Advanced Matrix Theory and Linear Algebra for Engineers

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Lecture No. # 12

Linear Independence and Subspaces

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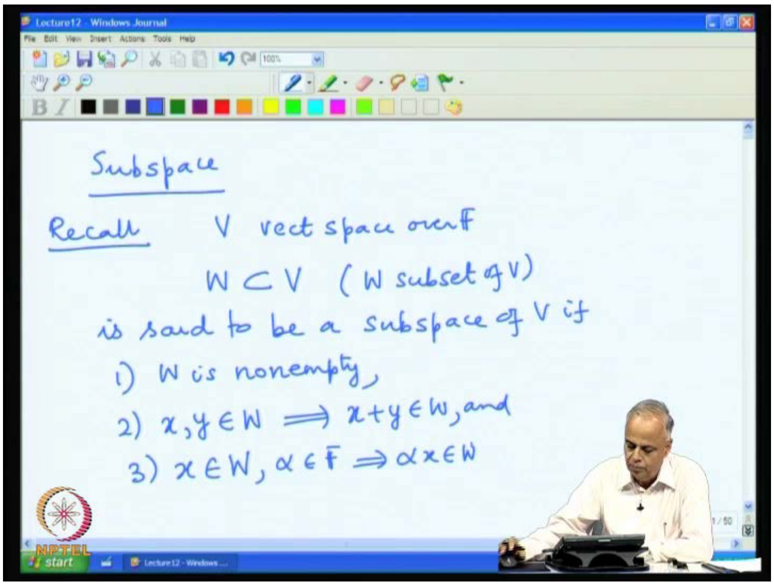

In the last lecture, we introduce the basic notion of the sub space of a vector space. Let us recall what is meant by a sub space. We have a vector space V over a field F, and we have a subset W subset of V is set to be a subspace of V. If one it is non empty should be non empty, two it is closed under addition that is x and y are in W implies x plus y is in W. And it is closed under scalar multiplication that is x is in W and alpha is any scalar then alpha x must be in W. These are satisfy then we say W is a subspace of V we have seen some example last time.

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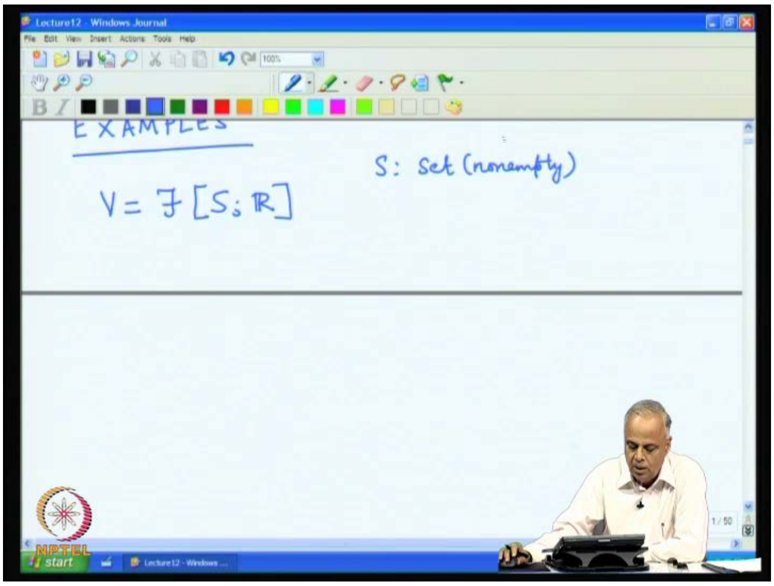

We look at more examples of subspaces the first example we had already seen in the last lecture. We shall see now another example let us look at the vector space V which is the collection of all functions from S to R. where S is an arbitrary set S is some non empty set. And we are looking at the collection of all functions real valued functions. Which associate with it every point in S a number in R. So we are looking at all such functions.

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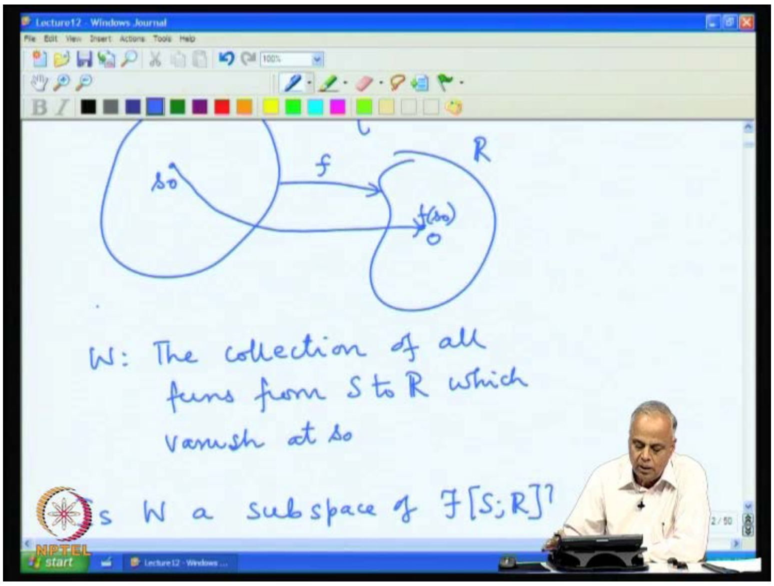

Now let S naught be a fixed point in S be a fixed point in S. we have S we have S naught a fixed point in S. Now what we want to look at is the collection of all the functions which make S to R so their functions from S to R such that at this point S naught the value of f must be 0, that is f S naught is equal to 0.

Here is R in the R the number 0 is there, we want to find we want all those functions f from S to R such that this S naught goes to 0 and that f. This f S naught must be equal to 0 so we are so W if the collection of all functions from S to R which vanish at S naught the collection of all functions from S to R which vanish at the point S naught. Now is W a sub space of the collection of all functions from S to R. In order to verify whether something is a subspace or naught we have to verify three things one whether it is non empty, whether is closed under addition and whether is closed under scalar multiplication. The first thing to observe is that the 0 function takes every point in the 0 value in particular the points S naught is map to the 0 value.

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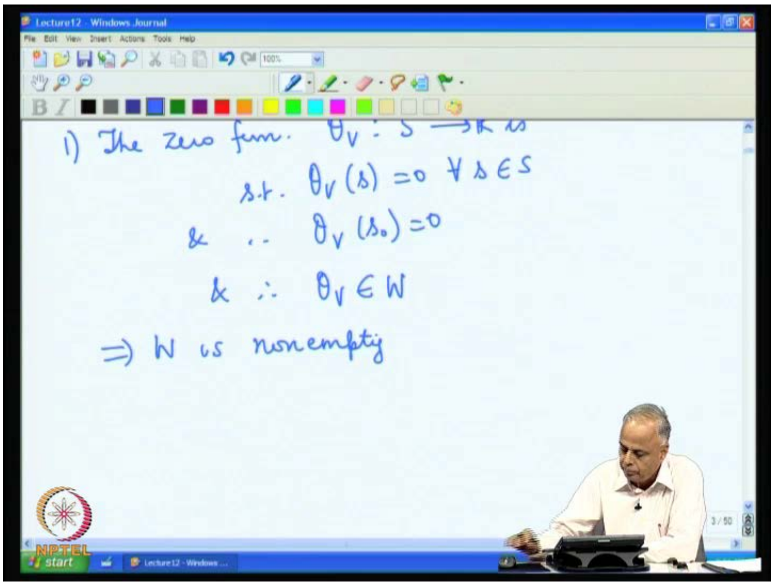

So the 0 function, which will denote by theta v maps S to R, is such that theta v of S is 0 for every S in S and therefore, theta v of S naught is 0 and theta v is the function with maps S naught to 0 and therefore, theta v belongs to this collection. Because this collection is the collection of all function which maps S naught to 0 and theta v is one such function and therefore, W is non empty. It has the 0 function is collection.

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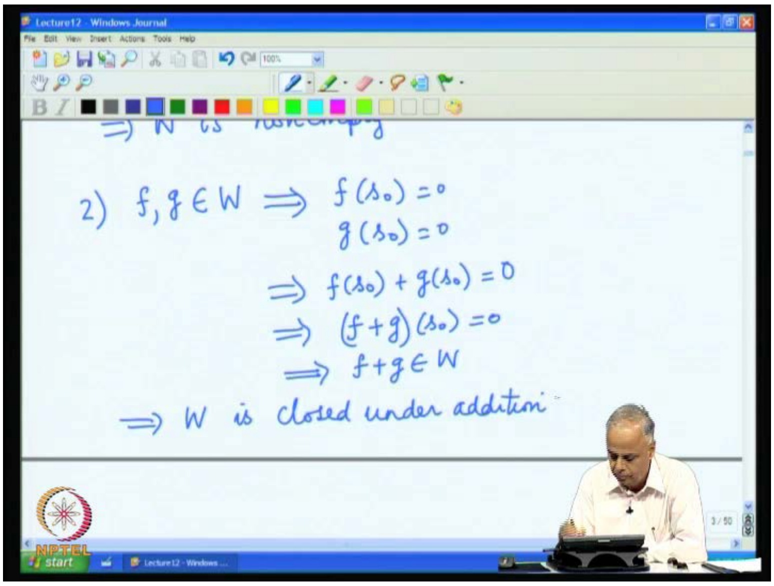

Next suppose we had 2 functions in this collection that f is in W means it has the value 0 at S naught, and the fact that g has using W means g is 0 at S naught. This means f S naught plus g S naught is 0 but, by the law of addition of functions this is the same at the value of the function f plus g at S naught this S the function f plus g also vanishes at S naught and therefore, f plus g is also in W. Thus W is closed under addition.

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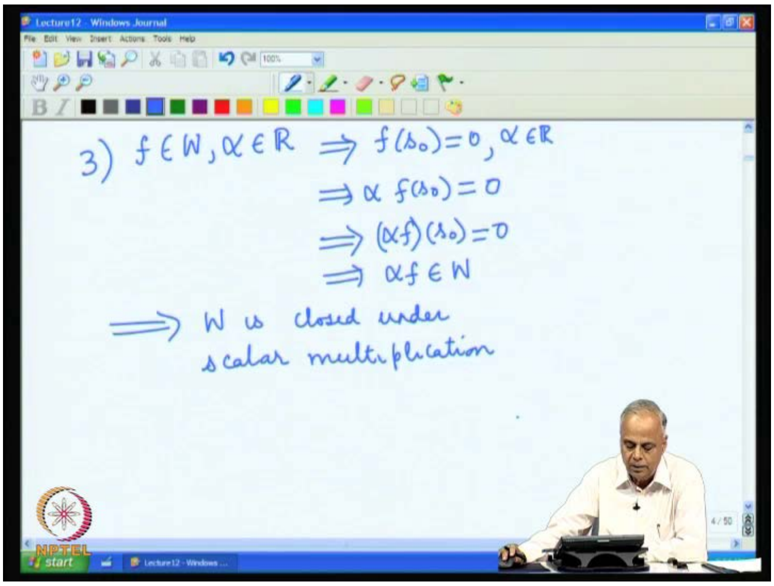

The next thing we have to verify is that W is closed under scalar multiplication. Suppose we have a function f in W and alpha is any scalar now our scalars are R because f is in the capital f of all functions to R in the vector space over R so scalars must be R. Take a function f in W and a scalar alpha in R now since f is in W it must take the value 0 at S naught and of course, we are given alpha S naught. And therefore, if I multiply f S naught by alpha, I will have alpha time 0 which is again 0. By the definition of scalar multiplication of functions this is the same as the value of the function alpha f at the point S naught is alpha times the value of f at S naught which is 0 of from above so we get alpha f text value 0 at the point S naught. This means alpha f is also qualifies to be in W. Thus whenever f is in W alpha is in R alpha f is in W that says W is closed under scalar multiplication.

