Advanced Matrix Theory and Linear Algebra for Engineers
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Lecture No. # 14
Basis- Part 1

In the last lecture, we looked at the two important notions, namely linear independence and linear dependence.

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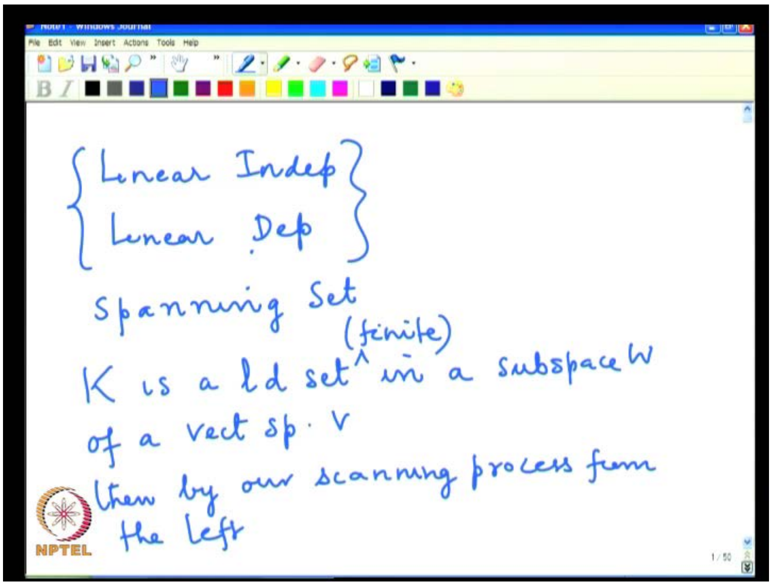

These are very important notions in the study of vector spaces; we also looked at the notion of a spanning set. One important property of linearly dependent set that we found was that, if K is a linearly dependent set in a subspace W of a vector space V then by our scanning from the left we saw this process towards the end of the last lecture, the scanning process from left; let us take the linearly independent set to be finite.

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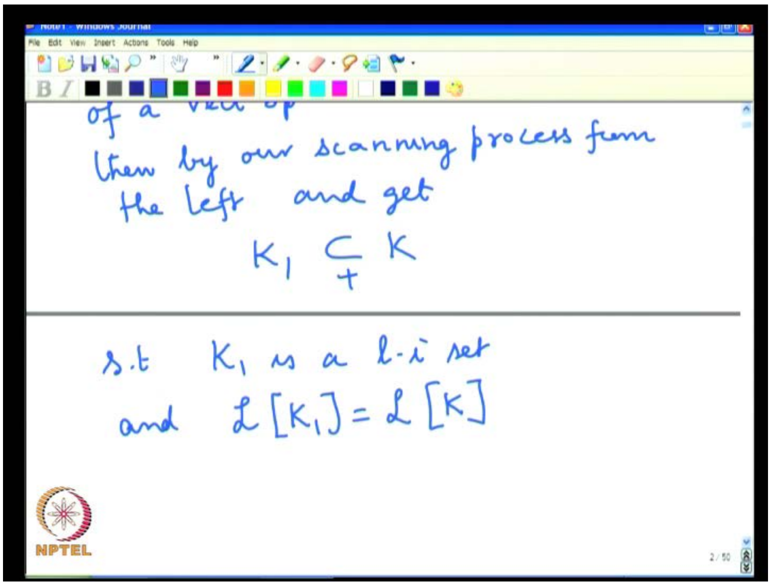

Suppose, we have a finite linearly dependent set K then by our scanning process, we can remove lot of unnecessary information and get a subset K 1 in K such that K 1 is a linearly independent set; and the subspace spanned by K K 1 is the same as the subspace spanned by K.

So, what this says is? That if, we have a linearly dependent set, you can remove the redundant information and get a smaller set and generate the same subspace that we had. We shall be using these properties repeatedly. Now, the question about the spanning set, what is a spanning set? A spanning set is the subset of the subspace W such that every vector can be expressed as a linear combination of these vectors.

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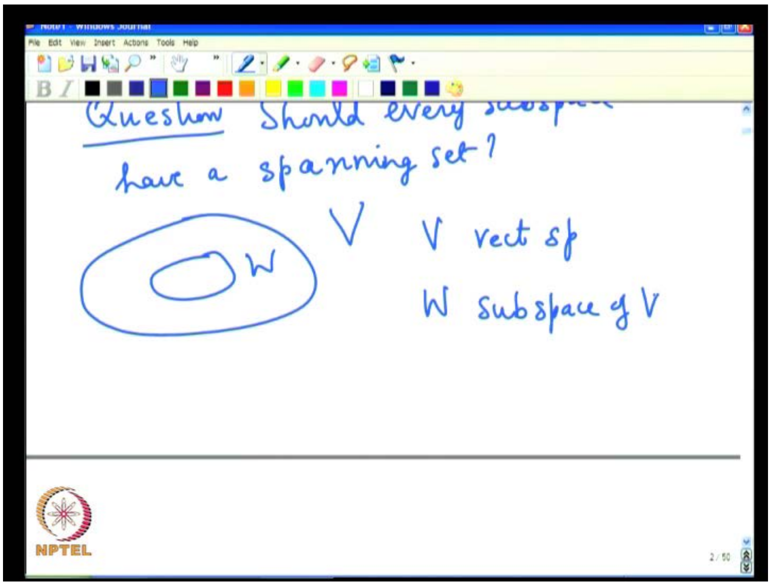

Now, the question is, should every subspace have a spanning set. We are not sure about this and because, we are no where proved that subspace should have a spanning set. Now, let us look at this question. Suppose, you have a vector space V and we have a subspace W. So, we have V is a vector space, we assumed that is a vector space over a field F and W is a subspace of V.

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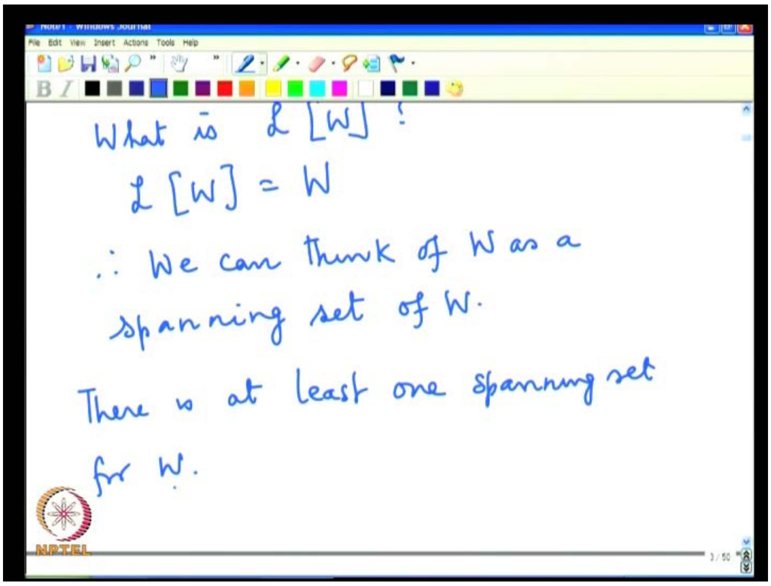

Now, we can think of W as a subspace of subset of W itself. Then what is L W, the subspace spanned by the set? We know that, L W is some smaller subspace containing

W, but W is already a subspace; and therefore, L W is indeed equal to W. And therefore, we can think of W as a subspace as a spanning set for itself; we can think of W as a spanning of W. And therefore, there is at least one spanning set for every W. There is therefore, at least one spanning set for W. But, are we satisfied with this spanning set?

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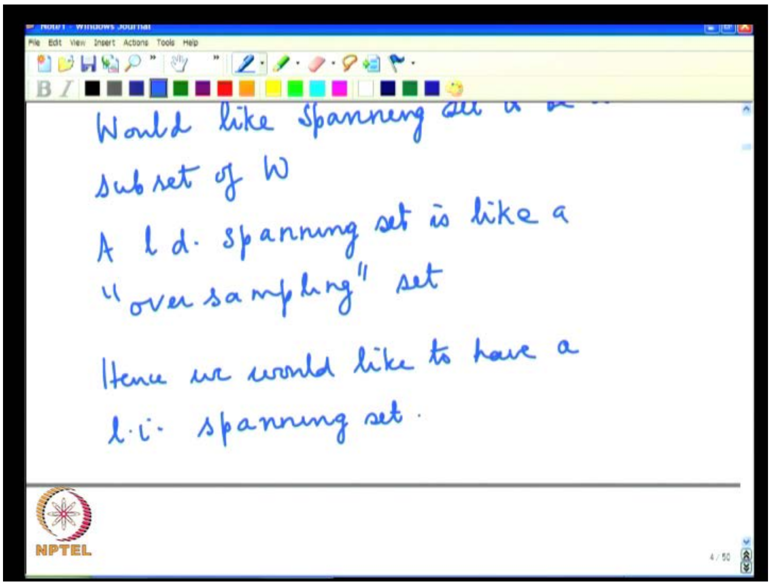

What is a spanning set? A spanning set is like a sampling set, so it is like a sampling set. What do we mean by this? When we say sampling set it means that by knowing these vectors in this spanning set, we can generate all the other vectors in the subspace. So, when you have a sampling set and you want to take W as a sample for itself, then we are sampling all the vectors and it is not a very good sampling. Therefore, from the point of view of sampling, our sampling set are the so called spanning set must be a subset of W.

So, would like spanning set to be a subset of W. Now, if that spanning set is linearly dependent that means he has a lot of redundant information. And therefore, we would be doing over sampling. Hence, a linearly dependent spanning set is like an over sampling set. And we would avoid over sampling, because we want to do minimal sampling. Hence, we would like to have non redundant information only that means we would like to have a linearly independent spanning set. This leads us to the notion of a basis.

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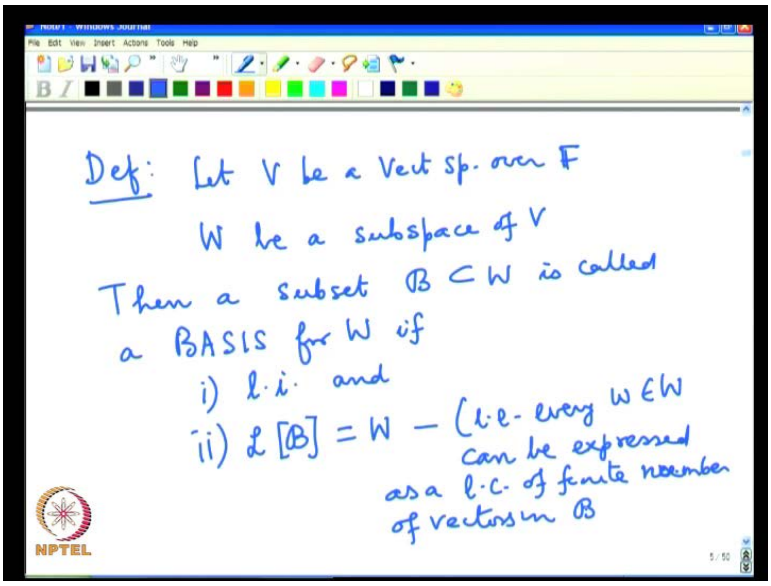

So, we give a definition now. Let V be a vector space over a field F, let W be a subspace of V. Then a subset B of V W is called a BASIS for W, if 1 we wanted to be a linear independent that is non redundant information. And it should be a good sampling set that means it must be a spanning set that means subspace span by L of S must be equal to W. We have used the notation B, so we call it as B, L of B must be equal to W; what does this mean? That is every w in W can be expressed as linear combination of finite number of vectors in B. So, that makes it a good sampling set, we do not have redundant information and by looking at them, we are able to take about all the vectors in W.

