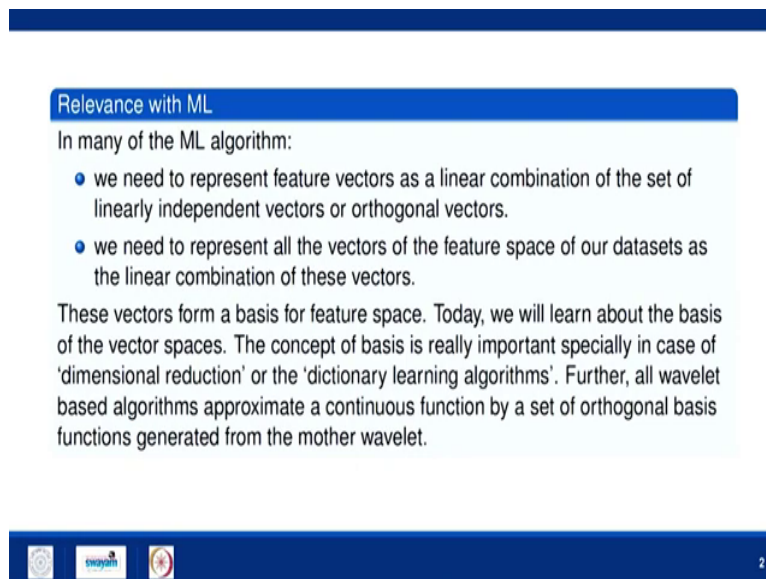


Essential Mathematics for Machine Learning
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Lecture – 04
Vector Subspaces: Basis and Dimensions

Hello friends. So, welcome to the lecture number 4 of the course Essential Mathematics for Machine Learning. If you remember in the last lecture we have talked about Vector Subspaces. So, today we are again going to talk about a very important concept from the vector spaces that is Basis and Dimension of Vector Subspaces.

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Relevance with ML

In many of the ML algorithm:

- we need to represent feature vectors as a linear combination of the set of linearly independent vectors or orthogonal vectors.
- we need to represent all the vectors of the feature space of our datasets as the linear combination of these vectors.

These vectors form a basis for feature space. Today, we will learn about the basis of the vector spaces. The concept of basis is really important specially in case of 'dimensional reduction' or the 'dictionary learning algorithms'. Further, all wavelet based algorithms approximate a continuous function by a set of orthogonal basis functions generated from the mother wavelet.

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So, in most of the machine learning algorithm we need to represent our input data that is in terms of feature vectors as a linear combination of certain vectors that all the feature vectors we want to write as linear combination of a set of vectors. Those set of vectors may be linearly

independent or they may be orthogonal. So, to write all the feature vectors or power data set in terms of linear combination of these vectors we have to find out that particular set of vectors.

So, these vectors form a basis for the feature space. Today we will learn about the basis of the vector spaces. The concept of basis is really important, especially in the case of dimension reduction or the recent algorithm like dictionary learning based algorithm. Further in wavelet based algorithms we approximate any function as the linear combination of set of orthonormal functions those we have generated from the mother wavelet.

So, let us come to the definition of basis.

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Basis

Definition: Let $V(F)$ be a vector space. A basis for V is a set of linearly independent vectors in V which spans the vector space V .

- The space V is finite-dimensional if it has a finite basis.
- A vector in basis is called basis vector.

Examples:

- (1) Standard basis for \mathbb{R}^2 is $S = \{(1,0), (0,1)\}$.
- (2) Standard basis for \mathbb{R}^3 is $S = \{(1,0,0), (0,1,0), (0,0,1)\}$.
- (3) Standard basis for \mathbb{R}^n is $S = \{(1,0,\dots,0), (0,1,0,\dots,0), \dots, (0,0,\dots,0,1)\}$.

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So, mathematically speaking the definition of basis is given as let F be a vector space. So, we are having a vector space V over the field F . A basis for V is a set of linearly independent vectors in V which expands the vector space V . Now, so, what there should be what should be the qualification to be a basis? The first all the vectors of that sets would be linearly independent.

Number 2: You take any vector from the vector space V that vector can be written as the linear combination of the vectors of the basis set. There are two types of vector spaces; finite dimensional vector spaces and infinite dimensional vector spaces. So, if the basis of a vector space V contains the finite number of vectors then we say that it is a finite dimensional vector space.