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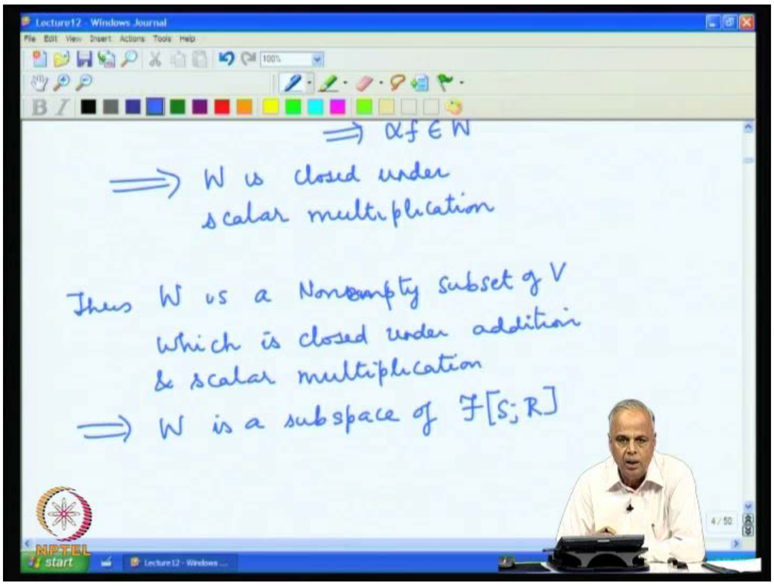

We have W is non empty, it is closed under addition and it is closed under scalar multiplication. W is a non empty subset non empty subset of V which is closed under addition and scalar multiplication. These are the qualifications was something to be a subspace of V and that says W is a subspace of a vector space was F S R. The collection of all functions from S to R which vanish at that point S naught is a sub space.

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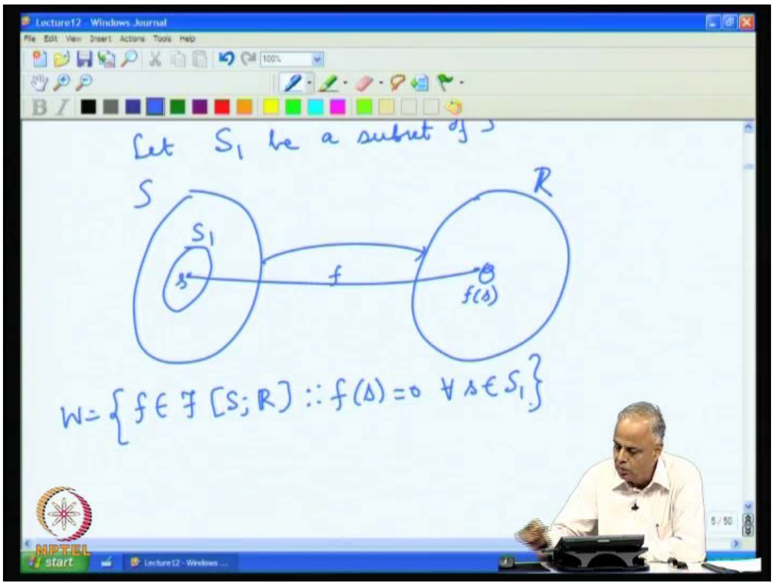

Now we can generalize this we can replace the point S naught by a subset of S so let S 1 be a subset of S. We have S and we have a subset S 1 inside that, and then we look at all the functions which map S to R. Function which map S to R such that if we take any point inside this S 1 it must go to 0.

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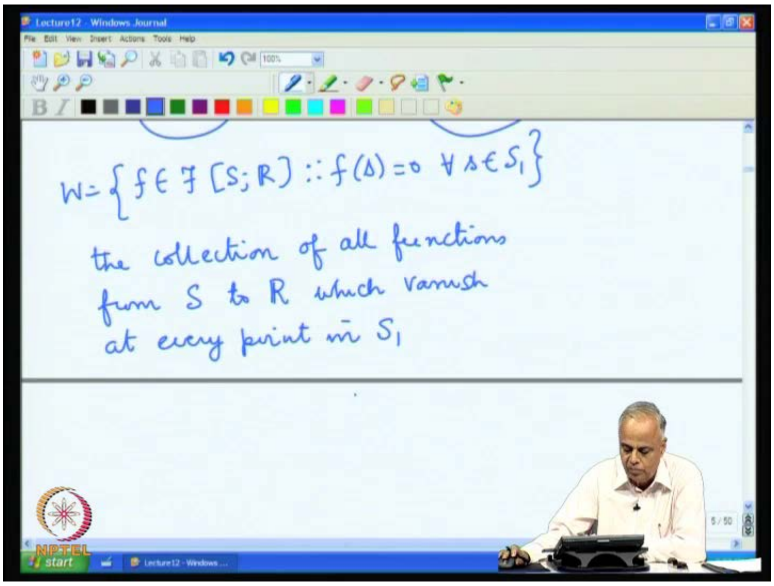

We are looking at all those functions f such that f S is 0 for every S in S1 .It is f S is 0 for every S in S1. It is a collection of all functions the collection of all functions in what is had a functions are these are from S to R and what else there must vanish at every point in S1.

The subset at any point in that subset the value of the function must be 0. We have taken instead of a single point S naught we have taken an arbitrary collection or arbitrary subset of a S.

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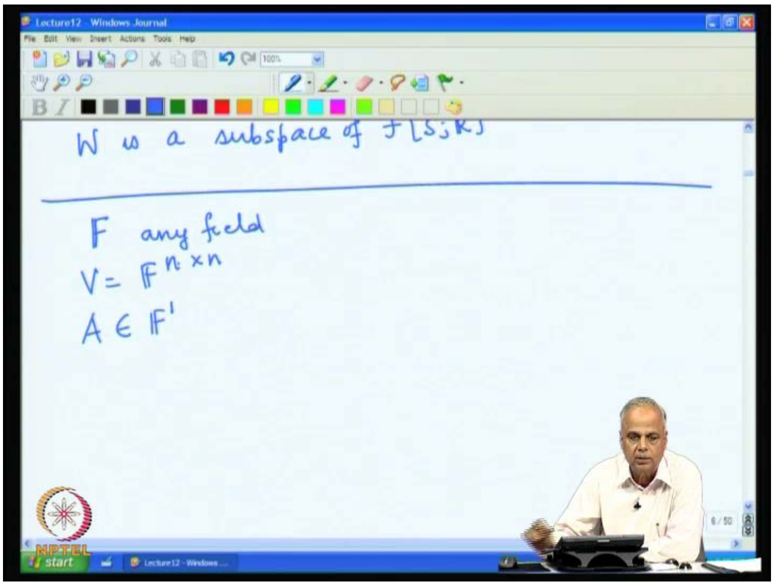

And looked at all those functions which and as before this collection W is a subspace of F S R. You look 1 more example now from the context of matrices. Let us take F any field and let us take the vector space V to do the collection of all m by n matrices is F m cross n means the collection of all matrices is over F. Now suppose a belongs to this vector space to make let us take all square matrices. Let us take the collection of all square matrices of size n by n. Take a matrix which belongs to this space.

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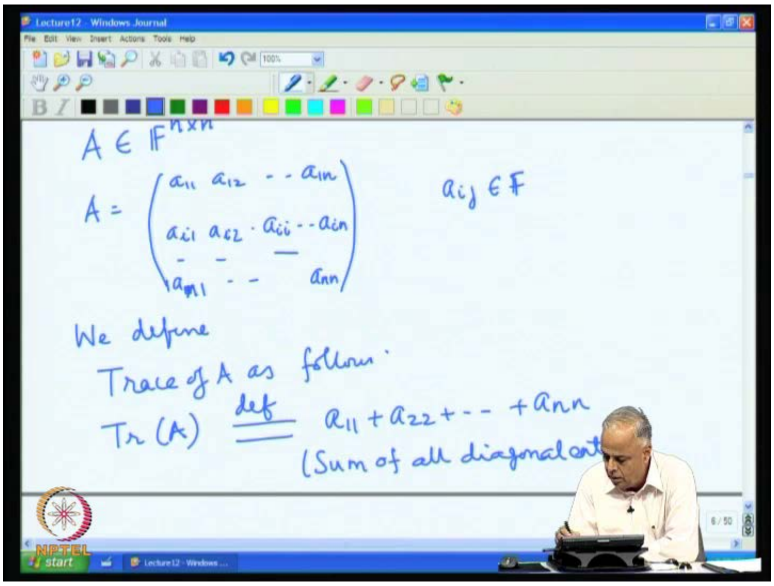

How does it look like it is a matrix a 1 1 a 1 2 a 1 n the i throw will be a i 1 a i 2 a i i and so on a i n and the last row will be a n 1 etcetera a n n. Where the a i j s are all from the field f. We take an n by n matrix over the field f. For such matrices we define trays of a as follows, we denote the trays of a by t R A.

The trays of a is defined this is the definition to be the just add all the diagonal entries a 1 1 plus a 2 2 plus a n n. It is just a 1 1 plus a 2 2 plus etcetera plus a n n. Sum of all diagonal entries. It is we will look come across trace later when we deal with matrices and non homogenous systems and so on and so forth.

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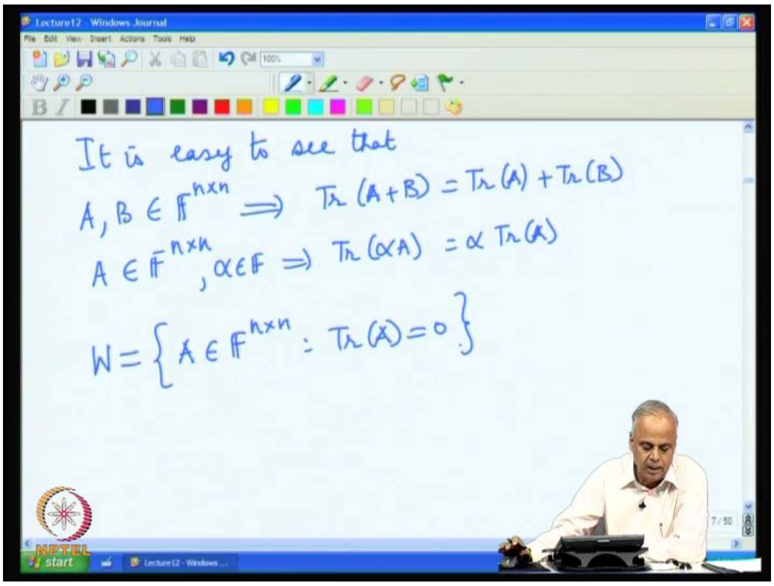

Let us look at this trace of A it is easy to see that if we take two matrices A and B in F n cross n, then the trace of A plus B is the same as trace of A plus trace of B. This is because when you add two matrices the diagonal entries of the sum is the same of the sum of the corresponding diagonal entries similarly, if A is an n by n matrix and alpha is any scalar, then that trace of alpha A is just alpha times the trace of A because when you multiply a matrix by a scalar that diagonal entries all get multiplied by alpha and therefore, the trace gets multiplied by alpha.

We have this vector space of n by n matrices over F and may now look at all those matrices which are n by n for which the trace is 0. Look at all those n by n matrices is for which the trace is 0.