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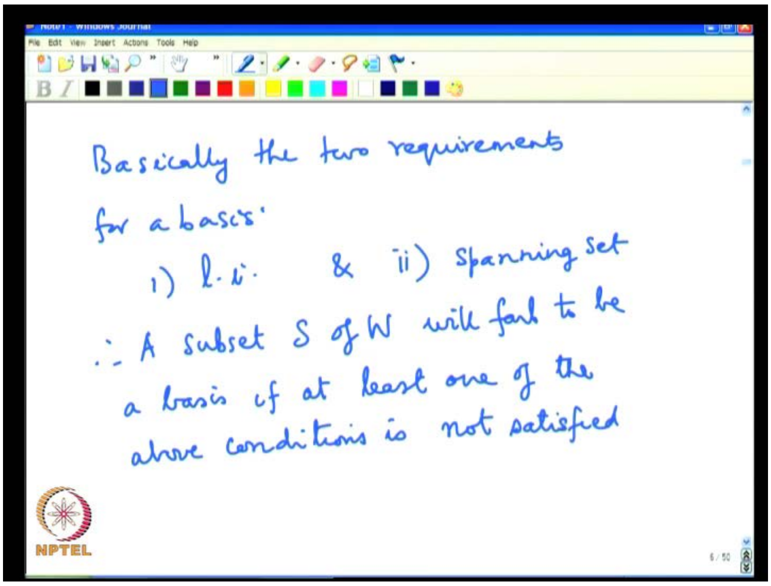

So, basically therefore, there are two requirements for basis we repeat are the following. So, the two basic requirements, one is linear independence and other is spanning set that every vector in W is linear combination of these vectors. Therefore, a subset S of W will fail to be a basis, if at least one of the above conditions is violated, if at least one of the above condition is not satisfied.

So, therefore, for example a basis a subset may failed to be a basis by not satisfying the condition 1 that means by being linearly dependent or it may be linearly independent, but may failed to satisfy the second condition namely spanning. So, are possibly it is neither linearly dependent nor spanning. What is important is? It will fail to be a basis, if at least one of the conditions is not satisfied.

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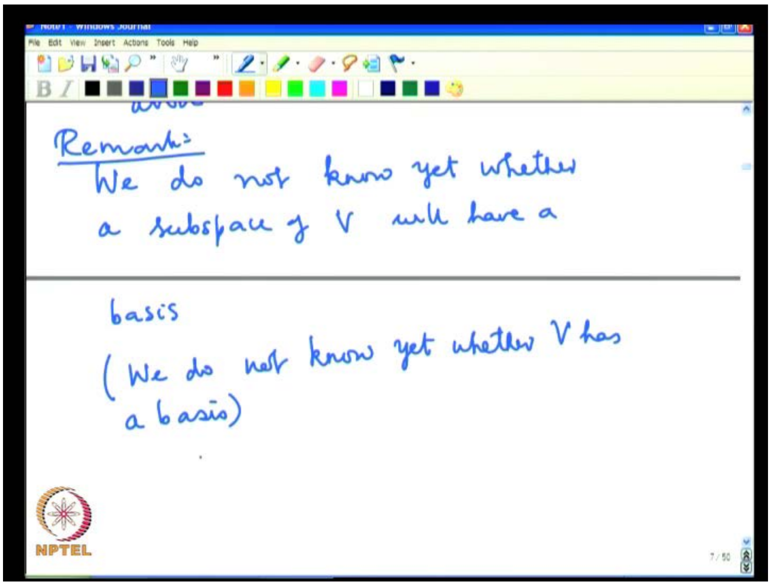

As of now, we are not sure whether a vector space will have basis. So, we do not know yet, so this we will call it as a remark; we do not know yet whether a subspace of V will have a basis. Therefore, now we must remember that, we can be thought of a subspace of itself and therefore, we can talk of a notion of a basis for V. And therefore, we do not know yet whether V has a basis, which we will get we should make a remark about this question little later. But, to begin with let us look at that some examples.

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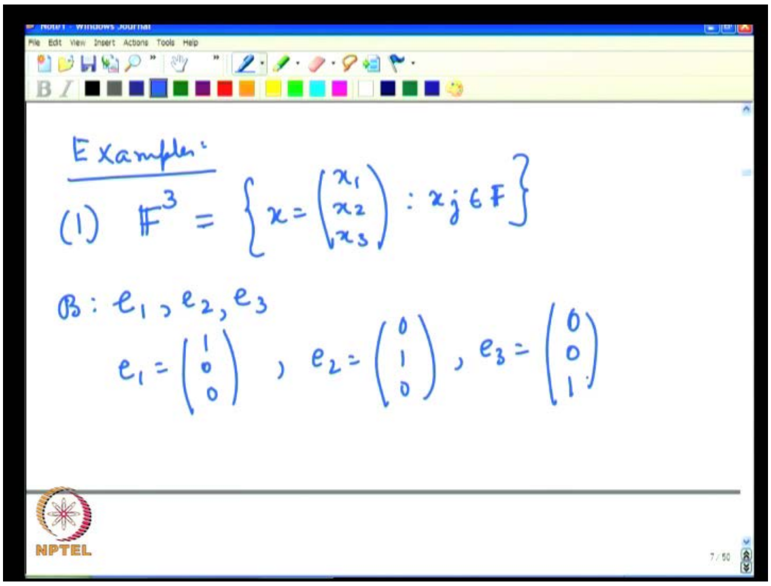

The first example that we see let us look at F 3. Now, in this where F is a spare field we can take F to be R then we get R 3, we can take F to be C we get C 3 and so on, and so forth. So, let F be any field and look at F 3, what is F 3? It is a set of all vectors which are of this form (x 1, x 2, x 3) column vectors such that the entries are all from the field F. Now, in this consider the vectors e 1, e 2, e 3, the set B consisting of three vectors, where e 1 is the vector (1 0 0), e 2 is the vector (0 1 0), e 3 is the vector (0 0 1).

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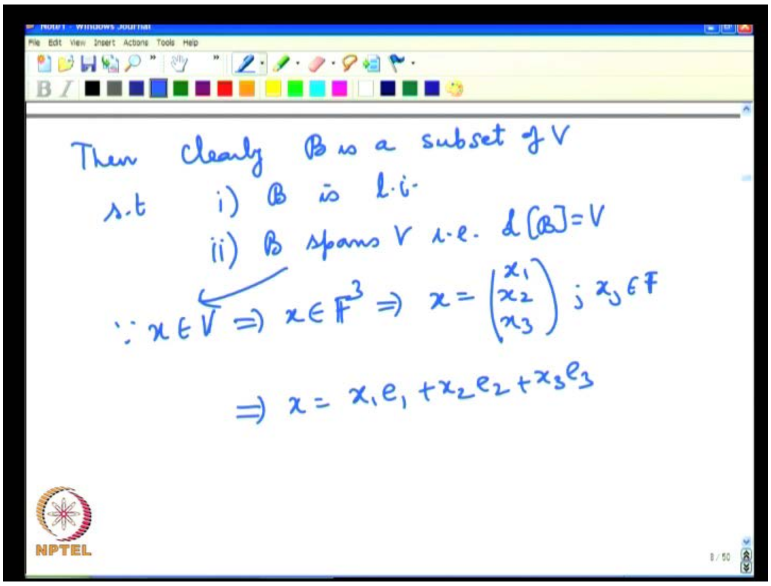

Then clearly B is a subset of V such that 1, B is linear independent; 2, B spans V that is L of B is equal to V. Why L B equal to B? Because, any vector x is equal to x 1 let us look at this, because x belongs to double V implies x belongs to F 3, because our V is now F 3 which means x is of the form (x 1, x 2, x 3), where this x j are in F. This means x can be written as x equal to x 1 e 1 plus x 2 e 2 plus x 3 e 3, which means x is a linear combination of e 1 e 2 e 3. And therefore, every vector is a linear combination of B vectors and B is already linearly independent.

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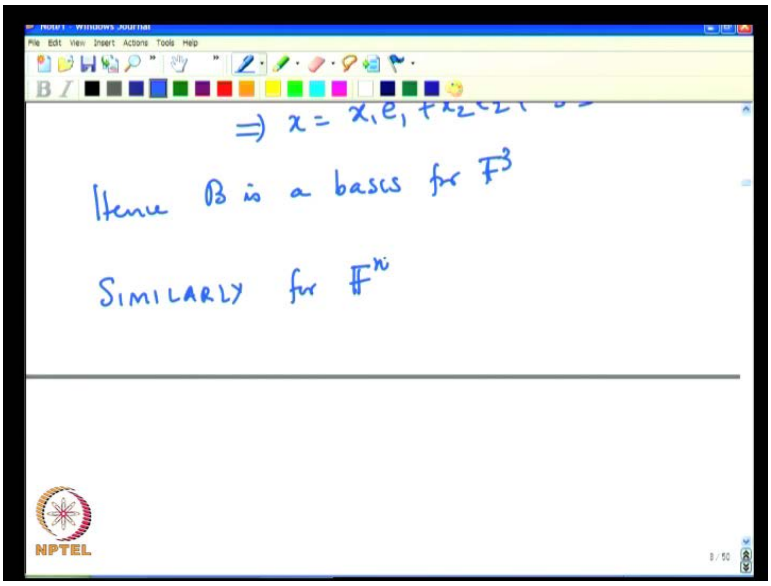

Hence, B is a basis for F 3.

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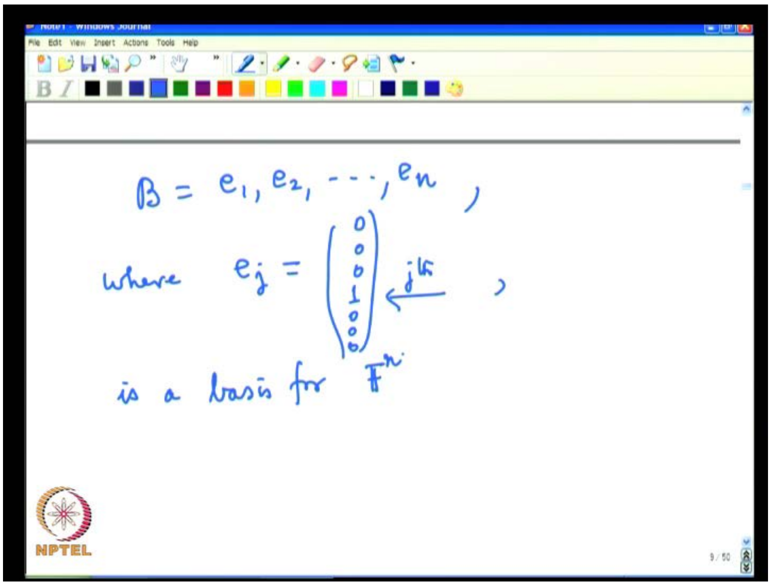

Similarly, instead of F 3 we can take F n for F n, we can take B equal to e 1, e 2, etcetera e n, where e j is the vector which has all the components 0, except that j th component which is 1 all others are 0. Do this for j equal to 1 to n, we get n vectors. So, if we take B to be equal to e 1, e 2, e n, where e j's are this, then this is a basis for F of n.

