Essential Mathematics for Machine Learning
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Lecture - 19
Minimal Polynomial and Jordan Canonical Form

Hello friends. So, welcome to the module of the course Essential Mathematics for Machine Learning. And the module name is Minimal Polynomial and Jordan Canonical Form. So, again it is very important topic from matrix story and we will see it is a generalization of what we have learn earlier that is of the diagonalization.

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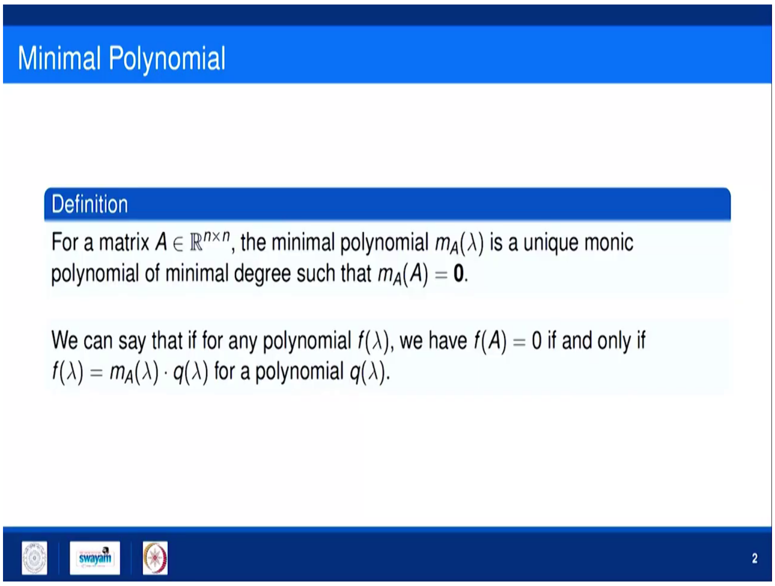

So, first we will discuss about minimal polynomial. So, the definition we can state like this, so for a matrix A which is n by n real matrix the minimal polynomial denoted by m A of lambda.

So, it is a polynomial of lambda is a unique monic polynomial of minimal degree, such that m A A equals to 0 matrix.

Means this polynomial annihilate the matrix A. One of such polynomial you know about the characteristic polynomial; however, if you are having a n by n matrix, then the degree of characteristic polynomial is n, where as if we talk about the minimal polynomial it is having degree n or less than n.

So, based on that we can say if for any polynomial f lambda, we have f of A equals to 0, if and only if f lambda is the product of minimal polynomial of a into another polynomial q. So, it means if a polynomial f annihilate the matrix A, then the minimal polynomial divides that particular polynomial. Now, see the example consider this 2 by 2 diagonal matrix.

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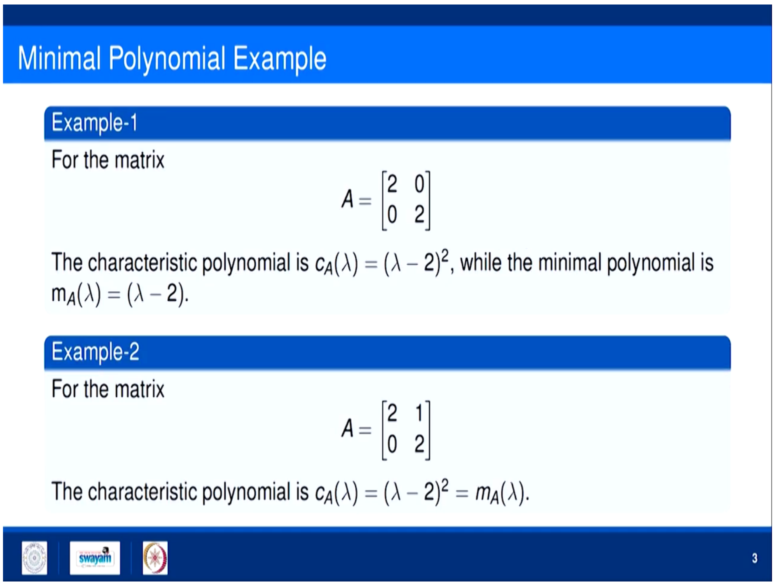

So, the characteristic polynomial of this is nothing just the lambda minus 2 whole square, while the minimal polynomial is lambda minus 2 because this particular polynomial annihilate the matrix.

