Essential Mathematics for Machine Learning
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Lecture - 07
Orthogonal Complements and Projection Operator

Hello friends, so welcome to the 7th lecture of this course on Essential Mathematics for Machine Learning. So, if you remember; in the last lecture we have learned about norm vector spaces, inner product spaces etcetera and we have seen different types of norms there like Manhattan norms, Euclidean norm; that is L 2 norm. Then we have seen P norm and maximum norm; that is L infinity norm.

In this lecture, we will continue from the inner product spaces and discuss about two of very important concepts related to machine learning that is Orthogonal Complements and Projection Operators. These projection operators are really very important when you do the machine learning algorithm, those are popular nowadays; especially like dictionary learning algorithm etcetera.

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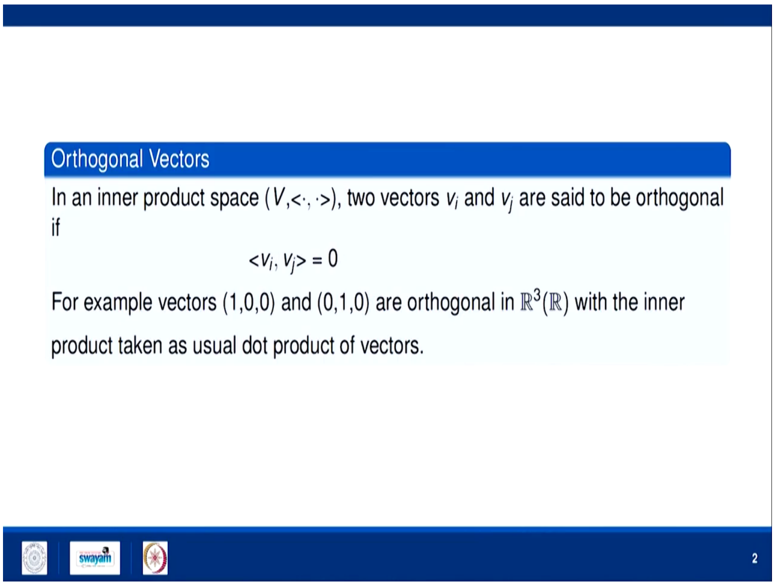

So, first let us define the orthogonal vectors. So, in an inner product space V together with norm; two vectors v i and v j are said to be orthogonal; if their inner product is 0. So, this orthogonalization is a generalization of the concept of perpendicular vectors. As I told you in previous lecture that one of the inner product on R n is usual dot products of vectors.

So, usual dot product; if you take two vectors and then their dot product is 0, we say that the two vectors are perpendicular to each other. So, in the same way these are the generalization; this orthogonal concept is a generalization of the perpendicular vectors, for any vector space where you can define orthogonality in different way.

So, if you want to see the; a simple example of orthogonal vectors. So, just take your vector space as R 3, define over the field of real numbers and take two vectors 1, 0, 0 and 0, 1, 0. Then, if you take the dot product of these two vectors which we are taking the inner product also in this case with R 3; then the dot product is 0; hence two vectors are orthogonal. And they are perpendicular also because first vector is defining the x axis, while the second vector is defining the y axis.

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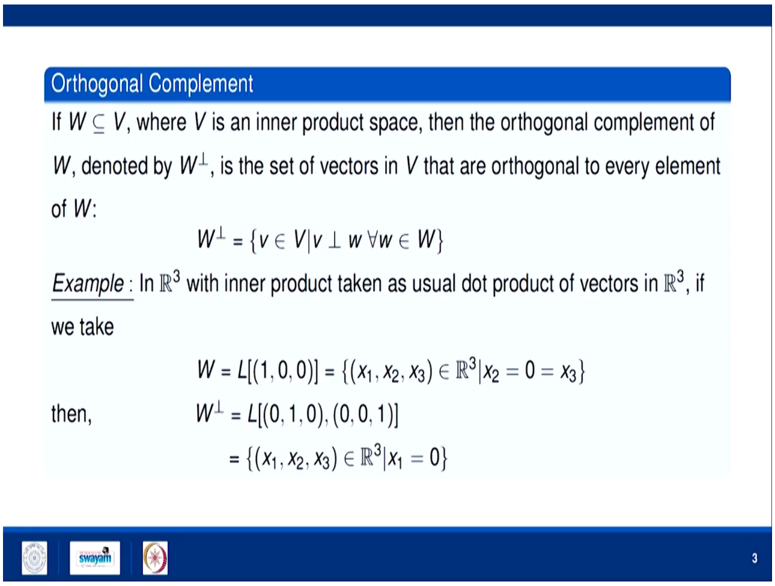

