Essential Mathematics for Machine Learning

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Lecture – 02
Vector Spaces: Definition and Examples

Hello friends, so welcome to the second module of this course. So, in first module we have learn about the importance of vectors in machine learning and then properties of vectors like linearly independent and linearly dependent. So, as you know most of the feature vectors share the common properties for a particular data set in machine learning and the features those are having those common properties we put in a single class.

So, based on that here in this module we will learn about Vector Spaces. So, basically vector spaces are set of vectors those share such kind of common properties. So, we start this lecture with the definition of vector spaces.

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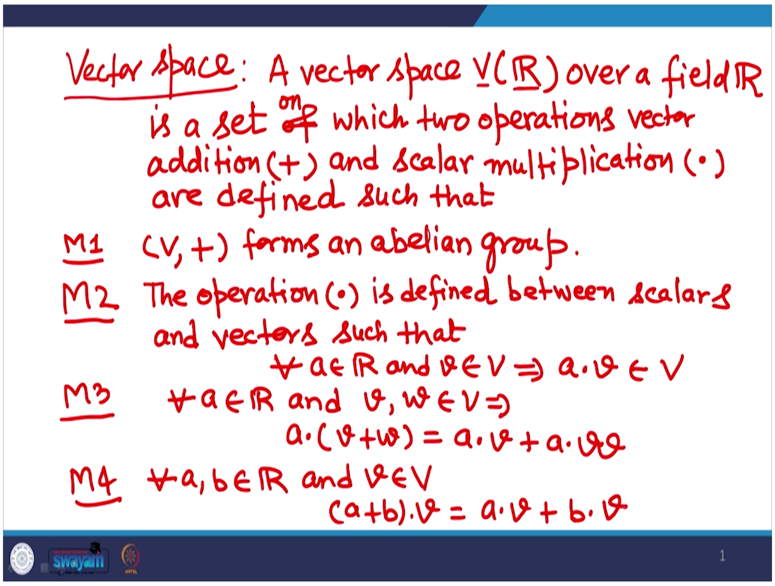

So, let us define vector space. So, a vector space V over the field R. So, here V is the set of vectors and R is a field of the scalars. In general we can define vector space over any arbitrary field, but in machine learning we usually have vectors those are defined over the field of real numbers. So, in this course we will focus on the vectors or vector spaces those are defined over the field of real number. So, vector space V R over a field R is a set of. So a set on which two operations let us say vector addition denoted by plus. So, this plus is defining the vector addition and scalar multiplication let us denote it by dot are defined.

So, these two operations are well defined on V and R, such that the first property the set of vector V together with the operation addition of vectors forms an abelian group. So, here forming abelian group means the set of vectors with respect to this vector addition operation satisfy certain properties, one is closure property, another one is commutative property, associative property, existence of additive identity and existence of additive inverse.

So, we will make it more clear by taking some example. The second property is the operation a scalar multiplication is defined between scalars that is the elements of the field R and vectors such that for all a belongs to R and v belongs to V we have a dot v belongs to V.

The third property is for all a belongs to R and any two vectors v w belongs to V, we have a multiplied with the addition of vectors v and w equals to a dot v plus a dot w. The fourth property is for any two arbitrary scalars belongs to the field R and v belongs to V, we have a plus b multiplied with v equals to a dot v plus b dot v.

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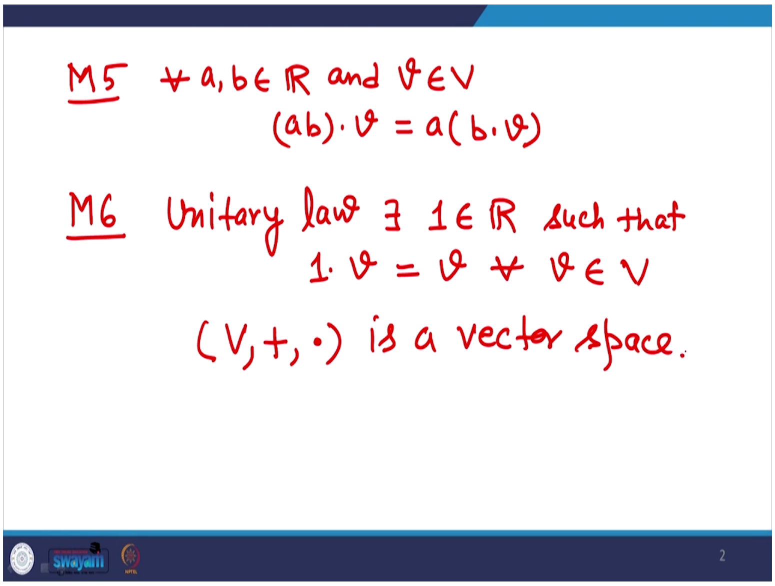

The fifth property is for all a b belongs to R and vector v belongs to the set V which is a set of vectors, the product of a b a scalar multiplication with v equals to a b dot v. So, in left hand side we are multiplying and b first and then we are multiplying this scalar to the vector v. While, on right hand side we are multiplying b first with v and then the vector b dot v we are multiplying with the scalar a and the last property is that is called Unitary law.

