Essential Mathematics for Machine Learning
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Lecture – 05
Linear Transformations

Hello friends. So, welcome to the 5th lecture of this course. So, in this lecture we will talk about a very important concept of linear algebra; that is called Linear Transformation. So, believe me linear transformation is the most useful concept if you are doing machine learning.

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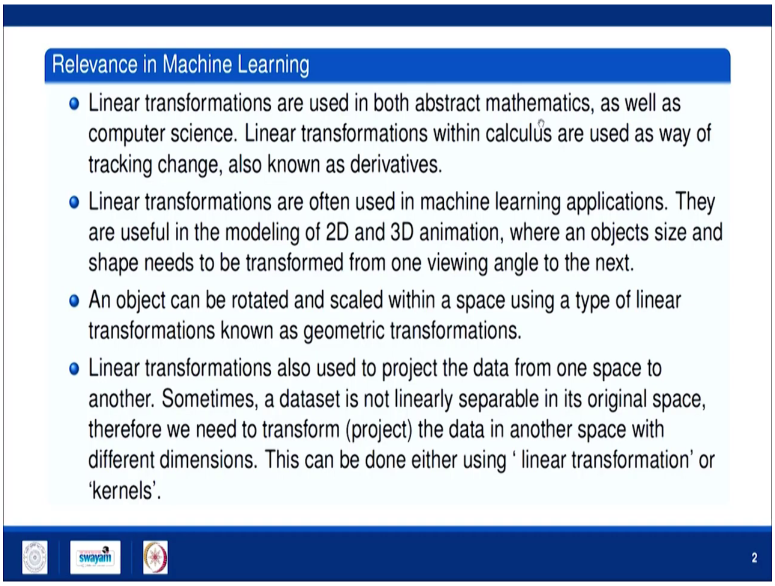

So, linear transformation are used in both abstract mathematics as well as in computer science. So, this concept is very very frequently used in many of the topics of computer science also. Even electrical engineering electronics etcetera.

So, linear transformations within calculus are used as a way of tracking change also known as derivative. So, if you want to define derivative from some R 3 to R 3 then you have to make use of linear transformation. So, linear transformations are often used in machine learning applications; they are useful in the moedeling of 2D and 3D animations.

So, very much useful in graphics, animation. How? Because by applying a linear transformation you can change the object shape in a plane or in a space and you can see from one viewing angle to another viewing angle.

Moreover an object can be rotated and scaled within a space using a type of linear transformations known as geometric transformations. So, in this lecture we will see later the scaling and rotation linear transformation. Linear transformations also used to project the data from one space to another. Suppose the data of your classification problem is not linearly separable in a it is original space.

So, what you can do? You can project the data to some other space where it can be linearly separable. And you can make use of some of the linear classifier like linear SVM or simple perceptron type of classifiers to classify the complex data with that is not linearly separable in original space.

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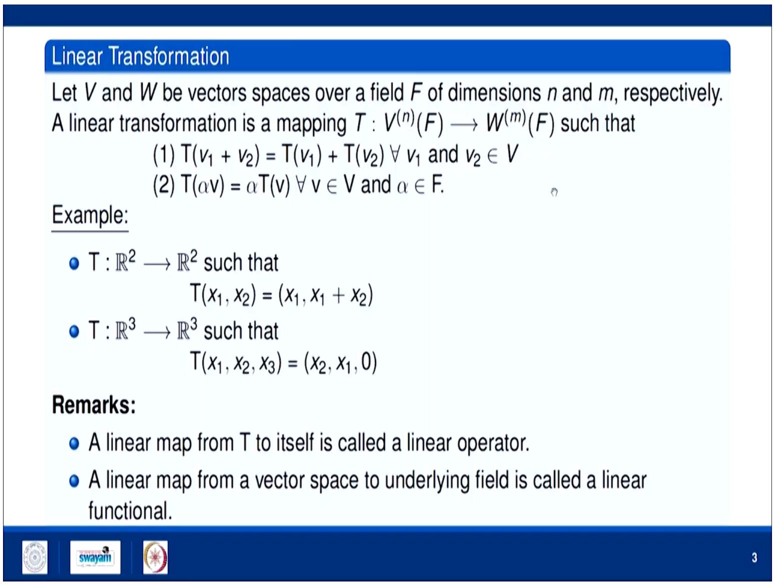

Let us come to the formal definition of linear transformation. So, let V and W be vector spaces over a field f of dimension n and m respectively. So, dimension of V is n dimension of W is m, a linear transformation is a mapping from vector space V to vector space W satisfying these two conditions. The first condition is additivity. That is T of v 1 plus v 2 equals to T of v 1 plus t of v 2 for all v 1 v 2 belongs to the vector space V.

