Lecture – 39
Solving m X n Games

Hello students, this is lecture number 39 and the title is we are going to study the most generalized m X n games. Now as you know, that we have first studied the simplest case that is the 2x2 game then we have studied by the mx2 game and then we have studied the 2xn game.

(Refer Slide Time: 00:53)

Solving m X n games

Method 1: Algebraic Method

Method 2: Linear Programming Method

Now this is the most generalized situation where we have a *m x n* game that is the payoff matrix is of the type a_{ij} which is an mxn matrix. Now we will study two methods for this kind of a scenario, the first one is called the algebraic method and the second one is the linear programming method.

(Refer Slide Time: 01:20)

Method 1: Algebraic Method

This method can be used to determine probability values by using different strategies by players A and B.

However, the method becomes lengthy when number of strategies for both the players is large.

So the first method that is the algebraic method is fairly simple and it means that we are going to convert the problem into a set of equations and then determine the value of the p_i's and the q_i's and the value of the game. So this method can be used to determine the probability values by using the different strategies by the players A and B. However, this method might become quite lengthy when the number of strategies for both the players is very large. So that is the limitation of this method.

(Refer Slide Time: 02:06)

Consider a game whose payoff matrix is $[a_{ij}]_{m \times n}$. Let $(p_1, p_2, ..., p_m)$ and $(q_1, q_2, ..., q_n)$ be the probabilities with which players A and B adopt their mixed strategies $(A_1, A_2, ..., A_m)$ and $(B_1, B_2, ..., B_n)$ respectively.

If V is the value of game, then expected gain to player A for this game when player B selects strategies $B_1, B_2, ..., B_n$ one by one is given by left hand side of simultaneous equations respectively

So look at this scenario suppose we have a game whose payoff matrix is given by the matrix $[a_{ij}]$ which is a m x n matrix and let $(p_1, p_2, ..., p_m)$ and $(q_1, q_2, ..., q_n)$ be the probabilities with which the players A and B adopt their mixed strategies $(A_1, A_2, ..., A_m)$ and $(B_1, B_2, ..., B_n)$ respectively.

If V is the value of the game then the expected gain to the player A for this game when the player B selects the strategies B_1, B_2, ..., B_n one by one is given by the left hand side of the simultaneous equations respectively.

(Refer Slide Time: 03:05)

Since, player A is the gainer player and expects at least V, therefore we must have

$$a_{11}\, p_1 + a_{12}\, p_2 + ... + a_{1m}\, p_m \geq V$$
$$a_{21}\, p_1 + a_{22}\, p_2 + ... + a_{2m}\, p_m \geq V$$

...

$$a_{n1}\, p_1 + a_{n2}\, p_2 + ... + a_{nm}\, p_m \geq V$$

Where $p_1 + p_2 + ... + p_m = 1$ and $p_i \geq 0$ for all i

Now since the player A is the gainer player that is it is the winning player and the player A expects at least V that is the value of the game. Therefore we must have these conditions that is $a_{11}\, p_1 + a_{12}\, p_2 + ... + a_{1m}\, p_m \geq V$ and like this we have n number of equations the last one being $a_{n1}\, p_1 + a_{n2}\, p_2 + ... + a_{nm}\, p_m \geq V$ with the condition that $p_1 + p_2 + ... + p_m = 1$ and all the p_i's should be ≥ 0 for all i.

(Refer Slide Time: 04:09)

Since player B is the loser player, therefore:

$$a_{11} q_1 + a_{12} q_2 + \dots + a_{n1} q_1 \leq V$$

$$a_{12} q_1 + a_{22} q_2 + \dots + a_{n2} q_2 \leq V$$

...

$$a_{1m} q_1 + a_{2m} q_2 + \dots + a_{nm} q_n \leq V$$

Where $q_1 + q_2 + \dots + q_n = 1$ and $q_j \geq 0$ for all j

Since the player B is the loser player see there are two players, one is the A player and the second one is the B player so we are assuming for all such cases that player A is the winner player and player B is the loser player. Since B is the loser player therefore we must have the following conditions $a_{11} q_1 + a_{12} q_2 + \dots + a_{n1} q_1 \leq V$ and similarly $a_{1m} q_1 + a_{2m} q_2 + \dots + a_{nm} q_n \leq V$ such that $q_1 + q_2 + \dots + q_n = 1$ and q_i's should all be ≥ 0.