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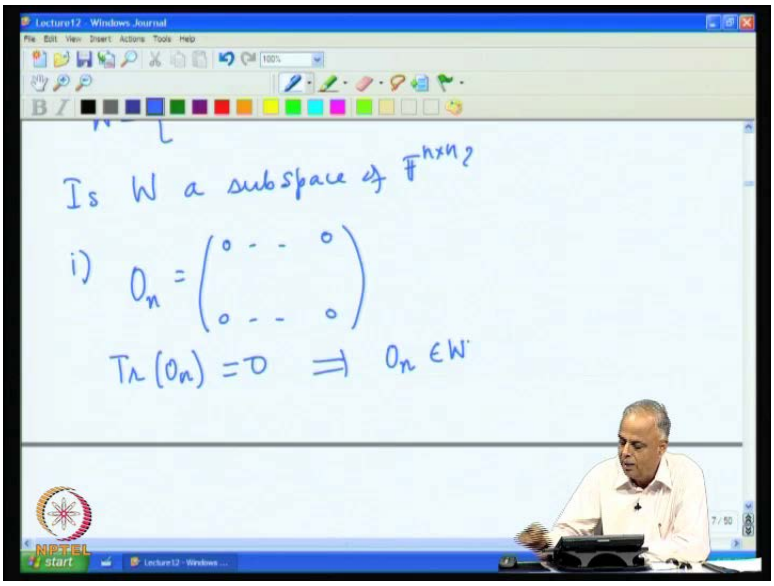

Let is W a sub space of F n cross n. Again in order to verify whether something is a subspace or not, we have to verify 3 things the fact that W is non empty, the fact that W is closed under addition and the fact that W is closed under scalar multiplication. Check each one of this the first thing is if you take the 0 matrix the n by n 0 matrix. All the entries are 0 and therefore the sum of the diagonal entries we certainly be 0 it has trace 0 and the moment a matrix has trace 0 it qualifies to be in W and therefore, the 0 matrix belongs to W which says W is non empty. There is the first thing that we have to verify that to see whether something is a subspace or not.

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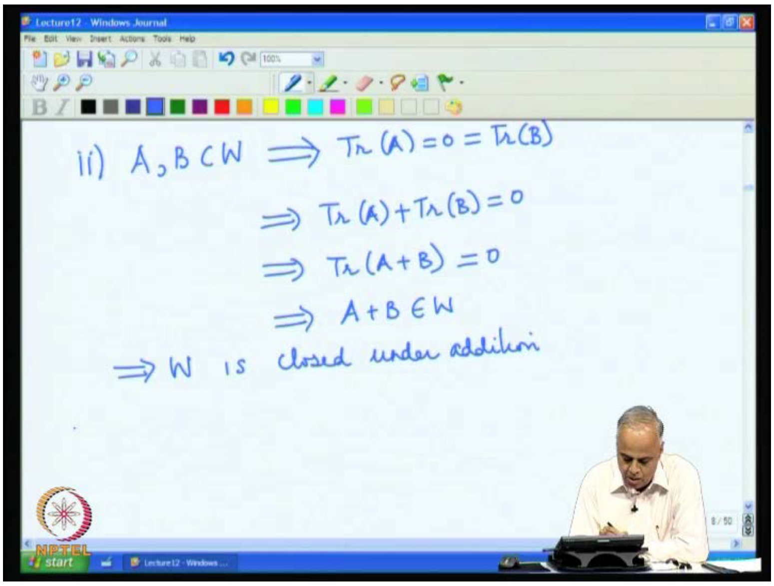

The second thing that we have to verify is whether it is closed under addition. If A and B are in W then they are in W mean the trace of A is 0 and the trace of B is also 0. And a trace of A and trace of B are 0 there is sum trace of A plus trace of B V also be 0, But we just now observed that the trace of A plus trace of B is the same as trace of A plus B so trace of A plus B is 0 and therefore, A plus B is a matrix is a trace of 0 and therefore, A plus B qualifies to be in W. Thus if A and B are in W A plus B is in W so W is closed under addition

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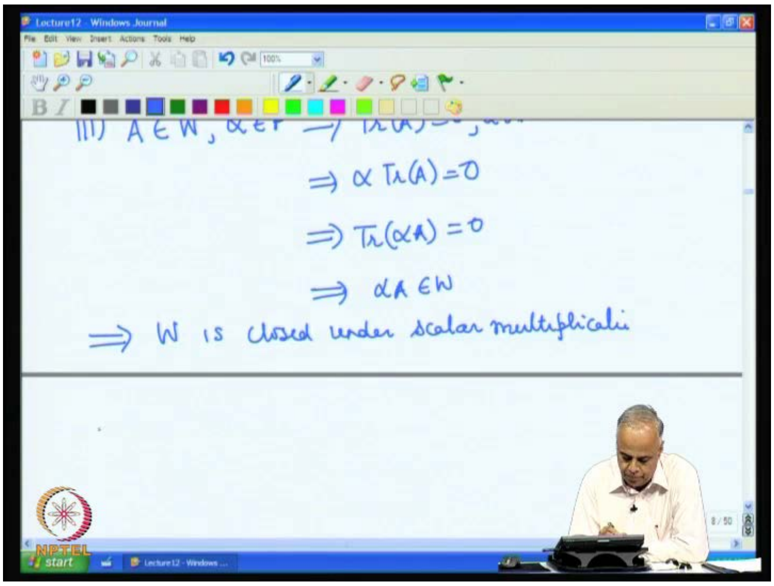

The next thing that we have to verify is whether closed under scalar multiplication so, let us take a in W and a alpha A scalar the fact that a is in W again tells us trace of A is 0. And we have alpha is in f and therefore, alpha times trace of A will be alpha times 0 which is 0 and again we just now observed that alpha times trace of A is the same as trace of alpha times a and therefore, that is 0 for alpha times a is a matrix. It has trace 0 and hence it qualifies to do in them so that says W is closed under scalar multiplication.

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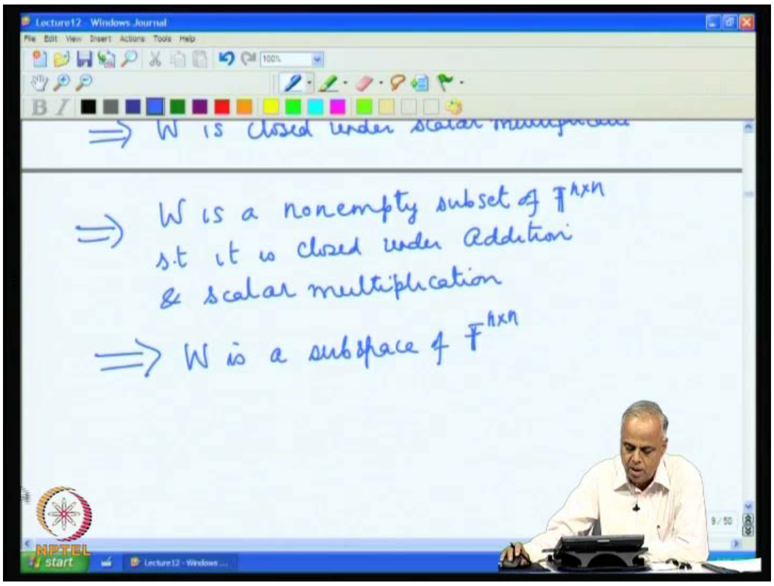

Thus we have seen the W all these imply W is a non empty is the first requirement which a subset of F n cross n the set of all n by n matrix is such that it is closed under addition and scalar multiplication. And these are the qualities that are required for something to be a sub space and hence W is a sub space of F n cross n. This is one example of a sub space in the vector space of matrices.

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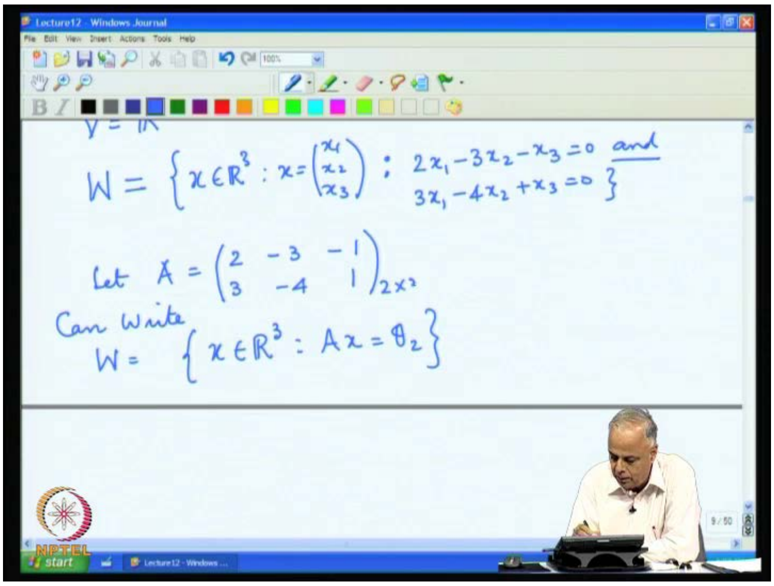

Let us look at one more example take V to be R3 for usual space again the standard space R3 with which we have we start all examples and, then now let us look at W to be the collection of all those vectors in R3 that means x is x 1 x 2 x 3 such that say 2 x 1 minus 3 x 2 minus x 3 is 0 and 3 x 1 minus 4 x 2 plus x 3 is 0.

We are looking for all those vectors is component x 1 x 2 x 3 are such that these 2 conditions are satisfy we can look at W as follows so let a be the matrix 2 minus 3 minus 1 3 minus 4 1 then these 2 conditions boiled out to saying A x is the 0 vector. So, we can write W as the set of all vectors in R 3 such that A x is theta a is a 2 by 3 matrix. W can be thought of now as the set of all x in R 3 said that A x equal to theta 2

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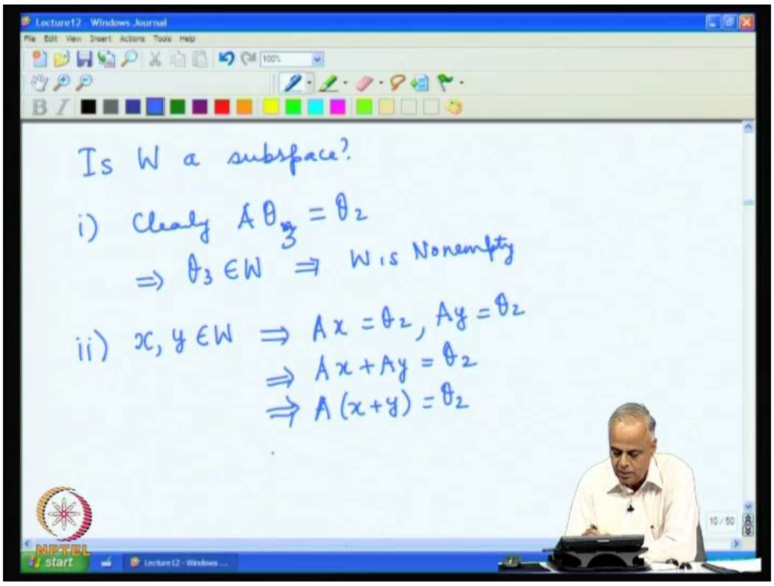

Is W a sub space. Again we have to verify three things that it is non empty, which closed under addition and it is closed under scalar multiplication. Clearly a times theta 3 is equal to theta 2. A is a 2 by 3 matrix theta 3 is A 3 by 1 0 vector. So, the result will be a theta 2 2 by 1 0 vector which implies that theta qualifies to be in W. Which implies that W is non empty. Next thing to verify is that W is closed under addition.