Now, what is an orthogonal complement? So, take W as a subset of an inner product space V, then the orthogonal complement of W denoted by this symbol. W and in superscript, you are having this orthogonal symbol; is the set of vectors in V that are orthogonal to every element of W, that is W orthogonal is v; belongs to all vectors V belongs to the inner product space V; such that the v is orthogonal to w, for all w belongs to the set capital W.

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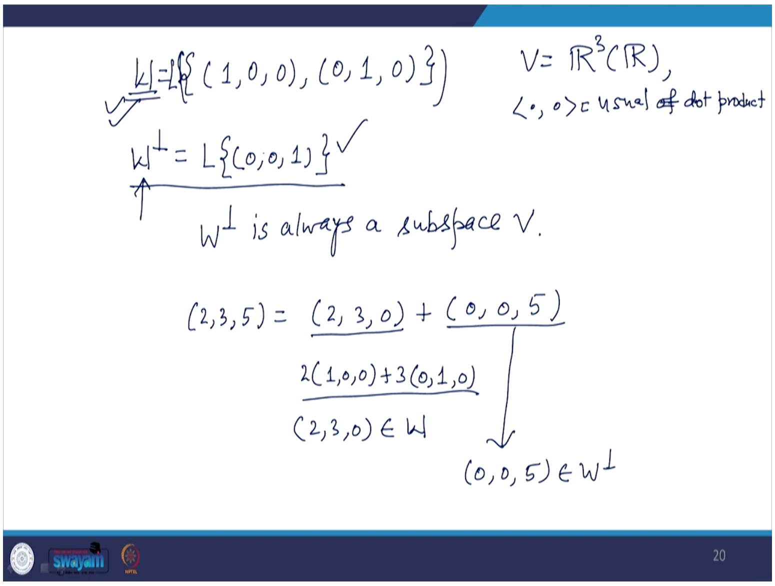

So, if you want to take an example of this; so let us take the W as 1, 0, 0 and 0, 1, 0. Now, you take an inner product space is here R 3 over the field of real number. And the inner product is usual dot product between of the vector of R 3. Now, what is the orthogonal complement of W? Then, it is a space; subspace of R 3 is spanned by the vector 0, 0, 1.

So, you take any vector of this subspace; that is orthogonal complement of W and take the inner product with the vectors of W; it will give you 0. So, as I told you; this orthogonal complement of a set W is always a subspace of V. Here, there is no restriction on W; W may be a set of V or it may be a subspace of V.

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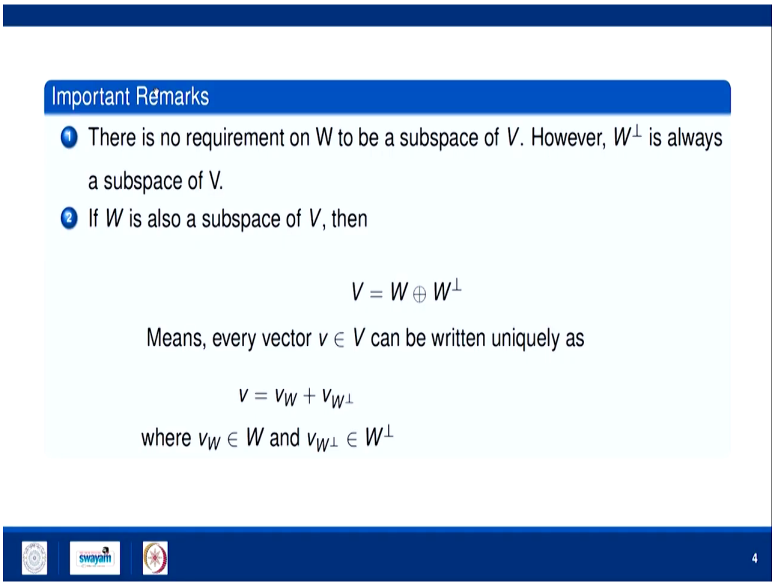

So, as I told you there is no requirement on W to be a subspace of V; however, orthogonal complement of W is always a subspace of V. If W is also a subspace of V, then the vector space V can be written as the direct sum of W and orthogonal complement of W.