Otherwise, we say that the vector space is an infinite dimensional vector space a vector in basis is called the basis vector.

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$B = \{v_1, v_2, \dots, v_n\}$ is a basis of a n -dimensional vector space V , if

- ① $\{v_1, v_2, \dots, v_n\}$ is linearly independent
- ② For any vector $v \in V$, we have
$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$
where, $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars from the field F

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So, if we talk this definition mathematically so, what we are having? A set B having vectors let us say v_1, v_2 up to v_n is a basis of a n dimensional vector space V , if the set of vectors v_1, v_2, v_n is linearly independent that is the first thing we need. And the second thing is for any vector v belongs to the vector space V , we have v equals to $\alpha_1 v_1$ plus $\alpha_2 v_2$ plus $\alpha_n v_n$, where $\alpha_1, \alpha_2, \alpha_n$ are scalars form the field F . So, this is mathematically we can define basis in this way also.

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The image shows a slide with handwritten text under the heading "Examples". It contains three numbered examples:

- ① If $V = \mathbb{R}^2(\mathbb{R})$, then $\{(1,0), (0,1)\}$
 $(\alpha, \beta) \in \mathbb{R}^2$, $(\alpha, \beta) = \alpha(1,0) + \beta(0,1)$
- ② $V = \mathbb{R}^3(\mathbb{R})$, $\{(1,0,0), (0,1,0), (0,0,1)\}$
- ③ $V = \mathbb{R}^n(\mathbb{R})$, $\{(1,0,\dots,0), (0,1,\dots,0), \dots, (0,0,\dots,1)\}$

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So, if you see some example of the basis, so, if you take V equals to \mathbb{R}^2 over the field \mathbb{R} , then if you take vectors like 1, 0 and 0, 1 in \mathbb{R}^2 then this set forms a basis for V .

The first thing both of these vectors are linearly independent and the second thing you take any vector V from \mathbb{R}^2 , we can write that vector; let us say you are taking some vector α , β arbitrary vector belongs to \mathbb{R}^2 then we can write α β is α times 1, 0 plus β times 0, 1.

So, what I want to say that this set expands whole \mathbb{R}^2 space. Similarly if you take V equals to \mathbb{R}^3 over the field of real numbers, then one of the possible basis is 1, 0, 0; 0, 1, 0 and 0, 0, 1. If you take V equals to \mathbb{R}^n over the field of real numbers then one of the possible basis is 1 0 0 0 0 1 0 0 0 1.

So, all these are n tuples vector having one of the element as 1 and rest of the elements are 0.
 So, all these 3 are called a standard basis for \mathbb{R}^2 , \mathbb{R}^3 and \mathbb{R}^n respectively.

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(iv) $V = \{ M_{2 \times 2} \mid \text{set of all } 2 \times 2 \text{ real matrices} \}$
 $V(\mathbb{R})$ forms a vector space.
 Now $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$
 4
 (v) $V = \{ M_{2 \times 2} \mid \text{set of all } 2 \times 2 \text{ real sym. matrices} \}$
 $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$
 3

If we talk if we take a vector space like this so, $\mathbb{R}^{2 \times 2}$ matrix is having real entries.

So, so, certainly this V over the field of real number forms a vector space. We have seen it in previous lectures. Now, what will be the basis of this? So, basis of this will be so, this is one of the possible basis for this vector space V . Here you can notice all the vectors those are 2 by 2 matrices are linearly independent and any 2 by 2 matrix can be written as the linear combination of these 4 matrices.

If I change this vector space let us say V equals to $M^{2 \times 2}$ set of all 2 by 2 real symmetric matrix then what will be the basis? Certainly this set over the field of real numbers will form a

vector space. And what will be the basis? One of the possible basis will be given as $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ sorry $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ because this is pairs of symmetric matrices.

So, these two elements will be equal to have symmetric to be the matrix symmetric. So, this basis will be having 3 vectors. So, while this is having 4 vectors. So, these are some of the examples of basis of different vector spaces.


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Dimension

The number of vectors in a basis of $V(F)$ is called the dimension of the vector space $V(F)$.