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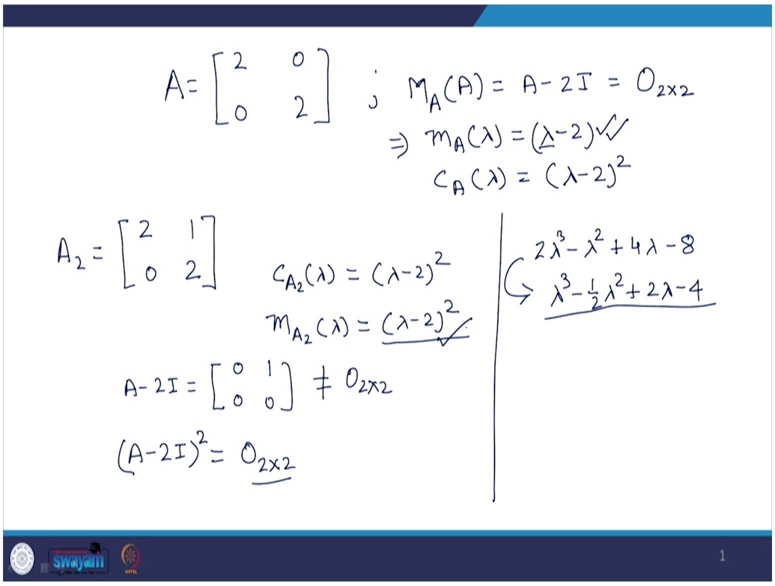

Here A equals to 2 0 0 2, then m A of A becomes A minus 2 I and this is equals to 0 2 by 2 null matrix. It means the minimal polynomial of A is given by lambda minus 2. While the characteristic polynomial of A is lambda minus 2 whole square.

So, if you observe here, here this minimal polynomial is having less degree when compare to characteristic polynomial as well as it is monic. Now, what is the meaning of monic? Monic means the leading coefficient of the polynomial is 1 and it is a polynomial of single variable.

So, for example, if you are having a particular polynomial, let us say 2 lambda 3 minus lambda square plus 4 lambda minus 8, then it is not monic. However, the monic form of this will become the leading coefficient that is the coefficient of highest degree term that is lambda cube here is 1.

So, in that way it will become 1 minus 2 lambda square plus 2 lambda minus 4. Another important thing about monic polynomial is that, it is a polynomial of single variable that it is a polynomial of lambda only.

So, if you take another example let us say I take another matrix A 2 which is 2 1 0 2, then what we can say here again characteristic polynomial of this matrix is lambda minus 2 whole square. And if we see the minimal polynomial of this matrix then it is again the same as the characteristic polynomial that is lambda minus 2 square.

Here if you simply take A minus 2 I which becomes 0 1 0 0 and it is not a null matrix of size 2 by 2. However, if you take A minus 2 I whole square it becomes 0 matrix of size 2 by 2.

So, here lambda minus 2 square is the monic polynomial which is having the least degree. And annihilate the matrix A hence the minimal polynomial is lambda minus 2 whole square. So, I hope with these two examples the concept or definition of minimal polynomial is clear to all of you.

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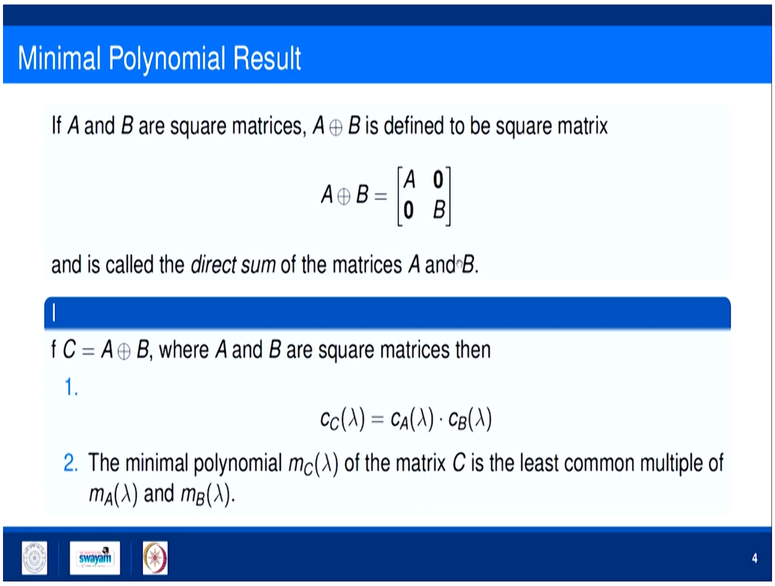

Now, come to a very important result about minimum polynomial that if A and B are square matrices, then A O plus B is defined to be square matrix of this form means, where a diagonal sum matrices is A and B and rest of the entries are coming from the 0 matrices of corresponding sizes. Now, this A o plus B is called the direct sum of the matrices A and B.