So, here v 1 plus v 2 is a vector in vector space V while T of v 1 and T of v 2 are vector are vectors in vector space W. The second condition is homogeneity if you multiply a vector of v from alpha where alpha is a scalar from the field F then T of alpha v equals to alpha times T of v. For all v belongs to v capital V vector space and alpha belongs to F.

So, if this mapping T satisfy these two conditions then we say that T is a linear transformation. Another name a popular name of linear transformation in some of the references you will find as linear map or linear mapping. So, these are two examples of linear transformations T of R 2 to R 2 such that T of x 1 x 2 equals to x 1 comma x 1 plus x 2. How to prove that it is a linear transformation?

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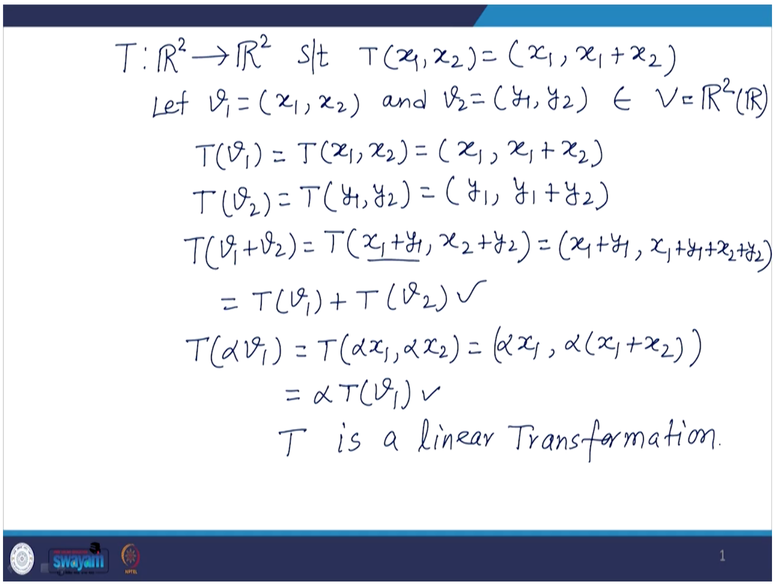

So, we are having T R 2 to R 2 such that T of x 1 x 2 equals to x 1 x 1 plus x 2. So, let v 1 equals to x 1 x 2 and v 2 equals to y 1 y 2 belongs to V.

So, these are two arbitrary vector from the vector space V that is your R 2 over the field of real number. So, again we are taking field as the field of real numbers, but this definition is true for any vector space over defined over any of the field. So, now, T of v 1 equals to T of x 1 x 2 which is x 1 comma x 1 plus x 2.

Similarly, T of v 2 equals to T of y 1 y 2, it is y 1 comma y 1 plus y 2. Now T of v 1 plus v 2 equals to T of x 1 plus y 1 comma x 2 plus y 2 and this will become x 1 plus y 1 that is your first component as per the definition of T and then sum of these 2 component.

So, x 1 plus y 1 plus x 2 plus y 2. Now this equals to T of v 1 plus T of v 2. So, first condition of additivity is satisfied. Now T of alpha times v 1 is T of alpha x 1 alpha x 2 this becomes alpha x 1 alpha time x 1 plus x 2 and this is alpha times T of v 1.

Because T of v 1 is x 1 comma x 1 plus x 2 and you are multiplying by alpha both the component. So, the second condition is also satisfied. So, hence T is a linear transformation. So, this is the working process for checking whether a transformation is linear or not. Similarly another example is T from R 3 to R 3 such that T of x 1 x 2 x 3 equals to x 2 x 1 0.

So, what is the geometrical interpretation of a linear transformation. So, we apply this transformation to a vector and it is scaled or rotate or change the shape of that. So, if we are having a set of vectors for example, suppose I am having x 1 x 2 plane in R 2.