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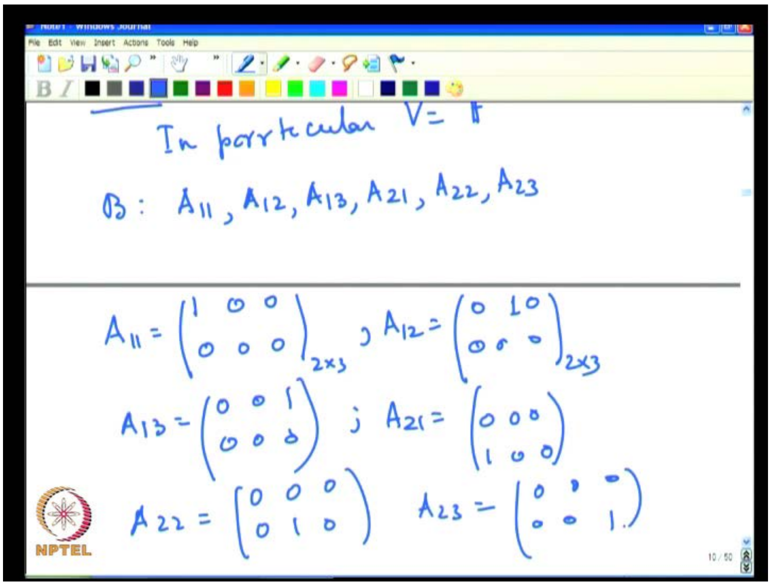

As the next example, let us consider V to be a set of all m by n matrices, whose are entries are from the field F. In particular, for simplicity first let us consider V to be F 2 3 that is all 2 by 3 matrices. Then we are going to look at a set B, which consist of the following matrices, A 1 1, A 1 2, A 1 3, A 2 1, A 2 2, A 2 3. All these A's are matrices and what are they? A 1 1 is the matrix 2 by 3 matrix which has 1 in the 1 1 place and 0 elsewhere. A 1 2 is the matrix, which has 1 in the 1 2 place and 0 elsewhere; similarly, we have A 1 3 is (0 0 1 0 0 0), it has 1 in the 1 3 place. A 2 1 has 1 in the 2 1 place; A 2 2 has 1 in the 2 2 place; and A 2 3 has 1 in the 2 3 place. So, we have these six matrices and consider these six matrices and call this collection of these six matrices as B.

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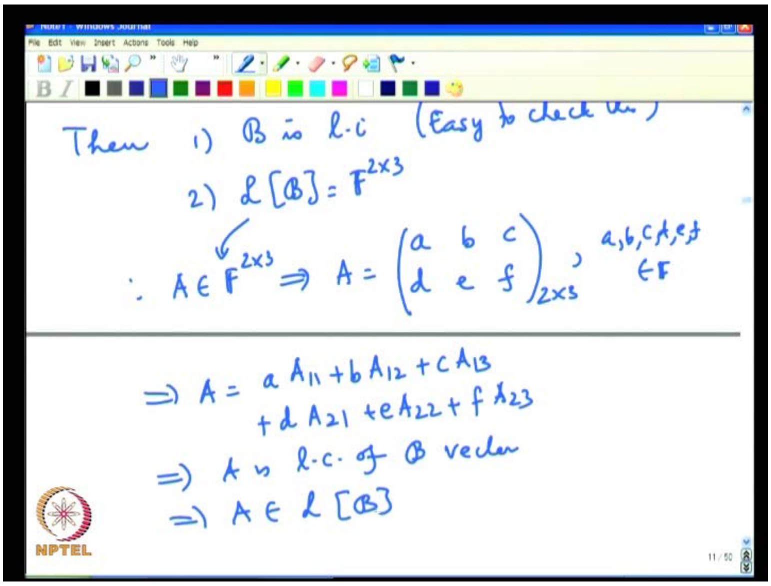

Then 1, B is linearly independent it is easy to see this, easy to check this. And L of B is indeed F 2 3, what do we mean by this? We want to show that every matrix in F 2 3 is a linear combination of these B vectors. So, because A belongs to F 2 3 imply A is of the form of (a b c d e f). Because, it is a 2 by 3 matrix, where all these (a b c d e f) in the field F.

Now, that says I can write A as a into A 1 1 plus b into A 1 2 plus c into A 1 3 plus d into A 2 1 plus e into A 2 2 plus f into A 2 3 that says A is linear combination of B vectors. That means A belongs to L; that is every vector in F 2 3 is a linear combination of this. So, therefore, we have L B is also B, B is in this case a space of matrices.

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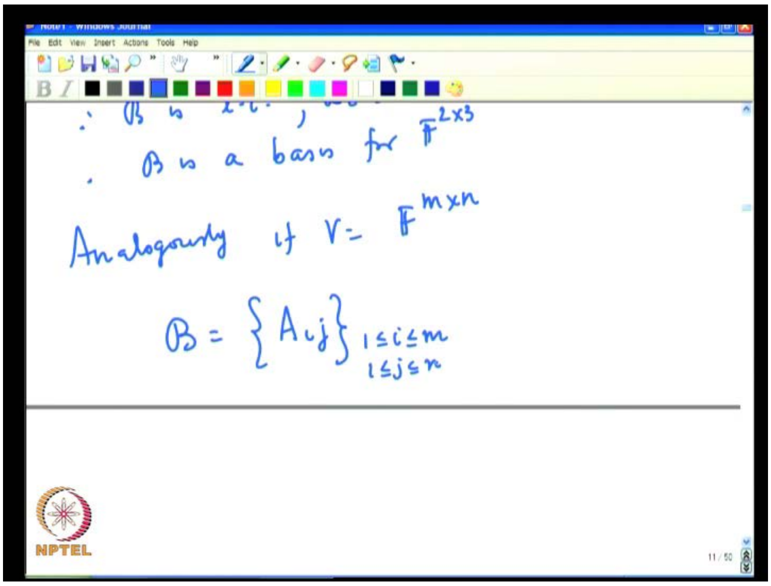

Therefore, B is linearly independent, L B is B. And these are the two requirements for a vector space to be a basis and therefore, B is the basis for F 2 3. Analogously, if we now consider all m by n matrices over the field F then, consider B to be the collection of all the matrices A i j, where i is 1 to m, j is 1 to n.

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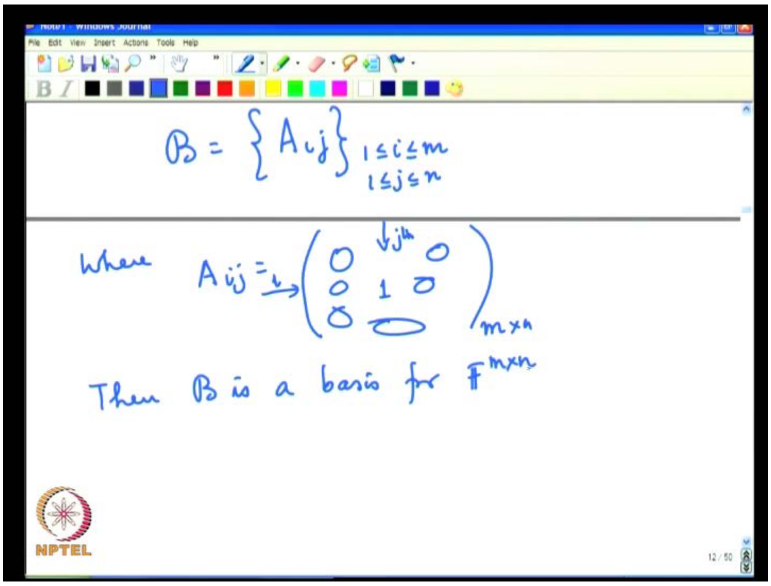

So, many matrices for each i and j we get one matrix, where A i j is a matrix which is m by n which has in the i th row and the j th column that place 1 and all others 0; i th row and j th column entries 1. And do this for every i and j then B is a basis for F m n.

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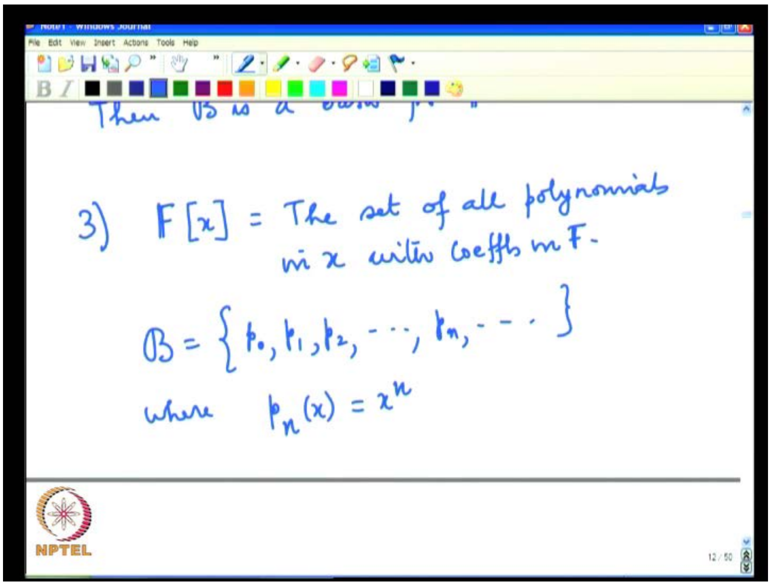

The next example will consider is that of the polynomials. Let us look at the set of all polynomials over x over the field F in a variable x over the field F. So, this is the set of all polynomials in x with coefficients in F. Let us consider this space, and now consider B to be the following polynomials, p naught, p 1, p 2, etcetera, p n, etcetera where p n of x is x to the power of n. So, p naught is 1, p 1 is x, p 2 is x square, p 2 is x cube and so on, and so forth.

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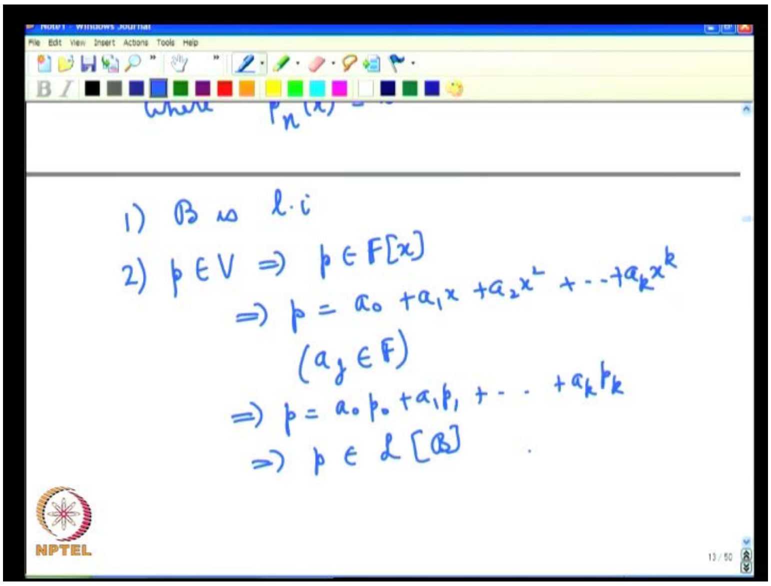

Now, this is an infinite set we have already seen that, B is linearly independent. And every polynomial is obviously a linear combination of a finite number of B vector; that is, if p belongs V that means p belongs to the polynomial space. Therefore, p must be of the form a naught plus a 1 x plus a 2 x square plus a k x to the power of k polynomial, say degrees k must be of this form, where a j's are all in F.

This immediately says that, p is a naught p naught plus a 1 p 1 plus etcetera a k p k, which means p belongs to L B. And therefore, every vector in V is a linear combination of finite numbers of vectors in B.