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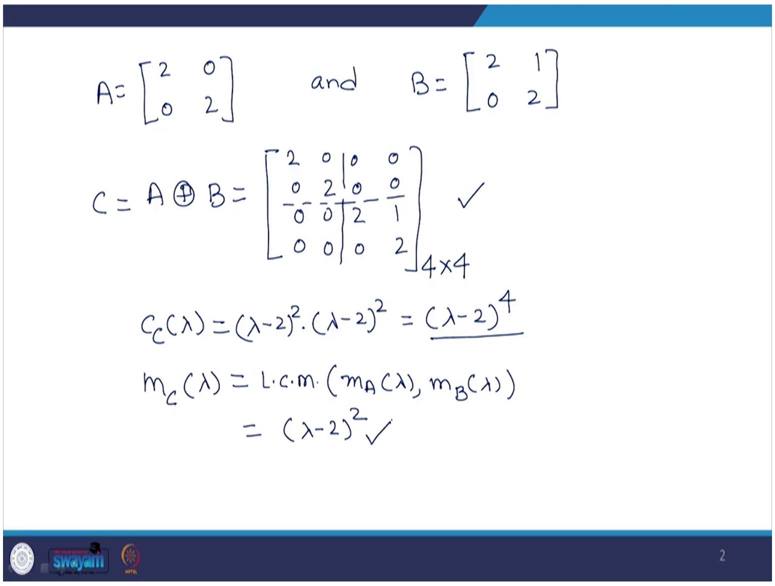

So, for example, if you take A equals to the same 2 0 0 2 and B equals to 2 1 0 2, then A o plus B that is the direct sum of A and B is 2 0 0 2. This is one of the diagonal matrix and another one is 2 1 0 2 and rest of the entries are 0.

So, it becomes 4 by 4 matrix, now what we will see that if C is a direct sum of A and B, where A and B are square matrices then the characteristic polynomial of C is just the product of the characteristic polynomials of A and B, while the minimal polynomial of C is the least common multiple of the minimal polynomials of A and minimal polynomial of B.

So, if this is I am saying let us say C. So, here characteristics polynomial of C is given by the product of characteristic polynomial of A and B. So, it is nothing just lambda minus 2 raise to power 4. While minimum polynomial of C is the least common multiple of minimal polynomial of A and minimal polynomial of B. The minimum polynomial of A is lambda minus 2, while the minimal polynomial of B is lambda minus 2 whole square.

So, L.C.M comes out to be lambda minus 2 whole square. So, here you can see if we talk about this 4 by 4 matrix here characteristic polynomial; obviously, is of degree 4, while the minimal polynomial is of degree 2.

So, this is I want to say about minimal polynomial, we will see the definition of Jordan Blocks and then Jordan Canonical Forms and then we will come back the relation of minimal polynomial with Jordan form.

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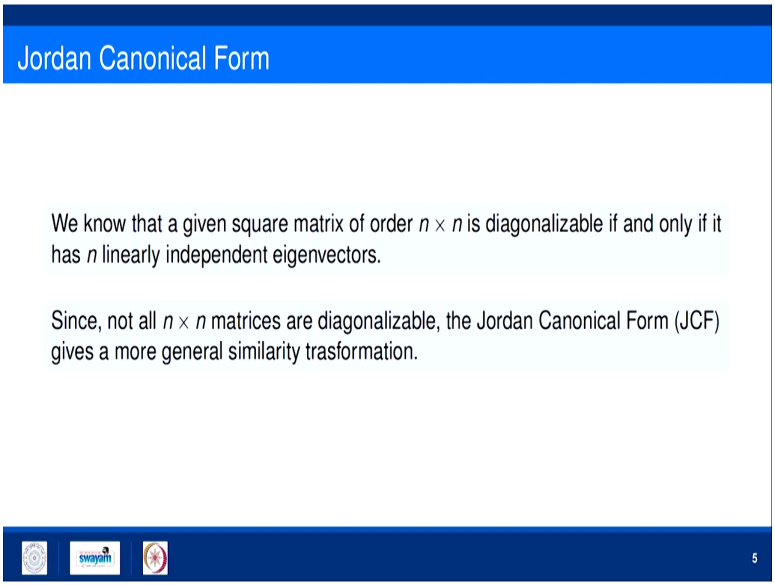

So, now come to Jordan Canonical form. So, we know that a given square matrix of order n by n is diagonalizable, if and only if it has n linearly independent eigenvectors.

So, if the number of l i eigenvectors equals to the size of matrix then, matrix is similar to a diagonal matrix where diagonal entries are eigen values. However, if this is not the case means if A n by n matrix are not diagonalizable means it is having less than n linearly independent eigenvectors.

Then the Jordan Canonical Form gives a more general similarity transformation. So, what we can say that diagonalization is a special case of Jordan Canonical Transformation.