Let us take x and y in W the fact that x is in W means a must nullified. It similarly, y is in W means a must nullified, it that is it A x must be theta 2 and A y must be theta 2 if A x and A y are theta 2 A x plus A y must be theta 2 plus theta 2 0 vector plus 0 vector is again the 0 vector. Matrix multiplication is distributed so, this can be written as A x plus y is theta 2.

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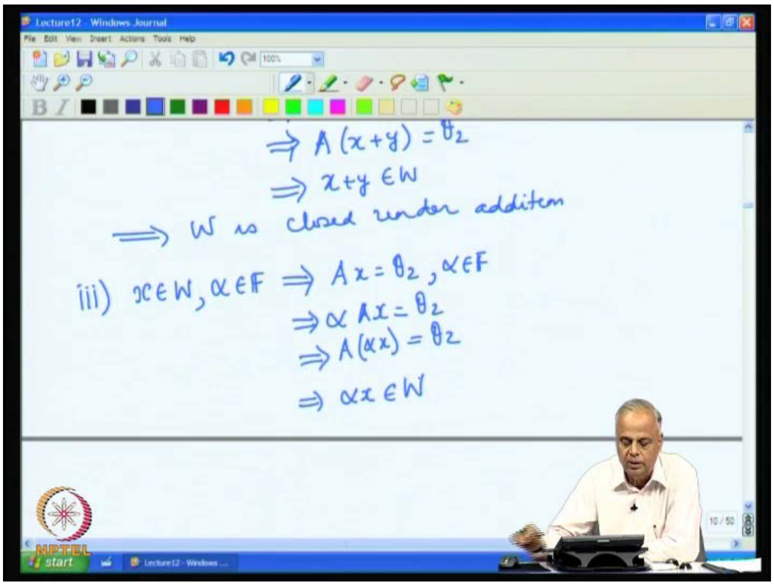

Which says that x plus y also get a nulled by a so it qualifies to be in W. We have W is closed under addition. The next thing we have to verify is the W is closed under scalar multiplication. Take x to be in W and alpha to be any scalar and the fact that x is in W says A x equal to theta 2 and alpha is in F but, if A x is equal to theta 2 alpha times A x will be alpha times theta 2 which is theta 2. Again in matrix multiplication constants can be moved and in and out of the multiplication so, A times alpha x equal to theta 2 let us says alpha x also gets a nulled by A. so, alpha x also qualifies to be in W

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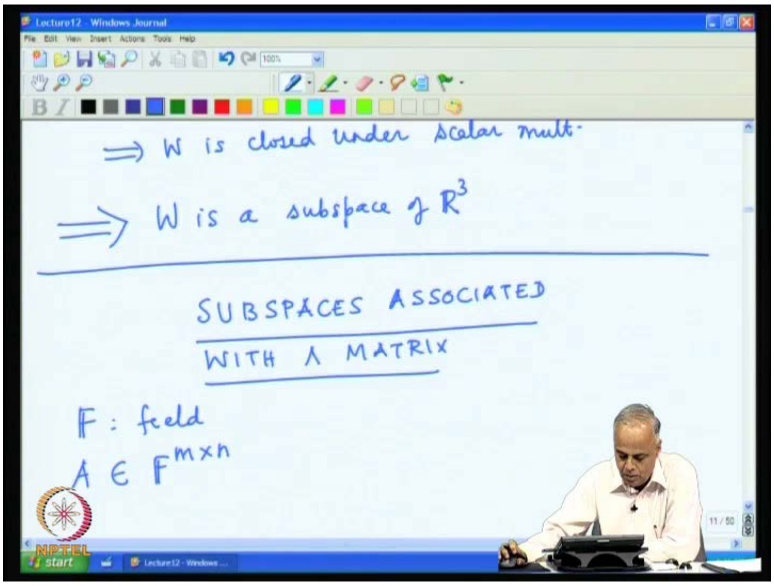

Therefore, W is closed under scalar multiplication. We have saying W is non empty is closed under addition and it is closed under scalar multiplication all this say the W is a subspace of R3. Now we can generalize this leads us to certain subspaces associated with the matrix. Let us going to look at the number a number of sub spaces associated with the matrix and these are the sub spaces that are going to help us in the analysis of the various questions that we raised about a matrix the systems of equations, diagonalization, decomposition, etcetera., We begin with F be a field and we look at a matrix which is in m cross n now we can generalize the previous example and then instead of R3 now we look at R n and look at all those vectors which that a nulled by A.

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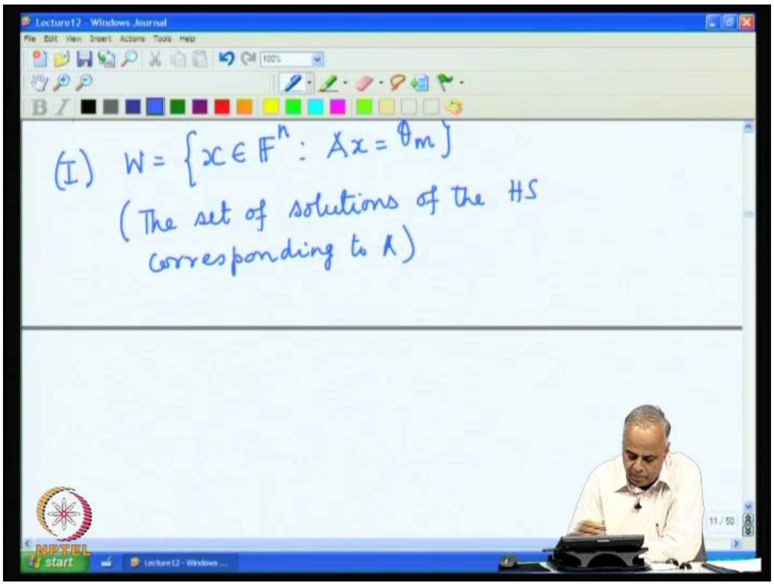

We look at the phase sub space that connected with the matrix A. If the following we look at W to be all these vectors in F n such that A x equal to theta M. This is the same as looking at all the solutions of the homogenous system corresponding to the matrix. This is the set of solutions corresponding solutions of the homogenous system corresponding to the matrix A. Hence, we look at the all these solutions of the homogenous system. We will see that this is the sub space. It will be a sub space of what it will be if this is the collection of vectors from F n? So, this is a subset of F n so it will be if totality a subspace it will be a sub space of F n.

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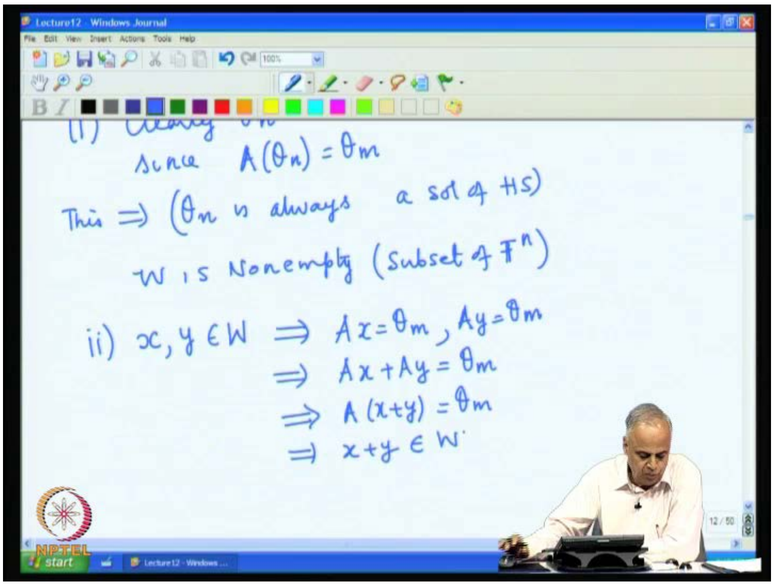

So, clearly the 0 vector in F n a solution of the homogenous system so clearly theta m belongs to W since, A theta n is theta m remember a is an m by n matrix, if you multiplied by the n by 1 0 vector you get the n by 1 0 vector this implies theta m is always this will because theta n is always this will because theta n is always in a solution of the homogenous system. Now this implies the W is non empty. Thus non empty and if it has to be a subspace it has to the closed under addition and scalar multiplication let us look at x and y in W what is that mean? Actually we must say W is a non empty subset of F n. Because all F be a for the definition of W we are considering some vectors in F n which has the additional property. W is the subset of F n which non empty subset of F n, now suppose x and y are in W because x is in W A x must be theta m because y is in W A y must be theta m.

Now because both are theta m there sum will be theta m plus theta m which will be theta m matrix multiplication is distributed therefore, x plus y is theta m that says x plus y belongs to W this proof is exactly allow to the last example we sum.

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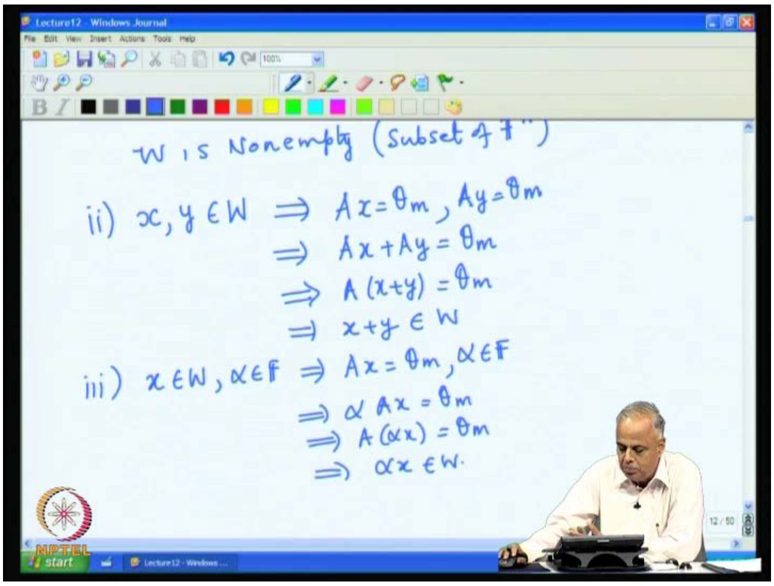

The next thing that we have to verify is the fact that W is closed under scalar multiplication so x is in W alpha in F so you get through this as in the previous example is says A x equal to theta m because x is in W and alpha is in F. That says alpha times A x is equal to theta m alpha times theta m is theta m and that says a of alpha x is theta m, and that says alpha x qualifies to be in W so thus we see that W is a non empty subset of F n. which is closed under.

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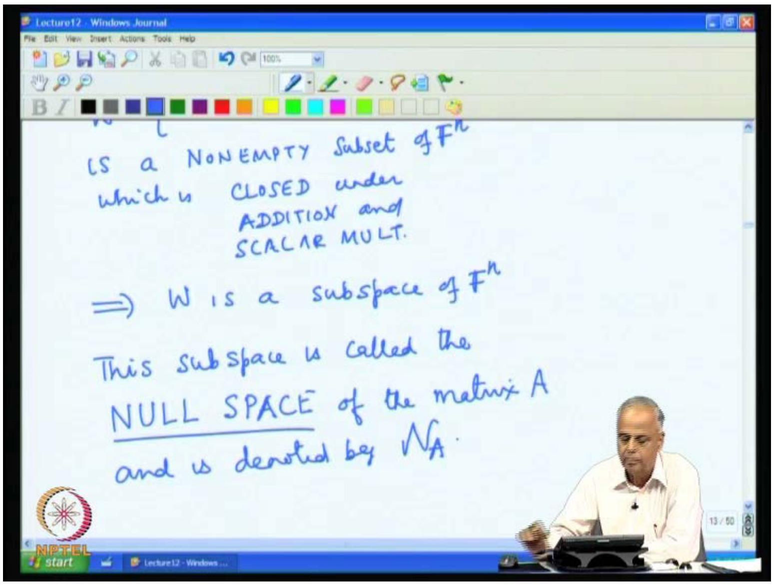

Here W is equal to the set of all x in F n such that A x equal to theta m is a non empty sub set of F n. Which is closed under addition and scalar multiplication and that is what qualifies something to be a subspace of F n which implies W is a sub space of F n. This subspace is called the null space of the matrix A. This subspace is called the null space of the matrix A and is denoted by n A n stands for null space.

