basis that you started with.

(Refer Slide Time: 25:15)



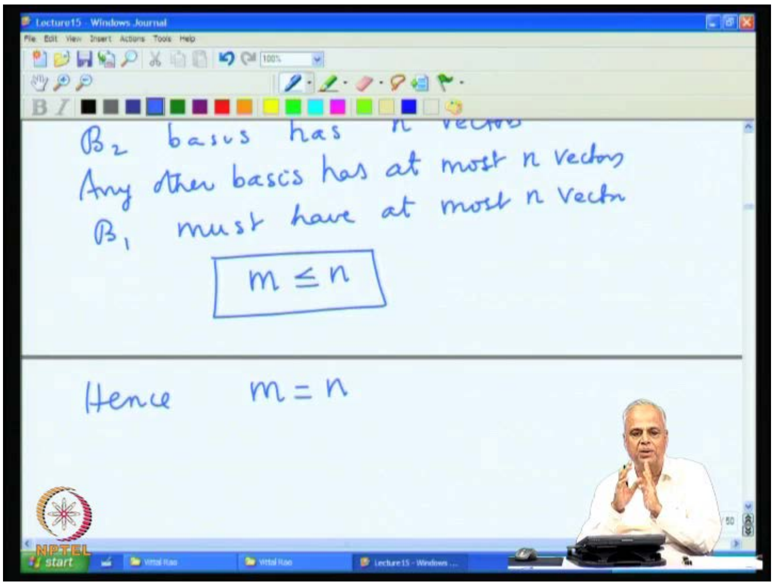
Let us x y Is further suppose W is a finite dimensional sub space. Suppose W is the finite dimensional sub space of the vector space V and B 1 is the basis and we now the movement is finite dimensional space is one finite basis and movement is the one finite basis all basis are finite movement is the B 1 is the basis is the finite number of vectors say it has n vectors having m vectors.

(Refer Slide Time: 26:08)



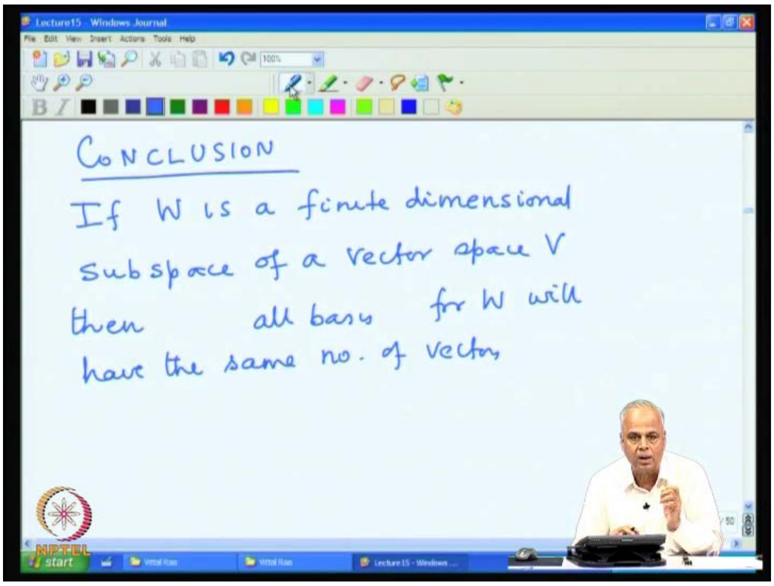
And suppose B 2 is the basis that can one also having finite number of vectors. Suppose it has n vectors suppose two basis for sub space one of the m vectors and other one is n vector. Now let us look as this basis B 1 the B 1 basis has m vectors we are just now saying that if you have one basis having m vector all other basis can have at most m vectors any other basis can have at most m vectors therefore, in particular B 2 must have B 2 is the basis B 2 must have at most m vectors that means n is must less than or equal to m.

(Refer Slide Time: 27:22)



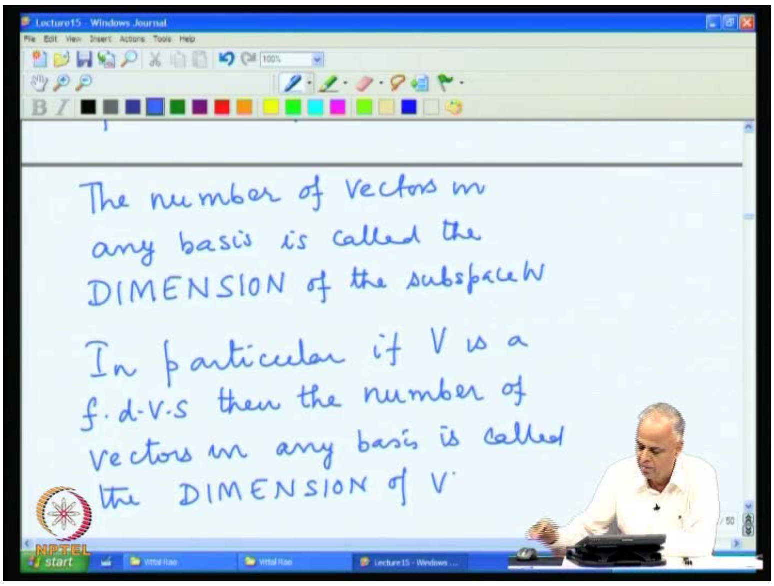
And other hand look at detail the basis has m vectors at the movement basis which has n vectors any other must have at most n vectors therefore, any other basis has at most n vectors that is show B 1 must have at most n vectors. That means m less than or equal to n therefore, if I have two basis one consisting m number of vectors and other consisting n number of vectors then n must be less than or equal to m and also m less than or equal to n comparing the two we get hence m equal to n. W hat is says that the movement of finite dimensional space movement which two basis you take we must have same number of vector which means all basis a finite dimensional space have the same number of vectors.