What is the meaning of this direct sum that; every vector is small v belongs to the vector space capital V, can be written uniquely as v equals to v; W plus v W complement, where v W is a vector in W and v; W complement is a vector in W complement. So, if you see the earlier example here as I told you; so here W is also a subspace and then you take any vector, let us say 2, 3, 5; then I can write it 2, 3, 0 plus 0, 0, 5.

So, now these 2, 3, 0 is a vector in this subspace and so W is here; the subspace expand by these two vectors because 2, 3, 0; I can write as 2 times 1, 0, 0 plus 3 times 0, 1, 0. So, it means 2, 3, 0 belongs to the subspace W and this belongs to the subspace W complement because, it can be given by this subspace it is an element of; it is a vector of this subspace.

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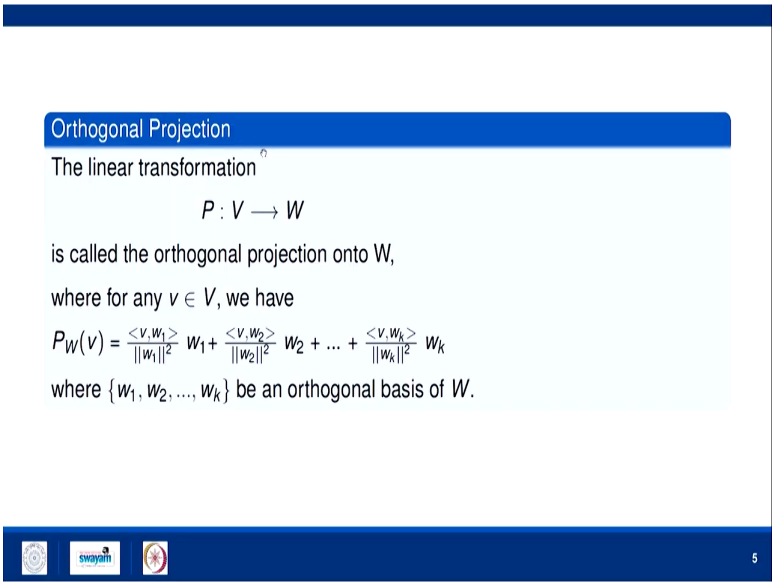

Now, come to another important; definition that is orthogonal projection.

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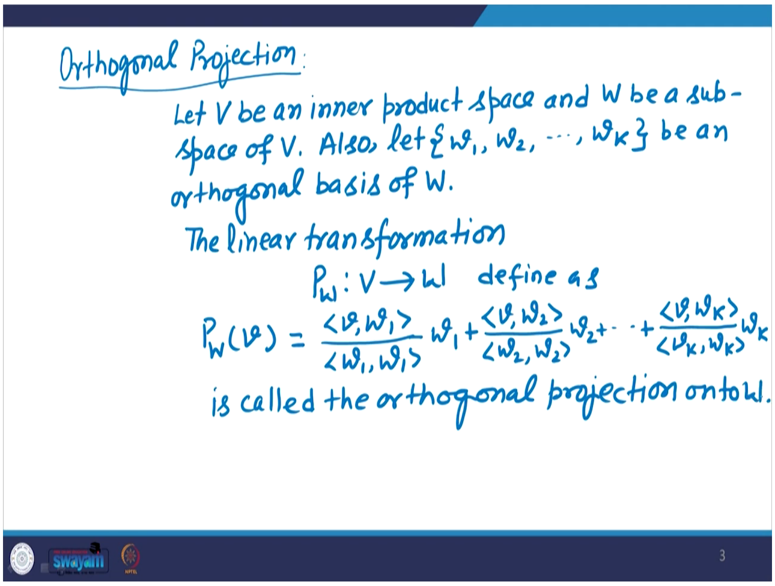

So, now we are going to define orthogonal projection. So, let V be an inner product space and W be a subspace of V. Also, let w 1, w 2, w k be a orthogonal basis of W. Here, meaning of orthogonal basis is that all these vectors w 1, w, w 1, w 2, w k are orthogonal to each other.

