Essential Mathematics for Machine Learning
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Lecture – 15
Polar Decomposition

Hello friends. So, welcome to the lecture number 15 of this course Essential Mathematics for Machine Learning. In this lecture we will talk about Polar Decomposition. In one of the previous lecture you have seen another decomposition algorithm for matrices that is singular value decomposition. And then we have seen some of the applications of singular value decomposition. Now, again polar decomposition is another decomposition technique for decomposing a square or rectangular matrix.

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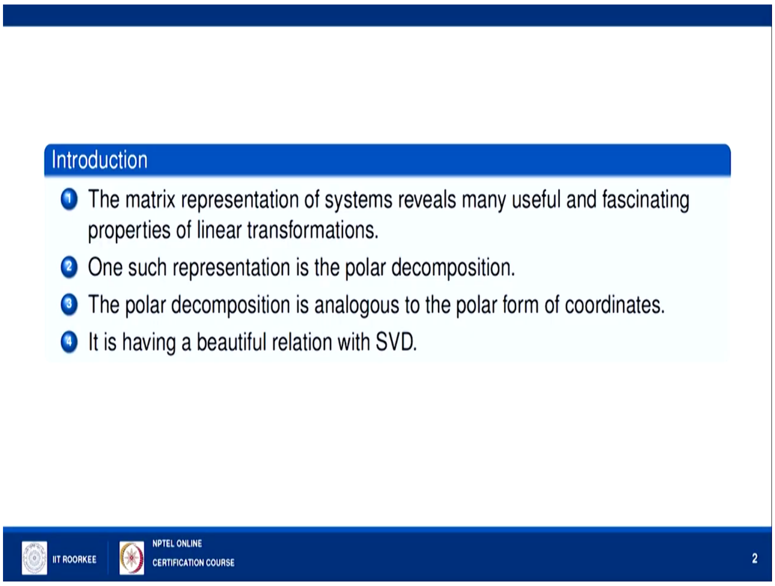

So, basically the polar decomposition is analogues to the polar form of coordinates. That is like, if you are having Cartesian coordinate like x and y then polar coordinates will be r and theta and then this particular decomposition again having a very beautiful relationship with singular value decomposition.

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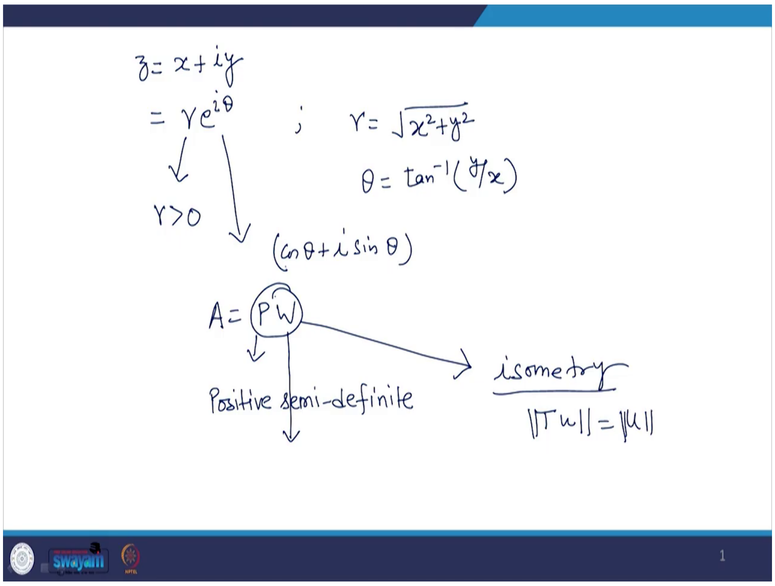

So, let us learn what is polar decomposition. So, in general we can write a complex number as x plus i y. Now why suddenly I am talking about complex number, because the polar decomposition is somewhat related to the polar form of complex number; that is the polar coordinates.

So, this particular complex number in polar form I can write as r into e raise to power i theta. Where r is nothing just the square root of x square plus y square and theta is tan inverse y upon x.