Examples:

- ① Dimensions of vector space $\mathbb{R}^2(\mathbb{R})$ is 2.
- ② Dimensions of vector space $\mathbb{R}^n(\mathbb{R})$ is n .
- ③ Dimensions of vector space $M_{m \times n}(\mathbb{R})$ is $m \times n$.
- ④ Dimensions of vector space $P_n(\mathbb{R})$ is $n+1$.
- ⑤ Dimensions of vector space P is ∞
where P is the set of all polynomials with coefficients from \mathbb{R} .


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My next definition is dimension. So, what is the dimension of a vector space? The dimension of a vector space is nothing just the number of elements in the basis or number of vectors in the basis.

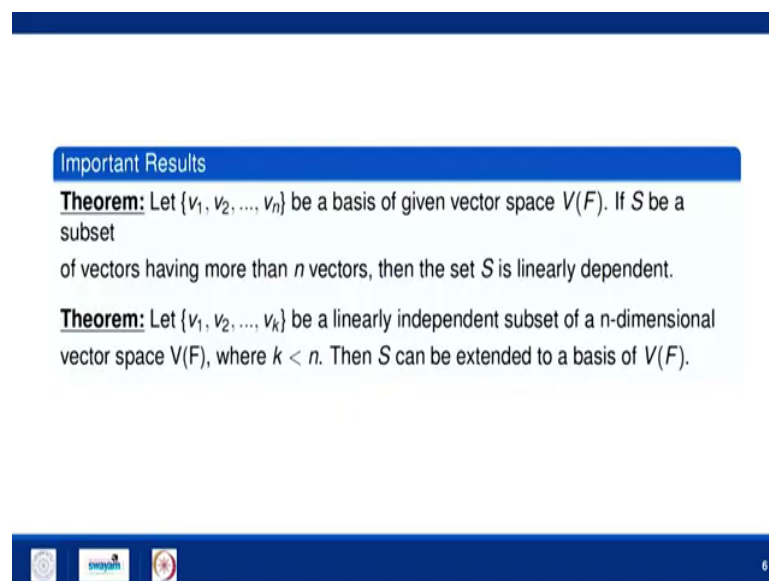
So, formally I can say the number of vectors in a basis of $V(F)$ is called the dimension of the vector space $V(F)$. So, as you have seen example, the dimension of vector space \mathbb{R}^2 over the

field of real number is 2. The dimension of vector space \mathbb{R}^n over the field of real number is n . The dimension of the vector space of all 2×2 real matrices over the field of real number is 4.

If you are taking the vector space of all m by n real matrices over the field of real number then dimension will become m times n . If you take the vector space of all the polynomials having degree n or less then the dimension of that vector space will be $n + 1$ because we will be having $n + 1$ elements in the basis. Similarly, dimension of vector space of all polynomials is infinite. Why? Because it is a infinite dimensional vector space.

We can write any polynomial from this vector space as a linear combination of finite number of vectors from that basis. However, basis will be containing the finite number of elements.

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Important Results

Theorem: Let $\{v_1, v_2, \dots, v_n\}$ be a basis of given vector space $V(F)$. If S be a subset of vectors having more than n vectors, then the set S is linearly dependent.

Theorem: Let $\{v_1, v_2, \dots, v_k\}$ be a linearly independent subset of a n -dimensional vector space $V(F)$, where $k < n$. Then S can be extended to a basis of $V(F)$.

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Now, come to some important results related to the basis and dimension. So, for my first result is let v_1, v_2, \dots, v_n be a basis of a n dimensional vector space V over F . If S be a subset of vectors having more than n vectors then the set S is linearly dependent.

So, what I want to convey that if the dimension of vector space is n , any set containing more than n vectors will be linearly dependent because in a set you can have at most n linearly independent vectors. And if you are having such a set where you are having n linearly independent vector then that set will be a basis of that particular vector space.

So, what in other words I can say that the basis is a maximal linearly independent set of a vector space. If you are having any other vector in that set that vector can be written as a linear combination of rest of the vectors. My second result is let v_1, v_2, \dots, v_k be a linearly independent subset of a n dimensional vector space V over the field F , where k is less than n . Then S can be extended to a basis of V over F .

For example, suppose I am having a vector space \mathbb{R}^5 and I am having 3 linearly independent vectors of \mathbb{R}^5 . So, what I can do? I can include two more linearly independent vectors to that set of 3 vectors and then I will be having 5 linearly independent vectors in \mathbb{R}^5 and then this set will be a basis of \mathbb{R}^5 .