So, what is this Unitary law? So, there exist 1 belongs to R such that if you multiply this identity scalar with any of the vector v belongs to V, then this is equals to v for all v belongs to capital V. So, if a set of vectors V satisfy all these 6 properties with respect to these two operations, then we say that V together with these two operation is a vector space.

So, now we will take some of the examples of vector spaces. So, my first example is coming from the vector space R 2.

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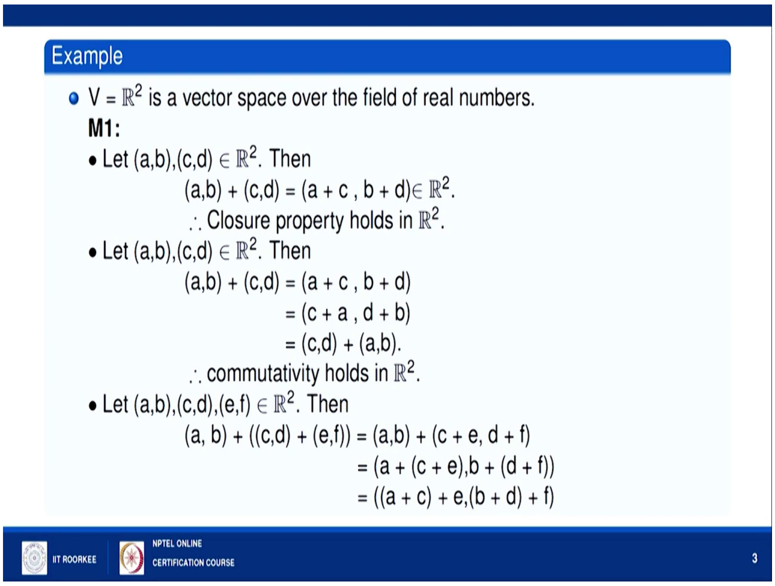

So, here R 2 is a plane you can take it as x y plane and each of the point of this plane is given by a vector, that is pair of point x and y. Where first component is the x coordinate of that point and the second component is y coordinate of that point. So, this R 2 forms a vector space over the field of real numbers.

So, to check this we have to see whether it is satisfying all these 6 properties. So let us see, so property 1 is first I am taking 2 vectors or 2 points in R 2 plane that is a b and c d. So, these are 2 vectors v and w let us say from the vector set capital V which is R 2, then a b plus c d. So, how we add the vector, we add the first component with the first component of the second vector and the second component of the first vector with the second component of the second vector.

So, a b plus c d becomes a plus c comma b plus d, again a plus c is a point in R 2 which is x x coordinate of a point in R 2, similarly b plus d is the y coordinate of a point in R 2. So, if you add these 2 points again we are getting a point in R 2. So, we can say that closure property holds in R 2. Similarly if I take again the a b plus c d it is equals to a plus c comma b plus d, I can write a plus c a c plus a comma d plus b and which is nothing just c d plus a b.

Hence the order of the vectors in addition does not matter and we are saying that commutative property holds in R 2. If you take three vectors let us say a b c d and e f, then first I am adding here c d plus e f and then whatever a result I am getting I am adding a b into it. So, it will become a b plus c plus e comma d plus f.

Now, I can write it a plus c plus e comma b plus d plus f. Since these are scalars these are real numbers, so they satisfy the associative property. So, a plus c plus e I can write a plus c plus e, similarly b plus d plus f I can write b plus d plus f.

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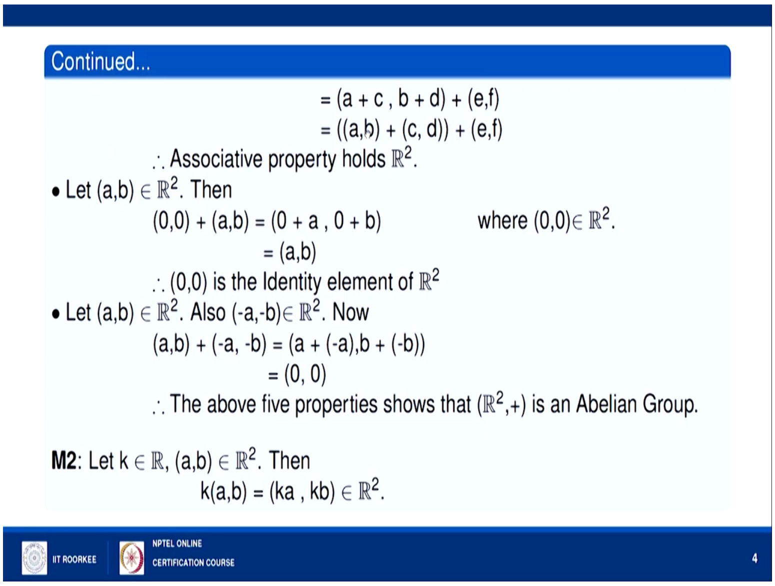

So, hence it becomes a b plus c d plus e f hence associative property holds in R 2. Further if you take the origin of R 2 plane that is 0, 0, so if you add this in any vector a b the result will come out to be a b only.

