Essential Mathematics for Machine Learning
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Lecture – 04
Vector Subspaces: Basis and Dimensions

Hello friends. So, welcome to the lecture number 4 of the course Essential Mathematics for Machine Learning. If you remember in the last lecture we have talked about Vector Subspaces. So, today we are again going to talk about a very important concept from the vector spaces that is Basis and Dimension of Vector Subspaces.

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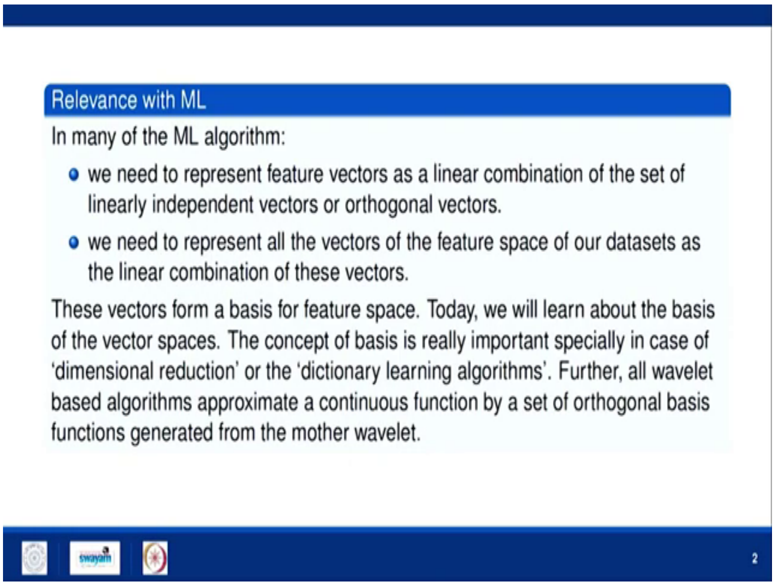

So, in most of the machine learning algorithm we need to represent our input data that is in terms of feature vectors as a linear combination of certain vectors that all the feature vectors we want to write as linear combination of a set of vectors. Those set of vectors may be linearly independent or they may be orthogonal. So, to write all the feature vectors or power data set in terms of linear combination of these vectors we have to find out that particular set of vectors.

So, these vectors form a basis for the feature space. Today we will learn about the basis of the vector spaces. The concept of basis is really important, especially in the case of dimension reduction or the recent algorithm like dictionary learning based algorithm. Further in wavelet based algorithms we approximate any function as the linear combination of set of orthonormal functions those we have generated from the mother wavelet.

So, let us come to the definition of basis.

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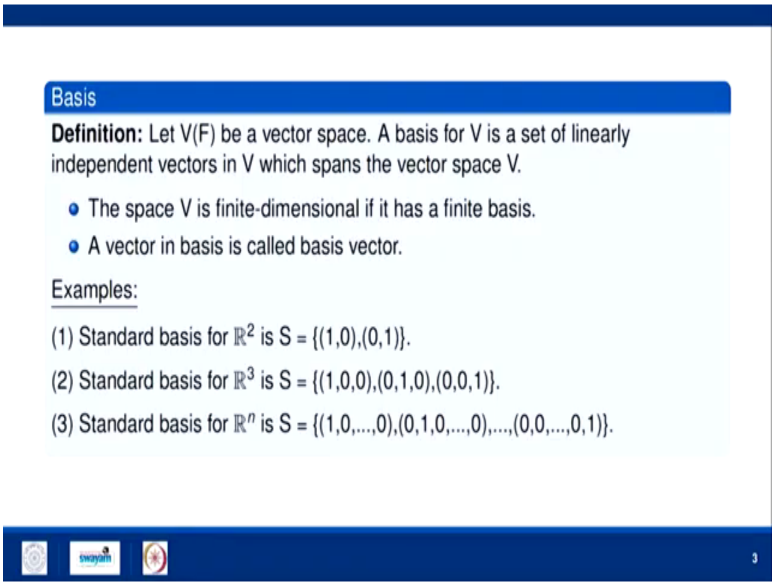

So, mathematically speaking the definition of basis is given as let F be a vector space. So, we are having a vector space V over the field F. A basis for V is a set of linearly independent vectors in V which expands the vector space V. Now, so, what there should be what should be the qualification to be a basis? The first all the vectors of that sets would be linearly independent.

Number 2: You take any vector from the vector space V that vector can be written as the linear combination of the vectors of the basis set. There are two types of vector spaces; finite dimensional vector spaces and infinite dimensional vector spaces. So, if the basis of a vector space V contains the finite number of vectors then we say that it is a finite dimensional vector space.