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Important Results

Theorem: Let $V(F)$ be a finite-dimensional vector space. Then any two bases of V have the same number of vectors.

Theorem: Let $V(F)$ be a finite-dimensional vector space and let S_1 and S_2 be two subspaces of V . Then

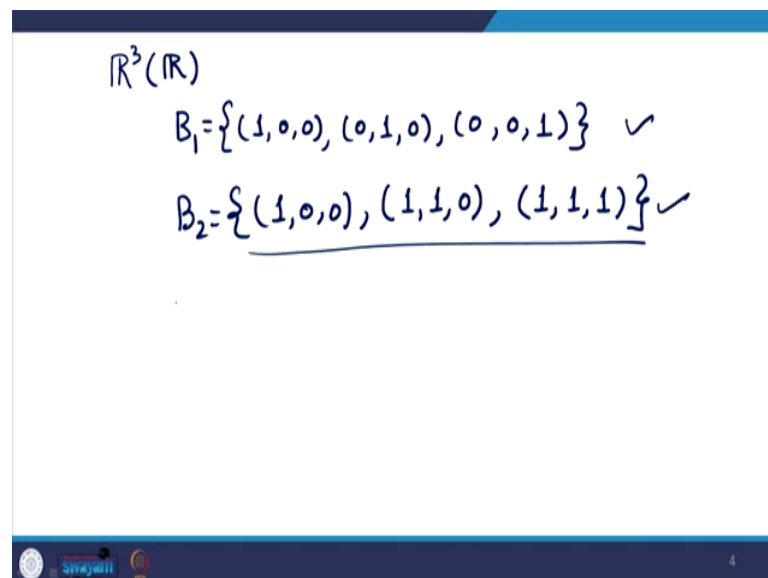
$$\dim(S_1) + \dim(S_2) = \dim(S_1 + S_2) + \dim(S_1 \cap S_2)$$

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Another important result is like let $V(F)$ be a finite dimensional vector space then any two basis of V have the same number of vectors.

We can have multiple basis for a vector space. However, each of the basis will be having the same number of vectors that is equals to the dimension of the vector space.

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The image shows a whiteboard with handwritten mathematical expressions. At the top, it says $\mathbb{R}^3(\mathbb{R})$. Below that, two sets of vectors are listed, each followed by a checkmark. The first set is $B_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. The second set is $B_2 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$, which is underlined.

$$\mathbb{R}^3(\mathbb{R})$$
$$B_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \checkmark$$
$$B_2 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\} \checkmark$$

So, for example, if we talk about \mathbb{R}^3 , you have seen that one of the set of vector $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

This particular set forms a basis for \mathbb{R}^3 over the field of real numbers. If I take another set of 3 vectors in \mathbb{R}^3 let us say $(1, 0, 0)$, $(1, 1, 0)$ and $(1, 1, 1)$. So, again these are 3 vectors in \mathbb{R}^3 and this is a linearly independent set. So, B_2 also a basis for \mathbb{R}^3 . So, we can have many other basis where we are having 3 li vectors from the set \mathbb{R}^3 from the vector space \mathbb{R}^3 as a basis, but the common thing is all these sets will be having 3 vectors.

My next result is let V be a vector space over the field F and it is finite dimensional. If you take two subspaces of V let us say S_1 and S_2 then dimension of S_1 plus dimension of S_2

equals to dimension of S_1 plus S_2 plus dimension of $S_1 \cap S_2$. So, we have written S_1 here it will be S_2 . So, dimension of $S_1 \cap S_2$.

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Examples

(1) Find a basis and dimension of the subspaces S of \mathbb{R}^3 given as


$$S = \{(x_1, x_2, x_3) \mid x_1 + x_2 - x_3 = 0\}$$

$$= \{(x_1, x_2, x_1 + x_2)\}$$

$$= \{x_1(1, 0, 1) + x_2(0, 1, 1)\}$$

Hence basis of S is $\{(1, 0, 1), (0, 1, 1)\}$.
Therefore $\dim(S) = 2$

(2) $W = \{(x_1, x_2, x_3) \mid x_1 = x_2 = x_3\}$
Basis of W is $\{(1, 1, 1)\}$ and $\dim(W) = 1$


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We will see this result by this example. Suppose we need to find out the basis and dimension of the subspaces S of \mathbb{R}^3 given as S equals to x_1, x_2, x_3 ; such that $x_1 + x_2 - x_3$ equals to 0. And we are having another subspace of \mathbb{R}^3 that is W , given by vectors x_1, x_2, x_3 belongs to \mathbb{R}^3 such that $x_1 = x_2 = x_3$. So, how to find basis of these?