So, hence 0, 0 is an element of R 2 and it is identity element. So, existence of identity holds here. Next if you take a b then certainly there will be minus a minus b also will be there R 2 and if we add these two points then result will come out to be 0, 0; which is nothing just additive identity of part 2. Hence, the existence of additive inverse holds in R 2.

So, these points satisfy and they implies that R 2 plus forms an abelian group. So, this is the first property which we have defined in the definition of vector space. Now come to rest of the properties. So, my second property was there that if you are having a scalar and a vector then the scalar multiplication between them also an element of the set R 2.

So, I am taking here k belongs to R which is a scalar, a b is a vector from R 2 then k multiply with a b will become k a comma k b. So, again k a comma k b is an element in R 2 and hence the second property holds.

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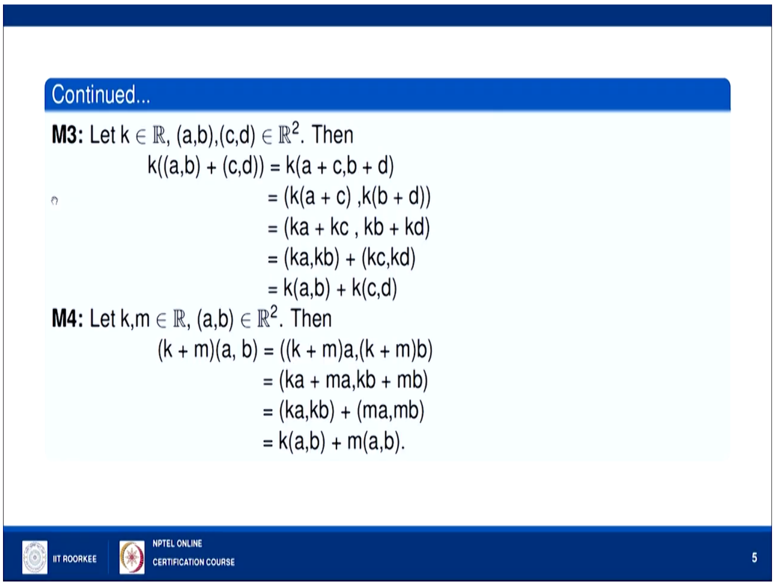

So now come to third property, I am taking a scalar k here and 2 vectors given by a b and c d. Then the multiplication of k with the addition of these 2 vectors a b and c d can be written in this form k times a plus c comma b plus d I can take k inside, so k a plus c comma k b plus d, which is k a plus k c comma k p plus k d. This I can write k a comma k b plus k c comma k d and which is nothing such k times a b plus k times c d.

So, hence third property hold in R 2, with respect to the operations vector addition and scalar multiplication. Now come to the fourth property I am taking two scalars k and m. Now come to fourth property I am taking 2 scalars k and m from the field R and a vector a b from the set of vectors R 2, then k plus m times a b can be seen as k times a b plus m times a b. So, hence fourth property holds in R 2.

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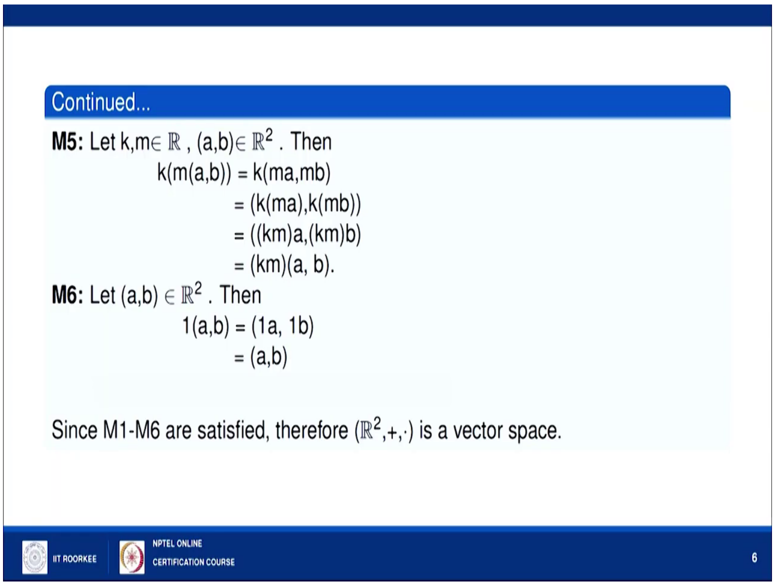

Now, come to fifth property. So, here again I am taking 2 scalars from the field of real numbers k and m and a vector element a b from R 2. Then k times m times a comma b can be written as k m a comma m b which is nothing just I can take k inside again the bracket k times m a comma k times m b and this finally comes out to be k m times a comma b, which is the fifth property.

