Essential Mathematics for Machine Learning
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Lecture – 03
Vector Subspaces, their Examples and Properties

Hello friends. So, welcome to the lecture 3 of this course Essential Mathematics for Machine Learning. So, in the last lecture, we have seen the definition of vector spaces, some of the example of vector spaces. In this lecture, we will learn about Vector Subspaces, their some of their Examples and Properties. Again, our field will be the field of real numbers that is R.

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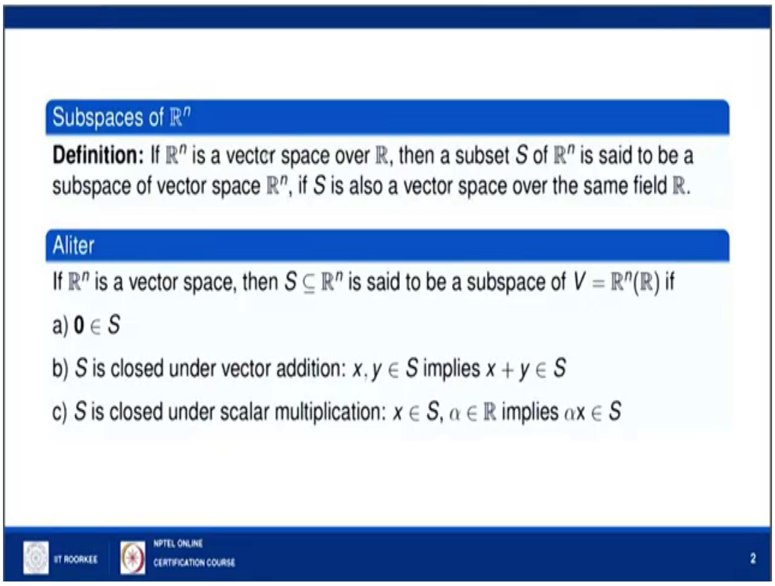

So, we start with the definition of subspaces. So, if R n is a vector space over R. So, here we are saying vector space R n, but in general it can be any vector space V. Then a subset of R n is said to be a subspace of vector space R n, if S is also a vector space over the same field R means the set of vectors of S also satisfy all those six properties which we have given in the definition of vector space.

As I told you, here I am writing R n due to machine learning application because in machine learning, we will see usually the vector space R n where n may be any integers. However, a the definition holds for any vector space V over the field f.

The alternate definition of this can be given as if R n is a vector space, then a subset S of R n is said to be a subspace of R n over the field R if the 0 vector belongs to S so, whatever 0 vector we are having for V or R n that is also part of S. S is closed under vector addition that is if x, y are two vectors from S, then their edition x plus y is also a vector in S and S is closed under scalar multiplication means if you take a vector x from S and a scalar alpha from the field R, then alpha time x which should be again a vector of S. So, this is the definition of subspaces.

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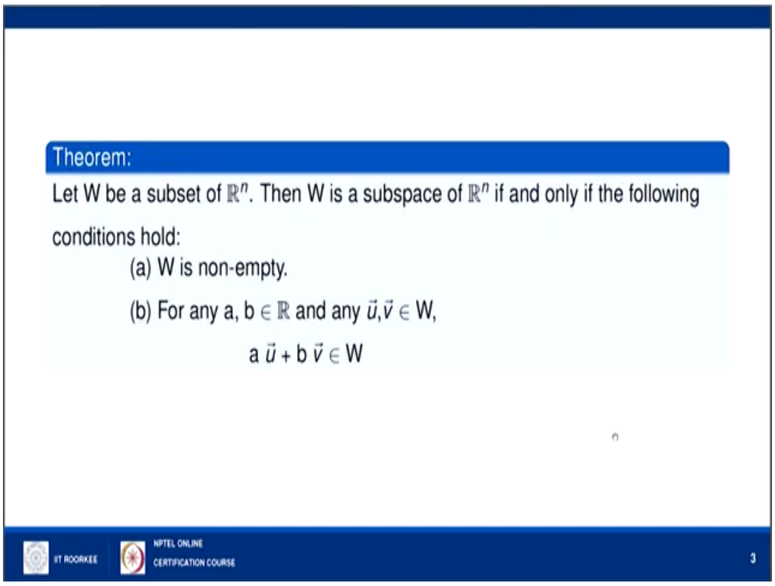

We are having a very important result about subspaces that is which is coming from the alternate definition only that let W be a subset of R n. Then W is a subspace of R n over the same field if and only if the following conditions hold. The first condition is W is non-empty and from where we are looking it that there should be 0 vector in W.

So, if there will be 0 vector, it will not be empty. Then, if you take any two scalars a, b belongs to R and two vectors u, v from W, then a times u from the previous definition a times u is again a vector of W. If W is a subspace, b times v is again a vector of W because W is a subspace, then a u plus b v is also a vector of w.

So, this particular theorem is just coming from the definition which we have seen in earlier slide that is the alternate definition. We can see from these three properties; we can show all six properties of a vector space. So, these three properties implies all those six properties.