Now, if you see here this is r is always positive and then this is e raise to power i theta that is basically cos theta plus i sin theta which is a complex number again. In the same manner I want some a factorization of a matrix somewhat similar to the what we have seen for the complex number. So, it is somewhat like P and W. So, where P is a positive thing. So, here P means positive semi definite matrix and then when we talk about W; W is somewhat an isometry.

So, a an isometry transformation means that; if T is an isometry then T of u norm equals to u. And then to restrict this W as an isometry what we are taking? We are taking this W as a matrix such that it is having the orthonormal columns; such kind of decomposition of a matrix is called polar decomposition. So, the main motivation is like that

Now, let us come to formal definition. So, here first I will talk about the square matrices and then we will go for rectangular matrices.

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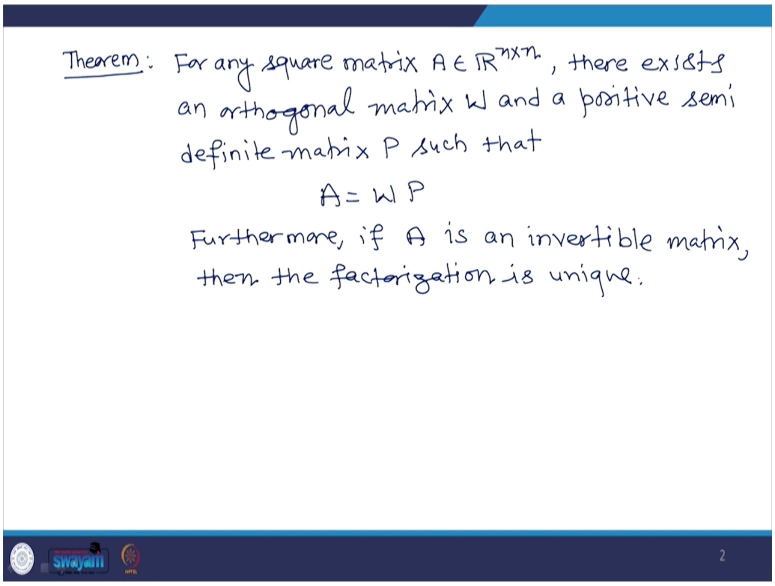

So, for any square matrix A, let us say a belongs to the vector space of n by n real matrices, there exists an orthogonal matrix W and a positive semi definite matrix P such that A equals to W P. So, here w is an orthogonal matrix means; the columns of W are orthonormal and P is a positive semi definite matrix.

Furthermore, if A is an invertible matrix, then the factorization is unique; means this kind of decomposition is unique. You can find out always unique W and P if A is an invertible matrix.

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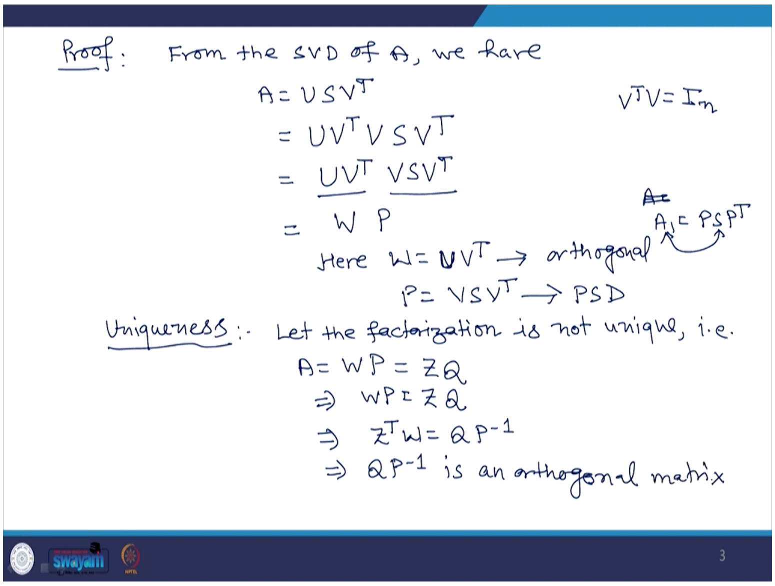

So, let us see a brief proof of this to understand the concept. So, here a is a square matrix of size n by n. So, if you remember the singular value decomposition from the singular value decomposition of A, we have. So, what was the singular value decomposition of A? A equals to U S V transpose; where U and V are orthogonal matrices and S is a diagonal matrix having the singular values of A as diagonal elements.