Further you know that 1 belongs to the field R, means 1 is a real number and if you multiply 1 with any vector a b it will be 1 comma a comma 1 comma b. And it is nothing just if you multiply 1 to any real number the real number will remain same. So, it is equals to a comma b here unitary property holds.

So, here all M1 to M 6 are satisfied and therefore we can say R 2 on the field of real number with respect to the operation set addition of vectors and scalar multiplication is a vector space. So, this is an example of part 2 similarly we can so that R 3 is also a vector space and any set of vectors like R n, where each vector is having the n components x 1 x 2 up to x n also forms a vector space.

So, this we have talk about R 2 R 3 and R n these kind of vector spaces, those are very common in machine learning.

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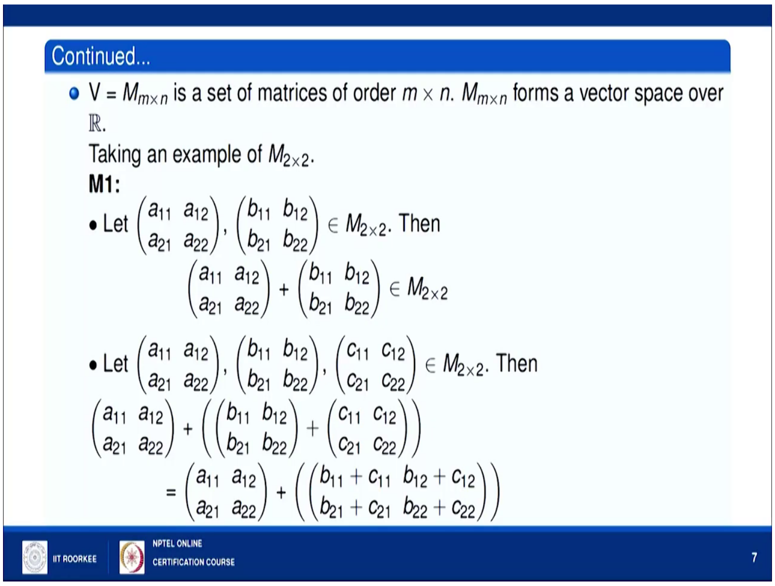

But the same time the set of matrices of order m by n, so where m and n are real integers. So, a positive integers in fact, so any matrix of order m by n forms a vector space over the usual addition of matrices and the scalar multiplication

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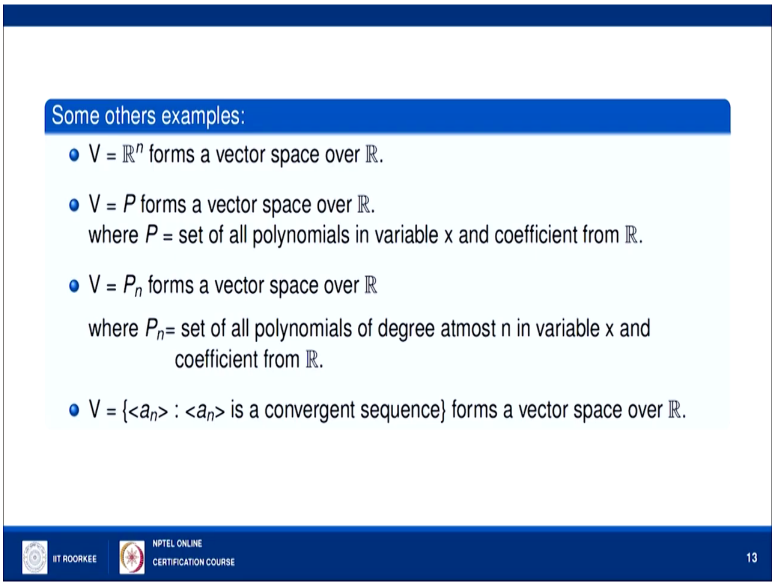

So, there are many other vector examples of vector spaces, as I told you R n forms a vector space over the field R. If you take the set of vectors of all polynomials in variable x and having real coefficient then this set also forms a vector space over the field R. Similarly, if you take a set of all polynomials of degree at most n; so, degree either n or less than n in variable x and having real coefficient; then this set also forms a vector space over the field of real numbers with respect to the operations usual polynomial addition and scalar multiplication.

If you take V as the set of all convergent sequences, then this set also forms a vector space over the field of real numbers where the operations are usual sequence addition and a scalar multiplication. So, so far we have seen many examples of vector spaces, let us see some of the set those do not form vector spaces.