(Refer Slide Time: 28:51)



So the main conclusion is if W is a finite dimensional sub space of a vector space V then any two basis for W will have the same number of vectors or we can say all basis all basis for W will have the same number of vector W is the finite dimensional space no matter which if you one basis you see how many are the 10 you go look at any other basis that will also have 10 vectors any other basis that will have 10 vectors and so an so there is this number which is any variant for all basis and this number gives as to the following definition.

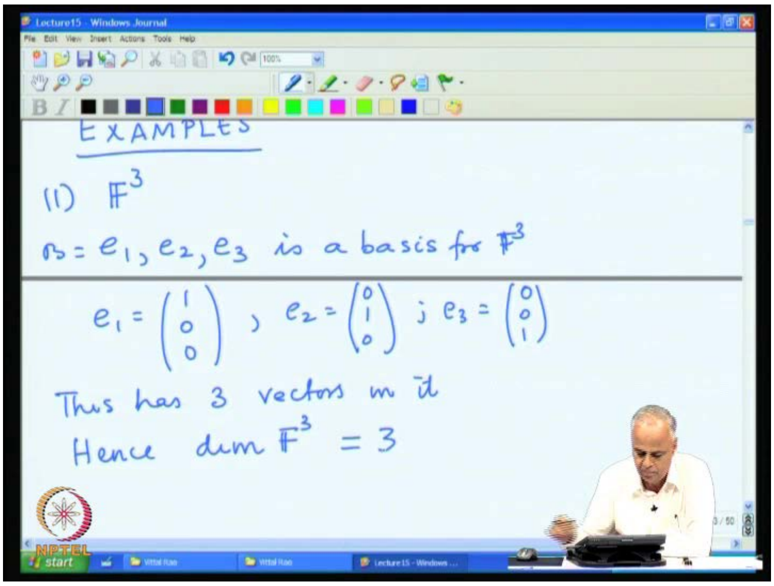
(Refer Slide Time: 30:17)



Let W be a finite dimension we right F B sub space a vector space V the number of vectors in m basis we know that is error level which our basis you will get the same number the number of vector many basis is called be dimensional of the.

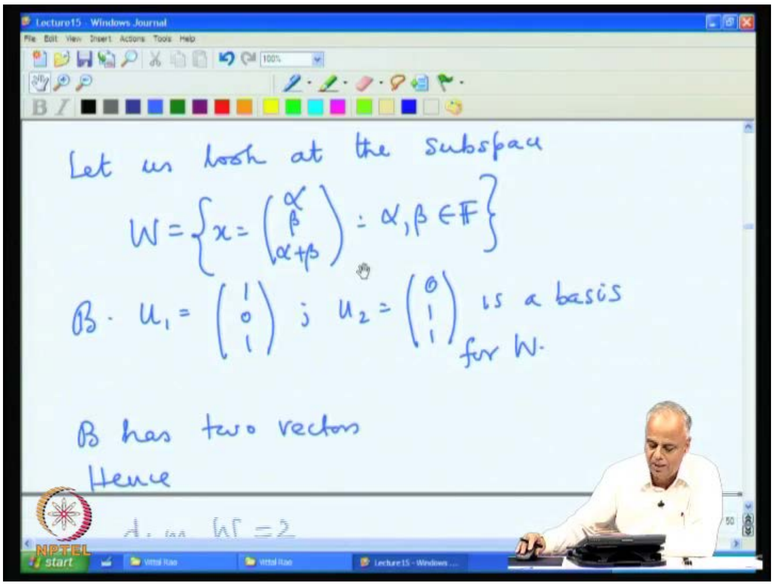
Sub space this is algebra notation of dimension there is various interpretation at the movement rustic only to the linear algebra is what is known as the hammed 10 dimension this as to be added it is called hammed dimension of the sub space. That we will call the only at dimensions because learn talking about any other dimension in discuss the number of vectors in a basis is called the dimension let us look at in particular make it even more specific in particular if V itself a finite dimensional vector space then two basis will have the same number of vector the number of vector in any basis is called the dimension of V.