So, now what you do? I can write U V transpose V S V transpose, why because, V transpose V equals to I V is an orthogonal matrix. So, V transpose equals to V inverse. So, hence V transpose into V equals to identity matrix of size n whatever be the size of A.

Now, what you do? U take V the product of U and V transpose together and then the product of these three matrices. So, now, see I am having now these two U into V transpose which is again an orthogonal matrix, because U is an orthogonal V transpose is an orthogonal. So, product will be an orthogonal matrix. And then I am having VSV transpose, which is a symmetric matrix as well as the eigenvalues of this matrix will be nonnegative. Why because you know if you are writing a matrix A as or some matrix let us say A 1 as P S P transpose then what will happen? The eigenvalues of A 1 and S are similar, if P is a unitary matrix.

So, by the unitary diagonalization kind of thing. So, what I am doing? I am writing this U into V transpose as W and this as P. So, here W equals to U into V transpose and P equals to V S V transpose. So, it is orthogonal and this one is positive semi definite. So, in that way using a concept of singular value decomposition we can find out the polar decomposition of a matrix A.

Now, what will happened? We will prove the other part; that is the uniqueness. That is if A is invertible then this decomposition is unique. So, let the factorization is not unique; that is we are having A equals to W P and another factorization is Z into Q where Z is an orthogonal transformation and Q is a positive semi definite matrix.

So, this implies W P equals to Z Q. I can write Z T into W equals to Q into P inverse, because Z is an orthogonal matrix. So, if i pre multiply by z transpose. So, it will become Z transpose into W while it will Z transpose into w into P equals to Q and then I post multiply by P inverse. So, Z transpose into W equals to Q into P inverse.

Now, since W is an orthogonal matrix as well as Z is an orthogonal matrix. So, Z transpose. So, left hand side is having a orthogonal matrix. So, right hand side is also an orthogonal matrix. So, this implies Q P inverse is an orthogonal matrix.

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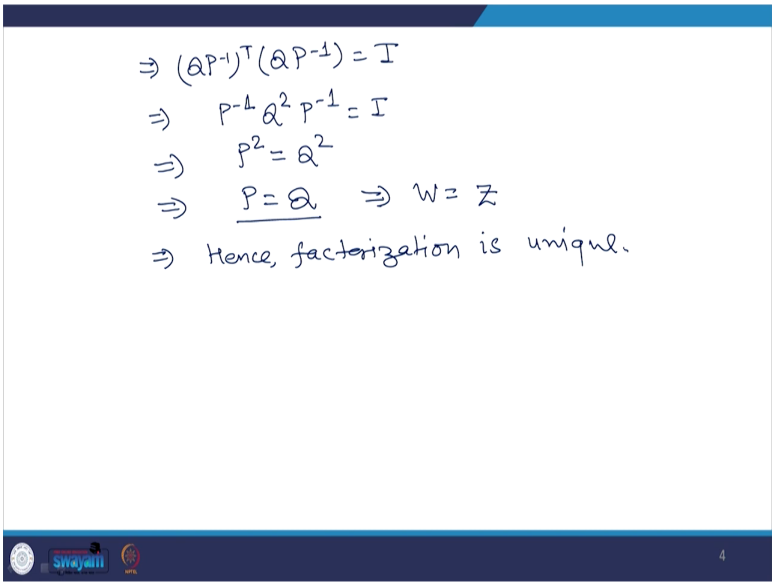

So, if it is an orthogonal matrix then what we are having QP inverse transpose into QP inverse equals to identity matrix. So, this means P inverse into Q square into P inverse equals to identity. This implies P square equals to Q square. This implies P equals to Q and if P equals to q then W equals to Z. So, hence factorization is unique. So, this is the proof of the polar decomposition theorem.