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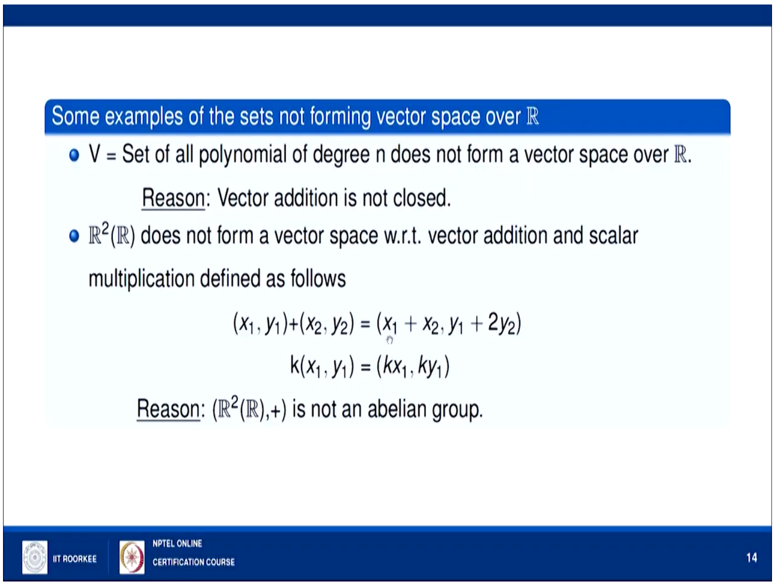

So, my first example is set of all polynomials of degree n does not form a vector space over R. Why? Because, vector addition is not closed.

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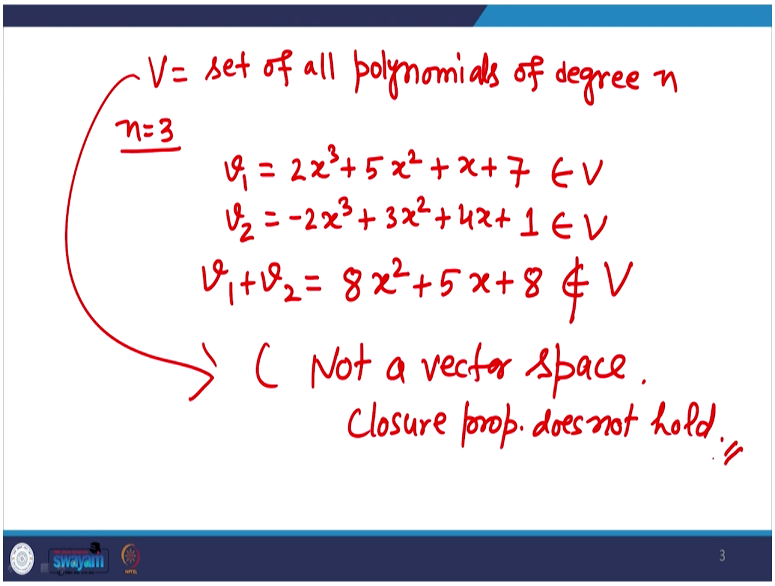

So, if you want to see this so let us say 2 elements from V. So, here V is set of all polynomials of degree n. So, let us take n equals to 3. So, I am taking V as the set of all polynomials of degree 3. So, let us take 2 vector 2 elements from this set or 2 vectors from this set.

So let us say v1, so v 1 I am taking let us say 2 x cube plus 5 x square plus x plus 7 and I am taking another vector minus 2 x cube plus 3 x square plus 4 x plus 1. So, v1 v2 are 2 polynomials of degree 3. So, it is an element of V it is an element of V, now if I take v1 plus v 2. So, what will be v1 plus v 2? So, it will become 8 x square because x cube term will be cancelled plus 5 x plus 8 and it is a polynomial of degree 2. So, hence it is not an element of V.

So, hence this set do not form a vector space. So, closure property; similarly if you take R 2 R we have seen in earlier example that if you take the usual addition of vectors in R 2 and this k usual multiplication of real numbers, then it forms a vector space. But, if you define your vector addition and scalar multiplication in this way like if you are having vector x 1 y 1 from R 2 and another vector x 2 y 2 from R 2 then their sum is x 1 plus x 2 comma y 1 plus 2 y 2.

In usual case it will be y 1 plus y 2, but we are defining this operation in this way one can define.

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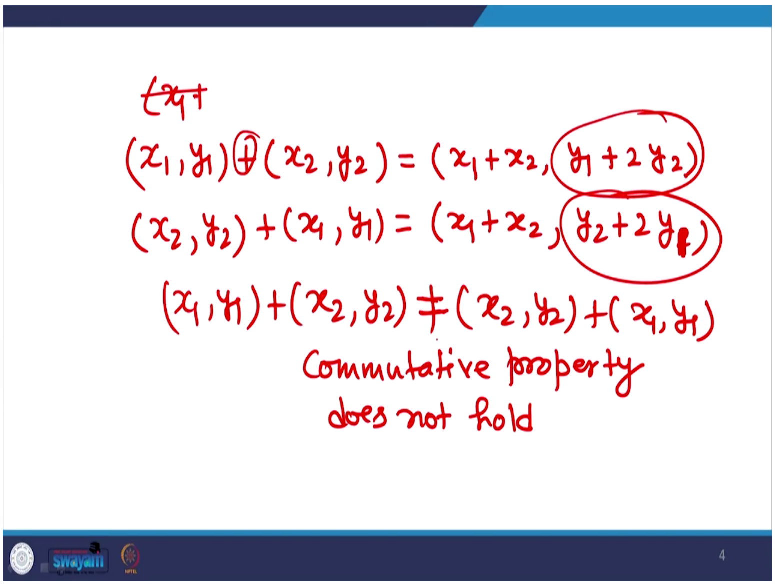

But, if you are defining in this way then what we will see that, so x 1 plus sorry x 1 comma y 1 plus x 2 comma y 2. So, these are two vectors from R 2 and we are defining their vector addition is x 1 plus x 2 comma y 1 plus 2 y 2 ok.