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To get the values of p_i's and q_j's from these two sets of inequalities, they may be considered as equations and solved for the given unknowns.

However, if the system of equations is inconsistent, then at least one of the inequalities must hold as strict inequality.

In general, the solution can be obtained only by trial and error method.

Now to get the values of p_i's and q_i's from these two sets of inequalities they may be considered as exact equations and solved for the given unknowns. However, if the system of equations is inconsistent then at least one of the inequalities must hold as strict inequality and in general the

solution can be obtained only by trial and error method. So let us hope that our system is satisfying consistency.

Ex: In a game of matching coins with two players, suppose A wins one unit of value when there are two heads, wins nothing when there are two tails and losses 1/2 unit of value when there is one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game for the player A.

So in order to understand this method let us look at this example which says that in a game of matching coins with two players, suppose A wins one unit of value when there are two heads, wins nothing when there are two tails and loses 1/2 unit of value when there is one head and one tail. So this is a simple two player game and we are required to determine the payoff matrix and the best strategies for each of the two players and the value of the game for the player A.

The payoff matrix for the given matching coin games is given by,

		B	
		B1	B2
A	A1	1	-1/2
	A2	-1/2	0

So first thing is we need to determine the payoff matrix, now the payoff matrix for the given matching coin game can be shown as in this table that is we have two players A and B and each of the player has two strategies. The player A has strategy A1 and A2 and the player B has two strategies B1 and B2 and according to the conditions given in the question, the payoff matrix can be written as 1 -1/2,-1/2 and 0.

See read the question carefully according to this it says that A wins 1 unit of value when there are two heads he wins nothing when there are two tails and loses 1/2 unit of value when there is 1 head and 1 tail. So that is the reason why along with the 1/2 we have to write negative sign because it is a losing case so that is the way to write the payoff matrix.

(Refer Slide Time: 07:56)

Determine if there is a saddle point:

Maxmin = -1/2

Minmax = 0

No saddle point

		B		
		B1	B2	Row min
A	A1	1	-1/2	-1/2
	A2	-1/2	0	-1/2
Col max		1	0	

And now we need to make sure whether there is a saddle point or not. So we will determine the saddle point as follows, we will find out the row min for each of the rows and we find that the minimum of the first row is -1/2. Similarly, the minimum of the second row is also -1/2 and therefore we can say the Maxmin = -1/2. Then comes the column maximum, the first column maximum is 1 because we have 1 and -1/2 and for the second column we have -1/2 and 0 so there we have 0. And since we want the minimum of 1and 0 so therefore the minimum is 0 and since -1/2 is not equal to 0 therefore there is no saddle point in this game.

(Refer Slide Time: 08:57)

For player A: Let p_1 and p_2 be probabilities of selecting strategies A_1 and A_2 respectively. Then expected gain to player A when player B uses its B_1 and B_2 strategies is given by:

$$p_1 + (-1/2)p_2 \geq V$$
$$(-1/2)p_1 + 0.p_2 \geq V$$
$$p_1 + p_2 = 1$$

Now let us look, at the case of the player A and let us assume that p_1 and p_2 are the probabilities of selecting the strategies A_1 and A_2 respectively of the player A. Then the expected gain to the player A when the player B uses its strategies B_1 and B_2 is given by the following equations, $p_1 + (-1/2)p_2 \geq V$ and similarly $(-1/2)p_1 + 0.p_2 \geq V$ and also we must have the condition that $p_1 + p_2 = 1$ because they are the probabilities.

I hope you have understood how did we get these equations we got these equations by looking at the coefficients that are given in the payoff matrix 1,-1/2,-1/2, 0.

(Refer Slide Time: 10:11)

Considering the first two inequalities as equations:

$$p_1 + (-1/2)p_2 = V$$
$$(-1/2)p_1 + 0.p_2 = V$$

Giving $p_1 = -2V$ and $p_2 = -6V$

substituting in $p_1 + p_2 = 1$, gives $V = -1/8$

That is: $p1 = (-2)(-1/8) = 0.25$

and $p2 = (-6)(-1/8) = 0.75$

All right, so now we will consider the first two inequalities as equations that is they are being satisfied exactly $=V$ and we have $p_1 + (-1/2)p_2 = V$ and similarly $(-1/2)p_1 + 0.p_2 = V$ and when you solve this from the 2nd equation you can get $p_1 = -2V$ and if you substitute this p_1 back into the 1st equation you will get $p_2 = -6V$ and we have the third condition that is $p_1+p_2=1$ so therefore we will substitute these values of p_1 and p_2 into this equation which will give us $V=-1/8$. Therefore, $p_1=0.25$ and $p_2=0.75$ so this is the way we have solved for the player A we have got the values of p_1 and p_2.