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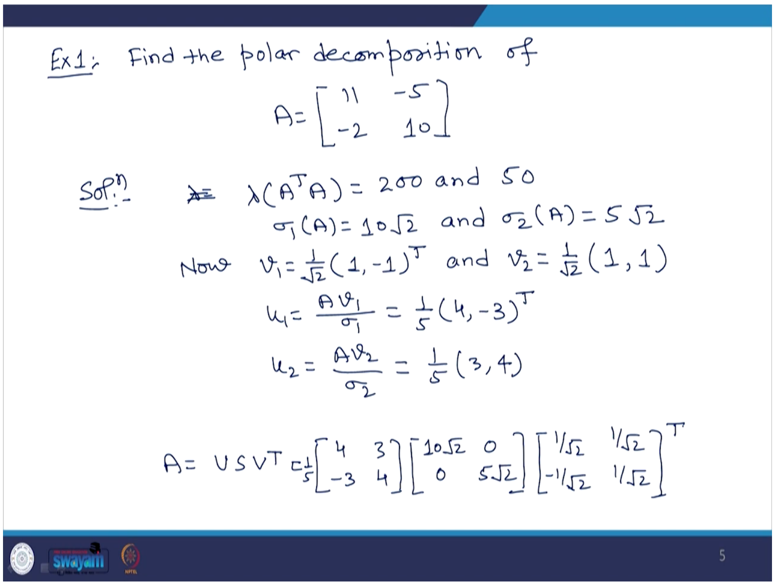

Now, let us take an example. So, find the polar decomposition of this matrix that is 11 minus 5 minus 2 10. So, as I told you, we will make use of singular value decomposition for calculating the polar decomposition in case of a square matrices. So, here the eigenvalues of the matrix A transpose A will be 200 and 50. So, it gives me the singular values of A as 10 root 2 and sigma 2 a is 5 root 2.

Now, the eigenvector of A transpose A corresponding to eigenvalue 200 will be 1 upon root 2 and then 1 minus 1 transpose. Similarly another eigenvector corresponding to eigenvalue 50 will be. So, I am writing here orthonormal eigenvectors sorry. So, 1 and 1.

Now, what I will be doing if I know v 1 v 2 I can have u 1 as A v 1 upon sigma 1 and this comes out to be 1 by 5 4, minus 3 transpose. Then u 2 will be A v 2 upon sigma 2 and this comes out to be 1 by 5 3, 4. So, now, what I am having A equals to USV transpose.

So, now, what is matrix U here? That is 1 by 5 4 minus 3 3 4 and then S will be having the singular values. So, 10 root 2 0 0 5 root 2 and then V transpose. So, V transpose will become 1 by root 2 minus 1 by root 2 1 by root 2 1 by root 2 transpose. So, this is the singular value decomposition of the matrix A.

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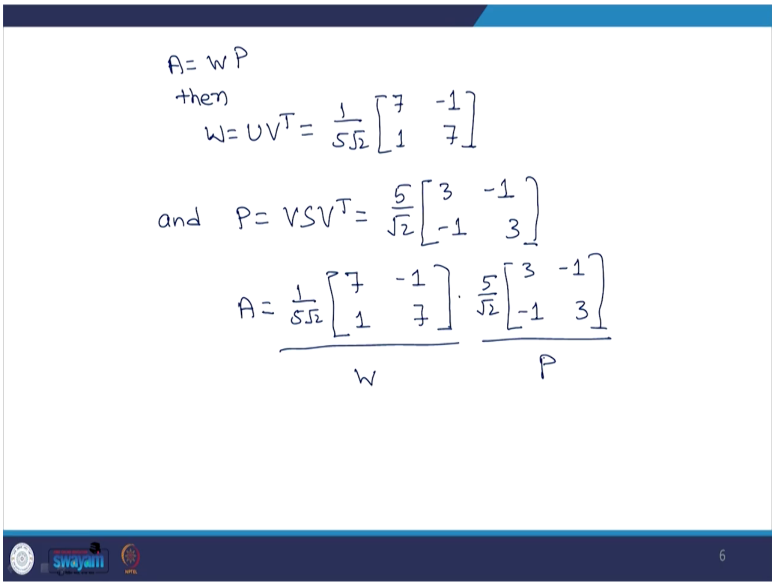

Now, from here we will obtain the polar decomposition. So, in polar decomposition if A equals to W P, then what is W? U into V transpose. So, this comes out to be 1 upon 5 root 2 7 minus 1 1 and 7. Similarly, P equals to VSV transpose. So, this comes out to be 5 upon root 2 3 minus 1 minus 1 and 3. And hence A equals to 1 upon 5 root 2 7 minus 1 1 7 into 5 by root 2 3 minus 1 minus 1 3. So, you can observe this is your W and it is P. You can verify that W is an orthogonal matrix while P is a positive semi definite matrix. So, this is about the square matrices.