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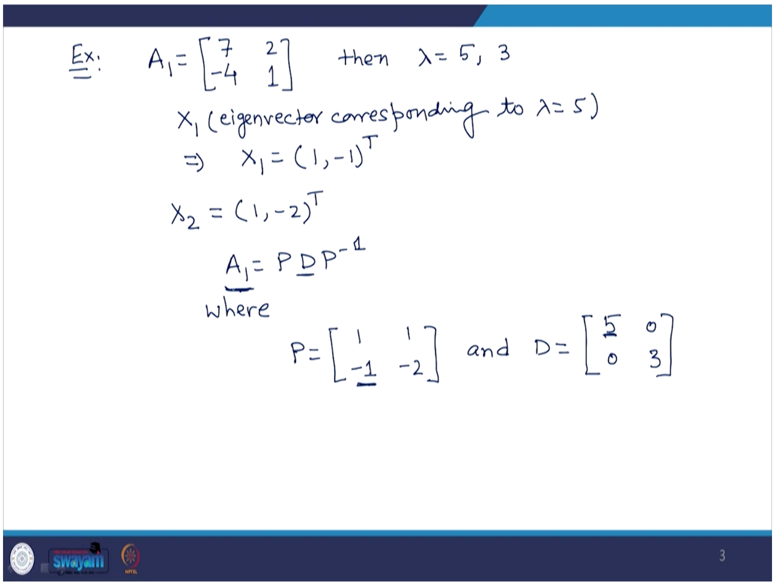

So, let us understand it with the help of an example. So, let we have a matrix A 1 which is a 2 by 2 matrix and entries of A 1 is 7 2 minus 4 1.

Then if you calculate the eigen values of A 1 it comes out to be lambda equals to 5 and 3. Now the eigenvector corresponding to, this I am saying eigenvector corresponding to lambda equals to 5. So, here X 1 comes out to be 1 minus 1 transpose. Similarly if X 2 is the eigenvector corresponding to the eigen value lambda equals to 3 then X 2 comes out to be 1 minus 2.

So, X 1 and X 2 are linearly independent because they are eigenvectors corresponding to distinct eigen value. Hence what I can do? I can write A 1 as P D P inverse, means A 1 is similar to a diagonal matrix D, where P is a matrix which is having column says the eigenvector of A 1 that is 1. And minus 1 is the first column that is your eigenvector X 1 and then second one is 1 and minus 2.

And here D equals to 5 0 0 3. Here just note that why I am taking 5 here because I have written the eigenvector corresponding to lambda equals to 5 as the first column. If you interchange these two columns then D will become 3 0 0 and 5.

And if you calculate P D P inverse it comes out to be A 1. So, this is an example of the diagonalization and we have learned it earlier also in this course; however, you consider another matrix.

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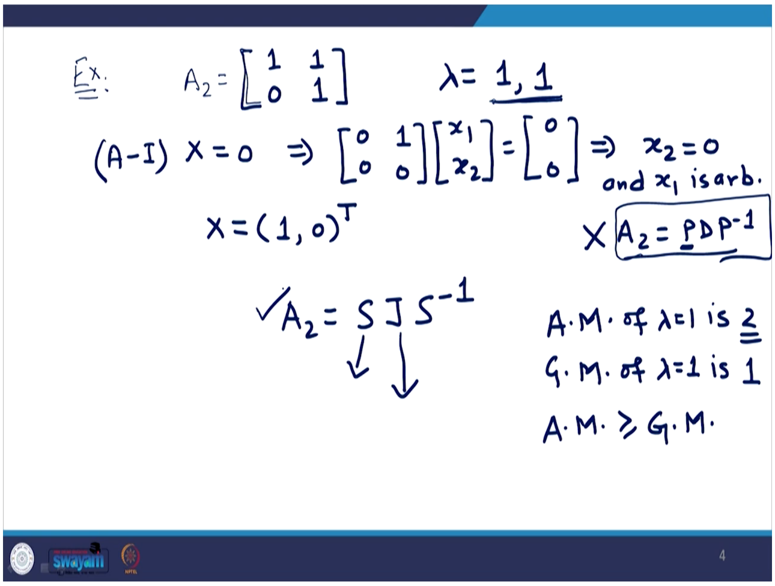

Let me take a matrix A 2 which is given by 1 1 1 0 1. Now it is a upper triangular matrix; so obviously eigen values are lambda equals to 1 and 1. Now, if you see the eigenvector corresponding to lambda equals to 1 then I can write A minus lambda I, lambda is 1 here into X equals to 0.