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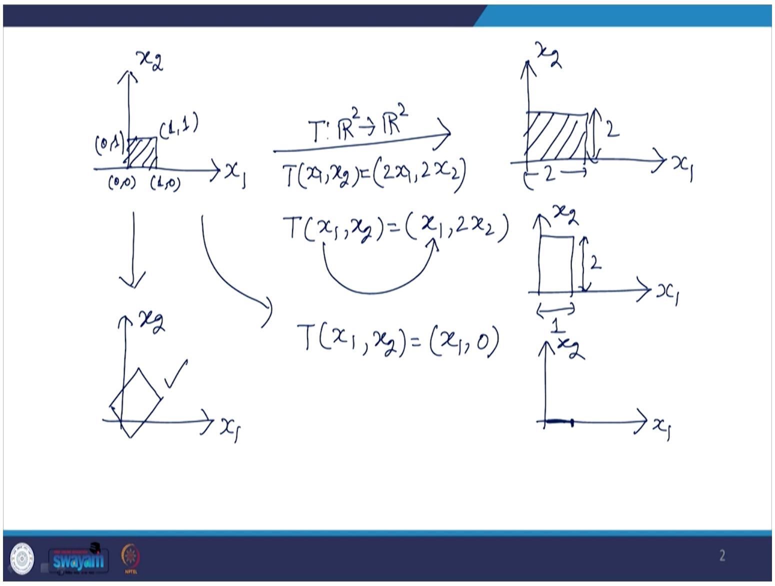

And in this I am having a square. So, let us say this is square of length 1. So, this is 0, 0 1, 0 1, 1 and 0, 1.

So, I apply a linear transformation T on it such that T of x 1. So, T R 2 to R 2 T of x 1 x 2 equals to 2 x 1 2 x 2. So, now, I am having what this transformation is doing it is scaling each dimension by 2. So, then what will be the output? The in the output space x 1 x 2. So, we will apply this transformation on to this square. So, I will get another square where the length of each side is now 2 say it is scaling.

If I apply a linear transformation T x 1 x 2 equals to x 1 2 x 2, then what I will get? So, x 1 and x 2. So, what this transformation is doing? It is not making any change in x 1 component; however, x 2 component is becoming twice. So, it will become a rectangle where this x 1 dimension remain same as 1; however, x 2 dimension become 2. If I am having a linear transformation let us say T x 1 x 2 equals to x 1 0.

So, what this linear transformation is doing? First component will remain as such while the second component become ze0ro. So, now, the output the result will become only a line from 0 to 1 on the x 1 axis. Similarly using the linear transformation I can rotate this square by an arbitrary angle in x 1 x 2 plane.

For example, I will rotate it by 45 degree like this. So, what will be the linear transformation for this?

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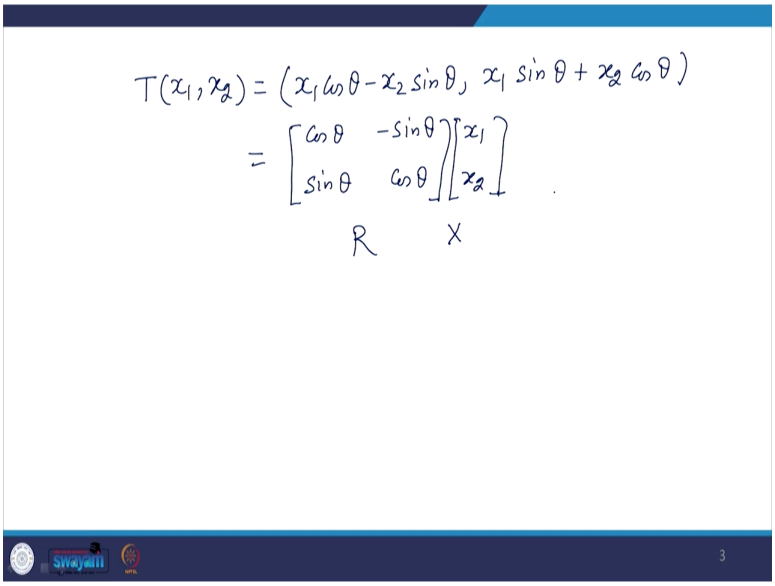

So, linear transformation for this will be T of x 1 x 2 will be x 1 cos theta minus x 2 sin theta and x 1 sin theta plus x 2 cos theta. Basically, it is the rotation matrix in x 2 x 1 plane given by cos theta minus sin theta sin theta cos theta and it is acting on x 1 x 2.