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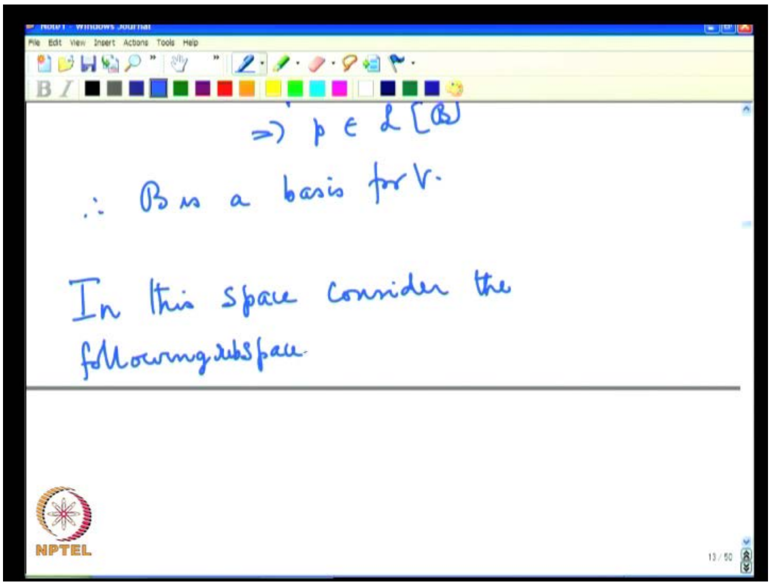

And therefore, B is a basis for V. Now, in this space consider the following subspace that is in the space of polynomials, we are going to look at a small subspace.

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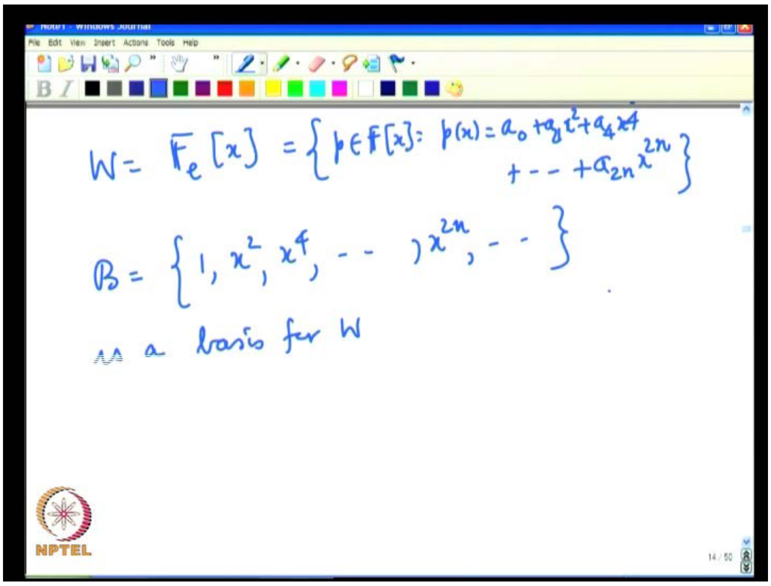

Let us say W I denote it by F e of x it is the set of all polynomials, which have only even powers of x that is the polynomial belonging to F of x such that p (x) of the form a naught plus a 2 x square plus a 4 x to the power 4 plus etcetera plus a 2 n x to the power of 2 n. So, it has only even powers of x, and that is why you call it as p (x). And this is a subspace and for this subspace, 1, x square, x to the power of 4, x to the power of 2 n, the polynomial consist of even powers of x is a basis for W; because, they are automatically linearly independent, because they are subset of a linearly independent set. So, therefore, linearly independent.

And as we seen about the definition of p, p is a linear combination and that to a finite number of these give rise to any p. And hence B is a basis. So, thus we have this important notion of basis. But, we should keep in mind that we are yet not sure whether every subspace should have a basis and whether every vector space is a basis.

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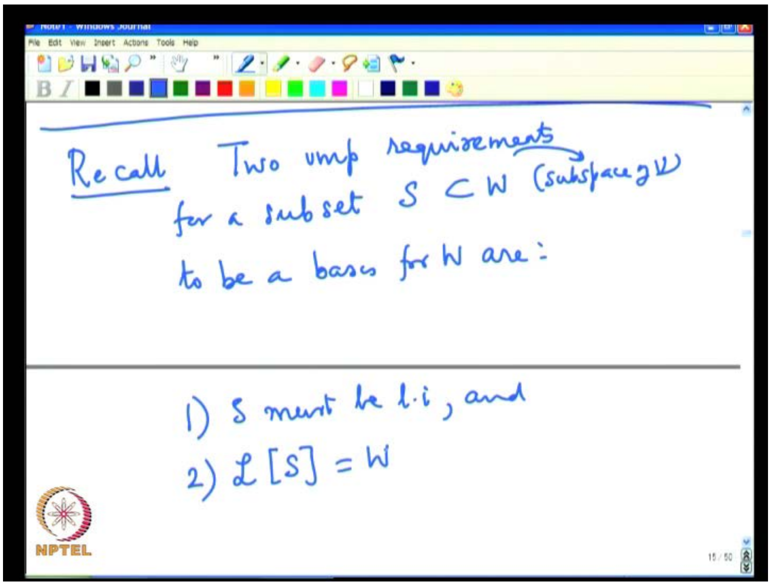

But, we should remember again that the two important requirements for a subset S of subspace W, W is a subspace of V. So, the two important requirements for a subset S of W to be a basis for W are: what are these two requirements? 1, S must be linearly independent and 2, linear span of S must be equal to W that is every vector W must be a linear combination of a finite number of vectors in S.

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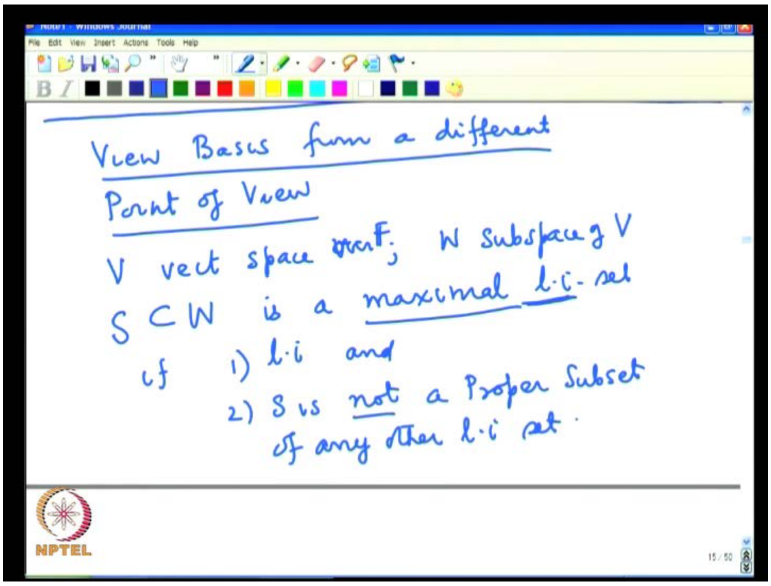

Now, we are going to look at view basis from a different point of view. Consider, now a vector space V, now we will introduce notion called a maximal linearly independent set.

So, let us say W is vector space over F, and W subspace of V then we say S is a subset of W is a maximal linearly independent set. If 1, there are two adjectives in this; 1, it must be linearly independent, so linearly independent. And 2, we have use the adjective maximal, what does maximal mean? It means that you cannot embedded it any other bigger linearly independent set; that is S is a not a proper subset of any other linearly independent set.

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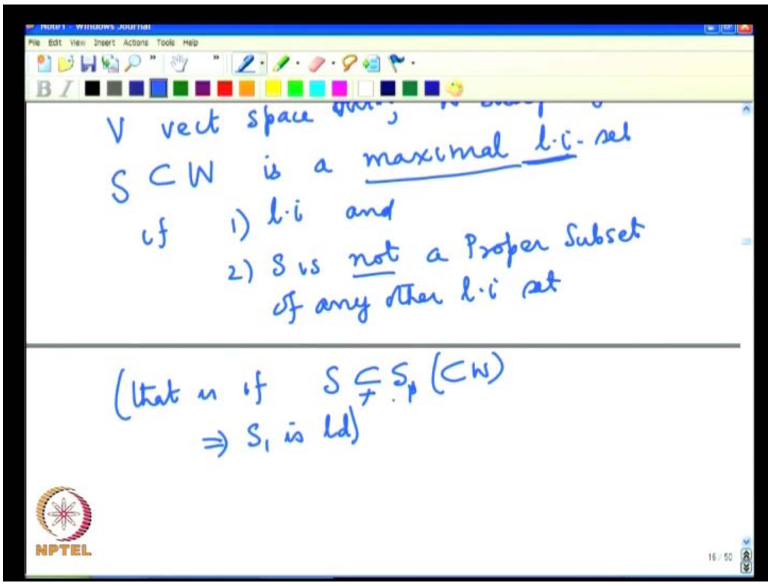

We can state it also like this that is, if S is proper subset of S 1 which of course is in W. So, if S is a proper subset of S 1 then S 1 cannot be linearly independent that means S 1 is linearly dependent. So, any one any set which encompasses S in W must be linearly dependent. So, the moment you try to make it bigger, it looses linear independence; So, that is a in a sense of maximal size you could reach for linear independence. And the moment you try to blow it up, it looses its linear independence.

So, we have this notion of maximally, a linearly independent set; that is a set, which is linearly independent and which cannot be embedded as a proper subset of any other bigger linearly independent set.

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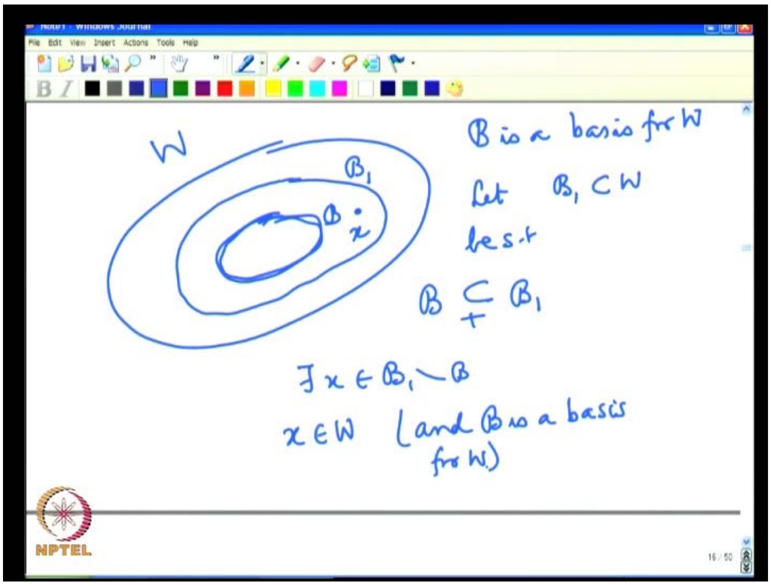

Now, let us consider W a subspace, and let us say B is a basis; suppose, B is a basis for W. Now, consider any set in B which is bigger than W; so, let B 1 contained in W be such that B is a proper subset of B 1. Now, B is a proper subset of B 1 and therefore, there must be a vector x in B 1, which is not in B. So, there exist an x, which is in B 1 not in B, so let us call that vector x.

Now, since x is in B 1, and B 1 is a part of W, we have x belongs to W. Now, x belongs to W and B is a basis we have started with the basis for W, and B is a basis for W. Now, B is the basis for W; it has two properties, one it is linear independent, and other one is that it spans W. Now, first we shall exploit the property that it spans W that means any vector in W is a linear combination of B vectors.