Otherwise, we say that the vector space is an infinite dimensional vector space a vector in basis is called the basis vector.

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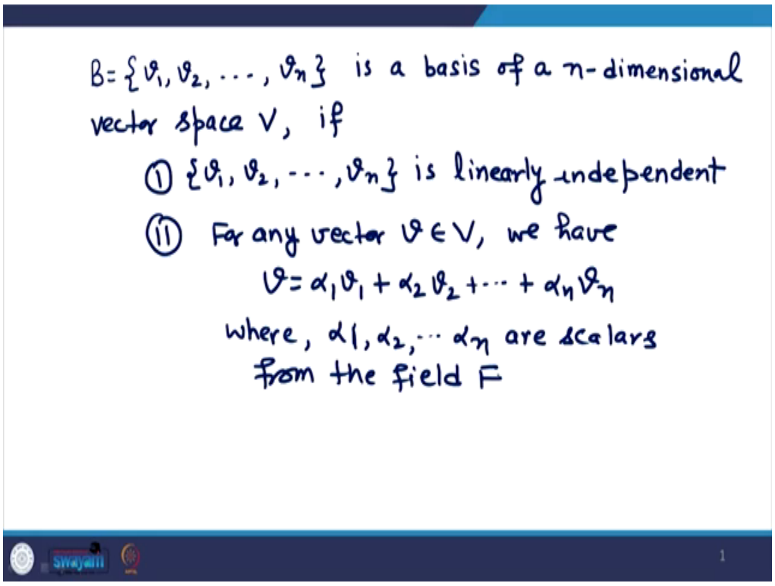

So, if we talk this definition mathematically so, what we are having? A set B having vectors let us say v 1, v 2 up to v n is a basis of a n dimensional vector space V, if the set of vectors v 1, v 2, v n is linearly independent that is the first thing we need. And the second thing is for any vector v belongs to the vector space V, we have v equals to alpha one v 1 plus alpha 2 v 2 plus alpha n v n, where alpha 1, alpha 2, alpha n are scalars form the field F. So, this is mathematically we can define basis in this way also.

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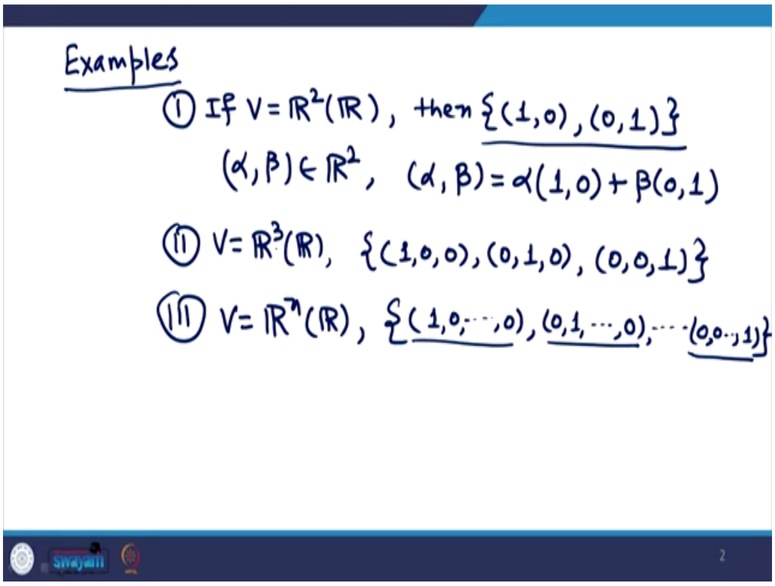

So, if you see some example of the basis, so, if you take V equals to R 2 over the field R, then if you take vectors like 1, 0 and 0, 1 in R 2 then this set forms a basis for V.

The first thing both of these vectors are linearly independent and the second thing you take any vector V from R 2, we can write that vector; let us say you are taking some vector alpha, beta arbitrary vector belongs to R 2 then we can write alpha beta is alpha times 1, 0 plus beta times 0, 1.

So, what I want to say that this set expands whole R 2 space. Similarly if you take V equals to R 3 over the field of real numbers, then one of the possible basis is 1, 0, 0; 0, 1, 0 and 0, 0, 1. If you take V equals to R n R then one of the possible basis is 1 0 0 0 0 1 0 0 0 1.

So, all these are n tuples vector having one of the element as 1 and rest of the elements are 0. So, all these 3 are called a standard basis for R 2, R 3 and R n respectively.