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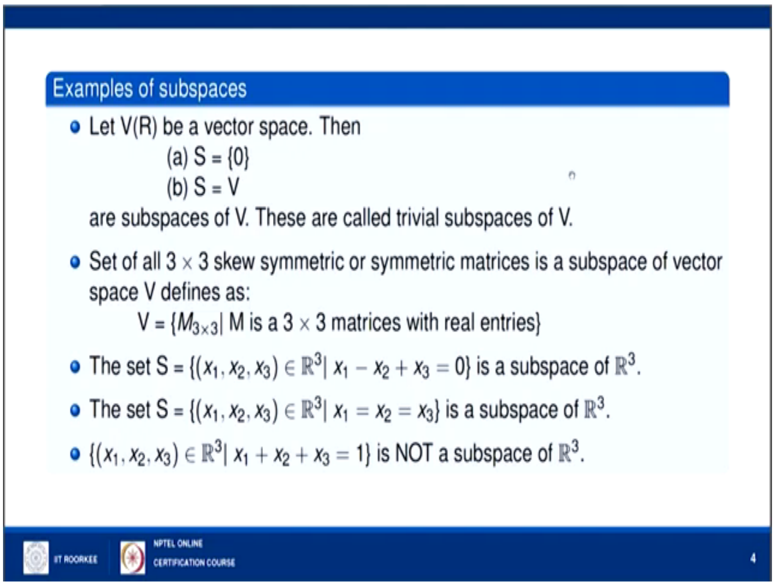

Now, come to the example of a subspaces. So, if V be a vector space over the field R then, S equals to 0 means S is the subset of V containing only 0 vector and S equals to V the whole set V are subspaces of V. This sub spaces are called trivial subspaces of v.

Set of all 3 by 3 skew schematic or symmetric matrices is a subspace of a vector space V define as V equals to M 3 by 3. So, like all 3 by 3 matrices having real entries forms a vector space if you take subset of the set of all 3 by 3 matrices, let us say a subset containing all the symmetric matrices, then this set will form a subspace of the earlier vector space. Similarly, the set S x 1, x 2, x 3 belongs to R 3 where x 1 minus x 2 plus x 3 equals to 0 is a subspace of R 3.

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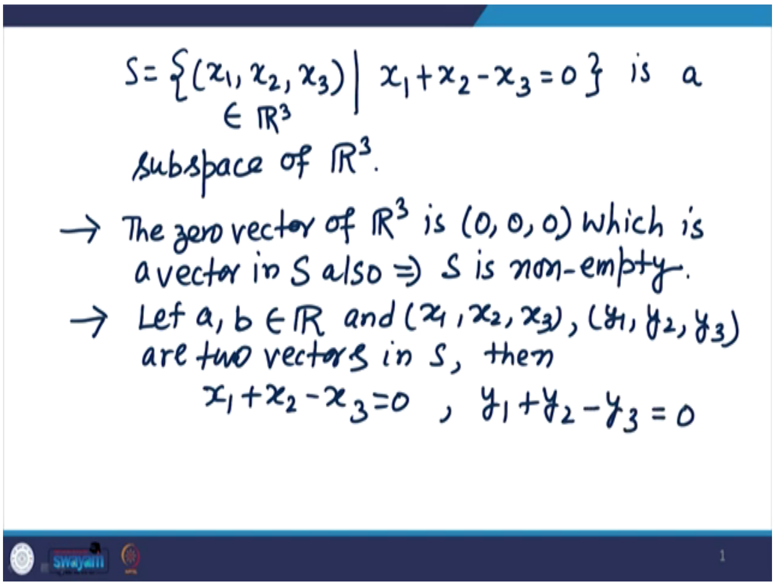

Let us try to show it. So, we are saying that the set S which is having element x 1, x 2, x 3 such that x 1 plus x 2 minus x 3 equals to 0. So, these belongs to R 3 is a subspace of R 3. So, how to show it? So, we will say first the zero vector of R 3 is given by 0, 0, 0 the origin of three-dimensional coordinate space.

Now, here x 1 is 0, x 2 is 0, x 3 is 0. So, if I take 0 plus 0 minus 0 it will be 0 only which is a vector in S also this implies S is non-empty. Now, let a, b are two scalars belongs to R and x 1, x 2, x 3, y 1, y 2, y 3 are two vectors in S, then what we are having since x 1, x 2, x 3 is a vector in S so, x 1 plus x 2 minus x 3 0 and y 1, y 2, y 3 is also a vector in S so, y 1 plus y 2 minus y 3 equals to 0.