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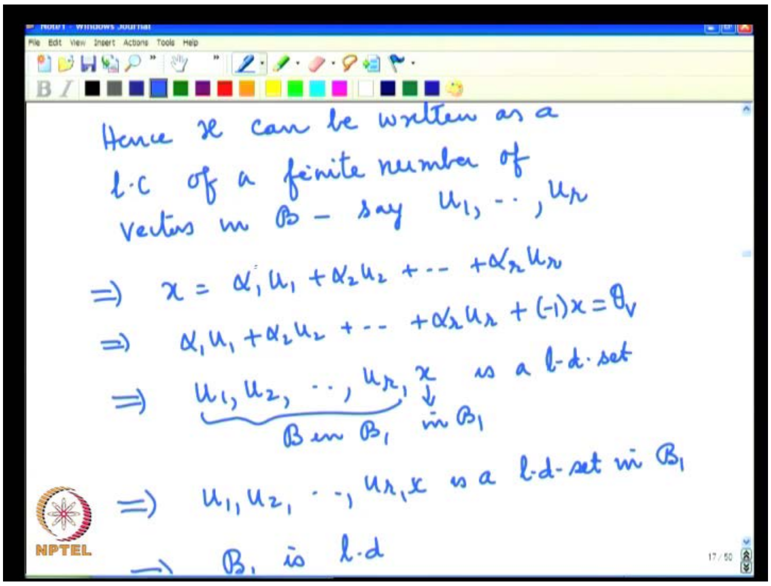

So, hence x can be written as a linear combination of a finite number of vectors in B - say the finite number of vectors are u 1, u 2, u r; what it that means? That means x is equal to some alpha 1 u 1 plus alpha 2 u 2 plus etcetera plus alpha r u r. Now, this says that alpha 1 u 1 plus alpha 2 u 2 plus alpha r u r plus minus 1 times x is theta. This says there is a non trivial linear combination of this u and x vector which gives the 0 vector. It is non trivial linear combination, because the co efficient of x is minus 1.

So, therefore, that says u 1, u 2, u r, x is a linearly dependent set. Now, where is the set? The vectors u 1, u 2, u r are all in B which is also B is itself is in B 1, and this also in B 1. And therefore, these vectors are in all B 1; and therefore u 1, u 2, u r, x is a linearly dependent set in B 1. The moment a set has a linearly dependent subset, the whole set must be linearly dependent. So, therefore, B 1 is linearly dependent set.

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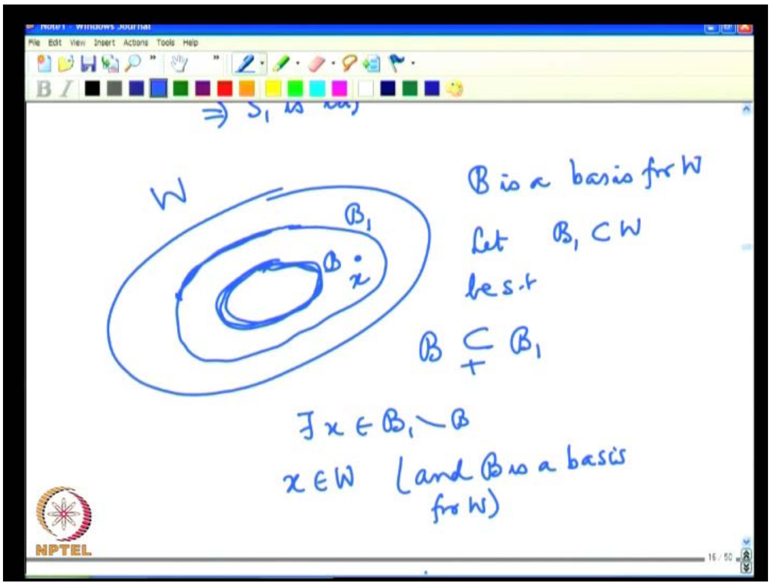

So, what have we concluded? We started with the let us go back, we started with the basis and then we looked at a set bigger than the basis; and then we showed that must be linearly dependent.

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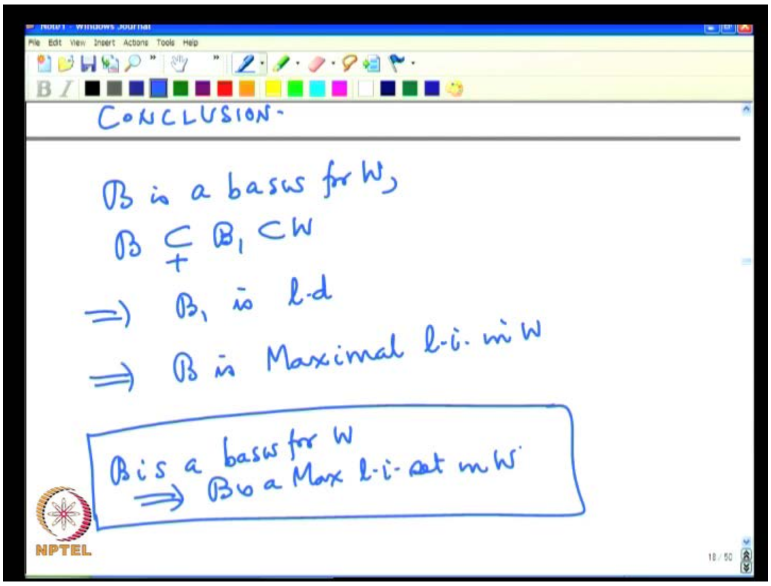

So, what it says is that B is basis. And hence, the conclusion is the following. B is basis for W, and B is proper subset of B 1 which is in W implies B 1 is linearly dependent. This means I am not able to embed B in a bigger linearly independent set which is precisely what is meant by saying B is maximal linearly independent set. Because it is a basis it is linearly independent. And we have seen just now, that you cannot put it in a bigger linearly independent set or anything bigger than that is linearly dependent. So, we have the first important observation that, B is a basis for W implies B is a maximal linearly independent set in W.

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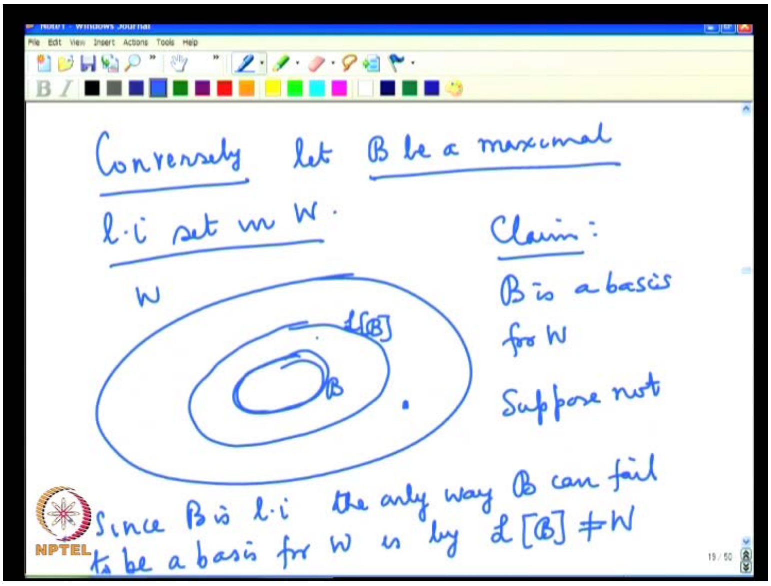

Now, let us look at the whether the converse is true. Conversely, let B be a maximal linearly independent set in W, so here is W and here is B which is a maximal linearly independent set. We want to claim that B is a basis for W. Now, how can B fail to be a basis? Suppose not. That means B has fails to be a basis. Now, B will be a basis, if it is satisfies two requirements, one is linearly independent, the other it is a spans the W.

Therefore, the failure of the basis can take place due to two reasons. 1, because it is fails to be a linearly independent, or B fails to be spanning W. Now, B is already given to be a maximal linearly independent set. So, it cannot fail to be a linearly independent, because already a linearly independent. So, therefore, the only way B can fail to be a basis is, that it does not span W. Therefore, since B is linearly independent, the only way B can fail to be a basis for W is by this span being not equal to W. So, that means the L B is in W, but not equal to W. So, let us now take L B.

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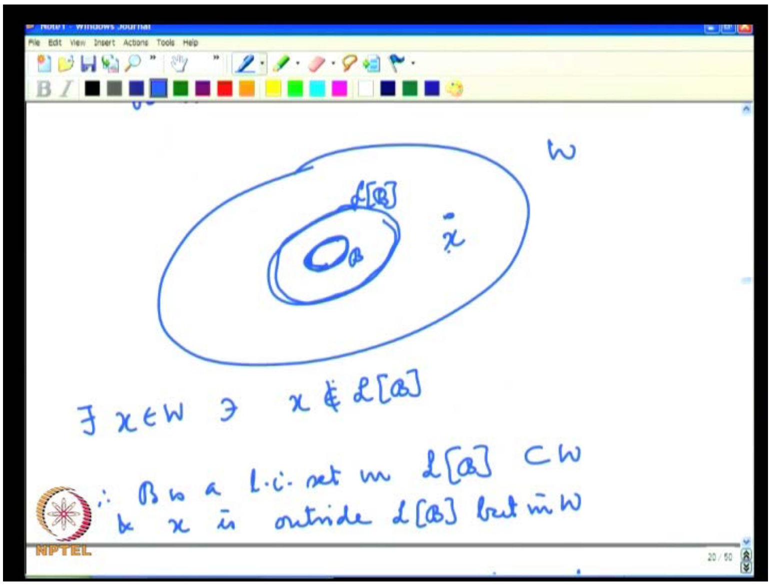

Now, L B is subspace of V it is sitting inside this subspace W and B is linearly independent set in L B. These two we observe. Because, now L B is a subspace by itself and B sitting inside L B, so B is a linearly independent set in W. But, now we got L B is not all of W and therefore, there is some vector. Let us draw the picture again, so here is W, here is B and here is L B and L B falls start of W. Because, we are assume that B is not a basis and therefore, there is vector x in W, which is outside x.

So, x there exist x in W such that x does not belong to L B. Now, B is the linearly independent set, this B is the linearly independent set in this subspace and x is outside that subspace; and therefore, B together with x we still be a linearly independent set in W we have seen this before. So, therefore, B is a linearly independent set in L B which is contained in W, and x is outside L B, but in W.

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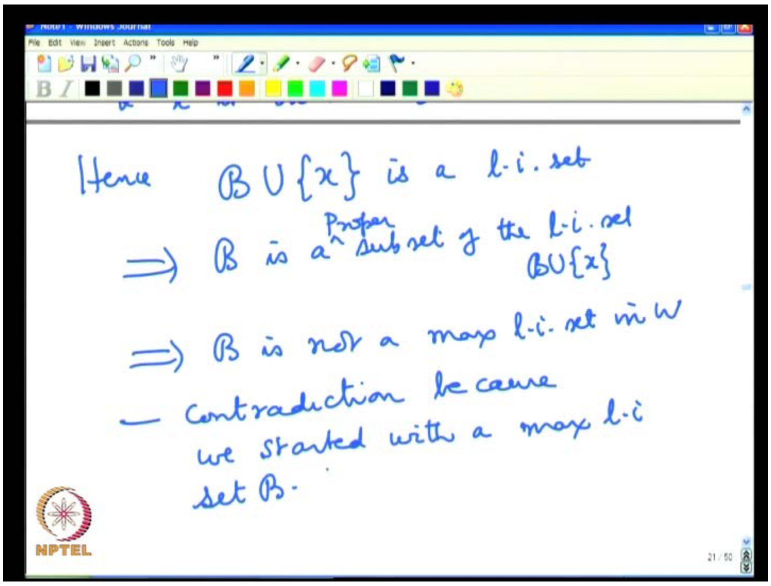

Hence, B union x is a linearly independent set which means B is a subset proper subset of the linearly independent set, B union x. And therefore, B is not a maximal linearly independent set in W, because we have embedded it in a bigger linearly independent set. But, we started with a maximal linearly independent set, we said let B be a maximal independent set. Remember, we started with fact that B is a maximal linearly independent set in W. And hence it is a contradiction, because we started with a maximal linearly independent set B. And therefore, our assumption that B is not a basis is false.