So, this is rotation matrix acting on the vector x where theta is the angle of rotation. Similarly we can define rotation matrices or rotation transformation in R 3. So, all these are geometrical interpretation of linear transformations. Some remarks, a linear transformation t from a vector space v 2 itself is called a linear operator. Similarly, a linear map or linear transformation from a vector space v to underlying field f is called a linear functional.

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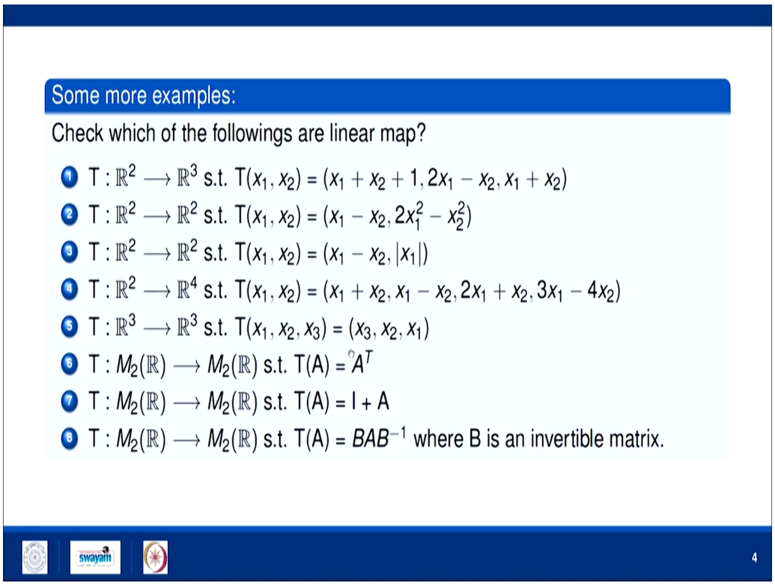

Some more examples; check which of the following are linear map. So, T R 2 to R 3 T x 1 x 2 x is going to x 1 plus x 2 plus 1 2 x 1 minus x 2 x 1 plus x 2. So, the first condition we have to check the necessary condition to be a linear transformation. And what is the necessary condition to be a linear transformation that? If you are having a linear transformation from vector space V to W. So, the zero vector of V map to the zero vector of W.

So, in this case, if you take x 1 0, x 2 0, it is going to 1 0 0. So, it is not a linear transformation because zero vector is not going to zero vector. If you take this 1 t of x 1 x 2 equals to x 1 minus x 2 2 x 1 square minus x 2 square again it is not a linear transformation. Because, additivity condition will fail here. So, for if you want to check it.

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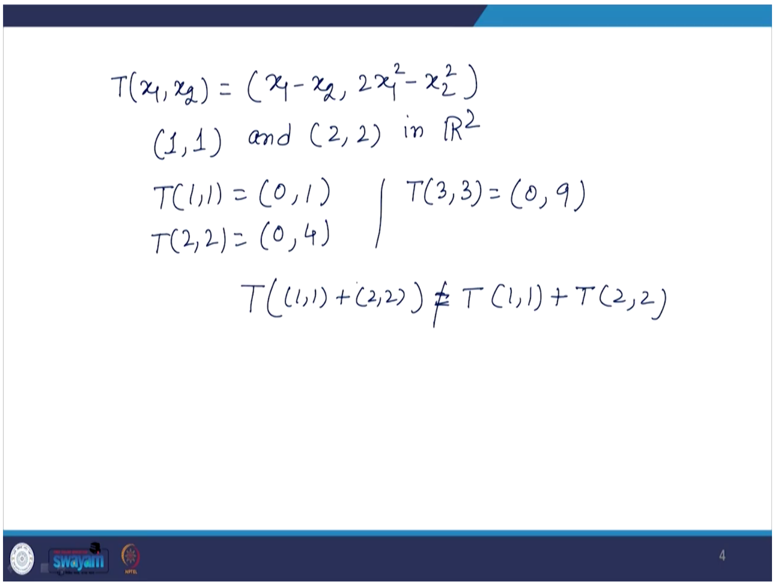

So, I am having T of x 1 x 2 going to x 1 minus x 2 2 x 1 square minus x 2 square. So, take 2 vectors, let us say 1 1 and 2 2 in R 2. So, t of 1 1 is going to 0 and 1. T of 2 2 is going to 0 and 4 while T of 3 3 that is the sum of 1 1 and 2 2 is going to 0 9.