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$$\begin{aligned} V &= \mathbb{R}^3(\mathbb{R}) \\ S &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\} \\ &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 = x_3\} \\ &= \{(x_1, x_2, x_1 + x_2) \in \mathbb{R}^3\} \\ &= \{x_1(1, 0, 1) + x_2(0, 1, 1)\} \\ \Rightarrow B_S &= \{(1, 0, 1), (0, 1, 1)\} \\ \dim(S) &= 2 \end{aligned}$$

So, my vector space is \mathbb{R}^3 over the field of real numbers and I am defining my set S as the space of all the vectors x_1, x_2, x_3 belongs to \mathbb{R}^3 ; such that $x_1 + x_2 - x_3$ equals to 0. So, in the previous lecture we have learn that how to prove that S is a sub space of \mathbb{R}^3 .

Now, we need to find out basis of S . So, here I can write x_1, x_2, x_3 belongs to \mathbb{R}^3 such that $x_1 + x_2$ equals to x_3 . So, I can write it $x_1 + x_2$ and since x_3 equals to $x_1 + x_2$. So, I can replace x_3 as $x_1 + x_2$ / this I can write as $x_1, 0$. So, 1 is coming from the first component.

In second component we did not have any x_1 . So, x_1 is 0 and then 1 plus x_2 $0, 1, 1$. So, here I can write the basis of S is containing two vectors $1, 0, 1$ and $0, 1, 1$. Hence dimension of S equals to 2 .

So, in that way we can find out the dimension basis and dimension of a vector space.

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$$\begin{aligned} V &= \mathbb{R}^3(\mathbb{R}) \\ W &= \{(x_1, x_2, x_3) \mid x_1 = x_2 = x_3\} \\ &= \{(x_1, x_1, x_1) \in \mathbb{R}^3\} \\ &= \{x_1(1, 1, 1)\} \\ B_W &= \{(1, 1, 1)\} \\ \underline{\dim(W) = 1} \end{aligned}$$

Take another example. There again V equals to \mathbb{R}^3 over the field of real number and we are having W as x_1, x_2, x_3 ; such that x_1 equals to x_2 equals to x_3 . So, what kind of vector we are having in W , where all the 3 components are equal.

So, for example, 1, 1, 1; 2, 2, 2; minus 1, minus 1, minus 1; all these kind of vectors. So, certainly the basis of I can write it x_1, x_1, x_1 because x_2 equals to x_1 and x_3 is also equal to x_1 belongs to R^3 .

So, what is this? It is $x_1, 1, 1$. So, the basis of W is having only one vector 1, 1, 1. So, any vector of W can be written as some scalar times this 1, 1, 1. So, dimension of W is 1.

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$$\begin{aligned}
 &\text{Basis of } S \cap W \\
 &S \cap W = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0 \text{ \& } x_1 = x_2 = x_3 \} \\
 &= \{ (x_1, x_2, x_1 + x_2) \mid \underline{x_1 = x_2 = x_3} \} \\
 &= \{ (x_1, x_1, 2x_1) \} \\
 &B_{S \cap W} = \{ (0, 0, 0) \} \\
 &\dim(S \cap W) = \underline{\underline{0}}
 \end{aligned}$$

If you need to find out basis of S intersection W then what is S intersection W ? It is all vectors x_1, x_2, x_3 belongs to R^3 such that x_1 plus x_2 minus x_3 equals to 0, this condition is coming from the subspace S and the condition from sub space W is x_1 equals to x_2 equals to x_3 .

So, these I can write $x_1, x_2, x_1 + x_2$, such that this I have written by this condition such that x_1 equals to x_2 equals to x_3 . So, if I take x_1 equals to x_2 then I can write it x_1 , now x_2 equals to x_1 . So, x_1 and then $x_1 + x_1$ is $2x_1$. However, what I need? I need all the 3 component should be equal.