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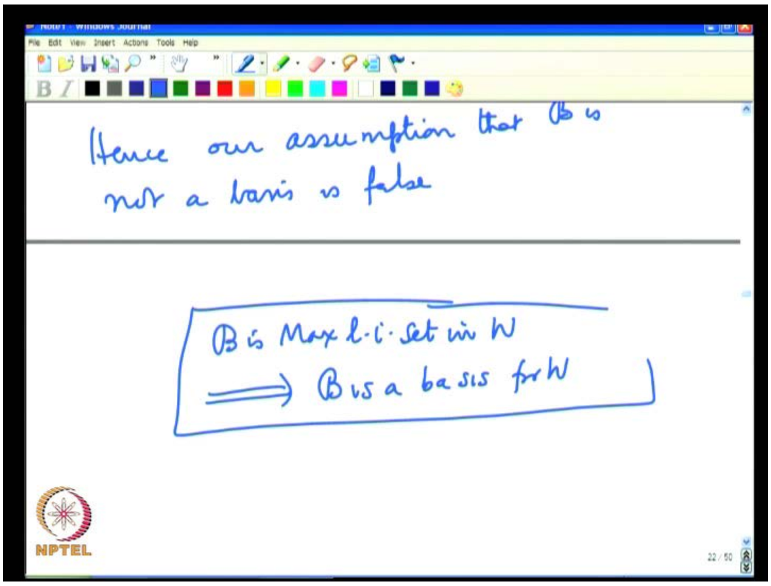

And hence, our assumption that B is not a basis is false. So, the conclusion we get now is, the converse conclusion we have that B is maximal linearly independent set in W implies B is a basis for W. Previously, we showed that, if B is a basis then it is a maximal linearly independent set.

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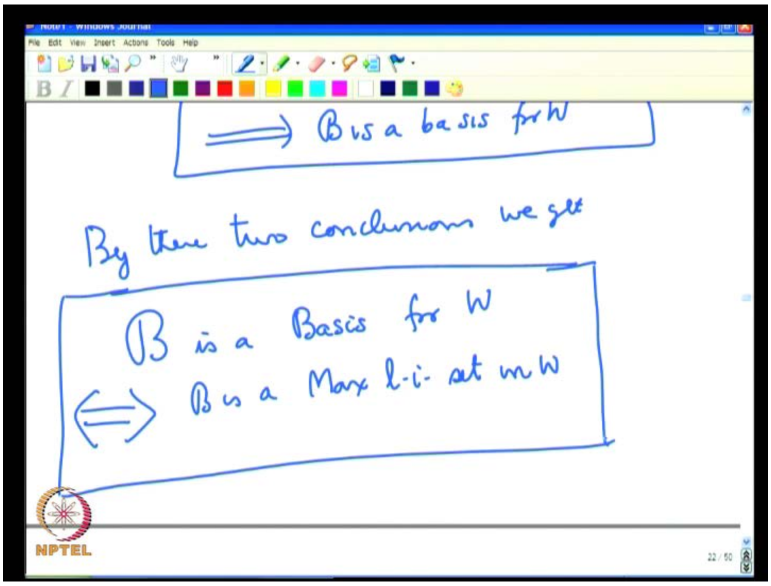

Now, putting these two together, putting by these two conclusions we get B is a basis for W, if and only if B is a maximal linearly independent set in W. So, this is the very important observation about basis. In other words, it is short of optimally sized linearly independent set. It is optimally sized sampling set, because it is going to span the whole space.

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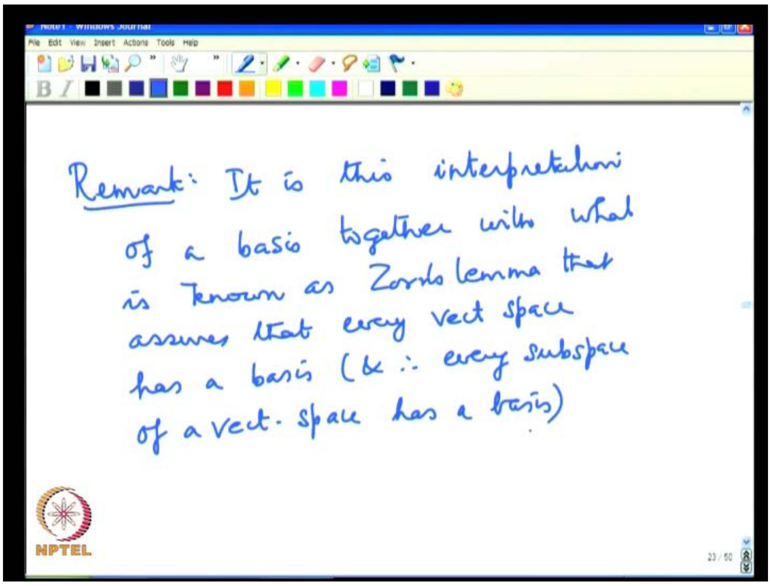

So, it is this property we will just make a note, we will not prove this. It is this interpretation of a basis together with what is known as Zorn's Lemma. That assures us that every vector space has a basis; and therefore, every subspace of a vector space has a basis, we will not go into the proofs of these statements using Zorn's Lemma we will take them for grant.

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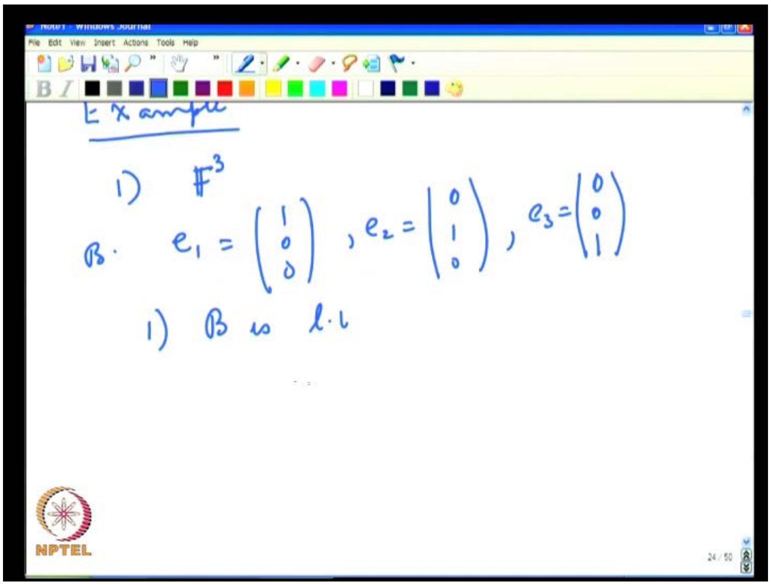

Now, let us look at the some examples again. Let us look at F 3 and we had the basis B e 1 is (1 0 0), e 2 is (0 1 0), and e 3 is (0 0 1). We have already seen and it is very easy to see that, B is linearly independent.

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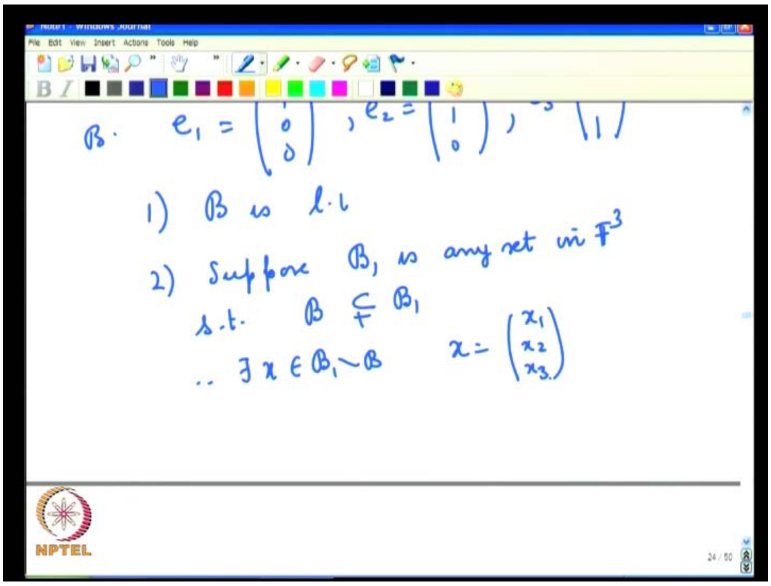

Now, suppose B 1 is any set in F 3 such that B is contained in B 1, but not equal to B 1; that is B is a proper subset of B 1. Therefore, there exist an x in B 1 minus B, call this as x 1 x 2 x 3.

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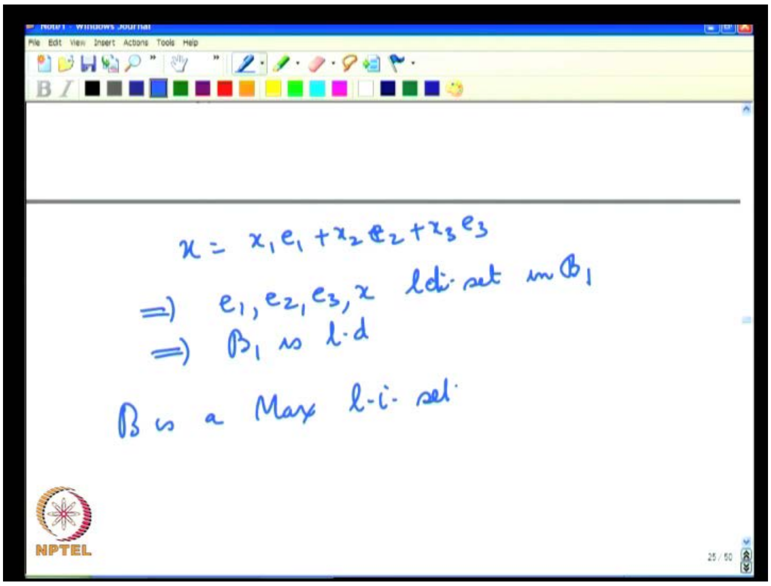

And we have x is equal to x 1 e 1 plus x 2 e 2 plus x 3 e 3; and that implies that e 1, e 2, e 3, x is a linearly dependent set and all of them are in B 1. And therefore, B 1 has a linearly dependent set inside it. And therefore, B 1 is linearly dependent, which says that anything bigger than B is linearly dependent and it is already a linearly independent B. And therefore, B is a maximal linearly independent set. So, thus we see that a basis has to be a maximal linear dependent set.