(Refer Slide Time: 32:36)



Let us look at some examples to compare the dimension all you have do is some basis for vector space and count how many vectors are they in the basis so is a relevant which basis you are capture as a long as capture basis that is enough look at the number you are going to same which over basis to capture let us look at the space example If F 3 the you are standard example and know but B e 1 e 2 e 3 is the basis for F 3 what is e 1 e 2 e 3 recall e 1 is this vector e 2 has the second entry 1 all other is 0 and e 3 and third entry 1 all other 0 this we are seeing basis for F 3 now this is the 3 vector and this has 3 vectors in it hence the dimension of F 3 is 3.

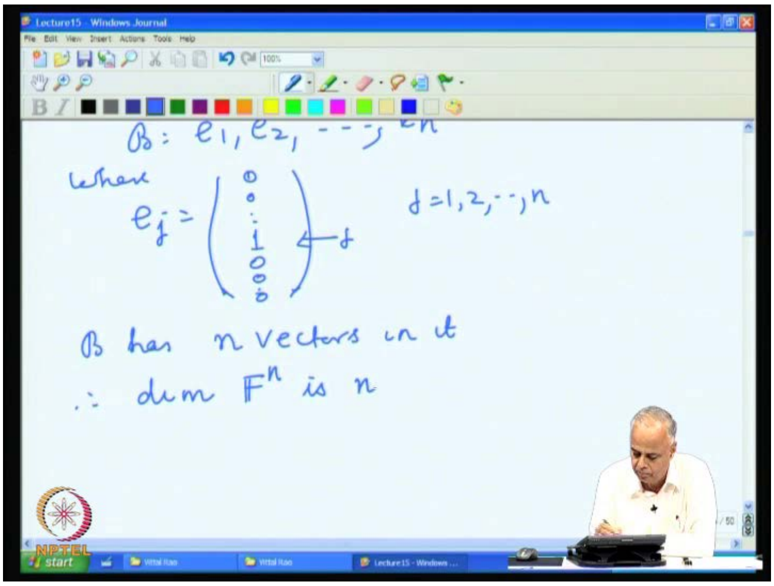
(Refer Slide Time: 34:10)



Let us look at the sub space W consisting of all base vector with are of the form alpha beta alpha plus beta where alpha beta are in F. This is collection of all those vector this third component of the.

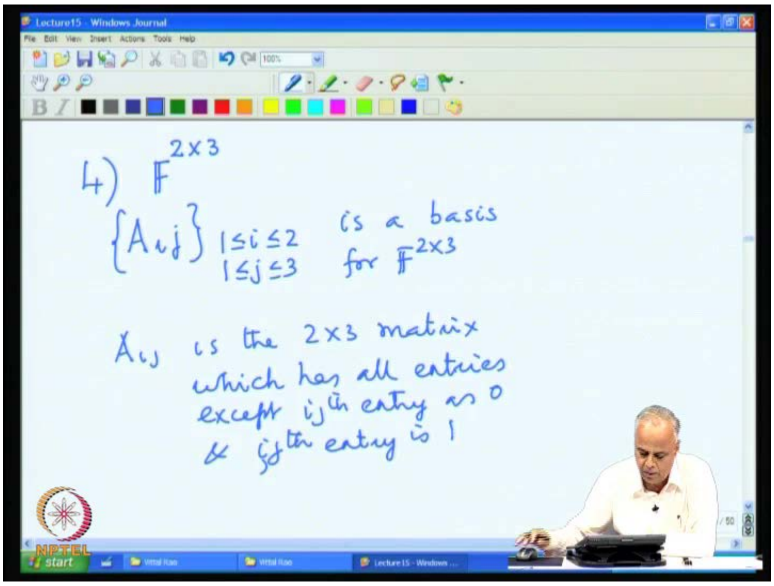
Third entry of the some of the plus to entry now we know the B consisting of this vector u 1 1 0 1 u 2 0 1 1 is a basis for W and we has two vectors in it and therefore, dimension of W. hence dimension of W 2 F 3 is a 3 dimensional vector space in that vector space is two dimensional sub space routine if a take F 2 be r r 3 will be usual space in which live in which is three dimensional space and this W you then become the place thus equal to x plus y which is two dimensional object the place geometrically is a two dimensional object so this is algebraic way of looking that fact.

(Refer Slide Time: 35:56)



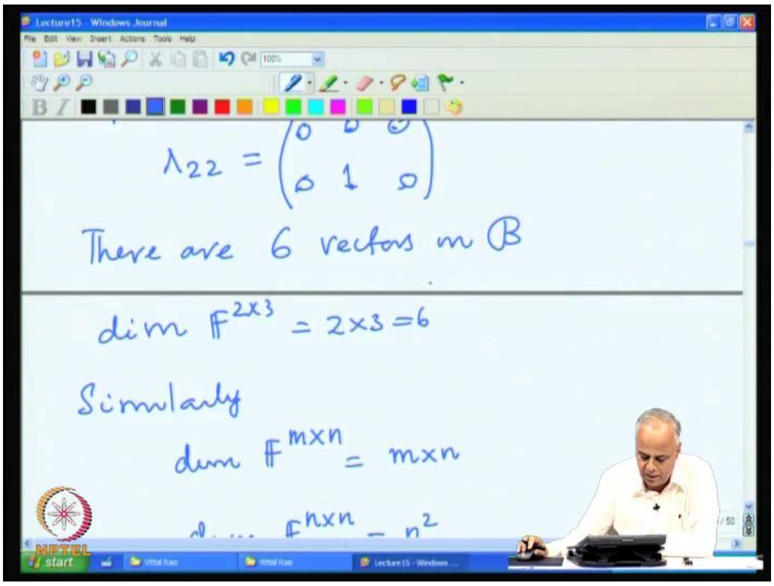
I now exciting the idea of F 3 F n if the vector take F n has basis e 1 e 2 etcetera e n where e j is that which as 0 0 etcetera and till you come to the j th component which 0 and all others are 0 where j equal 1 2 up to n this is basis for F n and since that B has n vectors in it and therefore dimension of F n is n.