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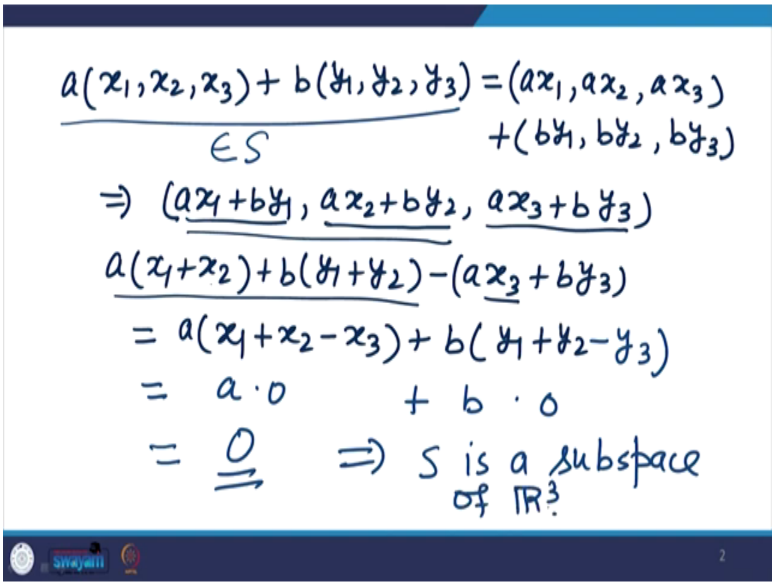

Now, a times x 1, x 2, x 3 plus b times y 1, y 2, y 3. So, what we are having here we have to show that it is also a vector of S. So, it will become ax 1, ax 2, ax 3 plus b y 1, b y 2, b y 3. It is ax 1 plus b y 1, ax 2 plus b y 2, ax 3 plus b y 3. So, to be this vector in S this component plus this component minus this component must be 0. So, let us see.

So, we are saying a x 1 plus x 2 plus b y 1 plus y 2 minus the third component. So, I have written these this particular thing as the sum of these two vectors. This equals to a times x 1 plus x 2 minus x 3 plus because x 1 plus x 2 from here and x 3 is coming from here b times y 1 plus y 2 minus y 3. This equals to a time 0 plus b time 0 because x 1, x 2, x 3 is a vector of S so, x 1 plus x 2 minus x 3 is 0. Similarly, y 1, y 2, y 3 is a vector of S so, y 1 plus y 2 minus y 3 is 0 and this equals to 0. It implies this addition is also belongs to S.

So, if I take any two arbitrary vector of S and the two scalars a and b, a times first vector plus b times second vector is also an element of S. So, from the previous result, we are saying that the set S is non-empty because zero vector is there and then, it is satisfying for any two vectors u and v and the two scalars a and b, a u times b v is also an element of S. So, hence S is a subspace of R 3.

In the same way, the set S x 1, x 2, x 3 belongs to R 3 where x 1 equals to x 2 equals to x 3 is also a subspace of R 3. So, what kind of vectors we will be having in this subspace where all the three components are equal means the first components, second components, third components alpha, alpha, alpha where alpha is a real number. So, certainly 0, 0, 0 when you take alpha equals to 0 will be there. So, S is non-empty.

Now, if you take two vectors alpha, alpha, alpha and beta, beta, beta then the sum will become alpha plus beta, alpha plus beta, alpha plus beta so, all the three components are equal. So, then the sum is also a vector in S. Similarly, if you take any x 1, x 2, x 3 and multiply with alpha let us say a so, a x 1, a x 1, a x 1. So, it will be all three components will be equal. So, all the properties hold for this particular example also.

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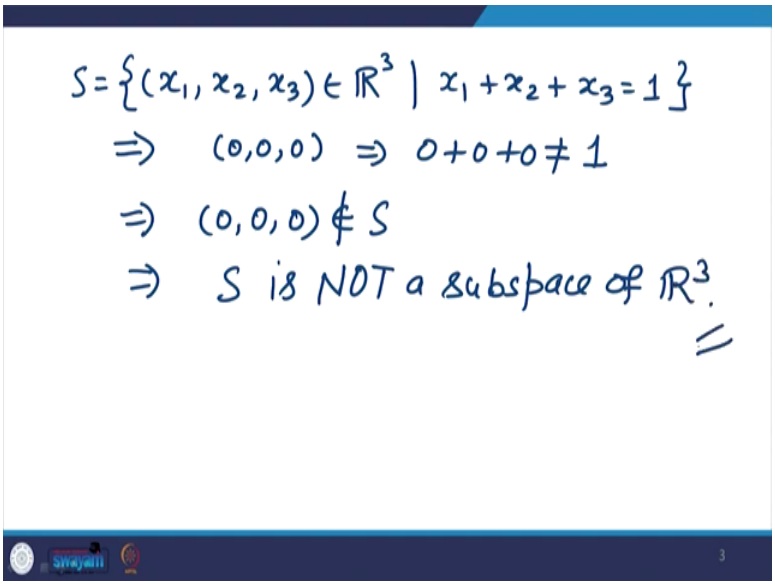

Now, come to the last one x 1, x 2, x 3 belongs to R 3 x 1 plus x 2 plus x 3 equals to 1 is not a subspace of R 3. So, what I am saying? I am taking a set S x 1, x 2, x 3 belongs to R 3 such that, x 1 plus x 2 plus x 3 equals to 1. So, what I am having from here and so, if I take zero vector, then 0 plus 0 plus 0 is not equals to 1. What it is saying that the zero vector of R 3 is not in S and hence, S is not a subspace of R 3.