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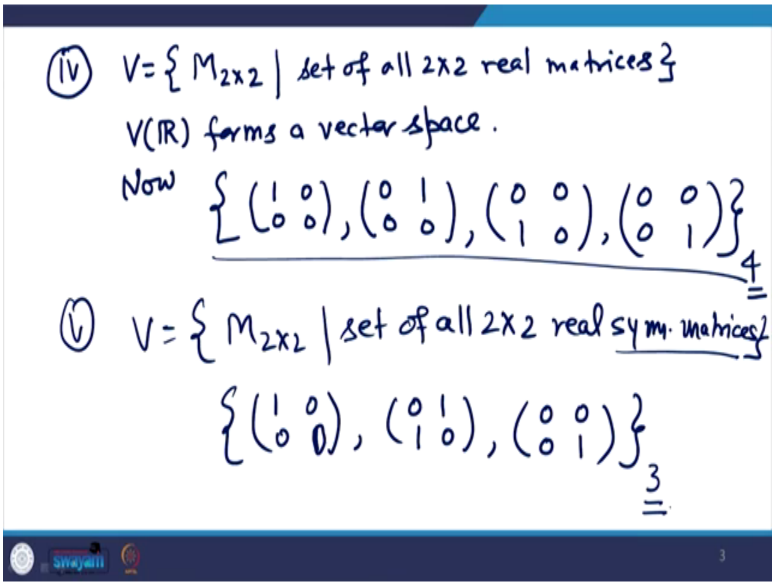

If we talk if we take a vector space like this so, R 2 by 2 matrix is having real entries.

So, so, certainly this V over the field of real number forms a vector space. We have seen it in previous lectures. Now, what will be the basis of this? So, basis of this will be so, this is one of the possible basis for this vector space V. Here you can notice all the vectors those are 2 by 2 matrices are linearly independent and any 2 by 2 matrix can be written as the linear combination of these 4 matrices.

If I change this vector space let us say V equals to M 2 by 2 set of all 2 by 2 real symmetric matrix then what will be the basis? Certainly this set over the field of real numbers will form a vector space. And what will be the basis? One of the possible basis will be given as 1 0 0 1 sorry 1 0 0 0, 0 1 1 0 and 0 0 0 1 because this is pairs of symmetric matrices.

So, these two elements will be equal to have symmetric to be the matrix symmetric. So, this basis will be having 3 vectors. So, while this is having 4 vectors. So, these are some of the examples of basis of different vector spaces.

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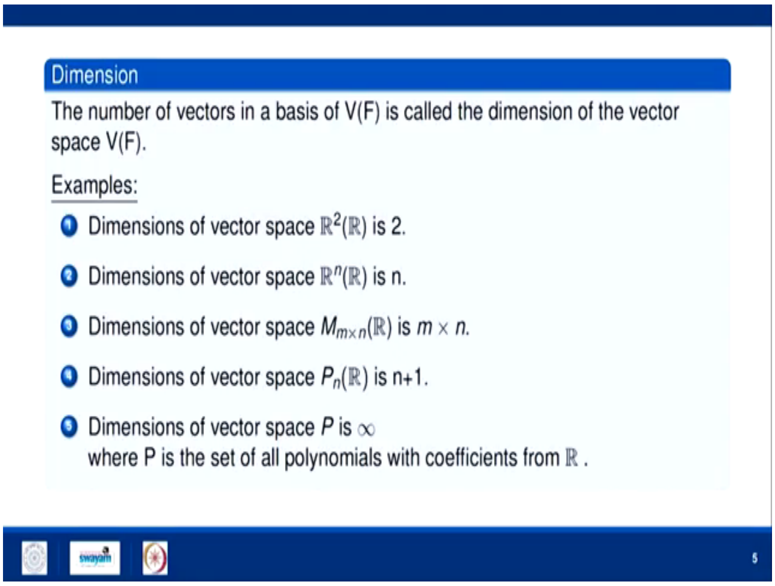

My next definition is dimension. So, what is the dimension of a vector space? The dimension of a vector space is nothing just the number of elements in the basis or number of vectors in the basis.

So, formally I can say the number of vectors in a basis of V F is called the dimension of the vector space V F. So, as you have seen example, the dimension of vector space R 2 over the field of real number is 2. The dimension of vector space R n over the field of real number is n. The dimension of the vector space of all 2 by 2 real matrices over the field of real number is 4.

If you are taking the vector space of all m by n real matrices over the field of real number then dimension will become m times n. If you take the vector space of all the polynomials having degree n or less then the dimension of that vector space will be n plus 1 because we will be having n plus 1 elements in the basis. Similarly, dimension of vector space of all polynomials is infinite. Why? Because it is a infinite dimensional vector space.