(Refer Slide Time: 36:52)



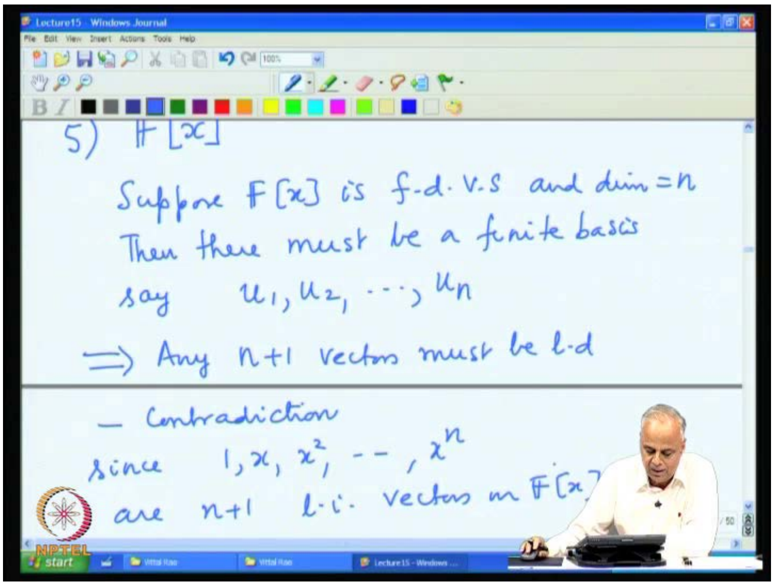
Let us make look at the examples of 2 by 3 matrix is over track we already seen that A i j 1 less than or equal to i less than or equal to 2 1 less than or equal to j less than or equal to 3 is a basis for r 2 3 what A i j Ai j is the 2 by 3 matrix which has all entries expect i j th entry as 0 and i j th entry is 1

(Refer Slide Time: 36:52)



For example, a 2 2 is expect 2 the second row second entry which must be one all other entry must be 0 if you look at this a j edges they form a basis how many vectors are their while going to put the one in this place one in this place and an and for one here one here one here one here one here one here and 6 place keep moving this 1 and therefore there are 6 vectors in r 2 3. Therefore dimension I am sorry 6 vectors in this basis and dimensions of F 2 3 is 2 in to 3 which is six similarly, dimension of F m equal to m times n this set of all m by n matrix is over F have dimension m times n in particular dimension of the set of all square matrix is n square. Let us now look at another example so we same vector space of F m we are seeing vector space of F 2 by 3 and generalize in to F by m.

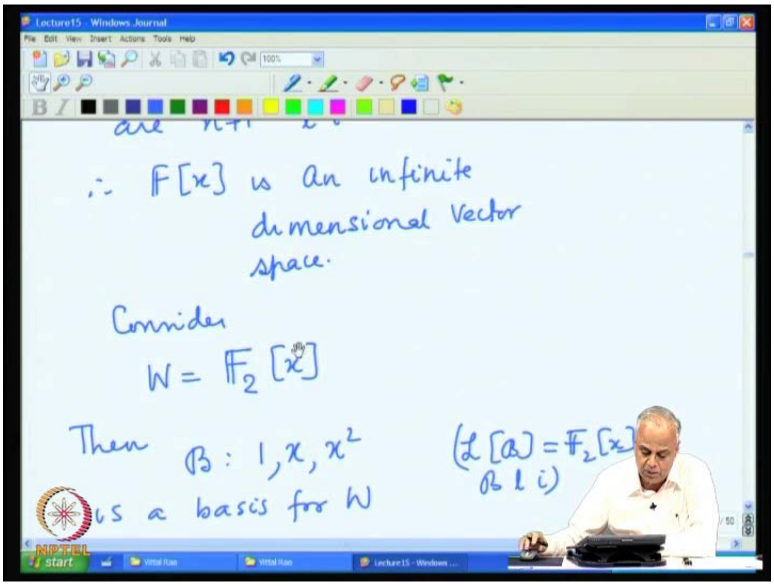
(Refer Slide Time: 39:45)



Now we look at the space of all the vector space of all polynomial over graph now this space is not finite dimensional because if it where suppose let say that is suppose F x is finite dimension that i S this then there must be the basis there must be finite basis say u 1 u 2 and u n say n is the dimension and say their n basis let suppose it is a finite dimension let is space and dimension is n then must have a finite basis consisting of in vectors and we seen that the movement a r basis consisting of n vectors any n plus 1 vector there must be linearly dependent implies any n plus 1 vectors must be.