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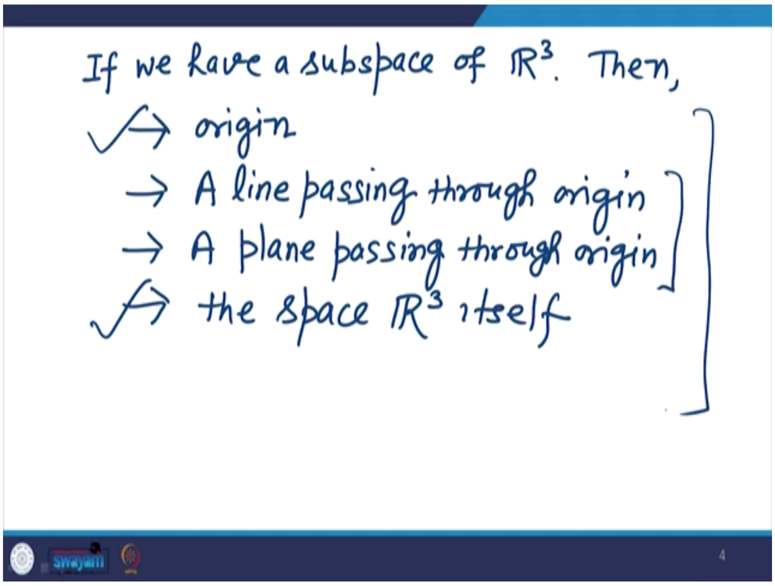

Further, if we have a subspace of R 3, then what are the possibilities? Either it will be origin, or it will be a line passing through origin or it will be a plane passing through origin. So, here I am telling you the geometrical interpretation of a subspace of R 3. So, if you are having a subspace of R 3, either it will be origin that is 0, 0, 0 because that will be a trivial subspace of R 3 or a line passing through origin or a plane passing through origin or the space R 3 itself.

So, first and last are trivial subspaces, while the middle two are nontrivial subspaces. So, only these four possibilities we are having geometrically for subspaces of R 3.

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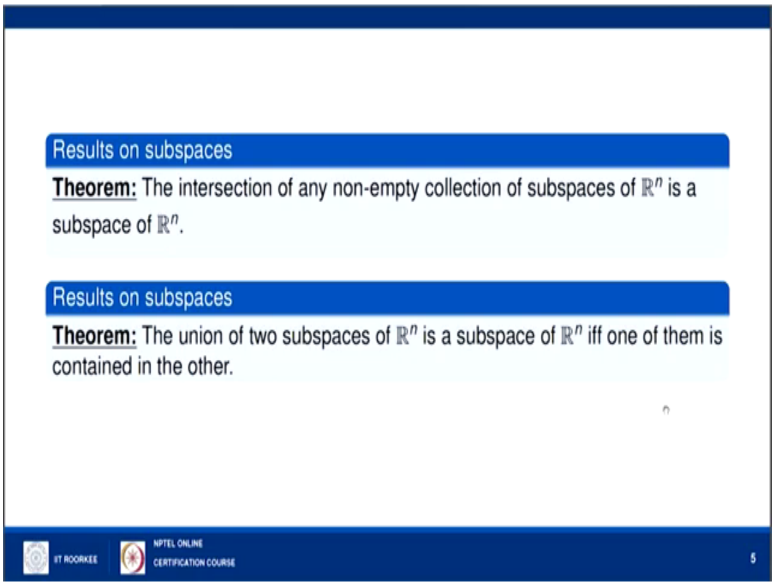

We are having some of the important results on the subspaces. One of the important result is the intersection of any non-empty collection of subspaces of R n is a subspace of R n or in simple language I can say, if you are having two subspaces of a vector space V, then their intersection will again be a subspace.

In case of union, what we can say? In general, the union of two subspaces of a vector space may not be a subspace. When it will be subspace then one of them is contained in other.

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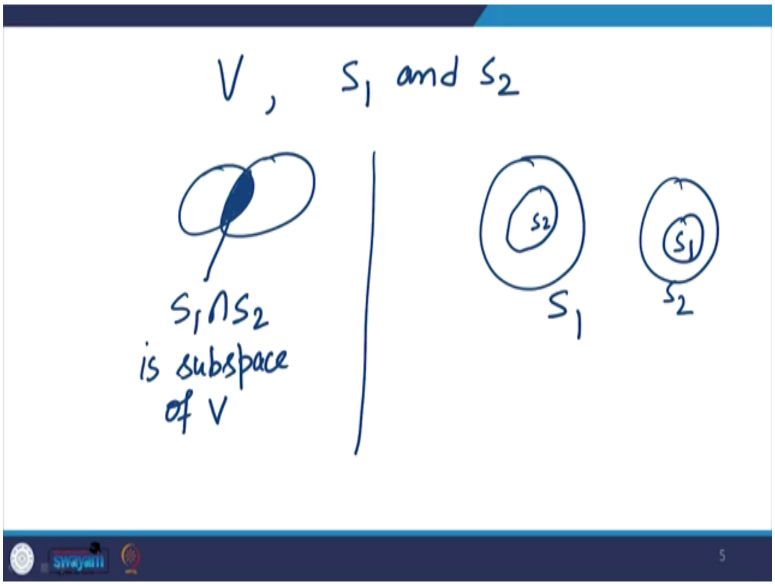