Now, the linear transformation P; W from the vector space V to W define as P W; v for a vector; a small v belongs to the capital V as the inner product of v w; 1 plus inner product between v w k; upon square of the norm of w k multiplied with w k is called the orthogonal projection onto W.

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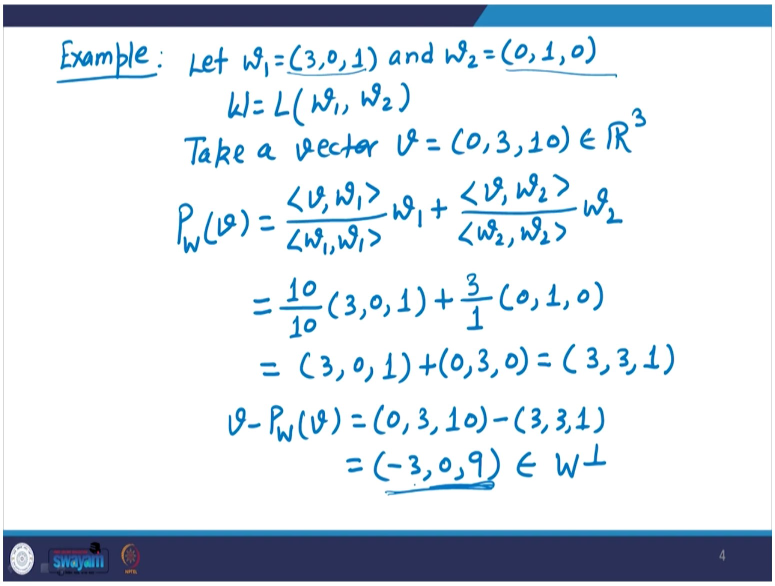

So, let us see an example of this orthogonal projection. So, let w 1 equals to 3, 0, 1 and W 2 equals to 0, 1, 0. Define W as linear span of these two vectors w 1 and w 2.

Now, take a vector v equals to 0, 3, 10 belongs to R 3; now what will be the orthogonal projection of this vector v onto the subspace W? So, it is P of W of v; as I define in earlier slide, v, w 1 upon; square of norm of w; times w 1 plus v, w 2; upon square of the norm of w 2. And here you can notice that w 1 and w 2 are orthogonal, where inner product is usual dot product of vector c R 3.

So, now what is dot product between v and w 1? So, it is a 3 into 0 plus 0 into 3 plus 1 into 10. So, 10 upon the square of the norm of w 1; that is 9 plus 1; 10, times w 1; that is 3, 0, 1 plus inner product of v and w 2. So, that is usual dot product of these two vectors; so 3 upon norm of w 2; that is 1 into w 2; 0, 1, 0.

So, it comes out to be 3, 0, 1 plus 0, 3, 0 which is 3, 3 and 1; so this is the projection of v onto subspace w, if you take a vector v minus the projection that is v is here 0, 3, 10 minus 3, 3, 1; so this is a vector minus 3, 0, 9; so these vectors belongs to the orthogonal complement of W.

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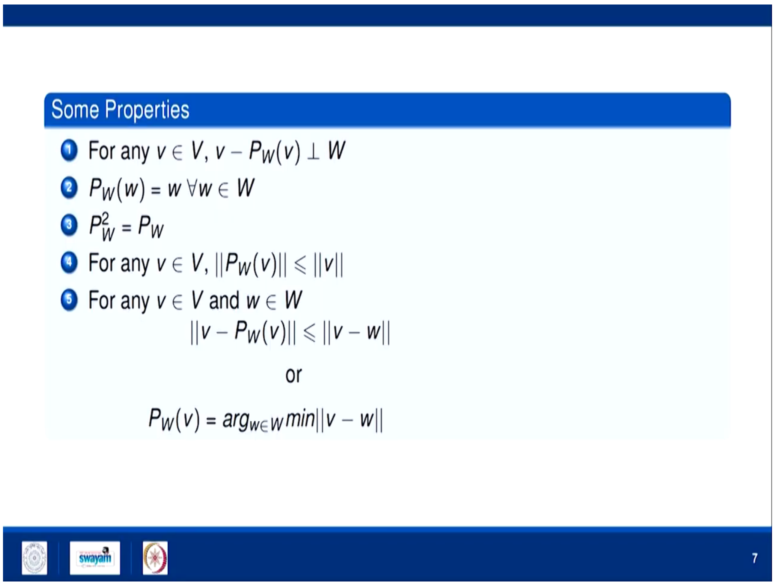

Now, we are having several properties of orthogonal projection. So, first property is for any v belongs to capital V; the vector v minus P W; v is orthogonal to W. So, you can see here; as we have calculated it here; so this is v minus P W; v. So, this vector is orthogonal to w 1, as well as w 2; you can easily verify; so, here 3 into minus 3 minus 9 plus 9, 0.