So, simply T of 1 1 plus 2 2 which is 0 9 not equals to T of 1 1 plus T of 2 2. Hence T is not a linear transformation. Similarly you can check about this. So, due to this modulus, absolute value of x 1 it is not a linear transformation.

While you can check that it will make a linear transformation, then this will make a linear transformation it will make a linear transformation; however, this 1 will not make a linear transformation. Why? Because 0 matrix here will go to identity 0 matrix should go to 0 matrix. So, in that way you can have all these checks whether a given transformation is linear or not.

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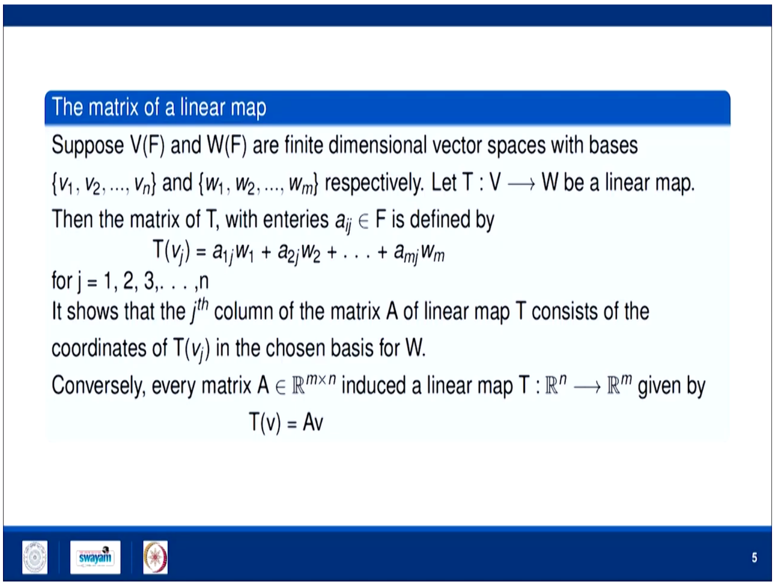

Now, every linear transformation is a matrix and every matrix is a linear transformation. So, what I want to say; linear transformation is nothing they are matrices only. So, how to find out the matrix corresponding to a linear transformation first of all a linear transformation is from a vector space to another vector space.

So, you have to decide the basis relative to what basis you want to find out the associated matrix. So, how to do it suppose V and W are finite dimensional vector spaces with over the field F with basis v 1, v 2, v n. So, dimension of V is n and dimension of W here is m and basis is w 1, w 2, w m. T from V to W be a linear transformation, then the matrix of T with enteries a ij belongs to F some it the entry of the matrix will come from the field F is defined by for any basis vector v j here T of v j will be a 1 j w 1 a 2 j w 2 a m j w m.

So, what first you have to take the basis vector from this vector space v. You have to find out it is image where it is mapping to space in W and then you have to write that particular vector as the linear combination of w 1, w 2, w m. Then the coordinates a 1 j, a 2 j, a mj will give the j th column of the associated matrix a.

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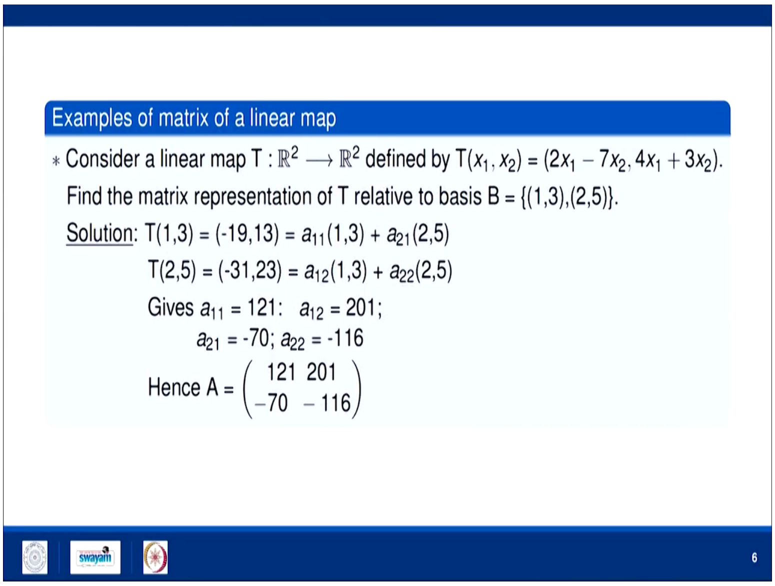

Suppose you are having a linear transformation from R 2 to R 2 defined by T of x 1 x 2 equals to 2 x 1 minus seven x 2 comma 4 x 1 plus 3 x 2. Find a matrix representation of t relative to basis b given by the vectors 1 3 and 2 5.