So, what I want to say if you are having let us say a vector space V and you are having two subspaces S 1 and S 2. So, if you take the intersection so, this is intersection. So, if S 1, S 2 are subspaces of V, then this sub intersection will be always a subspace of V.

But, we are talking about union so, if you are having a vector space sorry subspace S 1 and another vector subspace S 2, then their union may not be a subspace, but it will be a subspace certainly when one is contained in other, either this case or if this is S 2 and this is S 1. So, either S 2 is contained in S 1 or S 1 is contained in S 2.

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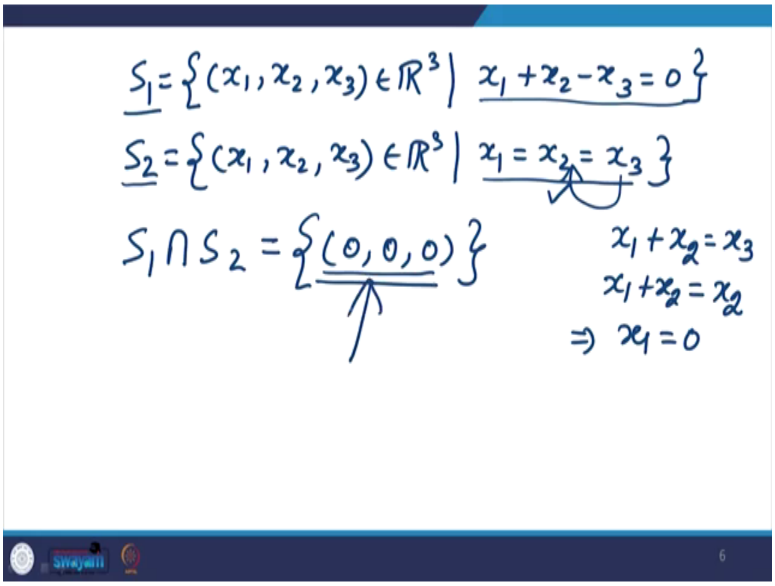

How to see the intersection of subspaces? I have given two examples after the definition of subspaces one was we are having x 1, x 2, x 3 belongs to R 3 x 1 plus x 2 minus x 3 equals to 0 so and we have shown that S 1 is a subspace of R 3. Another example I have given that time that x 1, x 2, x 3 belongs to R 3 such that, x 1 equals to x 2 equals to x 3. So, again this S 2 is also a subspace of R 3.

What kind of element we will be having in S 2? 1, 1, 1, 2, 2, 2, 0, 0, 0 where all the three components are equal and what kind of element we will be having in S 1 where the sum of first two components equals to third component. For example, 1, 1, 2, 3, 4, 7 like this so, 3 plus 4 equals to the third component 7 0, 0, 0.

What is the intersection of S 1 and S 2? So, here intersection will be only the zero vector. Because this vector is satisfying both of these properties. No other vector in R 3 will satisfy this particular property because this property saying that all the three components are equal and this is saying the sum of first two component equals to the third one. So, what we are saying from the first one, x 1 plus x 2 equals to x 3 and from the second one, x 1 equals to x 2 equals to x 3.

So, if I write x 1 plus x 2 so, from the second one, x 3 equals to x 2 so, equals to x 2. So, from here what I will get x 1 equals to 0 and if x 1 is 0 so, x 2 and x 3. So, only this vector will be in their intersection and it is a trivial subspace of R 3.

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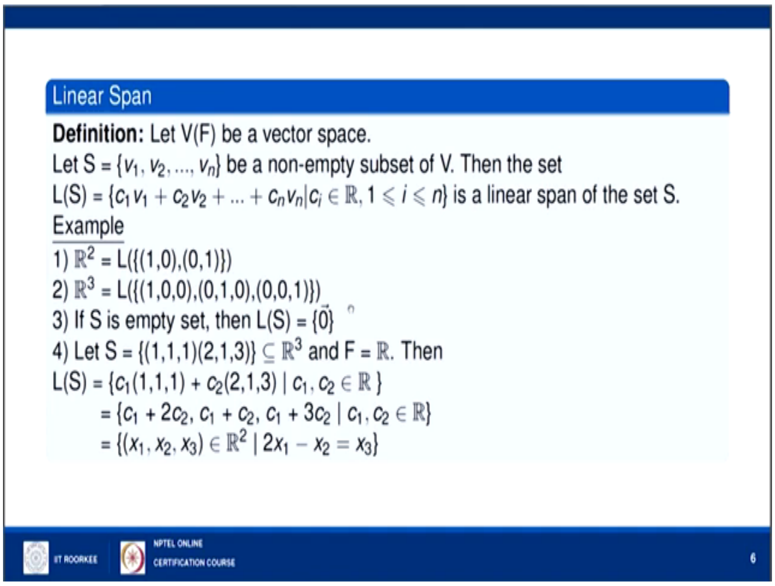