Similarly, here 0 plus 0 plus 0; so this v minus P W; v is orthogonal to w 1, as well as w 2; so it is orthogonal to W. Second property is the projection of W; on the subspace w is w, for all w belongs to capital W. The third property is P square W equals to P; W. Fourth property is for any v belongs to capital V; the norm of P; W, v will be always less than equal to norm of v.

The fifth property is for any v belongs to capital V and w belongs to capital W; v minus P W; v and norm of this vector will be always less than equals to v minus w; that is if P W; v is such w belongs to the subspace w which minimized the distance between v minus w. So, this particular property we can use to find a point in subspace w; a vector in subspace w, which is closest to the given vector v.

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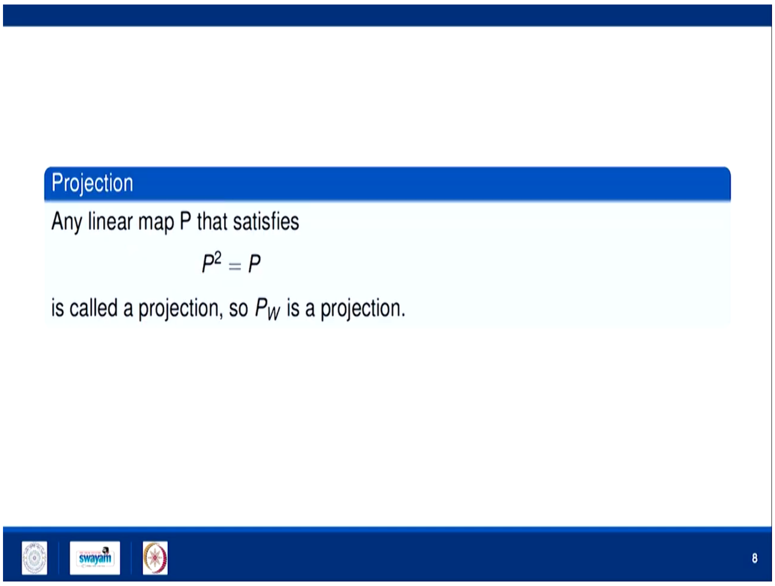

We will see it by an example, but before that let us see the definition of projection transformation. So, any linear transformation P that is satisfies P square equals to P is called a projection. So, here this orthogonal projection is also a projection. Another example is trivial example is identity.

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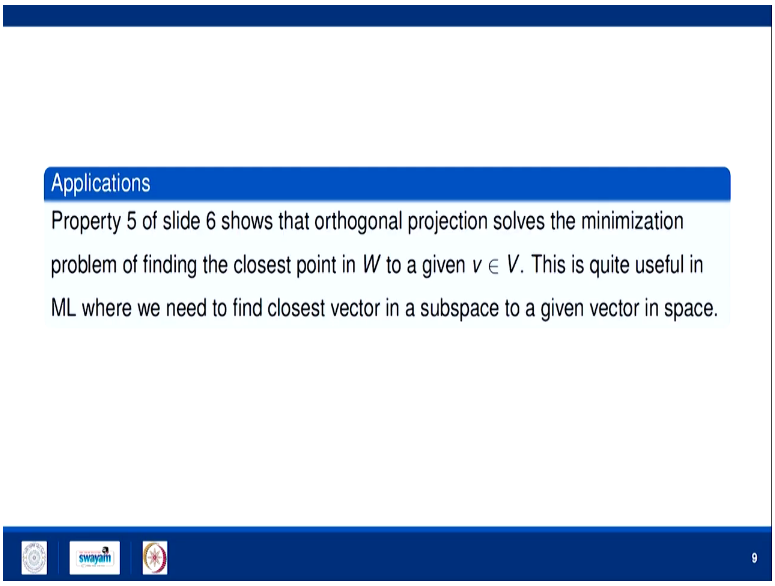

So, as I told you; the property 5 of slide 6 shows that orthogonal projection solves the minimization problem of finding the closest point in W to for a given v belongs to V. This is quite useful in machine learning where we need to find closest vector in a subspace to a given vector in space.