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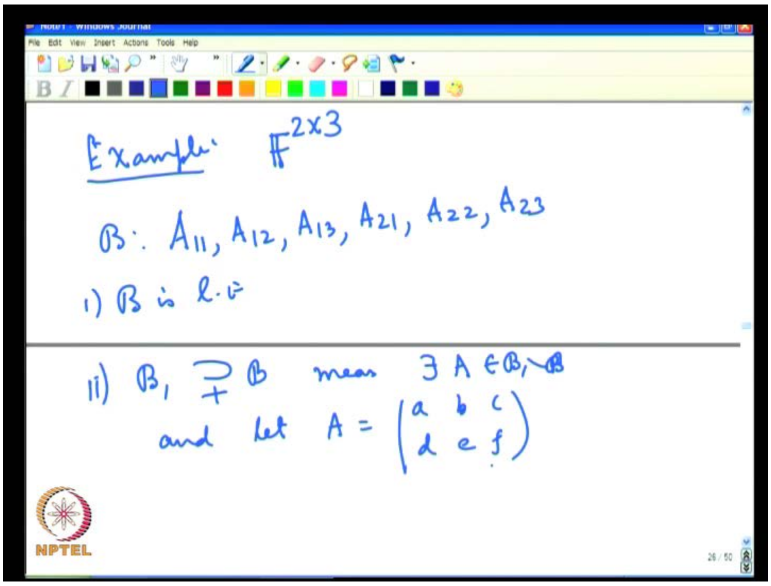

Now, let us look at the example of the matrices. Let us for simplicity look at F 2 by 3 the set of all 2 by 3 matrices, when we are looking at all the 2 by 3 matrices, we had these matrices A 1 1, A 1 2, A 1 3, A 2 1, A 2 2, A 2 3 remember these matrices. A 1 1 has 1 in the 1 1 entry and 0 elsewhere; A 1 2 has 1 in the 1 2 entry 0 elsewhere. In general, A i j has 1 in the i th row j th column and all other entries are 0. Then B is linearly independent, and B 1 contains B means there exist a matrix which is in B 1 which is not in B, and let A be is a b c d e f.

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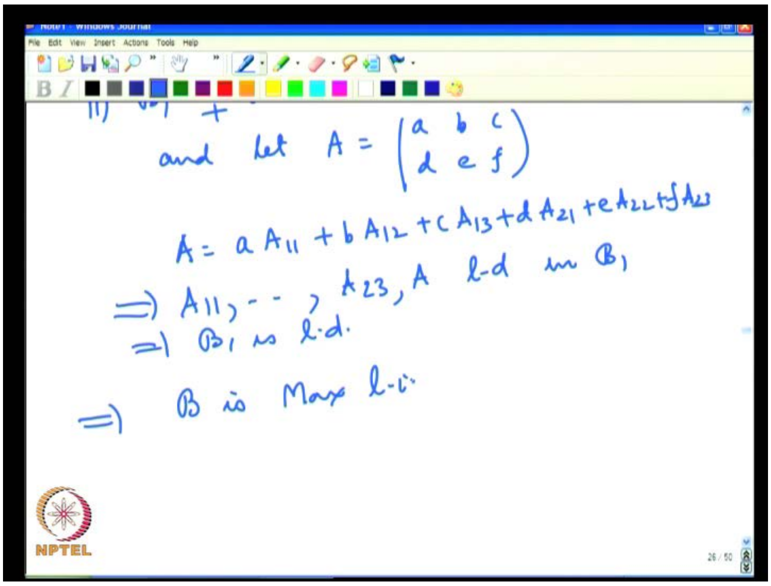

And as we have seen before, A can be written as a A 1 1 plus b A 1 2 plus c A 1 3 plus d A 2 1 plus e A 2 2 plus f A 2 3, which means the matrices, A 1 1 up to A 2 3 and A all linearly dependent and these are all in B 1. That says B 1 is linearly dependent. So, what we have is? B is linearly independent by 1 and B cannot be embedded in a linearly independent set or anything bigger than B must be linearly dependent. This is what is exactly meant by saying that B is maximal linearly independent set.

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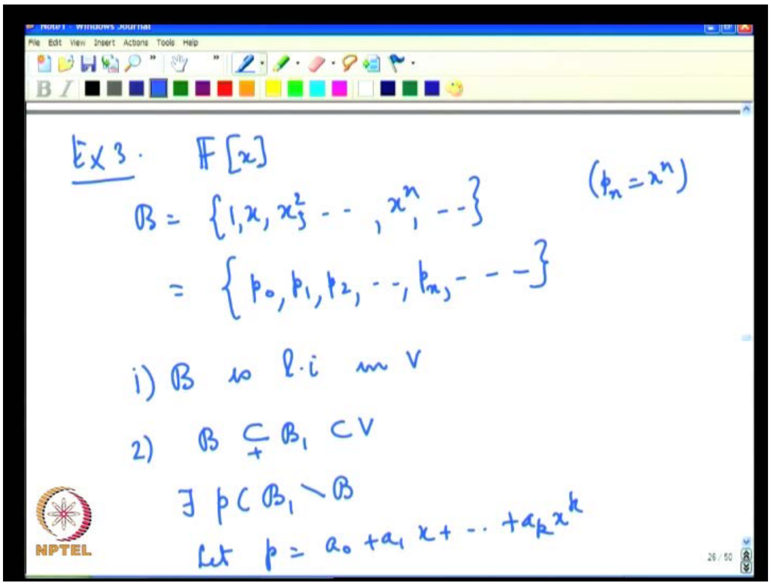

Now, let us now look at the example of the polynomials. Let us look at F x then we had the basis 1, x, x square, etcetera, x to the n we call p n as x to the power of n. So, the basis we can also write as p naught, p 1, p 2, p n, etcetera. Now, again as we have seen earlier many times, B is linearly independent in V. 2, suppose we have a set bigger than B 1 in V and again therefore, there must be somebody in B 1 who is not in B, they are all polynomials. So, there exist a p in B 1 minus B and since p is polynomial, so let p be equal to a naught plus a 1 x plus etcetera a k x power of k.

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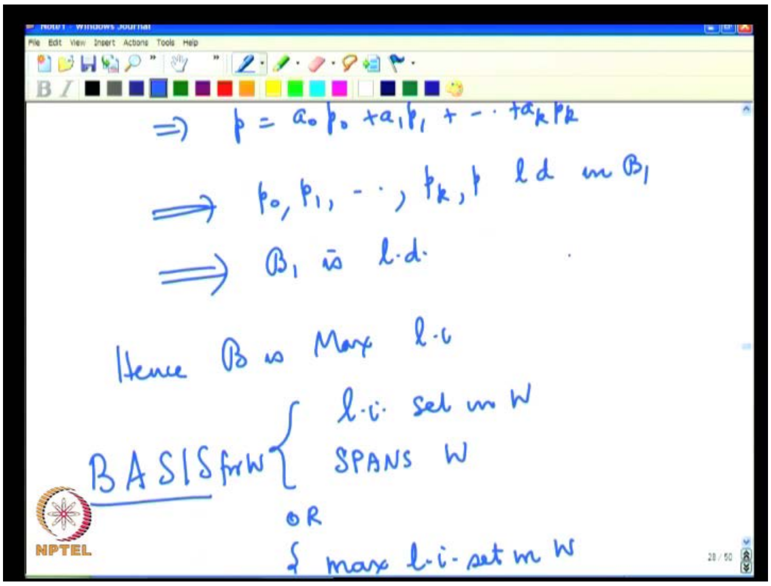

And that says, p is equal to a naught p naught plus a 1 p 1 plus a k p k. Now, it says p naught, p 1, etcetera, p k, p are all linearly dependent in B 1. And therefore, B 1 is linearly dependent. If your set has subset which is linearly dependent, the whole set must be linearly dependent. Therefore, B 1 is linearly dependent. And therefore, anything bigger than B is automatically a linearly dependent; and hence, B is a maximal linearly independent. Thus we see the basis is maximal linearly independent.

Thus the notion of the basis is in a sense an optimal sampling set. So, let us recollect that this is the most important aspects of basis. We can interpret as a linearly independent set spans, basis for V means spans V, so basis for W means it is linearly independent W and spans W; basis for V means linearly independent set in V and spans V; or we can look at it a equivalently as a maximal linearly independent set in W.

However, recall that our idea was getting into a basis was to get an optimal or reasonably meaningful sampling set. If you now look at the example of the polynomials, even though what to this notion of non redundant minimum requirement for spanning the space; this set B we got 1, x, x square, etcetera is an infinite set. So, we still have to deal with in certain vector spaces and infinite sampling set. The easiest cases are those or the simplest case are those, where we can get finite basis, because them we have a finite sampling set.

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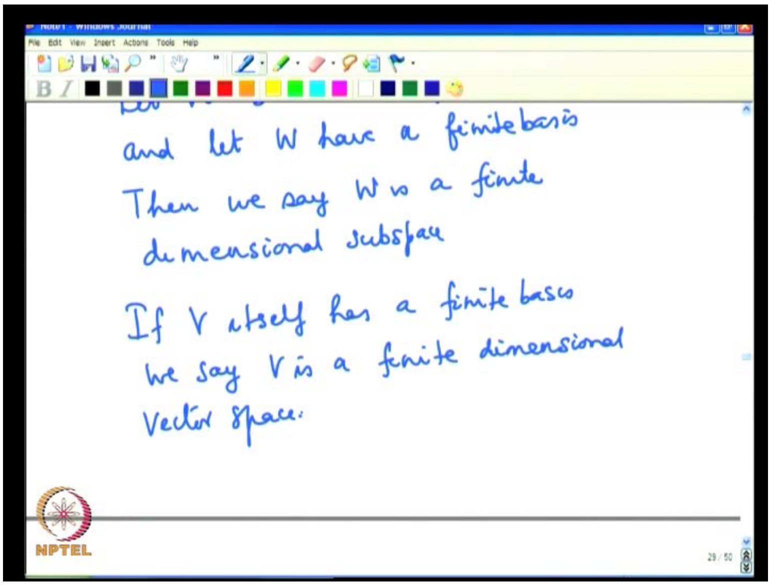

So, this leads us to the notion of finite dimensional subspaces. We have not yet introduced the notion of the word dimension, but we will come to it shortly, finite dimensional spaces. So, let W be a subspace of V and let W have a finite basis. So, we have a subspace which is having a finite basis then we say W is a finite dimensional subspace.

If V itself has a finite basis we say V is a finite dimensional vector space. So, we have a notion of finite dimensional vector space and finite dimensional subspace, we will be dealing mostly with this finite dimensional space in this course. Now, let us look at the finite dimensional space.

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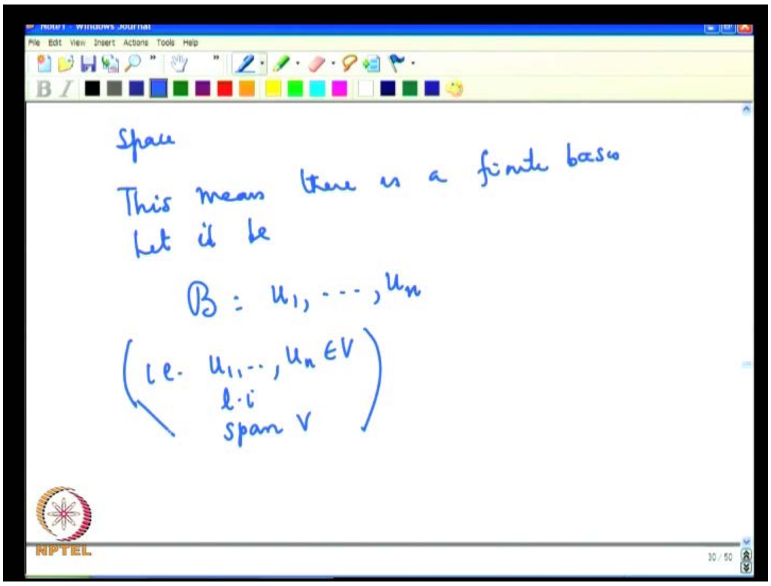

So, let V be a finite dimensional vector space. What does this mean? This means there is a finite basis, so let us call it let it be u 1, u 2, u n suppose there are n vectors which form a base. What does this mean? This u 1, u 2, u n are vectors in V they are linearly independent and they span V. That means every vector in V is a linear combination that is u 1, u 2, u n are vectors in V, they are linearly independent and they span V. This is what we mean, span V means everybody is a linear combination of these u 1, u 2.