So, first I will check the image of 1 3 under T. So, T of 1 3 will become minus 19 and 13. Now this vector I will write as the linear combination again of the basis of R 2. Here basis are same for both the vector space, it may be different in some of the examples. So, a 1 1 1 3 plus a 2 1 2 5. So, it will give the first column of the associated matrix a.

Similarly, T 2 5 is minus 31, 23. So, a 1 2 1 3 plus a 2 2 2 5. So, by solving these 2 I will get a 1 1 as 121, a 1 2 as 201, a 2 1 as minus 70 and a 2 2 as minus 116. Hence the matrix a is 121, 201, minus 70, minus 116. So, in that way you can find out the matrix representation of a given linear transformation.

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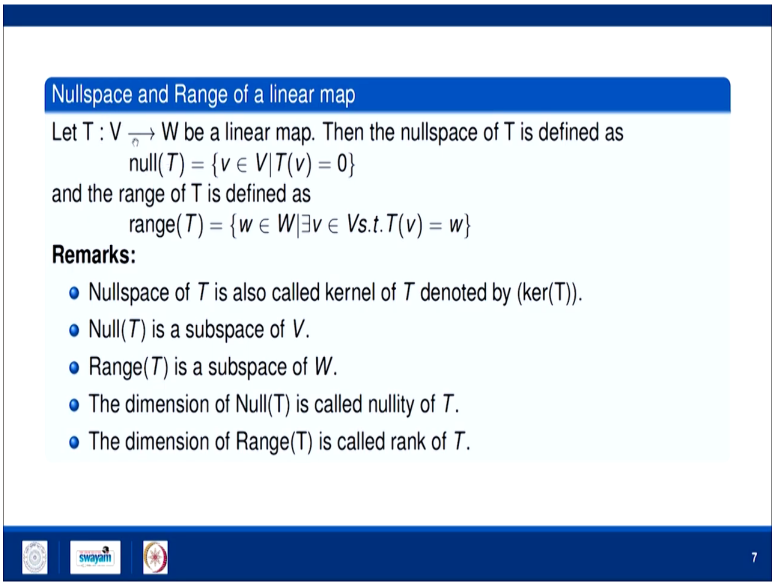

Similarly, if you are having a matrix; how to find corresponding linear transformation?

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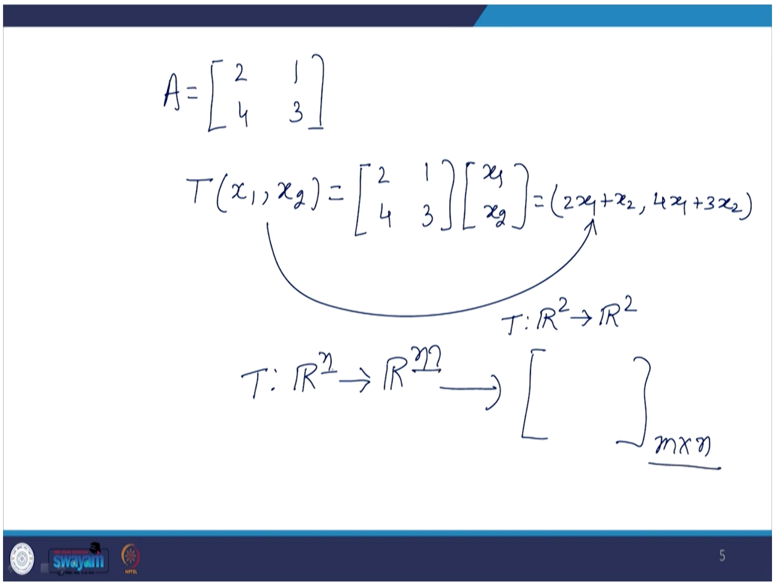

So, let us say you are having a matrix A equals to 2 1 4 3, then the linear transformation representation of this corresponding to it is relative to the standard basis will become 2 1 4 3 x 1 x 2. So, it will become 2 x 1 plus x 2 4 x 1 plus 3 x 2.