(Refer Slide Time: 11:35)

For player B: Let q_1 and q_2 be the probabilities of selecting strategy B_1 and B_2 respectively.

Then the expected loss to player B when player A uses his A_1 and A_2 strategies is given by

$$q_1 + (-1/2)q_2 \leq 1$$
$$(-1/2)q_1 + 0.q_2 \leq 1$$
$$q_1 + q_2 = 1$$

Now consider the player B and let q_1 and q_2 be the probabilities of selecting strategy B1 and B2 respectively. Then the expected loss to the player B when the player A uses his strategies A1 and A2 can be written by the equations $q_1 + (-1/2)q_2 \leq 1$ and similarly $(-1/2)q_1 + 0.q_2 \leq 1$ and again we must remember the conditions that $q_1 + q_2 = 1$.

(Refer Slide Time: 12:23)

Considering them as equations:

$$q_1 + (-1/2)q_2 = V$$
$$(-1/2)q_1 + 0.q_2 = V$$

This gives: $q_1 = 2V$ and $q_2 = -6V$

Substituting in $q_1 + q_2 = 1$ gives $V = -1/8$

Thus $q_1 = 0.25$ and $q_2 = 0.75$

Now again in the same manner, we will consider these inequalities as equations and we will get $q_1 + (-1/2)q_2 = V$ and similarly $(-1/2)q_1 + 0.q_2 = V$. This will give us $q_1 = 2V$ and $q_2 = -6V$, the value of q_1 can be obtained from the 2nd equation and once we get this q_1 we can substitute it in the 1st equation and we will get the value of q_2 and finally we will substitute both the values of q_1 and q_2 into the third equation.

That is $q_1 + q_2 = 1$ which will give us $V = -1/8$. And again we will put all these values into our q_1 and q_2 and we should get $q_1 = 0.25$ and $q_2 = 0.75$.

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Hence the optimal strategies for players A and B are (0.25, 0.75) and (0.25, 0.75) respectively and the value of the game is $V = -1/8$

This can be verified by the oddments method

Hence what we have got? we have got the optimal strategies of the players A and B as follows, the optimal strategies of the player A are (0.25, 0.75) and the optimal strategies of the player B are (0.25, 0.75) and the value of the game is V= -1/8 and I would like to ask you to verify this result using the oddments method that we did in the previous lecture because remember this is 2x2 situation. So you can just verify that the results that you get using the oddments method is also the same.

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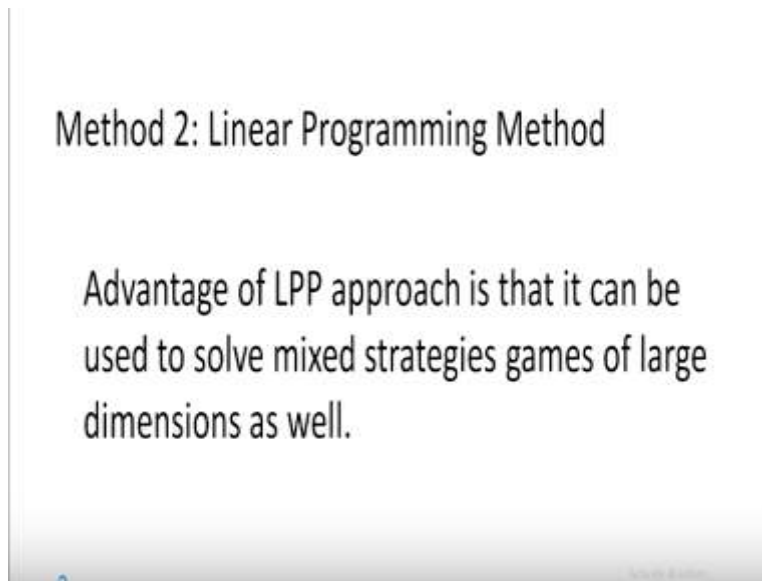
So now let us come to the 2nd method, in this method we will model the problem using a linear programming problem that is we will model the problem using the linear programming model and we will solve it using the simplex method. Now the advantage of using this LPP approach is that it can be used to solve the mixed strategies game for large dimensions as well. Remember the drawback of the method number 1 that is the algebraic method is that it can solve only very small sized problems, however, when the problem size is large then the 1st method that is the algebraic method might fail and here comes the 2nd method that is the linear programming approach.