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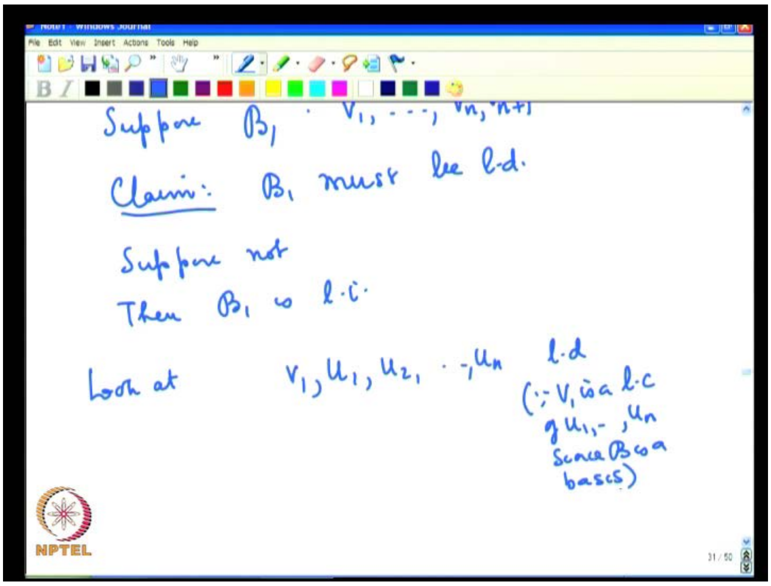

Now, suppose I have another set which has n plus 1 vectors, so I have a basis which consists of n vectors. Now, I am looking at any set which has n plus 1 vectors our claim is the following. We claim B 1 must be linearly dependent, why this so? Suppose not then B 1 is linearly independent. Now, look at here all these vectors u's, we now append the vector v 1. Since, u 1, u 2, u n is a basis, v 1 can be written in terms of u 1, u 2, u n; and therefore v 1, u 1, u 2, u n will be linearly dependent, since v 1 is a linear combination of u 1, u 2, u n since B is a basis; B is basis everybody must be a linear one.

Now, we have seen last time that if we have a linearly dependent set you can knock of redundant information from scanning from the left and get a subset which still spans the same space. Now, when we scan this v 1, u 1, u 2, u n by from the left, first we will be looking at v 1 we cannot knock of v 1, because v 1 alone is linearly independent being a part of a linearly independent set v 1. So, we cannot knock out v 1, so we will be knocking out some of the u 1's.

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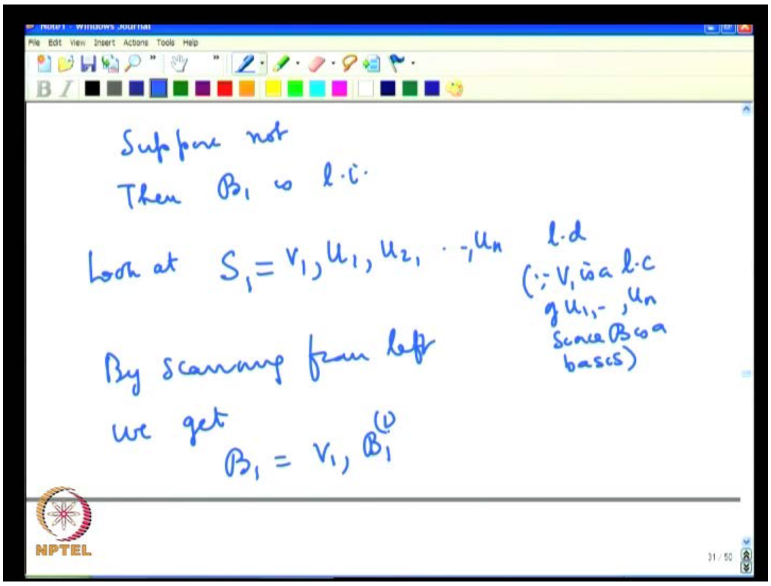

So, what we will get is? By scanning from left we get let us call this set as S 1, we get a set B 1 which will have v 1 and part of this B we will call it B 1 1.

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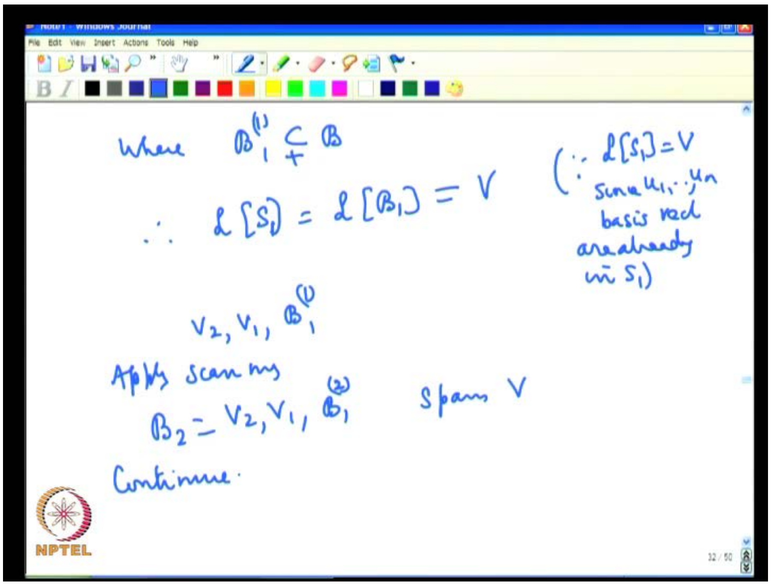

Where B 1 1 is contained in B and therefore, which is spans same space as S. So, therefore, we have L S is equal to L B 1. But, if you look at L S 1, the L S 1 already contains u 1, u 2, u n basis vectors who already span the whole space V. So, L S 1 will be V, so L B 1 will be equal to V; because, L S 1 is V since u 1, u 2, u n basis vectors are already in S 1.

So, we have got a subset of B 1 together with V 1 which forms a basis. Now, what we do is, we will look at appending V 2 to this and then apply scanning. As before we get V 2, V 1, B 2 1, another set will call it as B 2 which spans again V.

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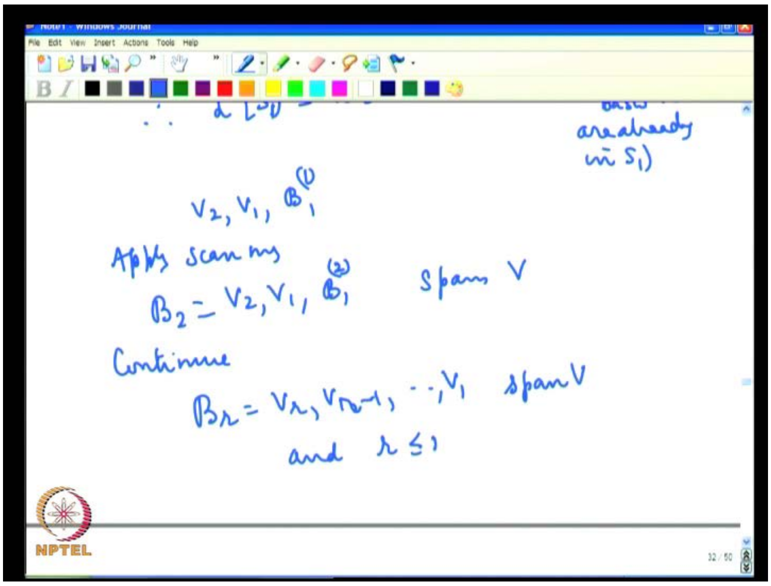

Continuing this way, we will knock of the vectors of u at each stage at least one will be knocked out. So, at the r th stage we would have got V r, V r minus 1, etcetera, V 1 and we would have knocked of all the vectors from B 1, span V. And since each stage, we knock out 1 u vector maximum stage we require is r is less or equal to n.

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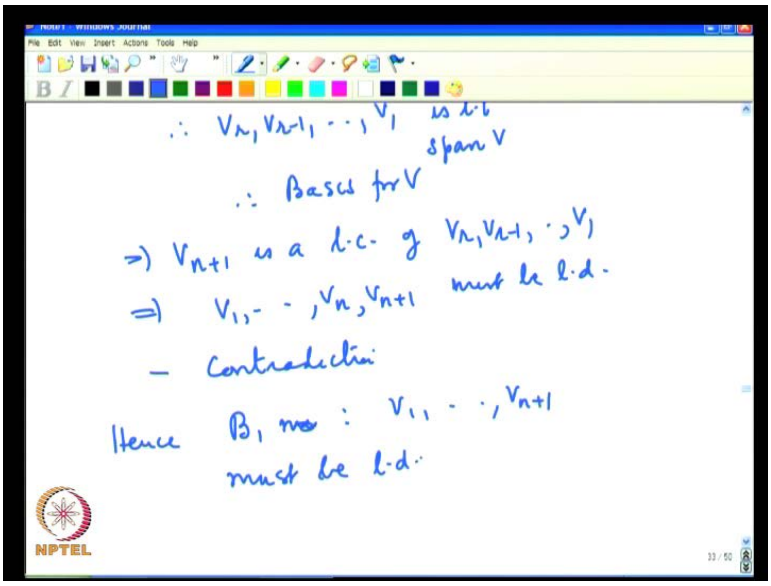

And therefore, we have V r, V r minus 1, V 1 is part of linearly independent set and they span V. And therefore, they form a basis for V. Since, they form a basis for V any vector must be a linear combination of that; in particular therefore, V n plus 1 which is out none of these values is a linear combination of V r, V r minus 1, V 1; which means that V 1, V 2, V n, V n plus 1 must be linearly dependent. Because, V n plus 1 is a linear combination of the previous values this must be linearly dependent.

This is the contradiction, because we started with hypothesis that suppose, V 1 is linearly independent. So, therefore, this is the contradiction. And hence, we must have B 1 must be set consisting of this V 1, V 2, V n plus 1 must be linearly dependent.

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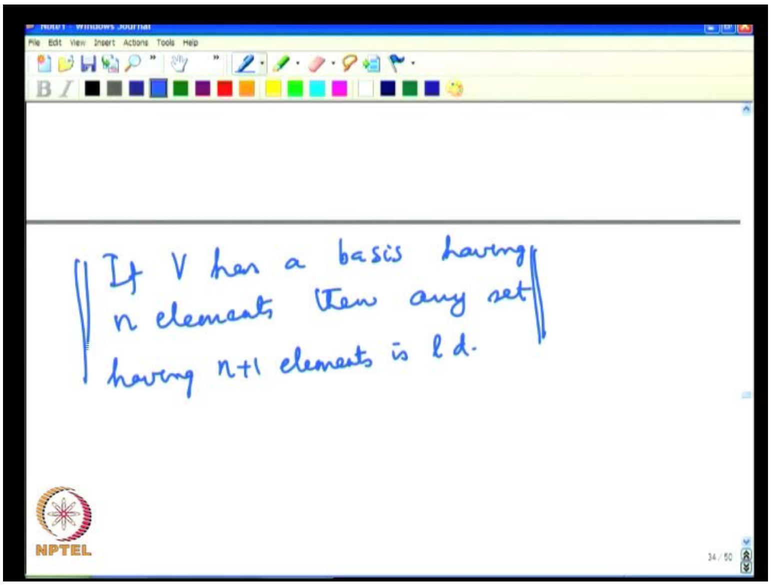

So, the conclusion that we have from this is that, if V has a basis having n elements then any set having n plus 1 elements is linearly dependent. This is a very important property, using this property we will able to show that once a vector space has a basis consisting of finite numbers of vectors. Then all basis will consist of finite number of vectors, and all basis have the same finite number of vectors. This will lead us to the notion of dimension, and we shall look at these aspects in the next lecture.