Now, we are coming another important definition that is linear span. So, let V F be a vector space. Let S v 1, v 2, v n be a non-empty subset of V. So, S is a set of vectors where vectors are coming from V. Then the set L s defined by c 1 v 1 plus c 2 v 2 up to c n v n that is the linear combination of vectors v 1, v 2, v n where all these coefficient c 1, c 2, c n are coming from set of real numbers is a linear span of the set S.

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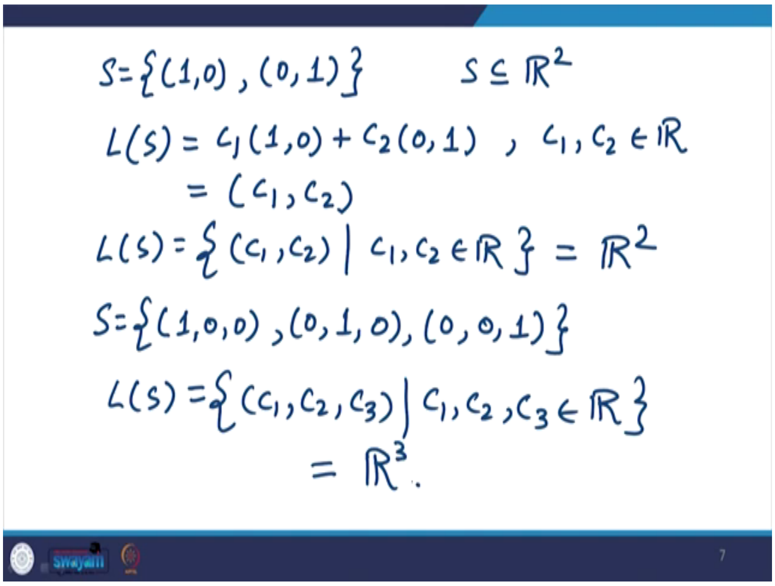

So, for example, if I take S equals to 1, 0 and 0, 1. So, S is a subset of R 2 we are having two vectors from R 2. Now, L s will be c 1 1, 0 plus c 2 0, 1 where c 1, c 2 belongs to R. So, what this will become? c 1, c 2. So, what is L s? L s is c 1, c 2 where c 1, c 2 belongs to R and what is this? This is nothing just the space R 2.

So, here linear span of 1 at, 0 and 0, 1 is R 2. Similarly, if I take three vectors 1, 0, 0, 0, 1, 0, 0, 0, 1 then linear span of S will become like the earlier one c 1, c 2, c 3 such that c 1, c 2, c 3 belongs to R and what is this? This is R 3.

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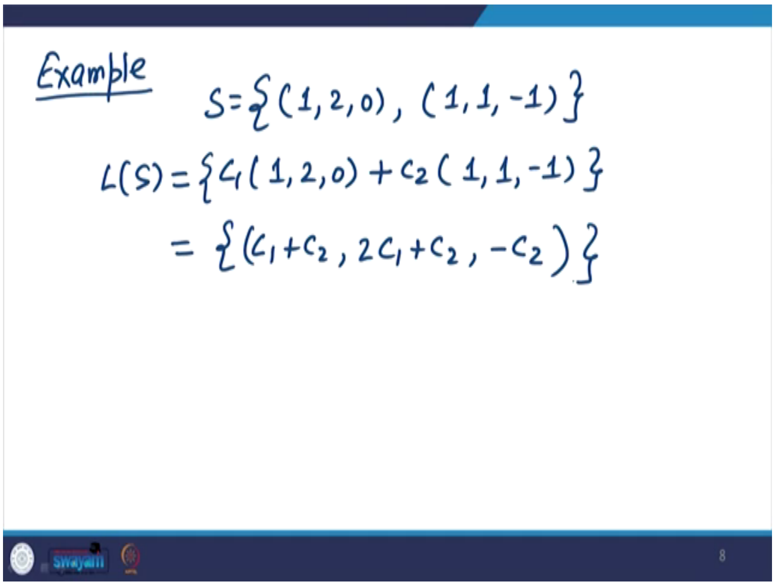

If I take any arbitrary example, in R 3 let us say I am having a set 1, 2, 0, 1, 1, minus 1 then linear span of S is c 1 times 1, 2, 0 plus c 2 times 1, 1, minus 1. So, what is this? I am having c 1 plus c 2, 2c 1 plus c 2, minus c 2. So, this is the linear span of this set.