We can write any polynomial from this vector space as a linear combination of finite number of vectors from that basis. However, basis will be containing the in finite number of elements.

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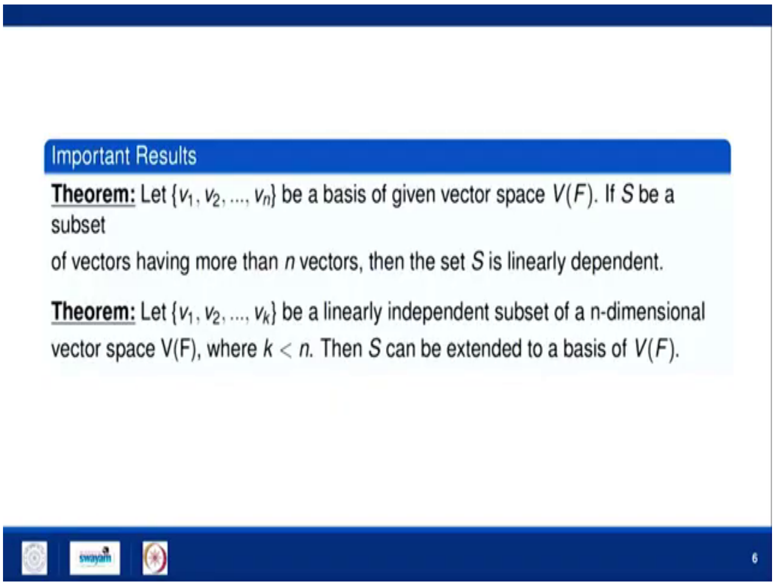

Now, come to some important results related to the basis and dimension. So, for my first result is let v 1, v 2, v n be a basis of a n dimensional vector space V F. If S be a subset of vectors having more than n vectors then the set S is linearly dependent.

So, what I want to convey that if the dimension of vector space is n, any set containing more than n vectors will be linearly dependent because in a set you can have at most n linearly independent vectors. And if you are having such a set where you are having n linearly independent vector then that set will be a basis of that particular vector space.

So, what in other words I can say that the basis is a maximal linearly independent set of a vector space. If you are having any other vector in that set that vector can be written as a linear combination of rest of the vectors. My second result is let v 1, v 2, v k be a linearly independent subset of a n dimensional vector space V over the field F, where k is less than n. Then S can be extended to a basis of V F.

For example, suppose I am having a vector space R 5 and I am having 3 linearly independent vectors of R 5. So, what I can do? I can include two more li vectors to that set of 3 vectors and then I will be having 5 linearly independent vectors in R 5 and then this set will be a basis of R 5.

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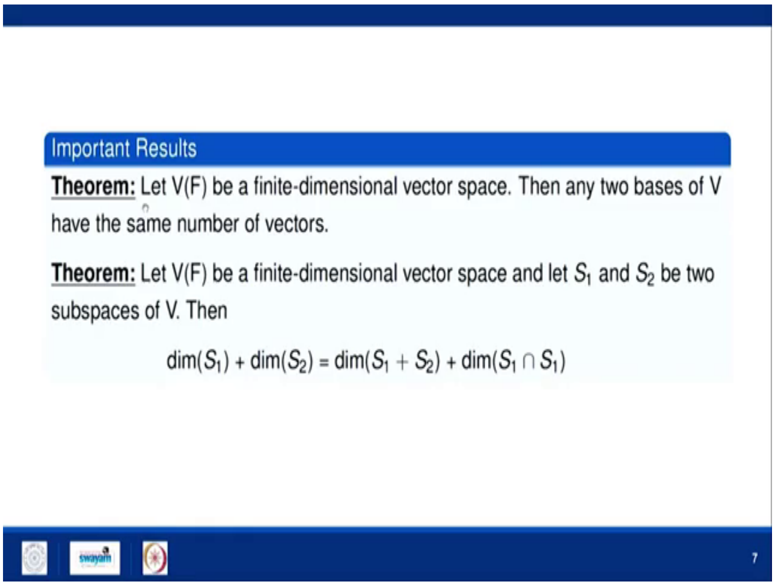

Another important result is like let V F be a finite dimensional vector space then any two basis of we have the same number of vectors.

We can have multiple basis for a vector space. However, each of the basis will be having the same number of vectors that is equals to the dimension of the vector space.