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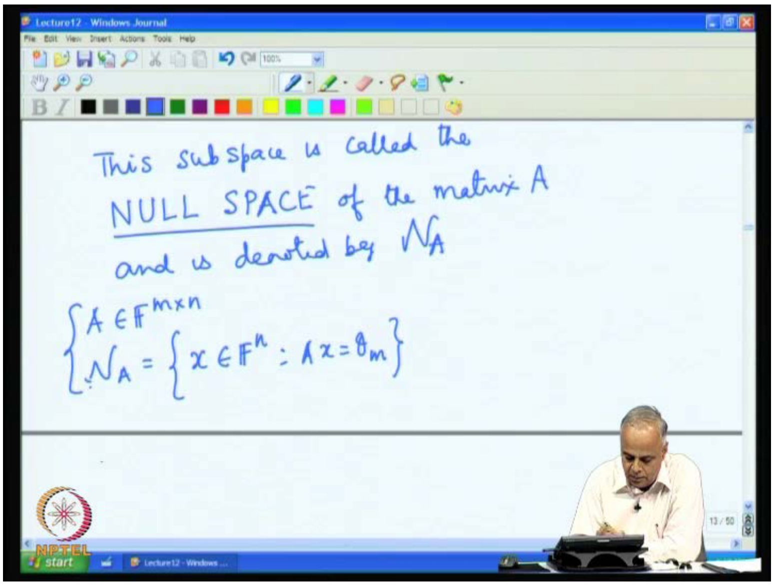

What do we have n a is equal to the set of all x in F n such that A x equal to theta m so the null space is nothing but, the set of all solutions of the homogenous system corresponding to the matrix A so if a is in F n by n matrix

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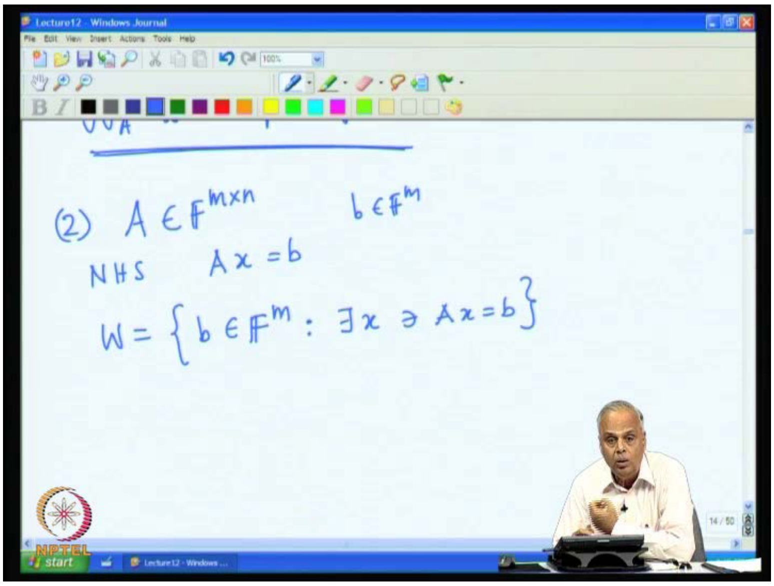

The null space is a sub space of F n so that is the first sub space that we have seen associated with the given matrix A. Now we shall look at another subspace which is associated with the matrix A. Again we have a F m n when we were solving non homogenous systems. A x equal to b we found that we may have a situation where for some b we will not have a solution for some b we have a solution. So, what we will like to do is collect all those b which are good in the sense we can solve the system A x equal to b so, we now look at W. Now where is this b this b is in F n because a is F n by n or 1 by n matrix x is n by 1 so, the product will be m by 1 so, we are looking at all those b in F m such that there x is a solution x satisfying A x equal to b. We are collecting all those b for with a system A x equal to b is consistent.

In the language of non homogenous system we are looking at all those b in F m for which the system A x equal to b is consistent.

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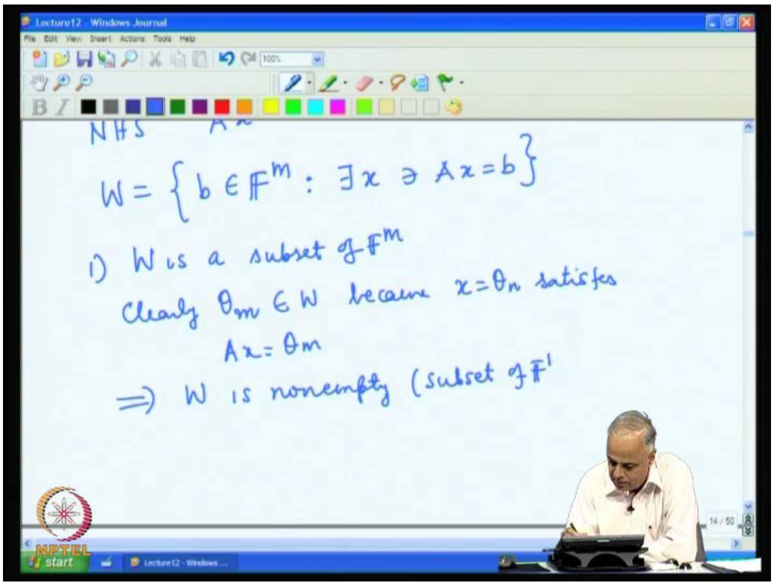

Now W is a subset of F m is it non empty. Clearly when we take b equal to theta m we have the homogenous system and we know the homogenous system always at the trivial solution therefore, clearly theta m belongs to W because x is equal to theta n satisfies A x equal to theta m. Therefore, W is non empty. Non empty subset of F m will keep remaining that.

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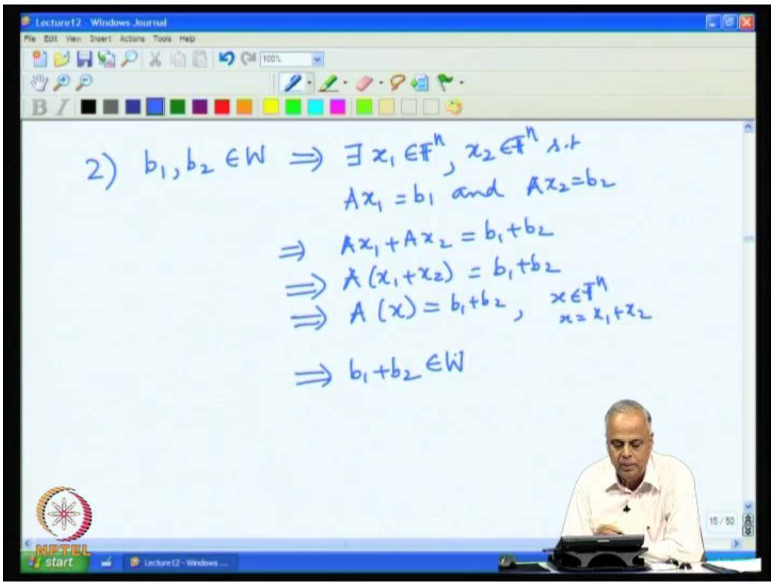

Then we have to look at whether it is closed under addition. Suppose b 1 and b 2 belong to W what does it meant to say that b 1 belongs to W, that mean the system A x equal to b consistent that mean there is the solution x such that A x equal to b 1. There exist an x 1 in F n similarly, for b 2 there is an x 2 in F n such that x 1 is the solution of A x 1 equal to b 1 and x 2 the solution of A x 2.

Now that says A x 1 plus A x 2 is b 1 plus b 2. Matrix multiplication distributive tells us A x 1 plus x 2 is b 1 plus b 2. Here x 1 is in F n x 2 is in F n and therefore, x 1 plus x 2 is in F n let us call it as x is equal to b 1 plus b 2 x belonging to F n x equal to x 1 plus x 2. therefore, it says the system A x equal to b 1 plus b 2 has a solution and that means b 1 plus b 2 is qualified to be in W therefore, W is closed under addition

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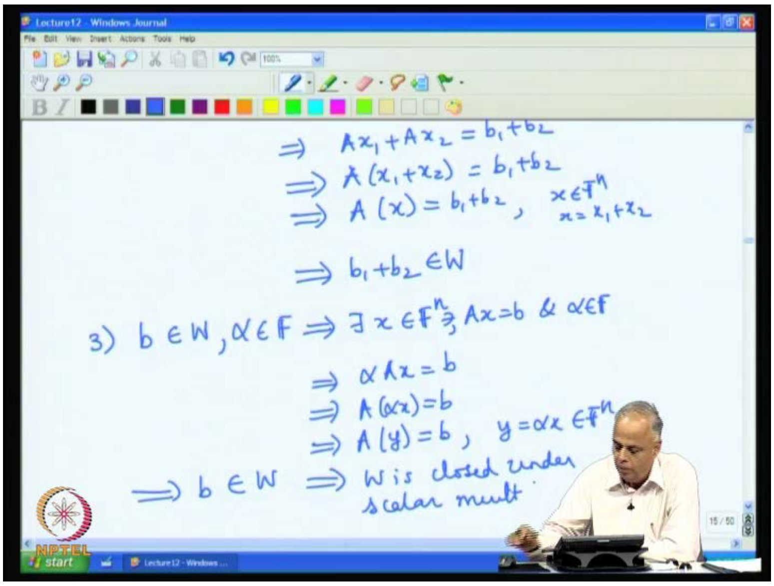

Similarly, we look at closed with respect to scalar multiplication suppose we used a consistent notation. Suppose b is in W and alpha is in F the fact that b is in W guarantees that there exist an x in F n such that A x equal to b and of course, where alpha is in F since A x equal to b we see that alpha times A x equal to b which in terms as a times alpha x equal to b. Now x is in F n alpha is in F and therefore, alpha times x is also in F n.

Let us call it as some y A y equal to b y equal to alpha x belongs to F n which means the homogenous system A x equal to b also as a solution namely the solution y therefore, b qualify to be in W which says W is closed under scalar multiplication.