Now, consider the rectangular matrices. So, rectangular matrix is may be a tall matrix where you are having more number of rows then columns or a fatty matrix where you are having more number of columns than row. So, for that case we are in both the cases we are having different polar decomposition; one is called right polar decomposition and the second one is called left polar decomposition.

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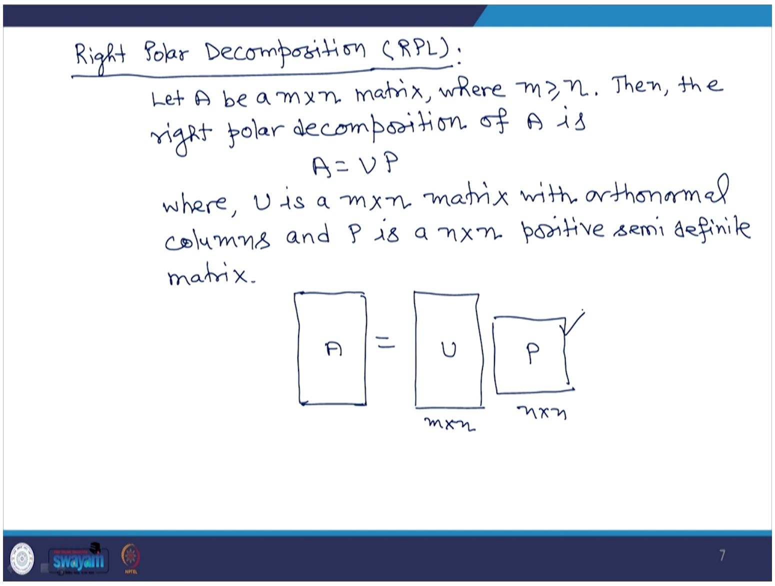

So, let us see what is right polar decomposition. So, let A be a m by n matrix, where m is greater than equals to n. Then, the right polar decomposition of A is A equals to UP. Where U is a m by n matrix with orthonormal columns and P is a n by n positive semi definite matrix.

Similarly, a so, what I want to say if A is a matrix which is having more number of rows. So, let us say this is your A. So, in right polar decomposition what I am having I am having a matrix U which is of the same size is A that is m by n and then I am having a square matrix n by n which is a positive semi definite matrix. So, such a decomposition is called right polar decomposition. And right means why right because here positive semi definite is in right in the decomposition.

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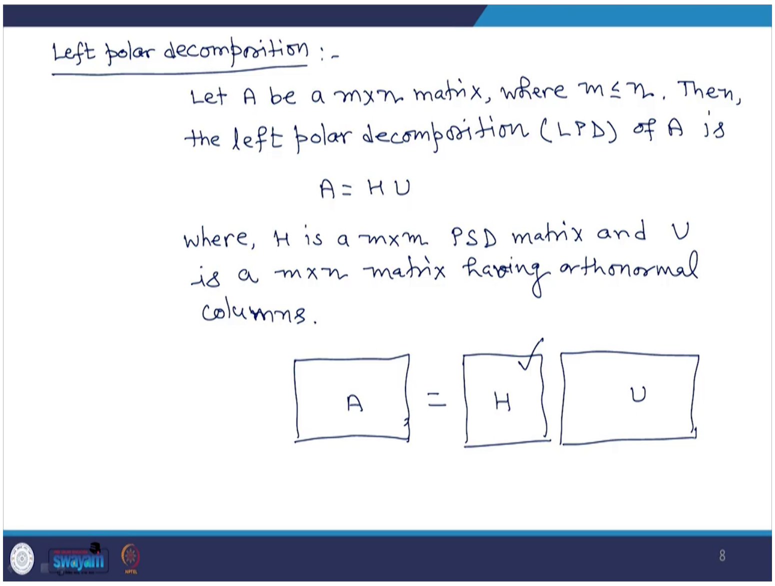

The another decomposition is left polar decomposition. So, let A be a m by n matrix where m is less than n then the left polar decomposition. In short I will write LPD of A is A equals to HU; where H is a m by m PSD means; positive semi definite matrix and U is a m by n matrix having orthonormal columns.