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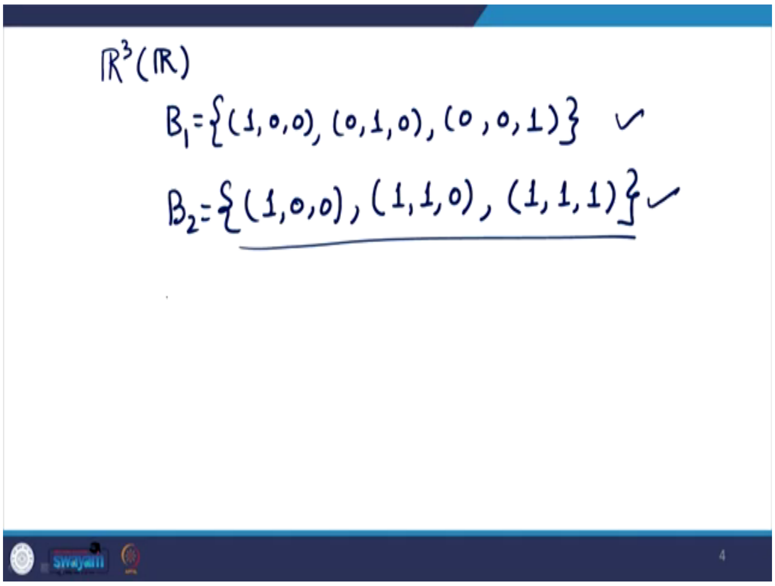

So, for example, if we talk about R 3 R, you have seen that one of the set of vector 1, 0, 0; 0, 1, 0 and 0, 0, 1.

This particular set forms a basis for R 3 over the field of real numbers. If I take another set of 3 vectors in R 3 let us say 1, 0, 0; 1, 1, 0 and 1, 1, 1. So, again these are 3 vectors in R 3 and this is a linearly independent set. So, B 2 also a basis for R 3. So, we can have many other basis where we are having 3 li vectors from the set R 3 from the vector space R 3 as a basis, but the common thing is all these sets will be having 3 vectors.

My next result is let V be a vector space over the field F and it is finite dimensional. If you take two subspaces of V let us say S 1 and S 2 then dimension of S 1 plus dimension of S 2 equals to dimension of S 1 plus S 2 plus dimension of S 1 intersection S 2. So, we have written S 1 here it will be S 2. So, dimension of S 1 intersection S 2.

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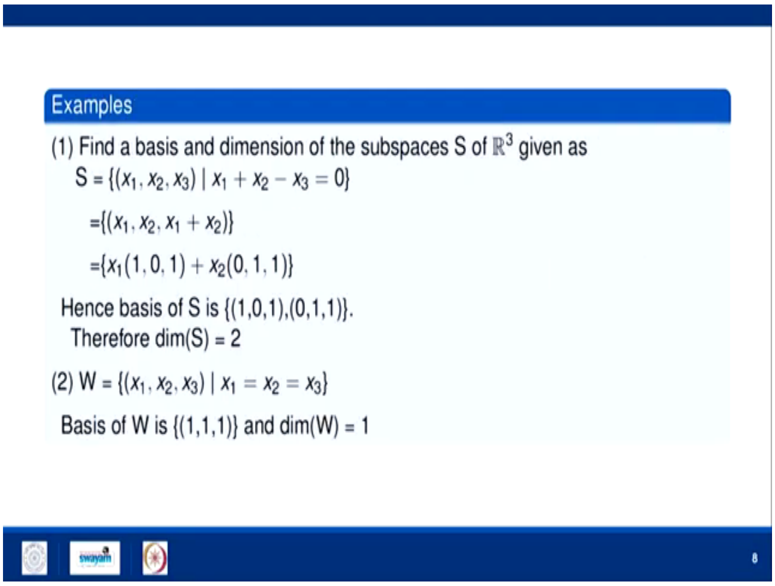

We will see this result by this example. Suppose we need to find out the basis and dimension of the sub spaces S of R 3 given as S equals to x 1, x 2, x 3; such that x 1 plus x 2 minus x 3 equals to 0. And we are having another subspace of R 3 that is W, given by vectors x 1, x 2, x 3 belongs to R 3 such that x 1 equals to x 2 equals to x 3. So, how to find basis of these?