So, it means 1 minus 1 is 0, 1 0 0 and then X is a vector in R 2 x 1 x 2 equals to 0 0. So, from here what I am getting x 2 equals to 0 and x 1 is arbitrary. So, from here I can write X equals to 1 0. So, what you are observing here we are having 2 eigen values means of this matrix both are equal that is lambda equals to 1; however, corresponding to 1 we are having only one eigenvector.

So, this matrix cannot be write as A 2 equals to P D P inverse because how to form this model matrix P, because we are having only one eigenvector. So, we cannot have this kind of diagonalization of this matrix A 2 hence we have to see a more general version for writing this kind of similarity Transformation.

So, here role of Jordan Canonical Transformation comes we write A 2 equals S J S inverse where S is again a matrix which is coming from the eigenvectors and generalize eigenvectors of A 2, while J is the Jordan Canonical Form of the matrix A 2. J is not a diagonal matrix here; however, it is having diagonal entries.

And one at super diagonal in some of the blocks, we will learn about it. Now, another important thing here I want to tell you that algebraic multiplicity and geometric multiplicity.

So, here eigen value lambda equals to 1 is repeating twice. So, here algebraic multiplicity of lambda equals to 1 is 2 that is the number of repetition of that particular eigen value. While geometric multiplicity is the number of linearly independent corresponding to that eigenvalue. So, how l i eigenvector I am having for lambda equals to 1 that is only 1.

So, algebraic multiplicity is the count of the repetition of eigen values. While the geometric multiplicity is the number of corresponding l i eigenvectors. Here you can notice that the geometric multiplicity never exceed algebraic multiplicity.

Now, if I talk about algebraic multiplicity and geometric multiplicity then I can write that, a matrix is diagonalizable only when algebraic multiplicity equals to geometric multiplicity. If it is not equal means algebraic multiplicity greater than geometric multiplicity for one or more eigen values of a matrix, then we have to go for Jordan Canonical Transformation.

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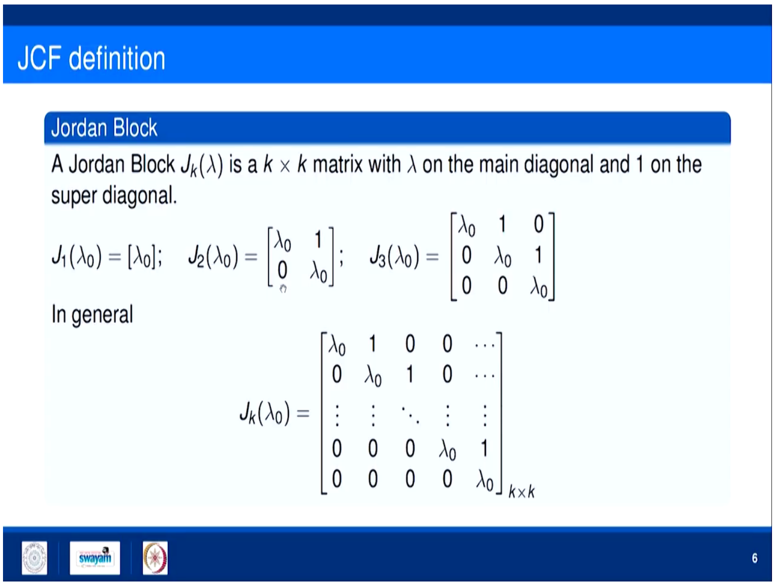

So, that is the motivation behind learning the Jordan Canonical Transformation. So, before going to Jordan Canonical Transformation let us learn about Jordan blocks.

So, a Jordan Blocks J k lambda, here k is the size of the block and it is corresponding to eigen value lambda is a k by k matrix with lambda on the main diagonal and 1 on the super diagonal. For example, a Jordan block of size 1 corresponding to eigen value lambda 0 is just lambda 0, A 1 by 1 matrix that, it is a Jordan block of size 1.

Similarly, a Jordan block is size 2 can be written as lambda 0 1 0 lambda 0 a Jordan block of size 3 lambda 0 1 0 0 lambda 0 1 like this. So, here you can notice we are having the eigen value at the main diagonal and at the supper diagonal we are having 1. Similarly a Jordan block of size k giving by this one.