Now, if I take the reverse order x 2 y 2 plus x 1 y 1, then it will become x 1 plus x 2 first component will be the same as in the previous case. However, in the second case it will become y 2 plus 2 y 1. So now this second y coordinate of these vectors. So, it is y 1 plus 2 y 2 it is y 2 plus 2 y 2 y 1 sorry y 2 plus 2 y 1, hence both are not equal. So, what we can say x 1 y 1 plus x 2 y 2 not equals to x 2 y 2 plus x 1 y 1 in general for all vectors in R 2.

So, what I can say that commutative property does not hold. So, hence if you are defining your vector addition in this way by this operation, then it R 2 on the field of real numbers does not form a vector space. So, what we can say from here? The only set is not important in the definition of vector space, but the operations also how you are defining the your operations.

For a particular set one set of operations will satisfy all the 6 properties of vector space, but if you change the operation, it may happen that the particular set does not form a vector space.

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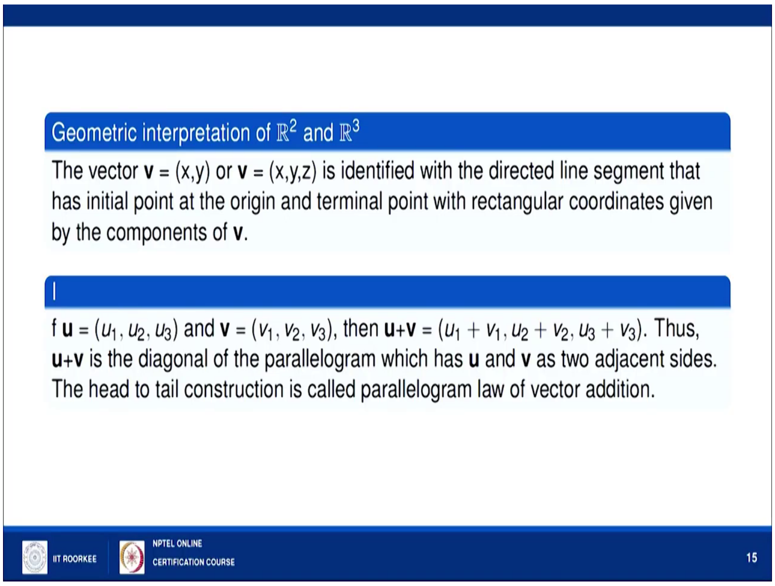

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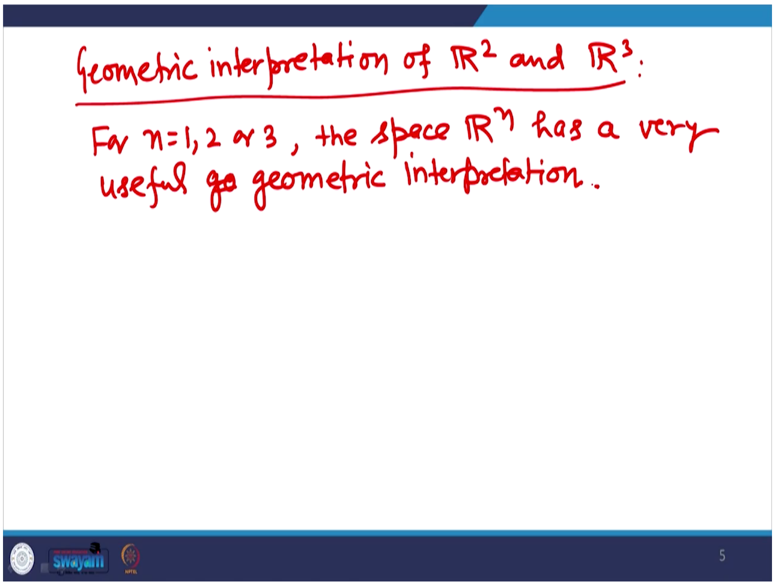

So, you have to be careful in this case. Now, we are coming to geometric interpretation of R 2 and R 3. So, what we can have for n equals to 1, 2 or 3. The space R n has a very useful geometric interpretation for any greater than 3, we cannot see in general. But for n equals to 1, 2 or 3 we can have this beautiful geometric interpretation. How? Let me explain it.