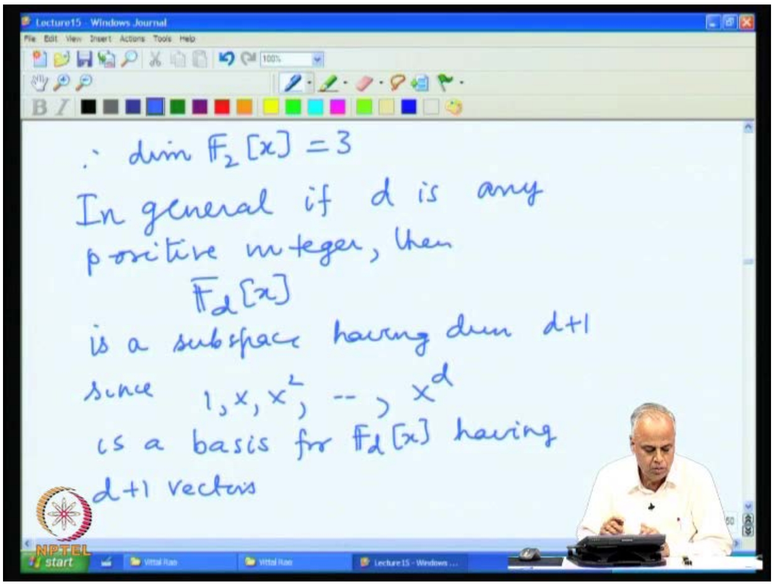
Linearly dependent in that space however this contradiction since 1 x x square etceteras x to the n are n plus 1 linearly independent. We do have m plus 1 whatever n you thing of I can later linearly independent set which is bigger than that therefore, we cannot have the basis consisting object n vector because you will have bigger linearly independent.

(Refer Slide Time: 41:46)



Therefore, F six the space of polynomial is an infinite dimensional space even infinite dimensional vector space now consider W to the sub state consisting of all polynomials whose degree is less than or equal to 2 then B consisting of polynomial 1 x x square is the basis for W for any polynomial degree two is linear combination of 1 x x square for 1 x x square span the space and then linearly independent and therefore, it is basis for w.

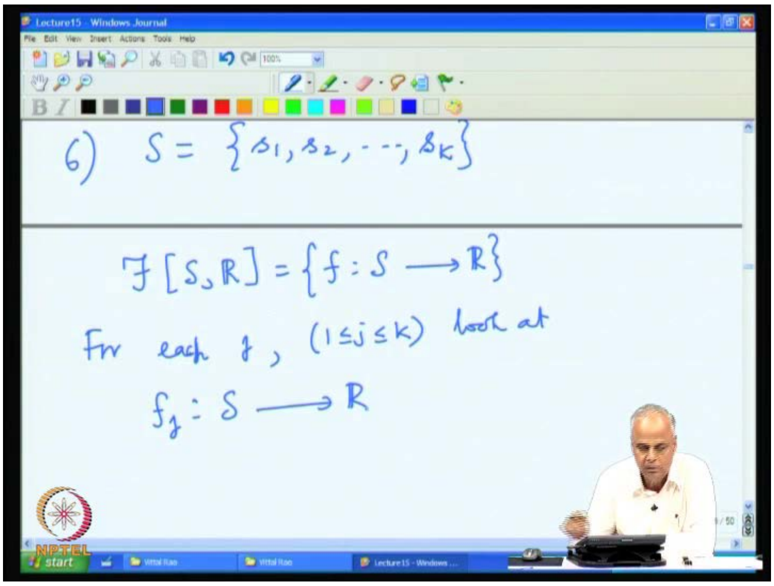
(Refer Slide Time: 43:01)



And there are 3 vectors in it B has 3 vector in it and therefore, dimension of this sub space F space F 2 x is 3. We have a infinite dimensional space namely F x the infinite dimension space of polynomial over a and inside that space in sitting finite dimensional sub space F 2 x of polynomial of degree less than or equal to 2 in general in be any positive integer any positive integer d and then F d x the collection or sub space all polynomial is degree is.