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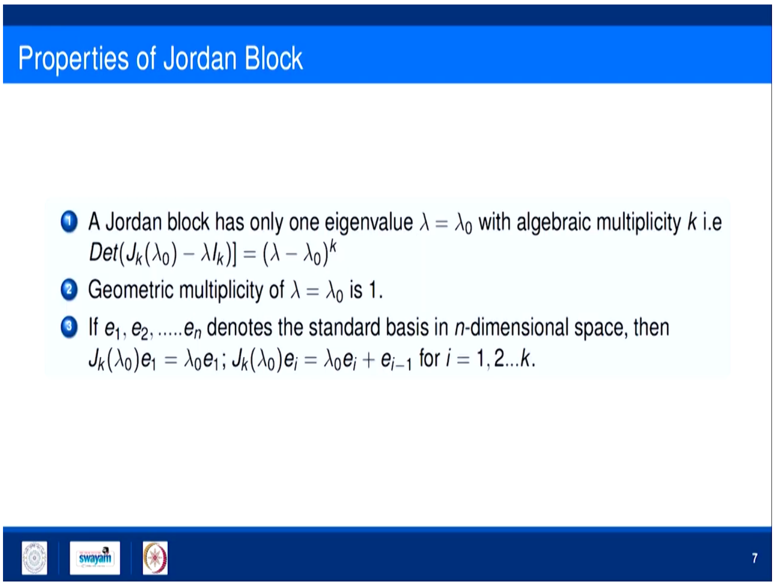

Now, if we talk about some of the important properties of Jordan block, then a Jordan block has only one eigen value that is lambda equals to lambda 0.

With algebraic multiplicity k that is the size of the Jordan block, the geometric multiplicity of lambda equals to lambda 0 is 1. Furthermore if e 1 e 2 e n denotes by standard basis in n dimensional space, then if you operate the Jordan block corresponding to eigen value lambda 0 on e 1 it equals to lambda 0 e 1, that is it is saying you that lambda 0 is an eigen value with algebraic multiplicity k of the Jordan block.

Furthermore J k lambda 0 if you operate it on any e i then it equals to lambda 0 e i plus e i minus 1. So, what I want to say from these properties that is first property is if you consider this.

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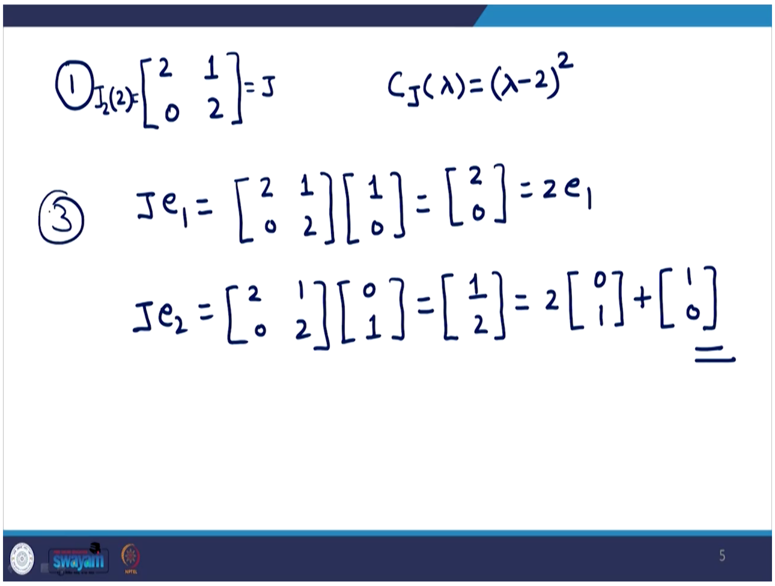

Let us say we have taken this matrix earlier also 2 1 0 2. So, it is a Jordan block of size 2, corresponding to eigen value lambda. So, here characteristic polynomial of this if I say it let say j simply C J lambda is nothing just lambda minus 2 square.

So, the eigen value of the Jordan block is 2 with algebraic multiplicity equals to 2. Similarly if I say about the geometric multiplicity of this the geometric multiplicity of a Jordan block equals to 1.

So, if you are having a eigen a Jordan matrix which is having different eigen values or if you are having a Jordan block which having the same eigen value.

Then based on this particular concept which I told you that the geometric multiplicity corresponding to an eigen value lambda of a specific Jordan block will be 1, tells you about how to write Jordan Canonical Form of a matrix. We will see in this example further in this lecture later on and furthermore what I want to say, if you are having this J e 1.

That is we are e 1 is a standard basis in R 2 a vector of the standard basis. Then it is 2 1 0 2 and what is standard basis in R 2, e 1 will be 1 0, e 2 will be 0 1, then it comes out to be 2 0 and is 2 times e 1. Similarly J e 2 is 2 1 0 2 and then 0 1, here what we are having it comes out to be 1 and then 2. So, this is equals to 2 times 0 1 plus e i minus 1 that is e 1, 1 0, this is the third property.