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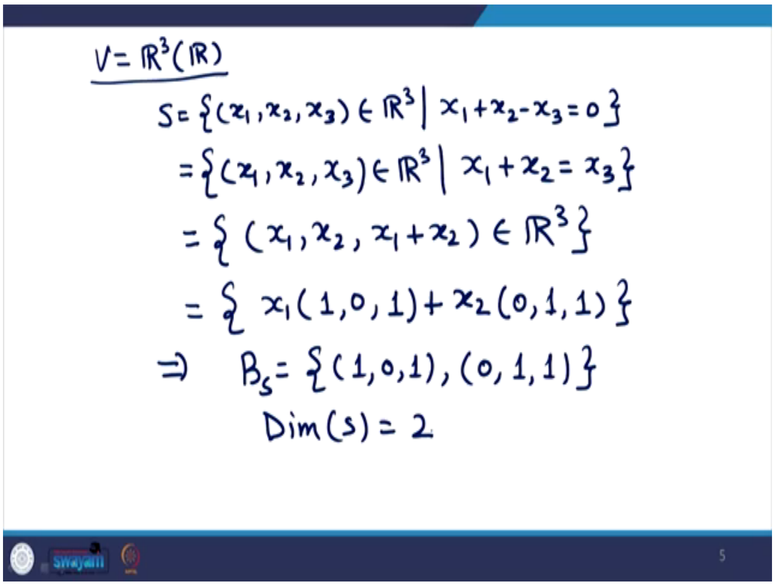

So, my vector space is R 3 over the filed of real numbers and I am defining my set S as the space of all the vectors x 1, x 2, x 3 belongs to R 3; such that such that x 1 plus x 2 minus x 3 equals to 0. So, in the previous lecture we have learn that how to prove that S is a sub space of R 3.

Now, we need to find out basis of S. So, here I can write x 1, x 2, x 3 belongs to R 3 such that x 1 plus x 2 equals to x 3. So, I can write it x 1 x 2 and since x 3 equals to x 1 plus x 2. So, I can replace x 3 as x 1 plus x 2/ this I can write as x 1 1, 0. So, 1 is coming from the first component.

In second component we did not have any x 1. So, x 1 is 0 and then 1 plus x 2 0, 1, 1. So, here I can write the basis of S is containing two vectors 1, 0, 1 and 0, 1, 1. Hence dimension of S equals to 2.

So, in that way we can find out the dimension basis and dimension of a vector space.

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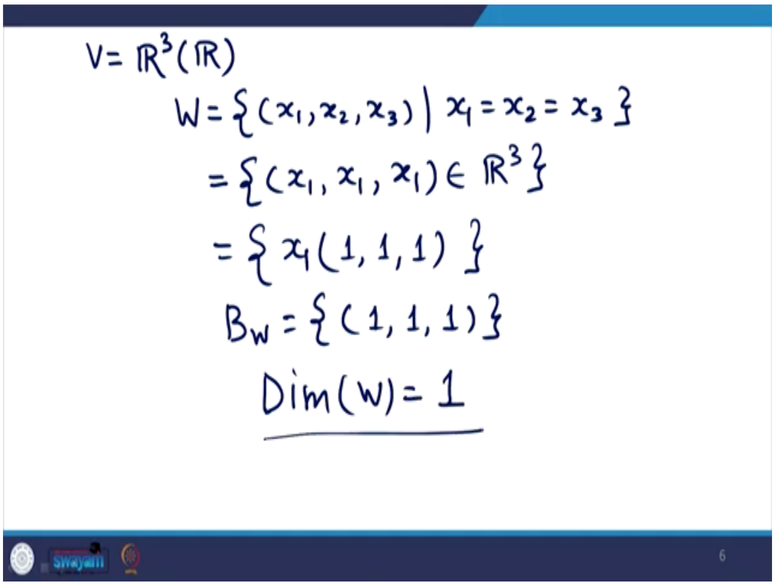

Take another example. There again V equals to R 3 over the field of real number and we are having W as x 1, x 2, x 3; such that x 1 equals to x 2 equals to x 3. So, what kind of vector we are having in W, where all the 3 components are equal.

So, for example, 1, 1, 1; 2, 2, 2; minus 1, minus 1, minus 1; all these kind of vectors. So, certainly the basis of I can write it x 1, x 1, x 1 because x 2 equals to x 1 and x 3 is also equal to x 1 belongs to R 3.

So, what is this? It is x 1, 1, 1. So, the basis of W is having only one vector 1, 1, 1. So, any vector of W can be written as some scalar times this 1, 1, 1. So, dimension of W is 1.