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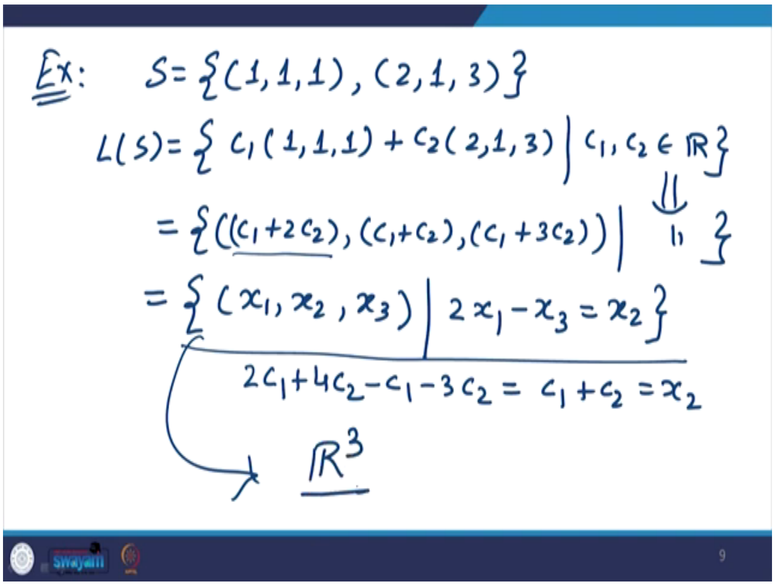

Another example I can take this one. S equals to 1, 1, 1 and 2, 1, 3. So, what will be linear span of S.? So, again I will take c 1 1, 1, 1 plus c 2 2, 1, 3 and c 1, c 2 belongs to R.

So, this will be c 1 plus 2c 2, c 1 plus c 2 and c 1 plus 3c 2 and c 1, c 2 belongs to R. So, what is this? How I can represent this set? I can represent this set as x 1, x 2, x 3 such that, twice of x 1 minus x 3 equals to x 2 because if you take 2 times of this so, 2 times of this will become 2c 1 plus 4c 2 minus x 3 so, minus c 1 minus 3 c 2 it will comes out to be c 1 plus c 2 and which is x 2 only. So, this is the linear span of S and we will say that it will be a you can easily prove that it will be a subspace of R 3.

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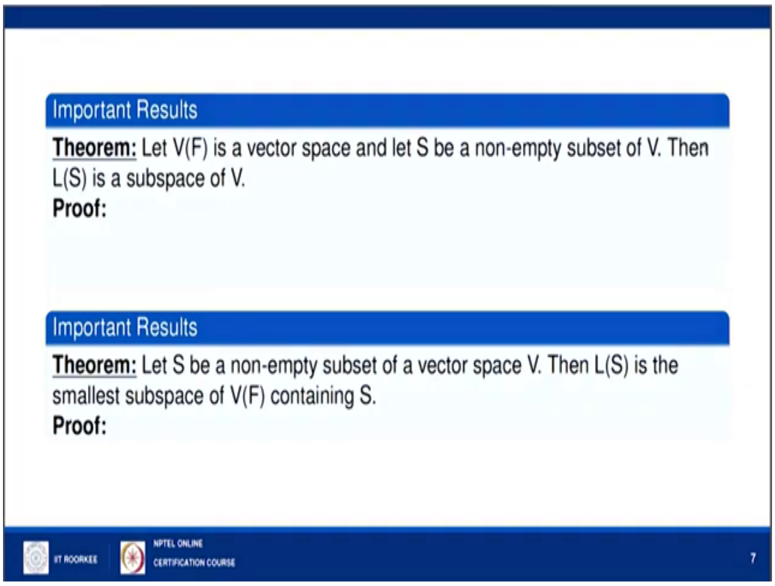

So, we are coming to these results only that let V F is a vector space and let S be a non-empty subset of V. Then the linear span of S is a subspace of vector space V. Moreover, this subspace is the smallest subspace containing S. What I want to say? If V is a vector space, S is a subspace of v, then the linear span of S is the smallest subspace containing the subspace S. So, I am leaving the proof of these two results as an exercise.

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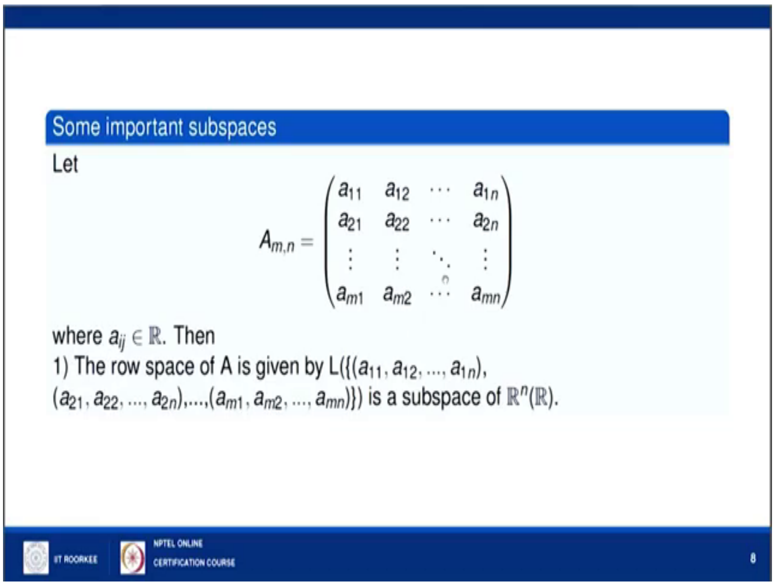