So, this is the linear transformation from T R 2 to R 2. So, if you are having a linear transformation from R n to R n, then matrix representation will be of a m by n matrix. So, this times this one you this you have to remember. Next, I am defining two important subspaces those are associated with linear transformations.

So, let T V to W be a linear map, then the null space of T is defined as all vectors V belongs to this vector space V such that the image of v under T is 0; that is 0 vector of W. So, all those vectors from the vector space V those are mapped to the zero vector of W will come in null space of T. Hence and you can check very easily that this null space of T is a subspace of V.

The range space of T is all vectors w belongs to this vector space W such that there exists v belongs to V such that T v equals to w. And you can easily verify that this range space or range of T is a subspace of W.

Null space of T also called kernel of T denoted by kernel of T. As I told you null space of T is a subspace of, range space of T is a subspace of W. So, the dimension of that null space is called nullity of T. Similarly, the dimension of range of t is called the rank of T.

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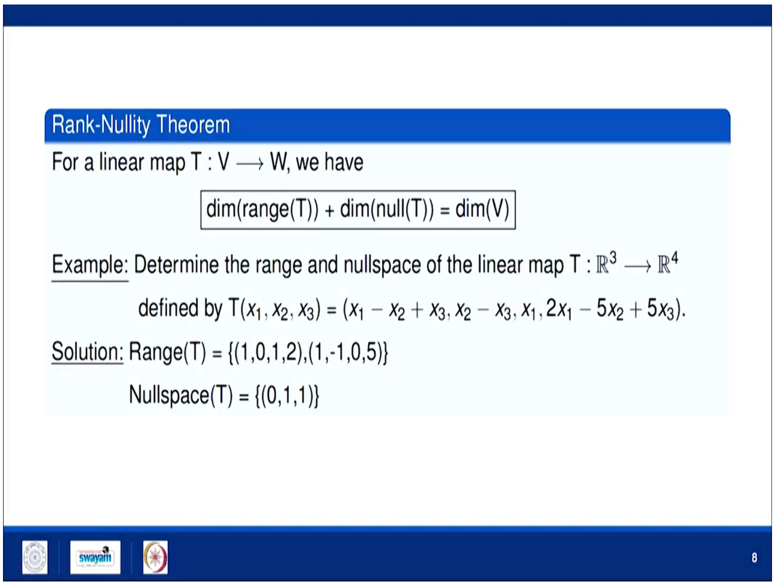

We are having a very important result here. So, for a linear transformation T which is defined from V to W. We have the dimension of range of T that is the rank of T plus dimension of null space of T; that is nullity of T equals to dimension of this vector space V.

So, let us take an example, how to find out range space and null space of a linear transformation. So, determine the range and null space of the linear map which is from R 3 to R 4 and here field is the field of real numbers defined by T of x 1 x 2 x 3 is going to x 1 minus x 2 plus x 3 x 2 minus x 3 x 1 and 2 x 1 minus 5 x 2 plus 5 x 3.

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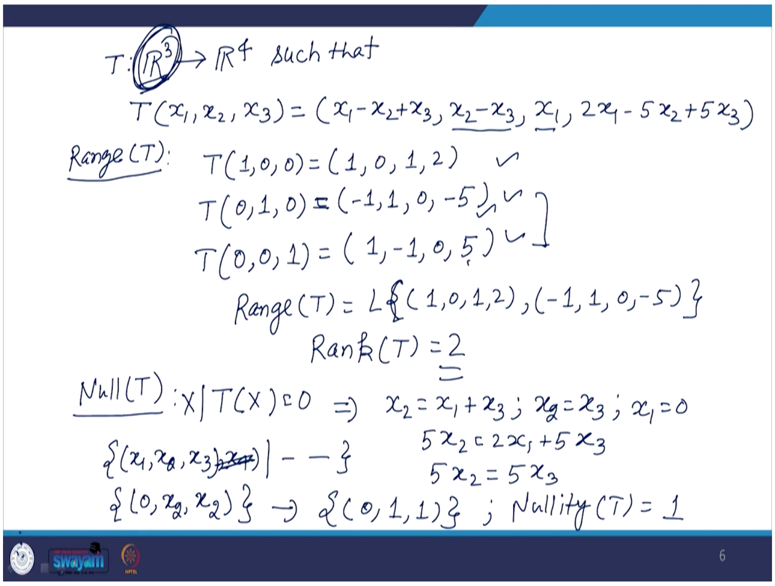