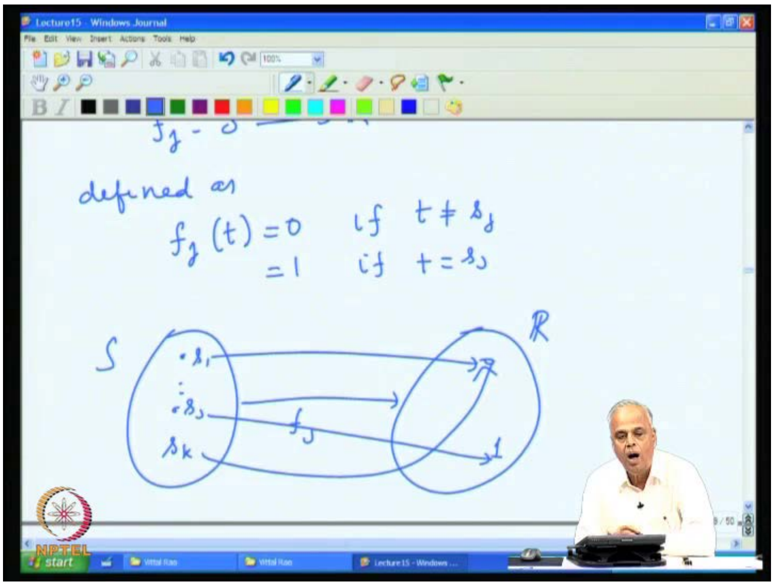
Less than or equal to d is a sub space having dimension when you add F 2 we add 1 x 1 x x square as a basis when you have x 3 1 x x square x cube the basis when you are F d 1 x x square x d will be a basis d plus 1 since 1 x x square x d is a basis for F d x and it has d plus 1 vectors. We are long number of finite number of sub space is sitting in the infinite dimensional sub space.

(Refer Slide Time: 45:12)



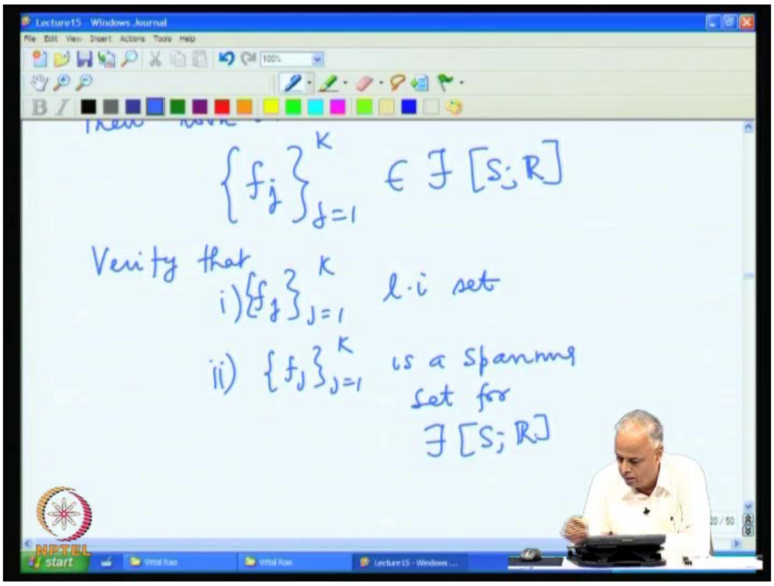
Now let us look at another interesting example, which may be let us take us set S which has a finite number of vectors elements say S k. Suppose we have set S having finite number of element then look at the vector space of all function from S noting in to the real life so it is set of all function which mark S 2 R let us look at where this is the finite dimensional space are a infinite dimensions space now for each j 1 less than or equal to j less than or equal to k look at the function f j mapping S 2 R.

(Refer Slide Time: 46:08)



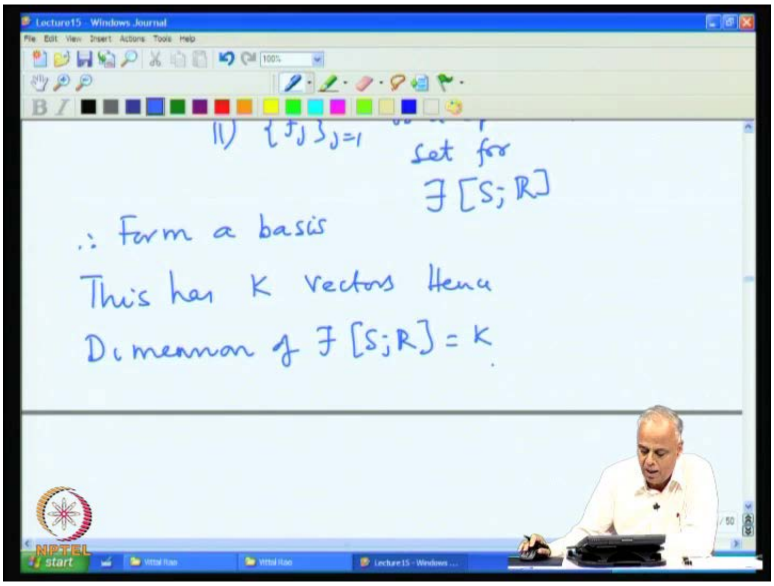
Defined as f j t is 0 if t is not S j 1 if t equal to S j What it means is here it is S said S the real line are error the point in the set now what is function f j does is it takes S 1 to 0 S k to 0 expect S j which goes to 1 and all others are going to go to 0.