Now we are going to define some of the important subspaces. So, if we are having a m by n matrix given by the entries a 11, a 12, a 1n, a 21, a 22, a 2n like that where a ij are real number so, all these entries are coming from the set of real numbers, then the row space of A is given by the linear span of the rows where each row is a vector.

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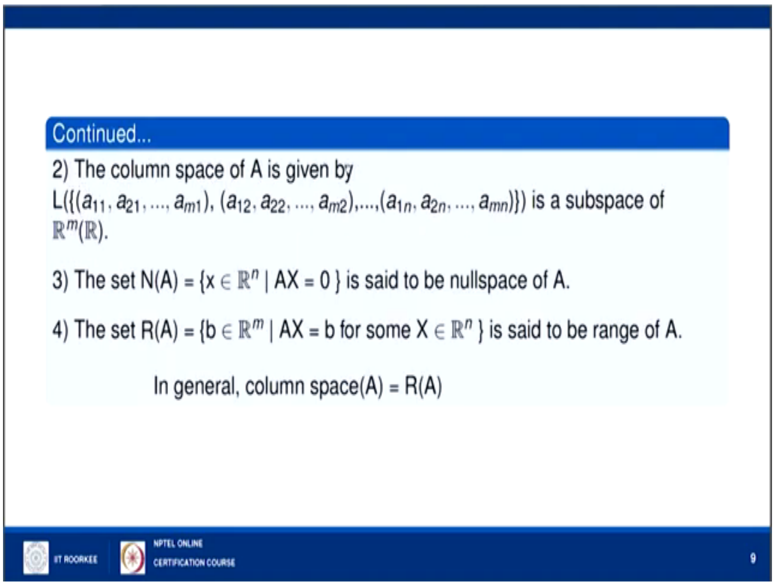

Similarly, the column space of A is given by the columns of the matrix A. Linear span of the column of the matrix of A. Then, we are having null space of A. So, nullspace of A is x belongs to R n such that AX equals to 0. So, all n dimensional vectors from R n if you multiply those vector with A like A into x, you get the zero vector. So, all those the space of all those vectors x from R n is called nullspace of A and it is a subspace of R n.

The set sorry it should be R here R A not null space A. So, the set R A is all those m dimensional vector b such that AX equals to b for some X belongs to R n and it is called range space of A.

In general, column space of A equals to range space of A. So, we are saying these are the four important subspaces related to a matrix and we frequently used these subspaces in machine learning especially for dimensional reduction concepts like principle component analysis and all kernel based dimensional reduction algorithms. So, one should know all these four subspaces.

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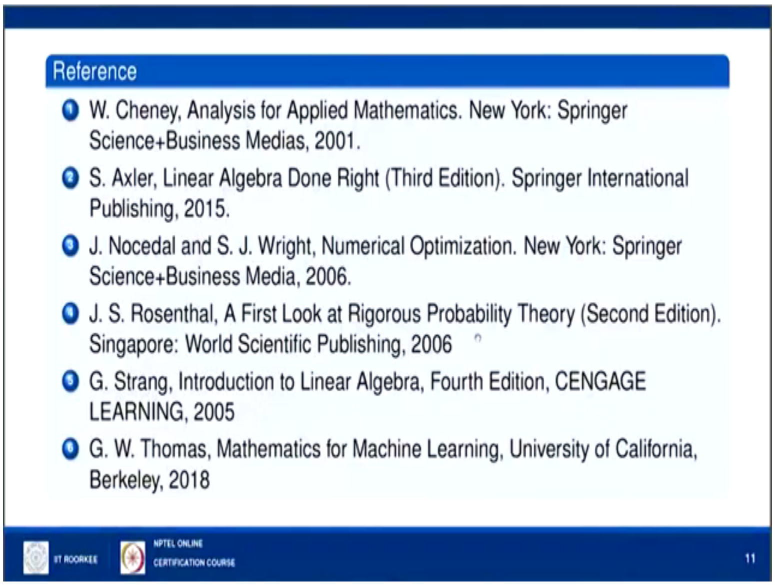

So, in this lecture, we have learn about vector subspaces. We start with a definition, then we have given an alternate definition, then we have given some of the example of subspaces, we have seen four important subspaces related to a matrix, we have seen some of the important result like the union of subspaces, intersection of subspaces and then, linear span of a subset of a vector space which is again a subspace. We have seen several examples. So, these are the references for this lecture. I hope you have enjoyed this lecture.

Thank you very much.