So, let us do it. So, I am having a mapping T R 3 to R 4 such that T of x 1, x 2, x 3 equals to x 1 minus x 2 plus x 3 x 2 minus x 3 x 1 and then 2 x 1 minus 5 x 2 plus 5 x 3. So, first we will find out the range of T. So, for finding the range of T; what I will take? I will take the standard basis of R 3 and I will check where this standard basis is mapping.

So, first take the standard basis of R 3. So, T 1, 0, 0 is mapping to 1 0 1 and 2. Now take another vector of the standard basis of R 3; 0, 1, 0 it is going to minus 1, 1 from here 0 and then minus 5. Now, take the vector 0, 0, 1.

So, 0, 0, 1 will go to 1 minus 1 0 and 5. Now, if you see these 3 vectors these 2 vectors are ld. Just this vector is minus 1 times this 1. And as I told you that in the basis of a sub space we include only.

Student: (Refer Time: 26:37).

Linearly independent vectors. So, here range of T will be linearly spanned by the vectors 1 0 1 2 and minus 1 1 0 minus 5.

Hence rank of T is 2. Similarly, we have to find out null space. So, null space of T is all vectors X such that T X equals to 0. So, it is giving me x 1 minus x 2 plus x 3 equals to 0. So, from here I can write x 2 equals to x 1 plus x 3. This equals to 0 gives me x 2 equals to x 3. x 1 is 0 from here and finally, the last 1 is giving me 5 x 2 equals to 2 x 1 plus 5 x 3. So, x 2 equals to x 3 gives me x 1 is 0. So, 5 x 2 equals to 5 x 3 because x 1 is 0 here. So, what I will get null space of v is x 1 x 2 x 3 x 4 such that all these are individually 0.

So, from here, what I will be having x 1 is 0 x 2 and then x 3 is sorry null space will come to the R 3. So, x 1, x 2, x 3 such that all these 4 equations equals to 0. So, x 2 this give me the range as 0 1 1. So, null space of t is spanned by the vector 0 1 1. Hence nullity of T equals to 1 because we are having only 1 vector in the basis of null space of T.

So, rank is 2, nullity is 1, rank plus nullity equals to 2 plus 1 3 that is the dimension of this vector space hence we are verifying the.

Student: Rank nullity.

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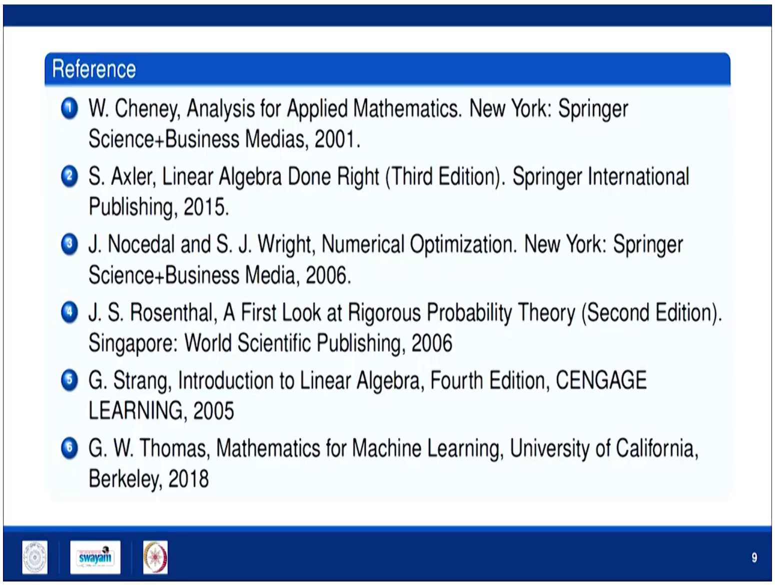

Rank nullity theorem. So, in this lecture, we have learned very important concept of linear algebra. In the next lecture; we will see some more properties of linear transformation. These are the references. I hope you have enjoyed this lecture.

Thank you very much.