So, in this case what we can have? The only possibility is that it should have the 0 vector because we need x_1, x_2 and $x_1 + x_2$, all 3 are equal. It will be only when x_1 is 0, x_2 is 0 and so, that $x_1 + x_2$ also become 0. So, this is the basis of $S \cap W$. Hence, dimension of $S \cap W$ is 0. Why 0? Because I told you earlier that the vector space containing the 0 vector can be spanned by the empty set by the basis ϕ empty basis.

So, there is no element in the basis. Hence, basis is the dimension of $S \cap W$ is 0. So, this is the way for finding the basis and then dimensional vector space.

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Continued...


(3) Find the basis and dimension of S_1 , S_2 , $S_1 \cap S_2$.

$$S_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 - x_3 + x_4 = 0, x_1 + x_2 + x_3 + x_4 = 0\}$$
$$S_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 - x_2 - x_3 + x_4 = 0, x_1 + 2x_2 - x_4 = 0\}$$

Solution: Basis of $S_1 = \{(1, 0, 0, -1), (0, 1, 0, -1)\}$
 $\dim(S_1) = 2$

Basis of $S_2 = \{(1, 0, 2, 1), (0, 1, 1, 2)\}$
 $\dim(S_2) = 2$

Basis of $S_1 \cap S_2 = \{(0, 0, 0, 0)\}$



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Let us take one more example. Find the basis and dimension of S_1 , S_2 , $S_1 \cap S_2$, where S_1, S_2 are the subspaces of \mathbb{R}^4 over the field of real numbers.

As I told you the intersection of two subspaces also a subspace. So, again $S_1 \cap S_2$ is also a subspace of \mathbb{R}^4 .

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$$\begin{aligned} S_1 &= \{ (x_1, x_2, x_3, x_4) \mid \begin{array}{l} x_1 + x_2 - x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 0 \end{array} \} \\ &\quad \downarrow \\ &\quad \begin{array}{l} x_4 = -x_1 - x_2 \\ x_3 = x_1 + x_2 + x_4 \\ \quad = 0 \end{array} \\ S_1 &= \{ (x_1, x_2, 0, -x_1 - x_2) \} \\ &= \{ x_1(1, 0, 0, -1) + x_2(0, 1, 0, -1) \} \\ B_{S_1} &= \{ (1, 0, 0, -1), (0, 1, 0, -1) \} \end{aligned}$$

So, let us find out that basis of S_1 . So, S_1 is given as x_1, x_2, x_3, x_4 , belongs to \mathbb{R}^4 such that the first constant is $x_1 + x_2 - x_3 + x_4 = 0$. The second condition is $x_1 + x_2 + x_3 + x_4 = 0$.

So, what we are having here by adding these two conditions what I can write x_4 equals to minus x_1 minus x_2 . Now, from the first condition I can write x_3 equals to $x_1 + x_2 + x_4$. If I substitute the value of x_4 from here that is minus x_1 minus x_2 I can get x_3 equals to 0.

So, let me write it here now using these result. So, $x_1, x_2, x_3 = 0$ and x_4 is minus x_1 minus x_2 . So, it is if I take x_1 , it is $(1, 0, 0, -1)$ plus x_2 $(0, 1, 0, -1)$. So, here basis of S_1 is $(1, 0, 0, -1)$ and then $(0, 1, 0, -1)$.


Hence the dimension of S_1 is 2. Similarly, you can find the dimension of S_2 using the same procedure and you will find that dimension of S_2 is coming 1, 0, 2, 1 and 0, 1, 1, 2. So, hence basis of S_2 is given by these two vectors and dimension of S_2 is 2. So, now, dimension of S_1 is 2, dimension of S_2 is 2. So, dimension of S_1 plus dimension of S_2 is 4 which is equals to the vector space dimension of the vector space \mathbb{R}^4 .

Hence dimension of $S_1 \cap S_2$ is 0. So, the basis of $S_1 \cap S_2$ contains only 0 element, which says that if you put all these 4 conditions together the solution will be x_1 equals to x_2 equals to x_3 equals to x_4 equals to 0. All 4 components are 0. So, in this lecture we have learned the concept of basis and dimension. In the next lecture we will learn another very important concept of mathematics related to the machine learning that is linear transformations.

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These are the references for this lecture. Hope you have enjoyed the lecture.

Thank you, thank you very much.