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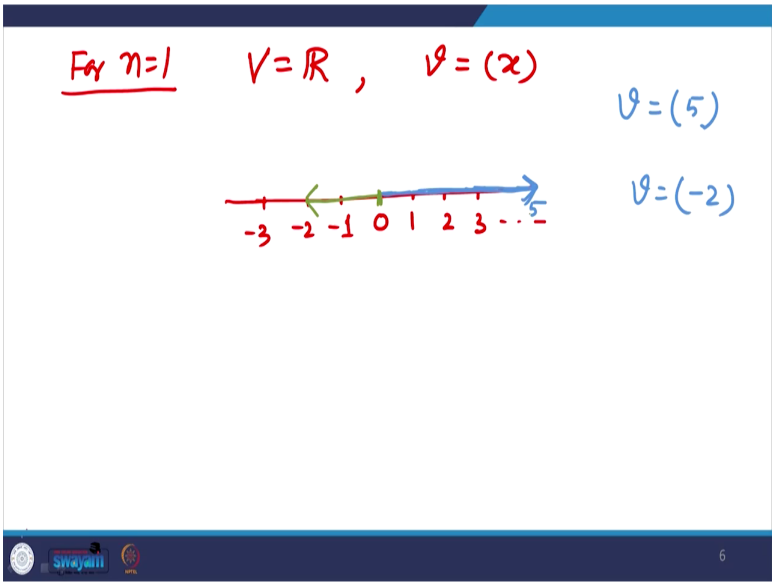

So, for n equals to 1 we are having the set V equals to R and the vectors will be like V equals to x. So, how you will see geometrically this vector? So, on the real line you will be having the origin let us say 0 here and then 1 2, 3 and so on; similarly, here minus 1 minus 2 minus 3 and so on.

So, we are having this real line, now if we are having a vector let us say v equals to x equals to 5. So, what will be this x equals to 5, so let us say 5 is here. So, the tail of this vector will be here at 0 and then we are having the head here at 5. So, this is the vector v equals to 5. Similarly if I am having something like v equals to minus 2, so V equals to minus 2 will be again tail will be here and I will be having this.

So, by the green color I am showing you the vector V equals to minus 2 and by the blue color I am showing you the vector V equals to 5.

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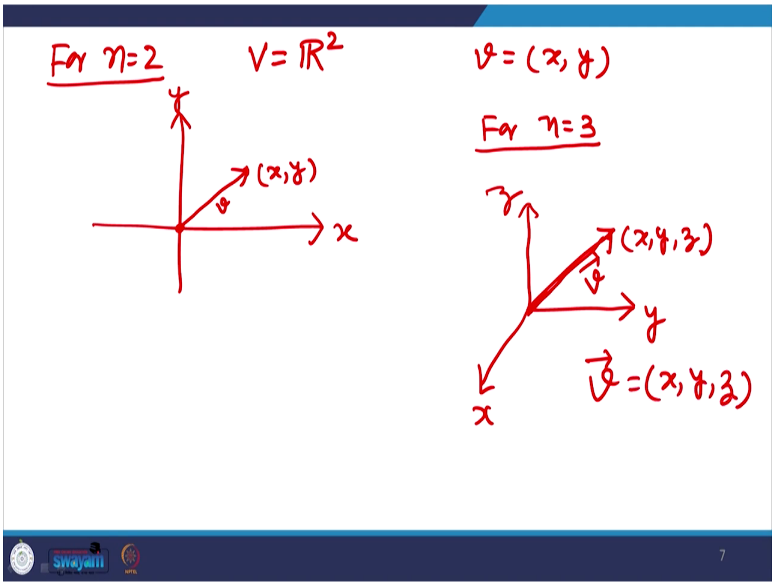

Similarly for n equals to 2, so now it is set of vector is R 2 and the vector will be something like v equals to x y. So, in coordinate plane x y this vector will be having tail at the origin and suppose this is the point x y. So, this straight line joining the origin and the point x y will show the vector v. For n equals to 3 we are having the 3 dimensional coordinate plane. So, let us say x y and z and I am having a point let us say x y z here.

So, again it will be a line in 3 D having tail at the origin and head at the point x y z. So, this is my vector v equals to x y z.

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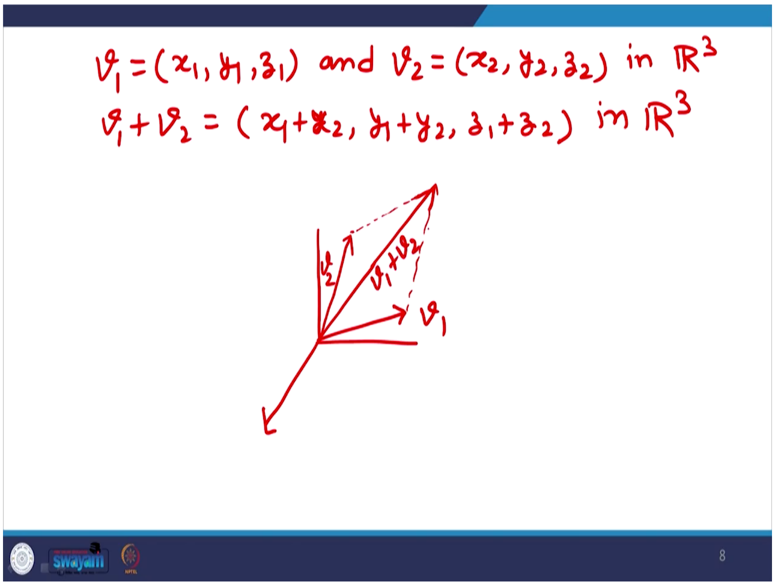

So, if you are having 2 vectors let us say v1 equals to x 1 y 1 z 1 and v2 equals to x 2 y 2 z 2 both are in R 3, then v 1 plus v2 will be x 1 plus x 2 y 1 plus y 2 and z 1 plus z 2; which is again a vector in R 3, because R 3 forms a vector space. And how to see these? So, let us say this is my vector v1 this is my vector v2, then the sum of v1 plus v2 is given by the diagonal of this parallelogram, so this is my v1 plus v2.