(Refer Slide Time: 46:52)



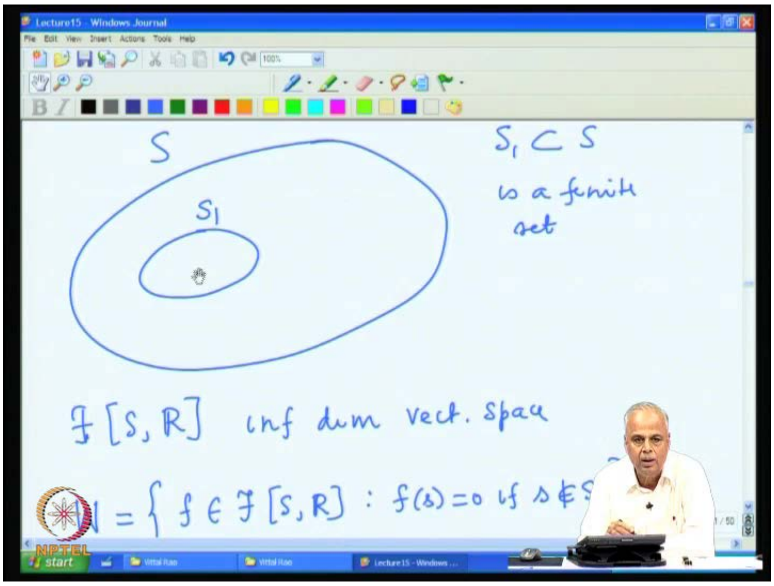
Then look at function we get like that first we get f 1 we take S 1 to and all other to 0 then we get f 2 this is S 2 and all other to 0 and we get the collection of collection of function f j j equal to 1 to k there are all in S R limit size exercise to verify that 1 f j j equal to 1 to k is a linearly independent set to this f j j equal 1 to k is a spanning set for W for spanning set for this space what our namely want to get this space f S i.

(Refer Slide Time: 46:52)



Now what is that mean is a linearly independent set is a spanning set and therefore, forms the basis how many vectors are there are precisely k vector namely function f 1 the function f 2 the function f k and this has k vectors vector are all function because the vector space are function therefore, the dimension hence dimension of S R is equal to k exactly k the size of the set x if you take a finite set S look at the function mapping S 2 R then dimension of the precisely k.

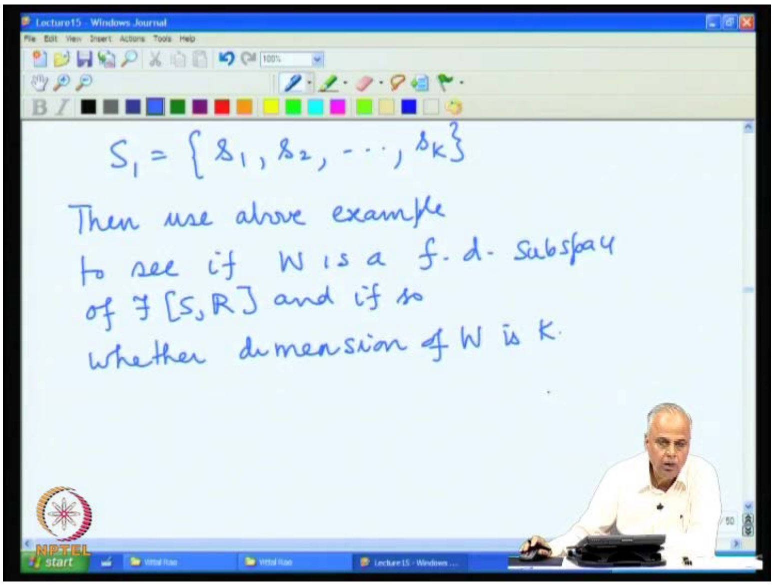
(Refer Slide Time: 49:05)



Now limitation exercise is verify the following suppose S is infinite set I have a infinite set S in say that there is finite set S 1 containing S is a finite set now look at the space of all function from S to R. This is going to infinite dimensional space this is going to infinite dimensional vector space now look at sub space W which consists all the function S to R set that f S.

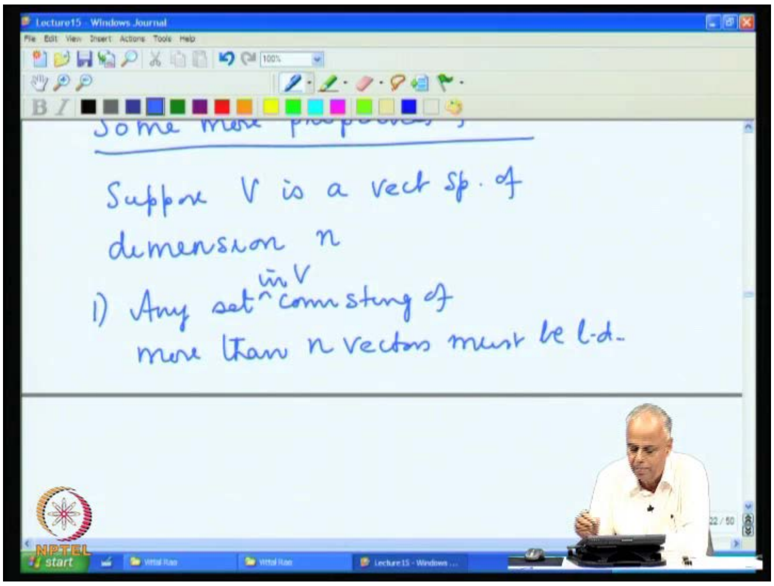
Is equal to 0 if S does not belongs to S 1 now where only when you a take a point in S 1 f will have you non 0 value the moment you go a outside S 1 F will take 0 value 1 will take their function which newin site set S 1 and dia out site S is we say technically say that the support of the function is contain in S 1 now in side S 1 the value may be 0 or 0 not we are not worried out what we are saying is outside S 1 it must necessarily the 0 so we are looking at all those function the dia outside the S .1