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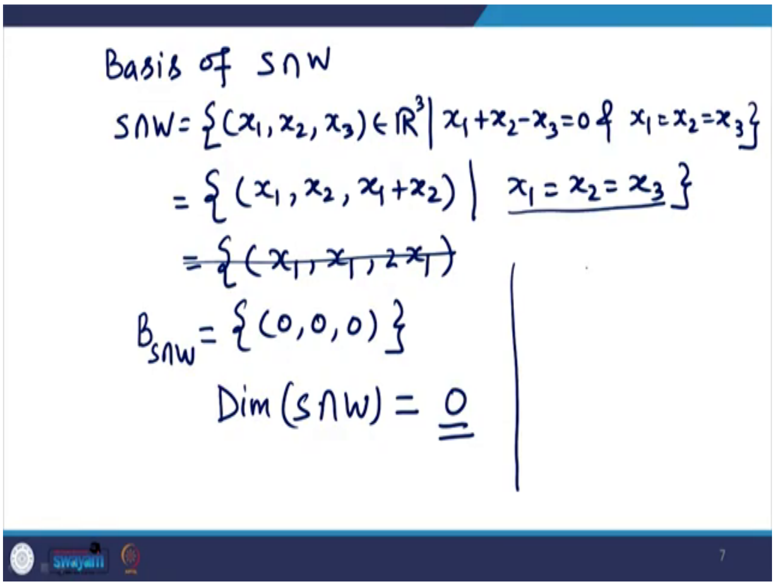

If you need to find out basis of S intersection W then what is S intersection W? It is all vectors x 1, x 2, x 3 belongs to R 3 such that x 1 plus x 2 minus x 3 equals to 0, this condition is coming from the subspace S and the condition from sub space W is x 1 equals to x 2 equals to x 3.

So, these I can write x 1, x 2, x 1 plus x 2, such that this I have written by this condition such that x 1 equals to x 2 equals to x 3. So, if I take x 1 equals to x 2 then I can write it x 1, now x 2 equals to x 1. So, x 1 and then x 1 plus x 1 is 2 x 1. However, what I need? I need all the 3 component should be equal.

So, in this case what we can have? The only possibility is that it should have the 0 vector because we need x 1, x 2 and x 1 plus x 2, all 3 are equal. It will be only when x 1 is 0, x 2 is 0 and so, that x 1 plus x 2 also become 0. So, this is the basis of S intersection W. Hence, dimension of S intersection W is 0. Why 0? Because I told you earlier that the vector space containing the 0 vector can be spanned by the empty set by the basis phi empty basis.

So, there is no element in the basis. Hence, basis is the dimension of x intersection W is 0. So, this is the way for finding the basis and then dimensional vector space.

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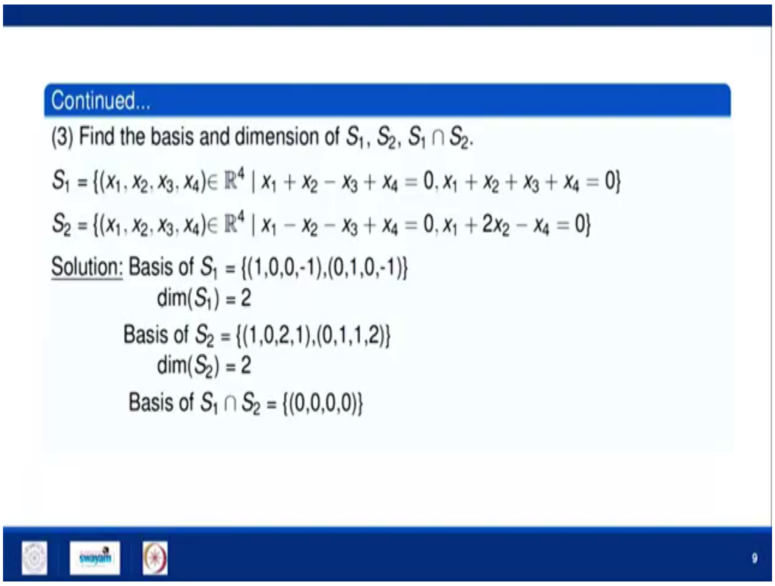

Let us take one more example. Find the basis and dimension of S 1, S 2, S 1 intersection S 2, where S 1 S 2 are the subspaces of R 4 over the field of real numbers.

As I told you the intersection of two subspaces also a subspace. So, again S 1 intersection S 2 is also a subspace of R 4.