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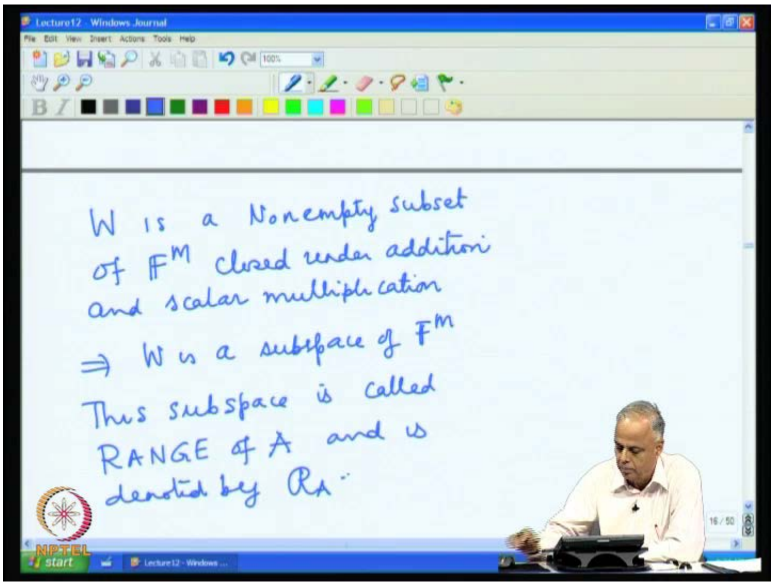

This what do we get W is a non empty subset of F m closed under addition and scalar multiplication. Therefore, it qualifies to be a subspace of F m. That says W is a subspace of F m. And this subspace is call the range of the matrix a this subspace is called the range of A and is denoted by R A.

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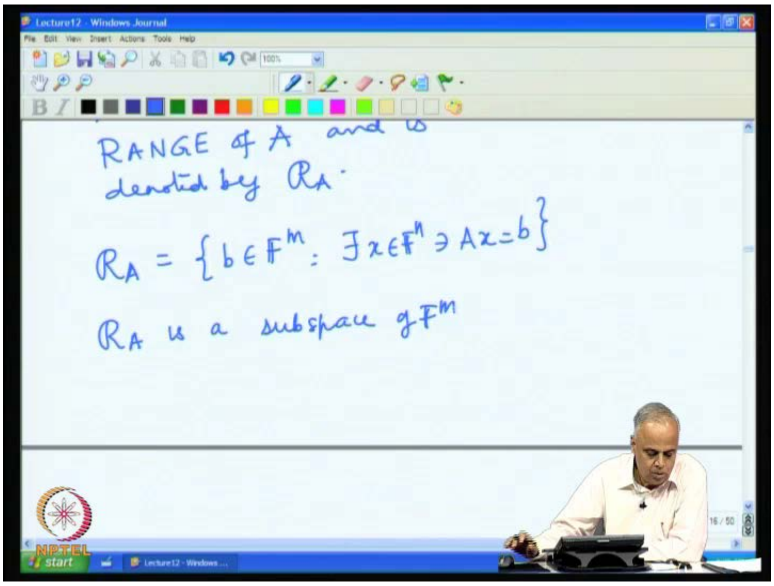

What is R A? R A is the set of all b in F m such that there exist an x in F n satisfying A x equal to b it is all those b for which A x equal to b is a consistent non-homogenous or homogenous b 0 is homogenous system. R A is a subspace of F m. So far, we have seen 2 sub spaces associated with the matrix

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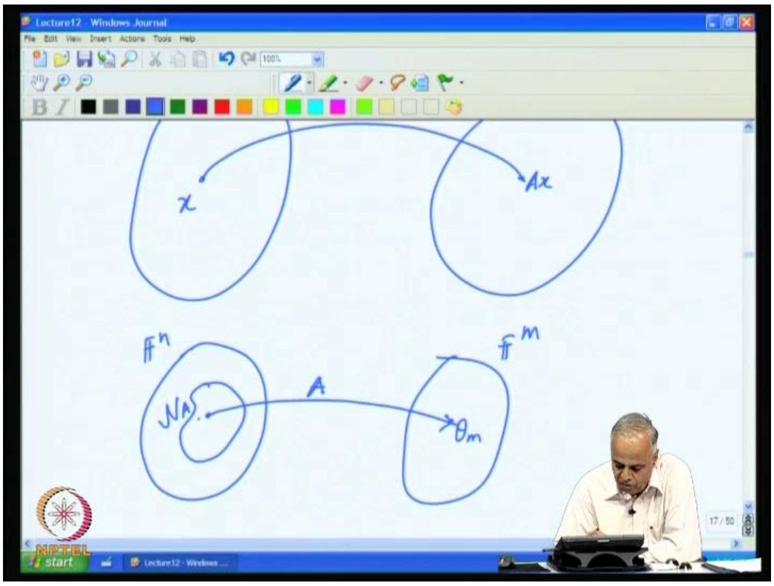

Let us say A is again F m n we have F m F and we have F m here. What matrix does says it? Takes a vector x in F n and maps it to a vector A x in F n now our null space in a is collection of all the vectors here. Which go and fall at the 0 vector here so how does the how do we get the null space to get the null space what we do is we look at F n? We look at F m in the F m there is the 0 vector and what we do is we collect all those follows we get go and fall their A times that becomes 0. This part is what is the null space of A so the null space of A if the collection of all those vectors in F m which get map to the 0 vector. The null space of A is the subspace of F n because we are looking at those vectors in F n is go and fall them on the other hand it get the range

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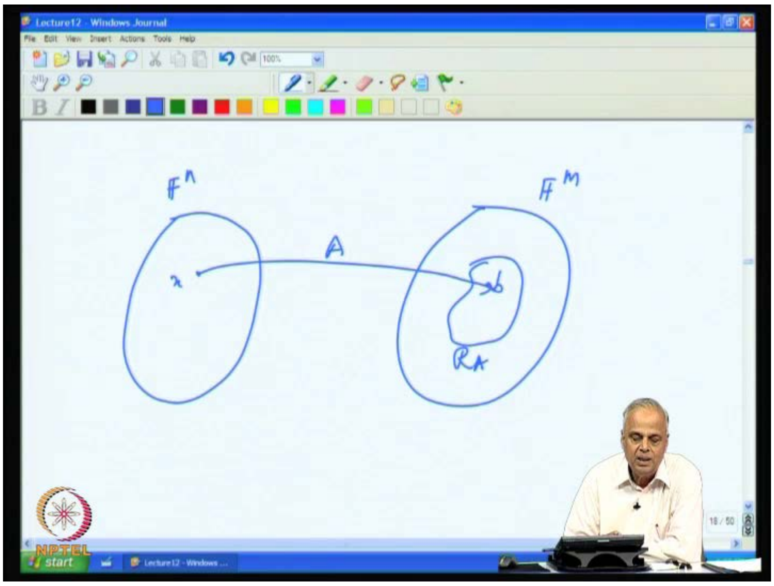

What we do is if we look at F n we have the F n here. Now with colour all this vectors b for which I can find in x A which goes a false there. Hence, we say b is the value taken by a at some point x if thing of a is a function from F m to F m b must be the value of the function A at some point x. So that this as the range of a so the range of A is sub space F m where as the nulls space of A is a sub space of A.

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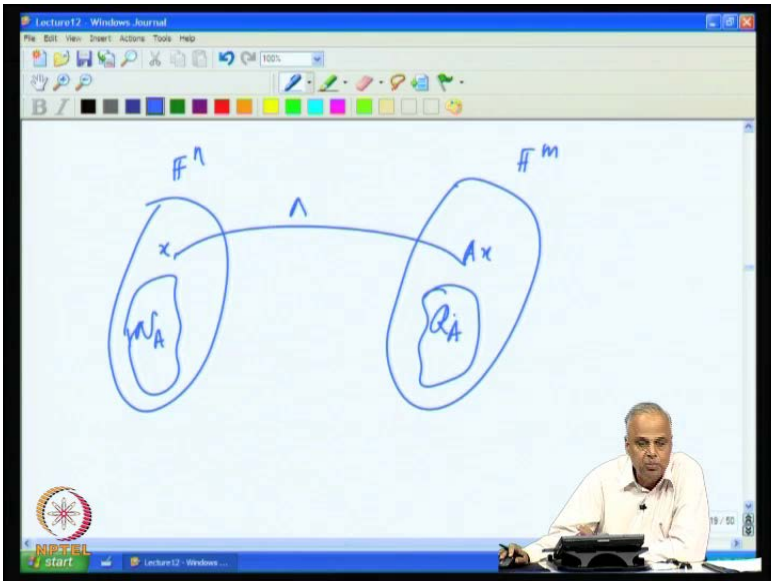

So, the net result is we have F n, we have F m in general a takes a vector x A R to A x and some part of it is the one that take in to all the 0 vector and some part of it here is the range of A that is those b for which solution exist for the non-homogenous system. So, we have two important sub spaces associated with the matrix and one is in F n and the other one is in F m.

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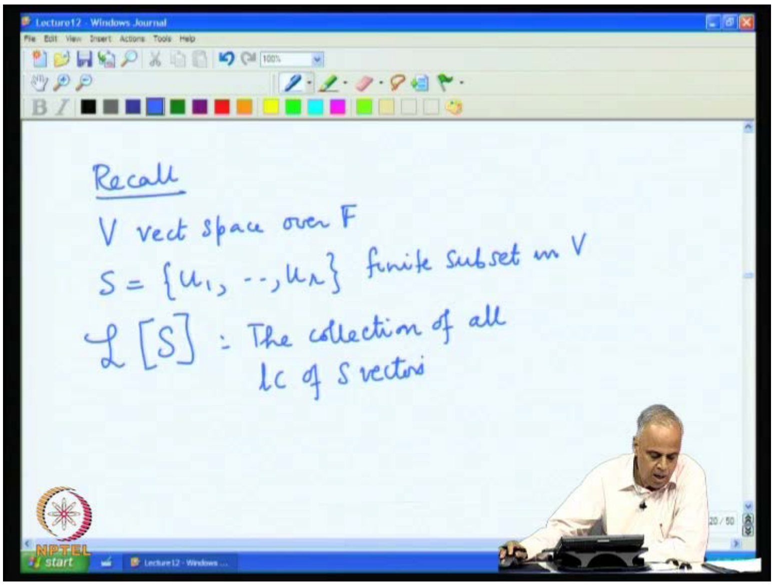

We should look at more examples of vector subspaces. Before we do that let us recollect that if we have vector space over F and if we take a finite subset in V, then we looked at L S the collection of all the collection of all linear combinations of S vectors. And this is a saturation limit we had reached the construction and this is what motivated as the definition of a subspace and L S is a subspace of V.

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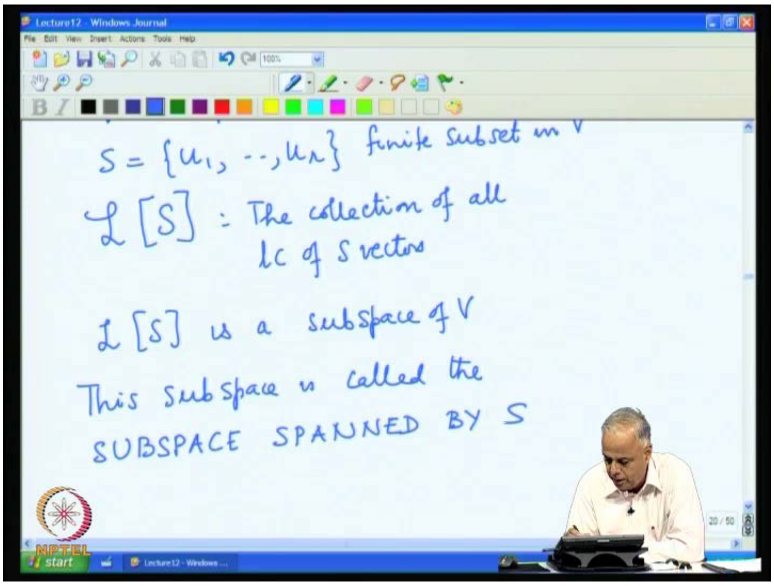

This subspace is called this subspace is called the subspace spanned by x subspace spanned by the set S.