So, what I want to say here A is a fatty matrix; that is more number of columns, then this equals to a positive semi definite matrix H into a matrix U which is having orthonormal columns. So, why left? Because here h is coming in left which is the positive semi definite matrix.

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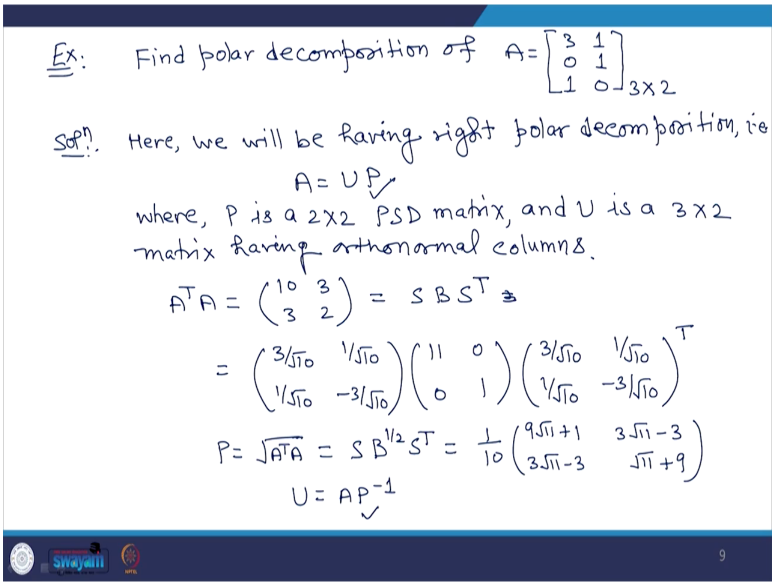

So, let us take some example of it. So, find polar decomposition of A equals to 3 1 0 1 1 0. So, it is a 3 by 2 matrix. So, let us come to the solution. So, here we are having more number of rows then column.

So, what we will be having? We will be having right polar decomposition. So, here we will be having right polar decomposition. That is A equals to U P where P is a 2 by 2 positive semi definite matrix and u is a 3 by 2 matrix having orthonormal columns. So, I will be having A transpose A. So, this is the process first you find out the A transpose A. So, it comes out to be 10 3 3 2

Now, it is a symmetric matrix. So, I can always write the unitary decomposition of this matrix. So, it will be SBS transpose where s is a unitary matrix and B is a diagonal matrix containing the eigenvalues of A and if I perform this decomposition this comes out to be 3 by root 10 1 by root 10 1 by root 10 and minus 3 by root 10, then B is here 11 0 0 1. So, this you can achieved by the diagonalization itself. And then the transpose of S; 3 by root 10 1 by root 10 1 by root 10 minus 3 by root 10 transpose.

Now, what is P? Actually, here P is a square root of A transpose A and then since A transpose A equals to SBS transpose. So, what is the A square root of A transpose A by one of the application of diagonalization I can write S B raise to power half S transpose. Because if a matrix A transpose A is having eigenvalue lambda 1 lambda 2 and so on. Then the eigenvalues of a square root of A transpose A will be square root lambda 1 square root lambda 2 and so on. And this comes out to be 1 by 10 9 root 11 plus 1. So, plus 1; 3 root 11 minus 3 3 root 11 minus 3 root 11 plus 9. So, this is your matrix P.

Now, you are having P. So, you have to calculate U. So, U will become A into P inverse. A is with U P inverse you can calculate from here. So, you will find out your matrix A also and in that way the polar decomposition will become A equals to U into P.