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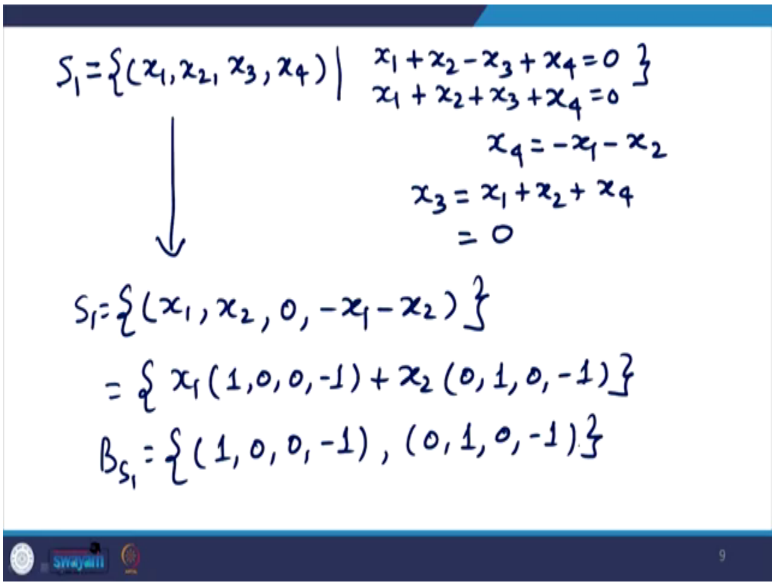

So, let us find out that basis of S 1. So, S 1 is given as x 1, x 2, x 3, x 4, belongs to R 4 such that the first constant is x 1 plus x 2 minus x 3 plus x 4 equals to 0. The second condition is x 1 plus x 2 plus x 3 plus x 4 equals to 0.

So, what we are having here by adding these two conditions what I can write x 4 equals to minus x 1 minus x 2. Now, from the first condition I can write x 3 equals to x 1 plus x 2 plus x 4. If I substitute the value of x 4 from here that is minus x 1 minus x 2 I can get x 3 equals to 0.

So, let me write it here now using these result. So, x 1 x 2 x 3 0 and x 4 is minus x 1 minus x 2. So, it is if I take x 1, it is 1, 0, 0, minus 1 plus x 2 0, 1, 0, minus 1. So, here basis of S 1 is 1, 0, 0, minus 1 and then 0, 1, 0, minus 1.

Hence the dimension of S 1 is 2. Similarly, you can find the dimension of S 2 using the same procedure and you will find that dimension of S 2 is coming 1, 0, 2, 1 and 0, 1, 1, 2. So, hence basis of S 2 is given by these two vectors and dimension of S 2 is 2. So, now, dimension of S 1 is 2, dimension of S 2 is 2. So, dimension of S 1 plus dimension of S 2 is 4 which is equals to the vector space dimension of the vector space R 4.

Hence dimension of S 1 intersection S 2 is 0. So, the basis of S 1 intersection S 2 contains only 0 element, which says that if you put all these 4 conditions together the solution will be x 1 equals to x 2 equals to x 3 equals to x 4 equals to 0. All 4 components are 0. So, in this lecture we have learned the concept of basis and dimension. In the next lecture we will learn another very important concept of mathematics related to the machine learning that is linear transformations.

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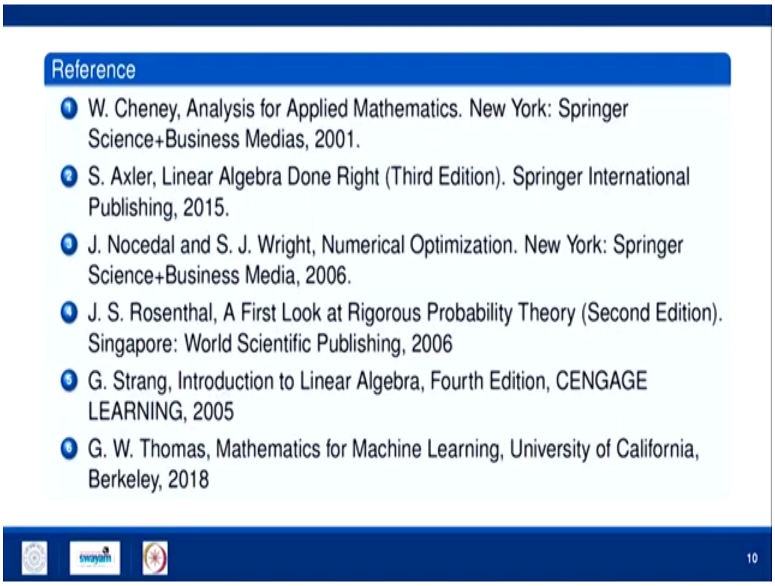

These are the references for this lecture. Hope you have enjoyed the lecture.

Thank you, thank you very much.