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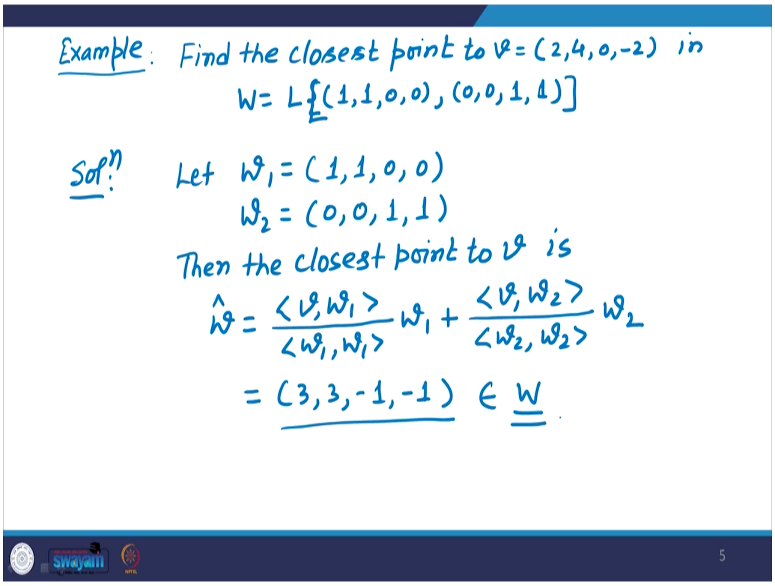

So, let us see an example of it; so find the closest point to v equals to 2, 4, 0, minus 2 in a vector space W which is a linear span of the vector 1, 1, 0, 0 and 0, 0, 1, 1.

So, let; how to find it? We will use property 5 of earlier slide. So, let w 1 equals to 1, 1, 0, 0 and W 2 equals to 0, 0, 1, 1; then the closest point to v is; let us say w cap and it will be the orthogonal projection of v, on the subspace w which comes out to be 3, 3, minus 1, minus 1 which is belongs to W.

So, this is an application of this orthogonal projection and this application is really useful in machine learning, when you are having the space of different classes and you have to find out for a given vector; the nearest vector of another class. So, in this lecture; we have talk about orthogonal projection, projection operator and how to find out the nearest vector into a subspace for a; to a given vector of the vector space.

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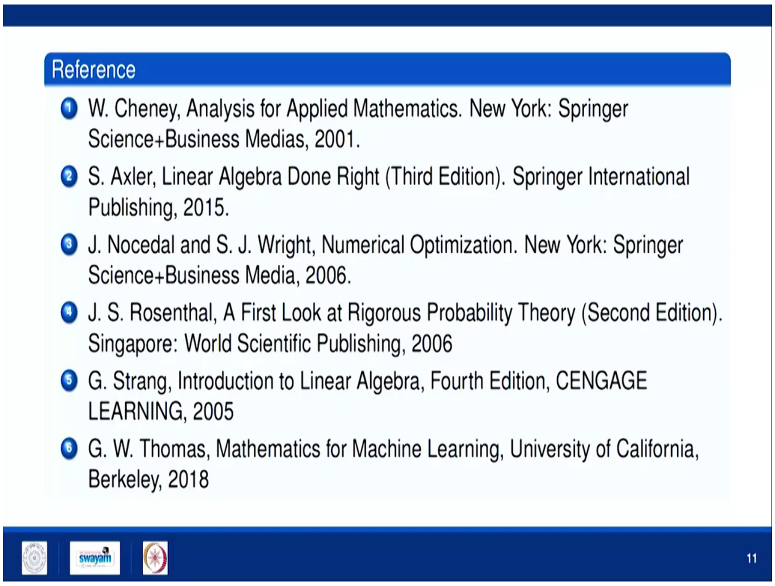

I hope you have enjoyed this lecture; you can follow these references for having more examples, for having more clear concepts. In the next lecture, we will talk about another very important property of linear algebra, those are related to the linear transformation or matrices; that is eigenvalues and eigenvectors and those are having tremendous applications in machine learning.

Thank you very much.