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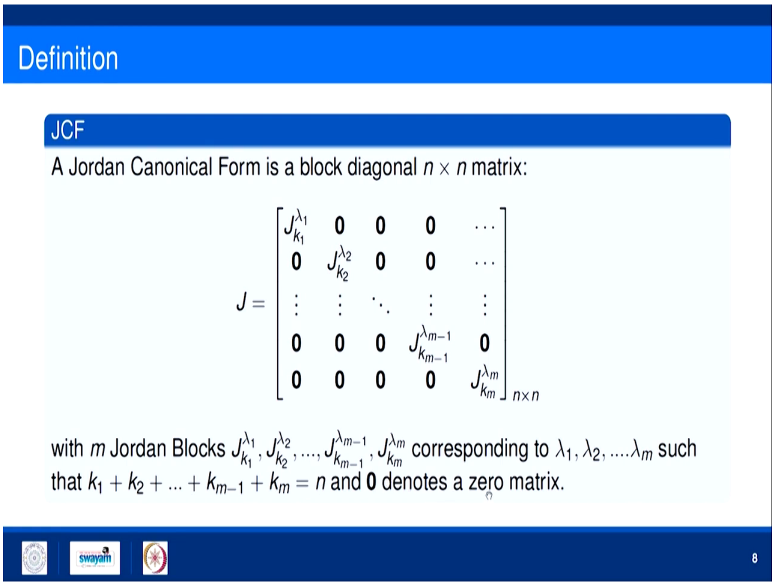

Now, when we talk about Jordan Canonical form, so Jordan Canonical Form is a matrix which is having many Jordan Blocks. So, formally we can define it a Jordan Canonical Form is a block diagonal matrix of size n in this form J k one lambda 1, J k 2 lambda 2.

So, here J k 1 lambda 1 is a Jordan block corresponding to eigen value lambda 1 and size is k 1. Similarly for this one and so on, with m Jordan Blocks as I told you corresponding to eigen values lambda 1, lambda 2, lambda m such that k 1 plus k 2 up to k m equals to n, that is the size of the matrix and 0 denotes a zero matrix.

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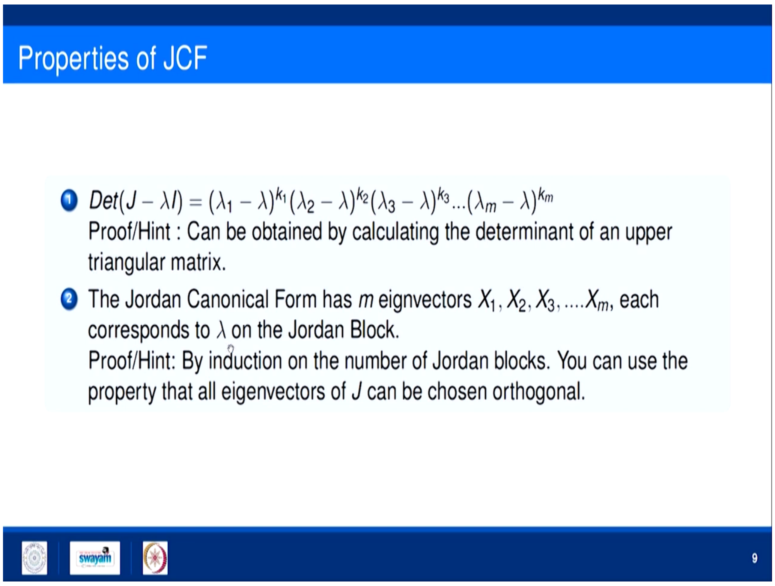

So, here if you are having a matrix a Jordan Canonical Form J, then determinant of this matrix is given by lambda 1 minus lambda raise to power k 1, that is coming from the first Jordan block, second Jordan block and so on.

So, it is the product of characteristic polynomial of e Jordan block. The Jordan Canonical Form has m eigenvectors because each Jordan block gives you only one l i eigenvector, here we are having m Jordan Blocks. So, m eigenvectors X 1, X 2, X m each.

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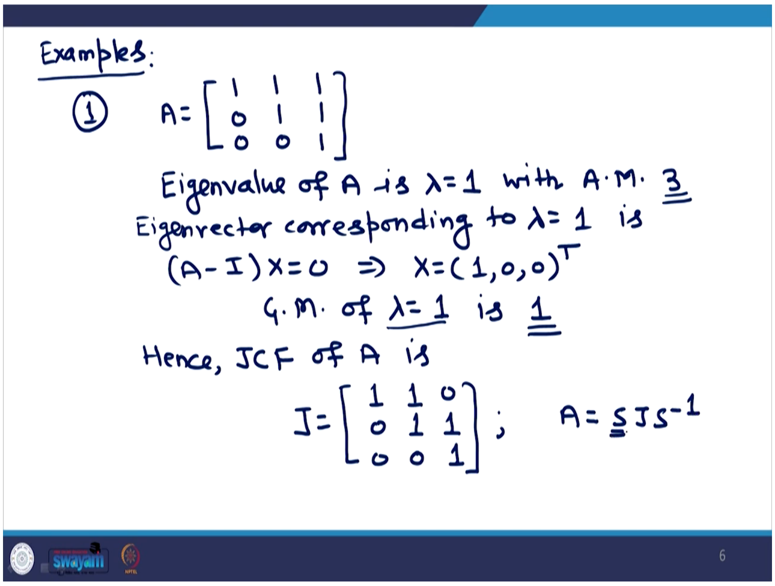