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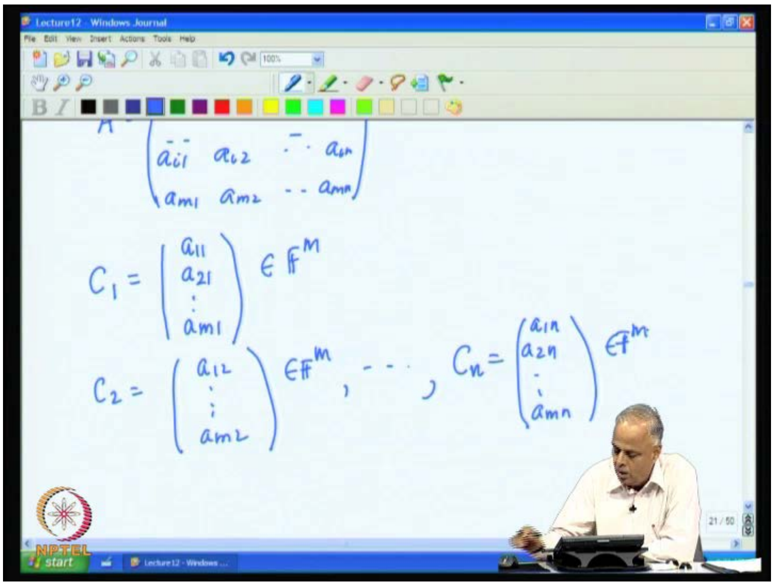

Now let us take a matrix A. Which is m cross n entries are from the field f so I have m cross n matrix and now is look at the matrix a let us write it as a 1 1 a 2 1 etcetera., a within a 1 n is the first row. So it should be a 1 2 a 1 n and so on the i th row will be a i 1 a i 2 a i n and the last row will be a n 1 a n 2 a n m.

Now consider an m by n matrix and look at the first column a first column as 1 2 3 m entries. If we now will write C 1 at the first column at the column vector a 1 1 a 2 1 a m 1 we get a vector in F m it is in F m because there are m entries here similarly, C 2 is a vector a 1 2 a m 2 that is also in F m and so on and in the m th column which is a 1 n a 2 n a m n that is also in F m. Each column belongs to F m. These are called the column vectors of F.

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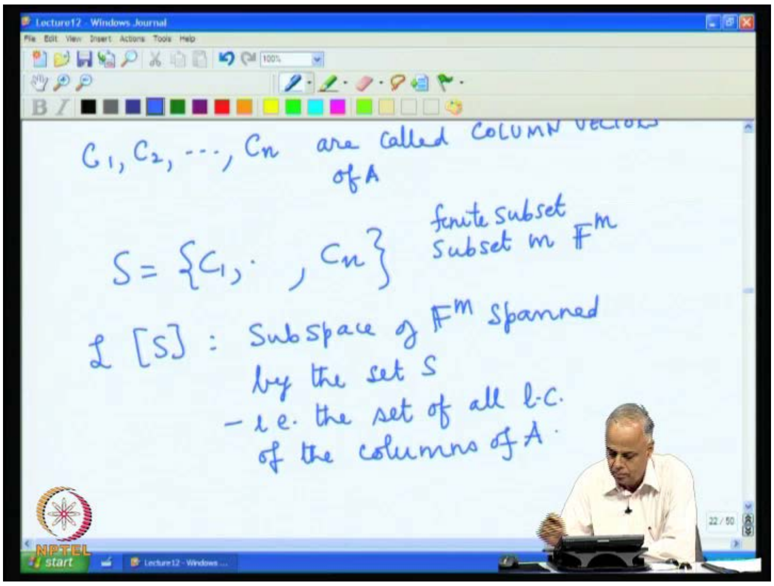

Here, C 1 C 2 C n are called column vectors of A. Let us consider the set of vectors C 1 C 2 C n this is the subset in F m. Because each one of these vectors has m components each one of the vector is in F m so, we have a subset of vectors in F m in the vector space F m and that is a finite subset. So, we have a finite subset of vectors in the vector space F m in the moment we have finite set of vectors when you we can look at L S this subspace of F m spanned by the S vectors. That is the set of all linear combinations of S vector. Vectors is thus are the column vectors so linear combinations of the columns of A. We have the subspace of a m spanned by the columns of F m. so, what is this subspace it is the subspace of all linear combinations of the columns of A.

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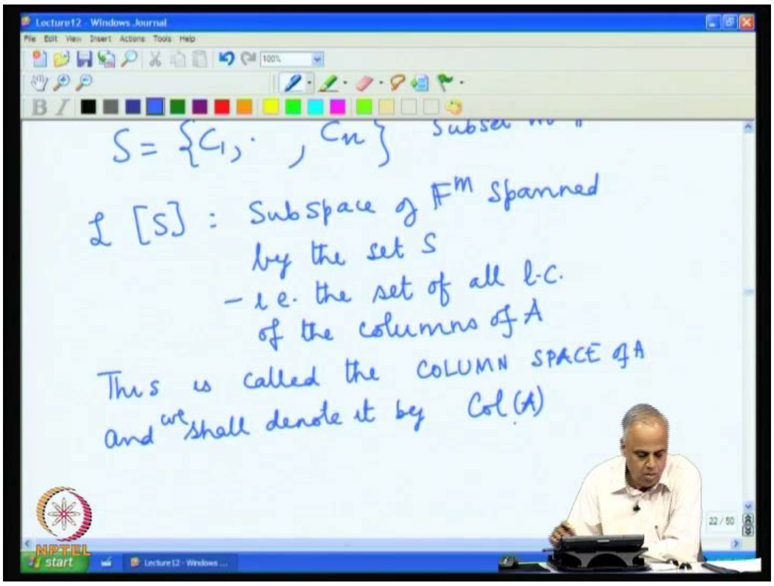

Let us write it as this is called the column space this is called the column space of A. And we shall denote by and we shall denote it by Col A. Column space of A

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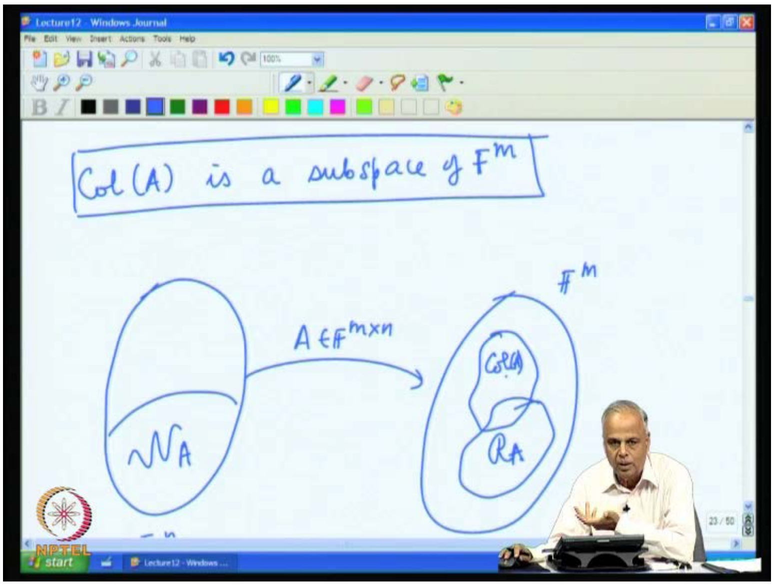

So what is Col A Col A is the set of all vectors in F m which are linear combinations of the columns of a where the alpha is are in F 1 to R.

The column A the collection of all linear combinations of the columns of A this the subspace spanned by the column vectors of A and now we have to verify whether this is subspace or not but, we already know that L S is always a subspace and therefore, column A of the column space of A which we denote by this what we call as the column space of A is a subspace of F m. Now, we are got another new subspace namely the column space of A so let us look at what are we got we have F n we have F m and we have the matrix a which belongs to F n which takes any vector in F m to a vector m F m now here we got one subspace which you called of the null space of A. Here we had 1 subspace which we called the range of A now we have got another subspace in F m known at the column space of A. We do not know what is relation with range of a say in general we will denote it like this it may or may not have an intersection with range of A or it may not be different from range of A at the moment.

We do not know so we do have a subspace in the column range F m so in F m we have got two subspaces and in F m we have got one subspace.

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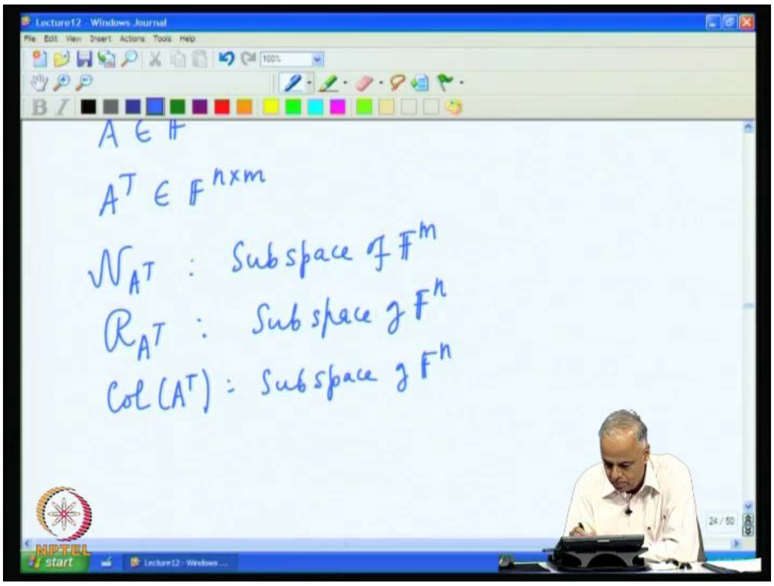

Now, given the matrix a we look at the matrix a transpose which is in F m n the A. Transpose you may recall is the matrix which is obtain from A by writing the rows as columns and A columns as the rows so the i th row of A transpose is simply the i th column of A written as a row so A transpose is an m. We can talk about the null space of A transpose we can talk about the range of A transpose we can talk about the column space of A transpose.

Now the nulls space of A matrix which was one by n if you have in m by n matrix the null space was a subspace of F m. Since, A transpose is F n cross m the null space will be a subspace of F m and if you have a matrix which is m by n the range is a subspace if you have m by n the range is a subspace of F m so similarly, if you have a matrix A transpose which is F n by m the range of a transpose will be a subspace of F n. And the column of A was a subspace of F m so, the column of A transpose will be a subspace of F n.

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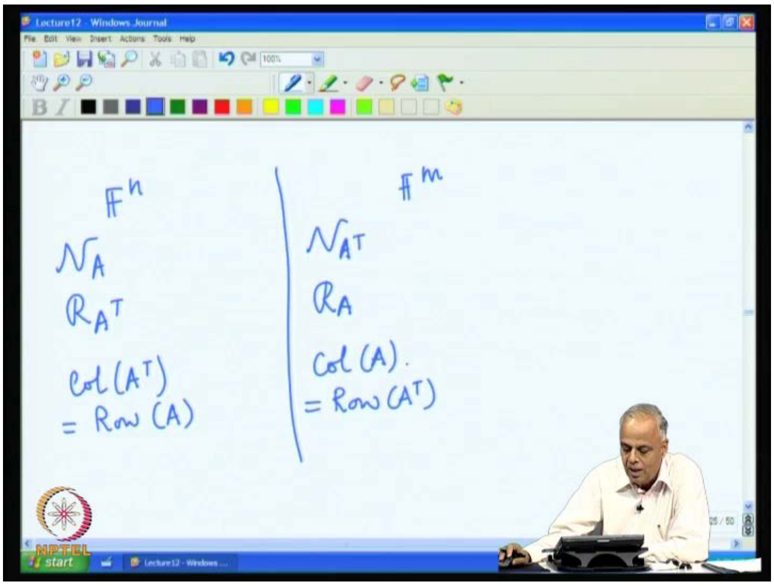

Totally we have got six subspaces and the F n site and the F m site. What are the subspaces we got here the null space of A the range of A transpose and the column space of A transpose. Here we had the null space of A transpose on range of A in the column space of A. Now the column space of A transpose is also called the row space of A because the columns of A transpose has the rows of A and similarly, this is called the row space of A transpose. Actually had 8 spaces out of which two are identical we end up this six spaces sub spaces associated with the matrix 3 of them are on the F m side. And three of them are on the F m side they will order relationship between the subspaces we shall be looking at these relationships one by one.