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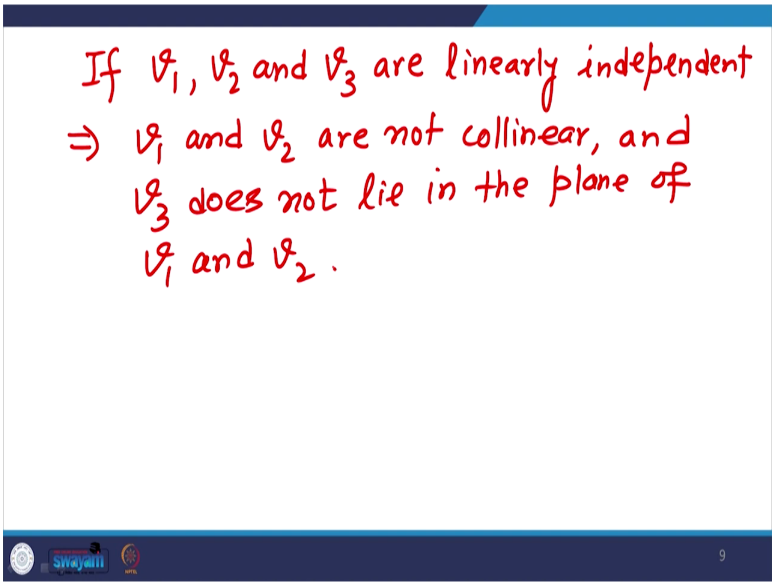

One more important remark here if v1 v2 and v3 are linearly independent. So, what is the meaning of geometrically of this sentence? So, it means v1 and v2 are not collinear and the vector v3 does not lie in the plane of v1 and v2. So, these are some geometrical interpretation of vectors in R 2 R 3.

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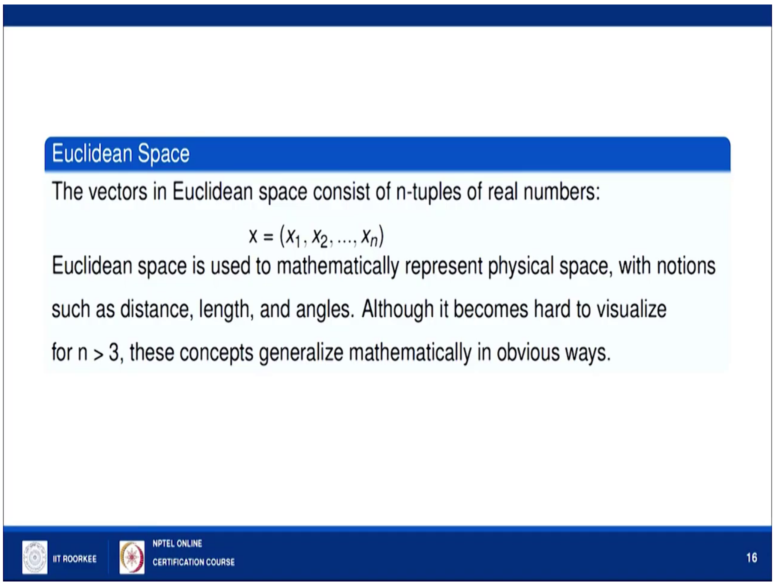

Similarly we are having Euclidean space R n, where we are having the vectors coming from n tuples like a vector X is X 1 X 2 up to X n. So, Euclidean space is used to mathematically represent physical space with notions such as distance, length, angles. Although it becomes hard to visualize for any when n is greater than 3. That is why I have given geometrical interpretation up to R 3 only.

However, these concepts generalize mathematically in obvious way and in machine learning also we will have vectors those are having huge dimensions. So, like dimension let us say n equals to 100 or let us say n equals to 1000. So, it will depend on your data set and the feature those you have been extracted from your data. So, how many attributes you are taking for a particular feature vector.

So, this is all about vector spaces, in the next lecture we will learn about subspaces that like we are having the set of vectors here V. So, if you are having subset of V, when there will be itself a vector spaces. Then we will see some of the properties of service spaces those are quite useful in machine learning and then we will see some of the numerical examples.

So, these are the references for this course especially for first part of this course, first 20 modules those are where topics are coming from linear algebra. So, I hope you have enjoyed the lecture.

Thank you very much.