Corresponding to lambda on the Jordan block so, let us take some example of Jordan Canonical Form. So, let us take a matrix A equals to 1 1 1 0 1 1 0 0 1, then eigen value of this matrix. So, eigen value of A is lambda equals to 1 with algebraic multiplicity 3.

Now eigenvector corresponding to lambda equals to 1 is A minus I, X equals to 0 and this comes out to be X equals to 1 0 0.

So, hence geometric multiplicity of lambda equals to 1 is 1. So, it cannot be diagonalized. So, we have to go for Jordan Canonical transformation, but before that how to write Jordan Canonical Form of A, hence Jordan Canonical Form in short I am writing JCF of A is.

So, you see here geometric multiplicity gives you the number of Jordan Blocks. So, what is the number of Jordan Blocks corresponding to lambda equals to 1 that is 1. And what is algebraic multiplicity algebraic multiplicity tells you that what will be the total size of the Jordan Blocks corresponding to eigen value lambda equals to 1.

So, here 1 Jordan block and size is 3. So, what will be Jordan Canonical Form 1 1 0 0 1 1 0 0 1. And then we will find out a matrix S such that A equals to S J S inverse. We will see how to find out this matrix s in next lecture.

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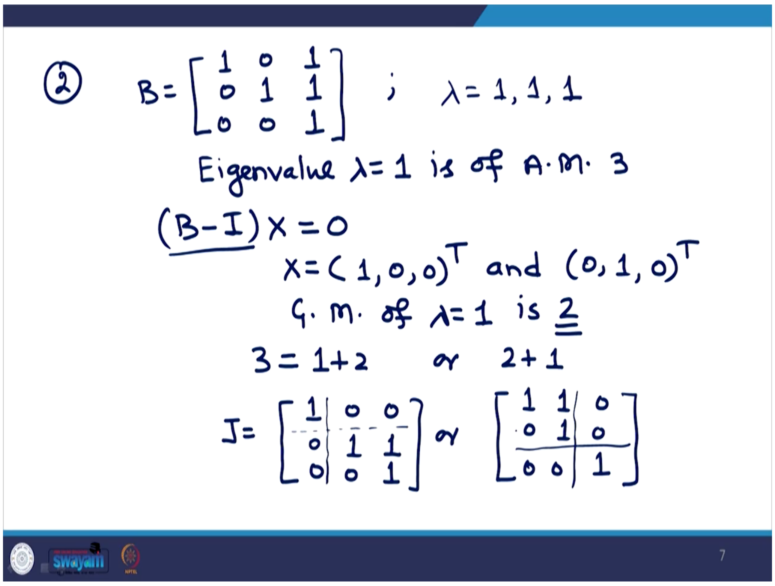

However, if you take another example let us say another matrix B which is 1 0 1 0 1 1 and 0 0 1, then again eigen value lambda equals to 1 is of algebraic multiplicity 3 because eigen value is 1 repeating three times.

Now, if you find out the eigenvectors corresponding to lambda equals to 1 then you have to solve B minus lambda I, X equals to 0. And in the null space of B minus I we are having two alive solutions means, analytics 2 of B minus I and those 2 vectors are X equals to 1 0 0 transpose and 0 1 0 transpose, you can check you can calculate it.

So, now geometric multiplicity of lambda equals to 1 is 2. So, here you will be having two Jordan Blocks and total size of those two Jordan Blocks will be sum will be 3. So, how you will decompose 3 in 2 blocks? So, obviously 1 plus 2 or 2 plus 1; so what will be Jordan Canonical Form?

So, here a block of size 1 another block of size 2. So, 1 0 0, this is a block of size 1 and then 1 1 0 1, or first this Jordan block of size 2, 1 1 0 0 1. So, it is a Jordan block of size 1.

I forgot to write here 0 0 and then here you are having of size 1. So, this is a Jordan Canonical form of B; however, generally we consider these two are same up to reordering of the Jordan Blocks.

So, in this lecture we have learned the definition of minimal polynomial, then we have learned about the Jordan Blocks and how to write the Jordan Canonical Form of a given matrix, based on it is eigen values and eigenvectors, basically based on the algebraic multiplicity and geometric multiplicity. In the next lecture we will learn about Jordan Canonical transformation, so with this.

Thank you very much.