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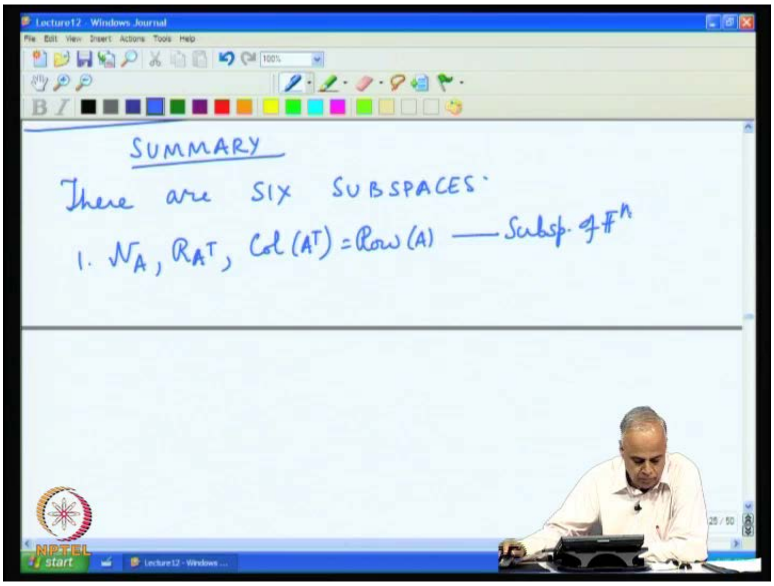

So, there are if you summaries there are six subspaces as of now associated with a matrix 1 the null space of A the range of A transpose and the column space of a transpose which is the same as the row space of A these are all subspaces of F n.

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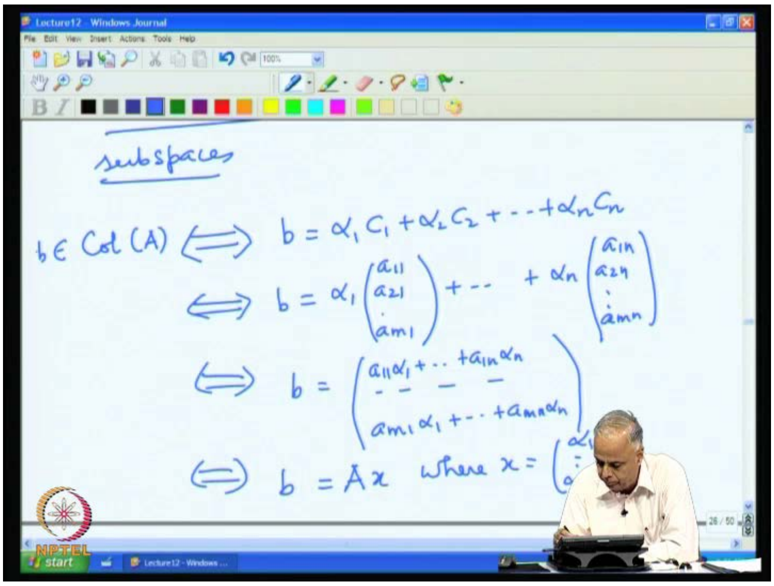

And then we have the null space of a transpose the range of a and the column space of A which is the same as the row space of a and these are all subspaces of F m. Now we shall look at some relationship between these spaces these subspaces. Let us begin with the column space of a suppose we have a vector b in the column space of A. What is that mean? This is true something qualifies to be in the column space of A. If and only if it is a linear combination of the columns of a so it must be of this forms there are n columns of a so b must be of this form what is that mean this means b is equal to alpha 1 the columns looks like a 1 1, a 2 1, a n 1 and so on plus alpha n into a 1 n, a 2 n, a m n. If we now wrote this b is equal to a 1 1 alpha 1 plus etcetera a n 1 alpha n and so on a m 1 alpha 1 plus a m n alpha m. Which is the same as say b is equal to A x where x is equal to alpha 1 alpha n this means since b be the linear combination this alpha 1 alpha 2 alpha m is exist which mean there is a vector x at the b is equal to A x.

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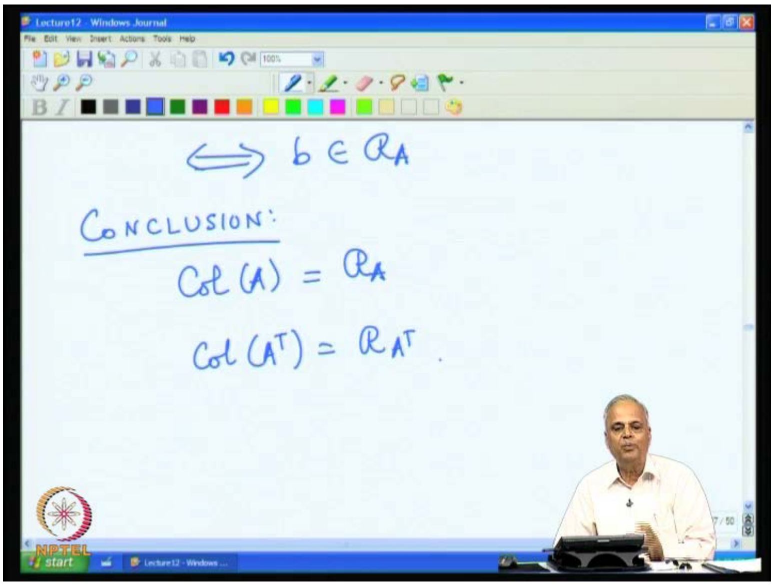

This is the qualification for a vector to the range of A. so, a vector belongs to the columns space of A. If and only if it belongs to the rank space of A conclusion is the columns space of A is the same as ranks space of A. So for any matrix a the column space and the ranks space are identical apply in this to the matrix A transpose look at the columns space of A transpose is the same as the rank of A transpose now we are the six sub spaces out of which tow here collapse into the same, and two here collapse into the same.

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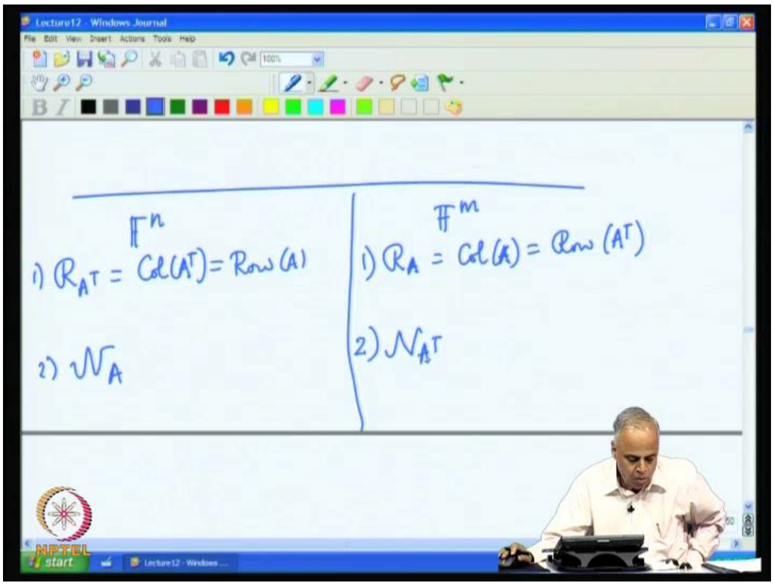

So, eventually what we are got is on the F n side on the F n side we are rank of A transpose. Which is the same in the column space of A transpose which is the same at the row space of A that is the first subspace on F n site the second subspace on the F m site is the null space of A on the F m site.

We have the range of A which is the same in the column space of A which is the same as the row space of A transpose and the null space of A transpose. Thus we have four important subspaces associated with the matrix.

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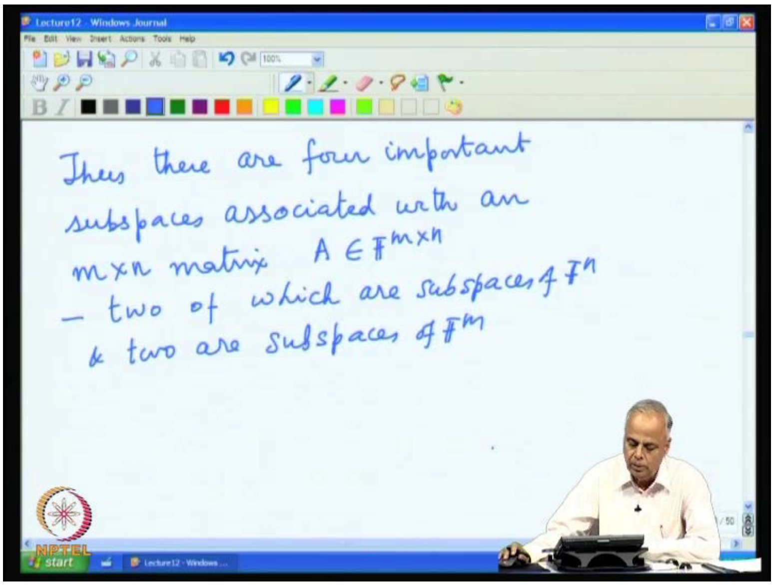

Thus there are four important subspaces associated with an m by n matrix A belonging to F m cross n two of which are subspaces of F n and two are subspaces of F m.

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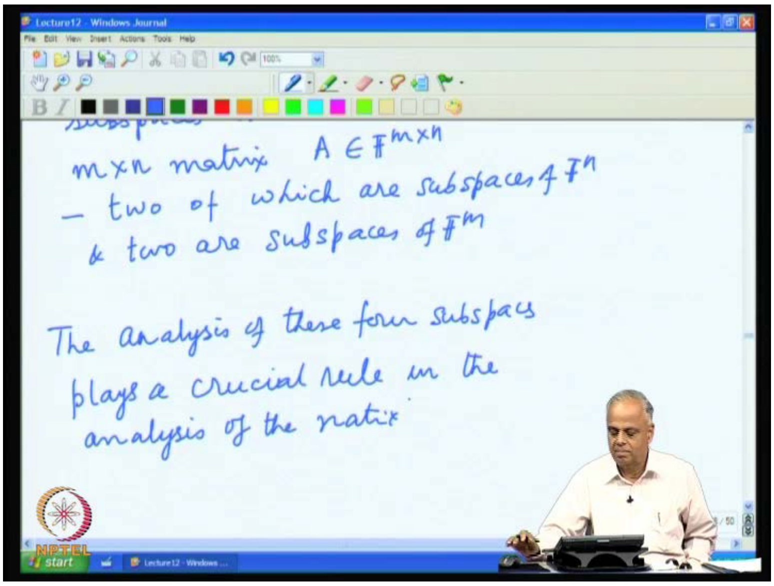

Now the analysis of these four subspaces. The analysis of these four subspaces plays the crucial role in the analysis of the matrix.