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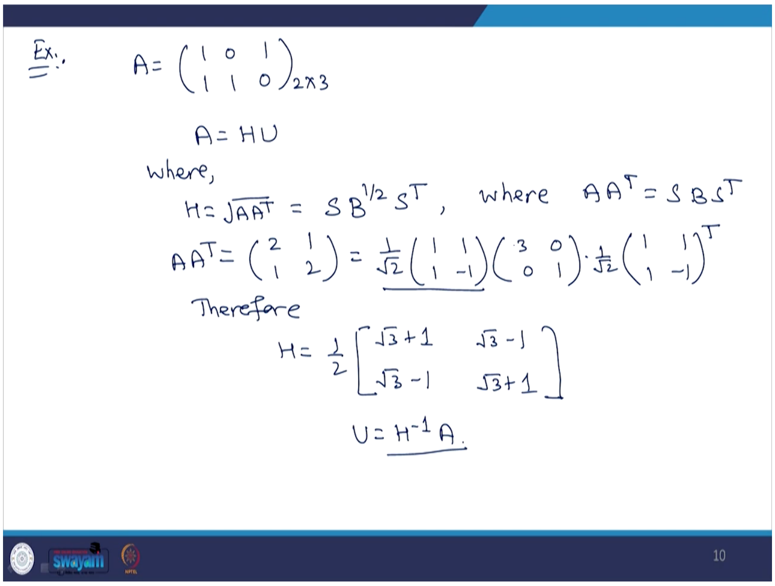

So, this is the way for finding the right polar decomposition. Example, if you are having a matrix let us say A equals to 1 0 1 1 1 0. So, it is a 2 by 3 matrix and I need to find out the polar decomposition. So, polar decomposition will be here left polar decomposition H into U; where h is a positive semi definite matrix of size 2 by 2 and U will be a 2 by 3 matrix having orthonormal columns.

So, here H will be square root of A into A transpose which is SB raise to power half into s transpose where A into A transpose equals to SBS transpose. So, if I calculate A into A transpose here; it comes out to be 2 1 1 2 and then 1 by root 2 1 1 1 minus 1 3 0 0 1 and then 1 by root 2 1 1 1 minus 1 transpose. So, then therefore, what will be H? H will become this into the B raise to power half.

So, B raise to power half will be having this diagonalization matrix has root 3 0 0 root 1; that is the 1. So, and then this matrix. So, it comes out to be 1 by 2 root 3 plus 1 root 3 minus 1 root 3 minus 1 and then root 3 plus 1. So, once you are having H then your U will become H inverse into A and by calculating this U you can write the polar decomposition of the matrix A.

So, in this lecture we have learn about polar decomposition; means how to do polar decomposition of A square matrix, how to do polar decomposition of a rectangular matrix and what is the relevance of polar decomposition with singular value decomposition.

So, we are having many applications of this polar decomposition like in machine learning in computer graphics and so on. Because it is a matrix factorization something like in terms of a orthogonal matrix and a positive semi definite matrix. And that you can utilize in many of the learning algorithm. So, these are the references

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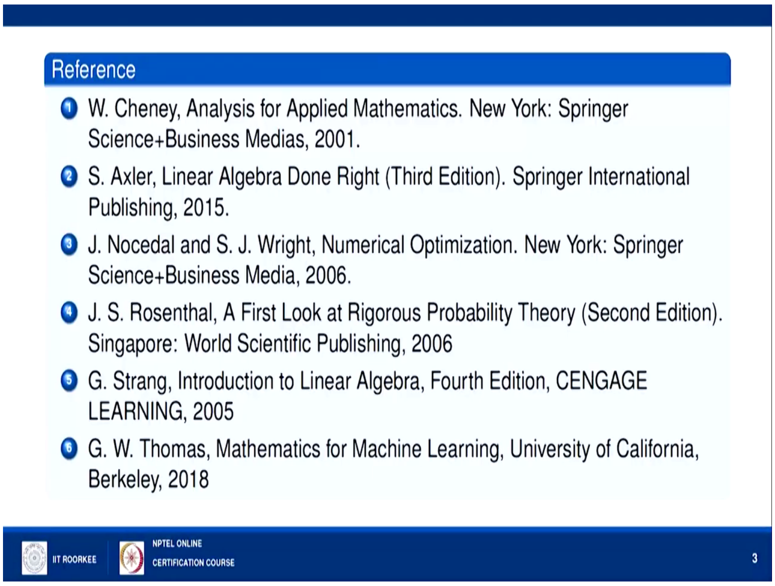

Thank you.