(Refer Slide Time: 51:00)



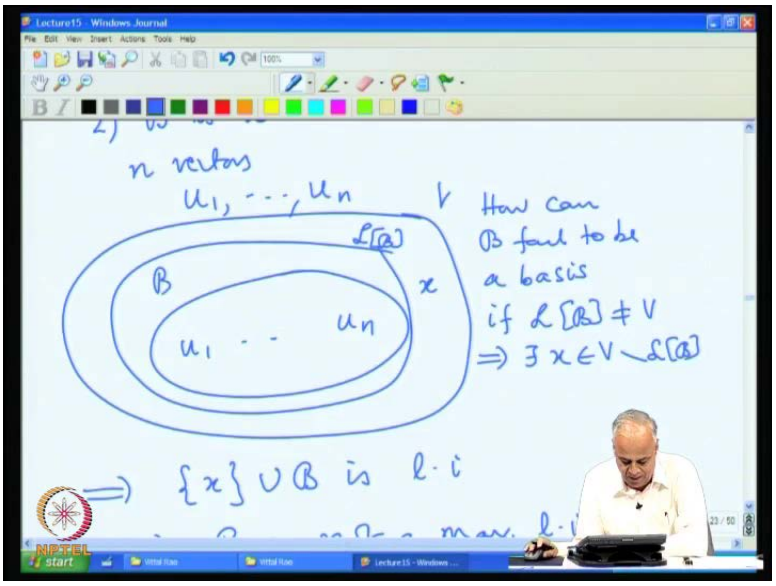
Now S 1 is finite set so let us call it us S 1 S 2 S k before then above example to see if W is a finite dimensional sub space of f S R and if weather dimension of W is k. We are seeing number of example of dimension.

(Refer Slide Time: 51:00)



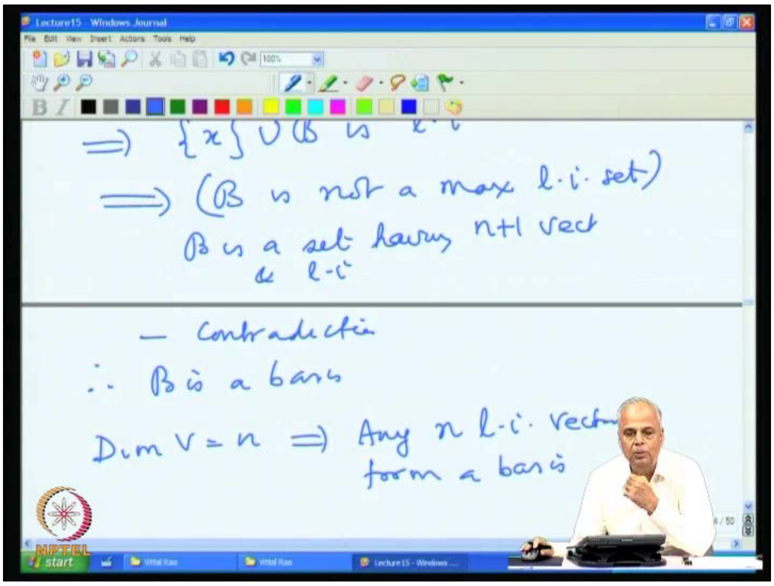
Let us move on look at some more properties of finite dimensional vector space we should space look at the properties so suppose V is vector space of dimension n suppose we are vector space of dimension n then we are observe that movement of basis any basis must have consist of n vector and moment of basis consisting of n vector any set consisting any n plus 1 set vector it must be linearly dependent any set consisting any set V consisting of more than n vector must be linearly dependent.

(Refer Slide Time: 53:07)



And secondly suppose B is the set having n vectors u 1 u 2 u n and suppose B is the linearly independent set B is the set in B because n vectors suppose this is linearly independent can this be basis well how can be fail to the basis how can B fail to the basis it can fail to the basis if fails to span be the hole space if L B is not equal to V here is V and this is L B L B not equal to V if it show then there is vector x outside L B the moment choose the vector x out the sub space and linearly independent set inside the sub space will there forget the set x union B is linearly independent which means B is not a maximum lineally independent because of the fact that it is contain a big a linear independent but how are we know but anything consisting of n plus 1 vector must be linearly dependent that we have a set consisting of B is a set having n plus 1 vector and linearly independent.

(Refer Slide Time: 55:22)



Which is contradiction because dimension is in therefore, our assumption that L B is not equal to V falls and hence B must be the basis therefore, B is the basis so what is the conclusion dimension of V equal to n implies any n linearly independent vector form of a basis and n dimensional space any linear independent form a space we shall see how we sample vector using basis in a next lecture.