(Refer Slide Time: 15:29)

Consider a payoff matrix of size $m \times n$.

Let a_{ij} be the element in the ith row and j^{th} column of the payoff matrix, and

let p_i be probability of m strategies ($i = 1, 2, ..., m$) for player A.

Then the expected gains for player A, for each of player B's strategies will be:

$$V = p_1 a_{1j} + p_2 a_{2j} + ... + p_m a_{mj} \text{ for } j = 1, 2, ... n$$

So let us look at a situation where we have a payoff matrix of the size mxn and let a_{ij} be the elements of the ith row and the jth column of the payoff matrix and let p_i be the probability of m strategies that is i=1,2 up to m for player A. Then the expected gains for player A for each of the player B's strategy can be written like this $V = p_1 a_{1j} + p_2 a_{2j} + ... + p_m a_{mj}$ for j = 1, 2, ... n.

(Refer Slide Time: 16:22)

The aim of player A is to select a set of strategies with probability p_i ($i = 1, 2, ..., m$) on any play of game such that he can maximize his minimum expected gains.

To obtain the probabilities p_i, the value of the game to player A for all strategies by player B must be at least equal to V.

Now the aim of the player A is to select a set of strategies with the probabilities p_i where i goes from 1,2 up to m on any play of game such that he can maximize his minimum expected gains this is what we have been doing till now. Now in order to obtain the probabilities the value of the

game to the player A for all strategies by player B must be at least equal to V. So it is the other way round for B as compared to player A.

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Thus to maximize the minimum expected gains, it is necessary that

$$a_{11} p_1 + a_{21} p_2 + \ldots + a_{m1} p_m \geq V$$
$$a_{12} p_1 + a_{22} p_2 + \ldots + a_{m2} p_m \geq V$$
$$\vdots \qquad\qquad \vdots$$
$$a_{1n} p_1 + a_{2n} p_2 + \ldots + a_{mn} p_m \geq V$$

where $p_1 + p_2 + \ldots + p_m = 1$;
$p_i \geq 0$ for all $i = 1, 2, \ldots\ldots$ M. m

Thus to maximize the minimum expected gain we must have the following conditions that is a_{11} $p_1 + a_{21} p_2 + \ldots + a_{m1} p_m \geq V$ and similarly $a_{12} p_1 + a_{22} p_2 + \ldots + a_{m2} p_m \geq V$ and the last equation will be $a_{1n} p_1 + a_{2n} p_2 + \ldots + a_{mn} p_m \geq V$ and of course the conditions that is $p_1 + p_2 + \ldots + p_m = 1$ and all the $p_i \geq 0$ where i goes from 1,2 up to m.

(Refer Slide Time: 18:05)

Dividing both sides of the m inequalities and the equation by V

Division is valid as long as V > 0.

In case V < 0, the direction of inequality constraints must be reversed.

If V = 0, add a constant to all the entries of the matrix ensuring that the value of the game (V) for the revised matrix becomes more than zero.

After optimal solution is obtained, the true value of the game is obtained by subtracting the same constant value.)

Now dividing both sides of the m inequalities and the equation by V, division is valid as long as V>0. So we can take the liberty of dividing the inequalities by V. However, in case V<0 the direction of the inequality constraints should be reversed and if V=0 then we can add a constant to all the entries of the matrix making sure that the value of the game V for the revised matrix becomes more than 0.

So I will be taking some examples to illustrate this situation. Now after optimum solution is obtained, the value of the game is obtained by subtracting the same quantity from the entire results.

(Refer Slide Time: 19:07)

Let $p_i/V = x_i$, (≥ 0). Then we have

$$a_{11}\frac{p_1}{V} + a_{21}\frac{p_2}{V} + \cdots + a_{m1}\frac{p_m}{V} \geq 1$$

$$a_{12}\frac{p_1}{V} + a_{22}\frac{p_2}{V} + \cdots + a_{m2}\frac{p_m}{V} \geq 1$$

$$\cdots\cdots\cdots\cdots$$

$$a_{1n}\frac{p_1}{V} + a_{2n}\frac{p_2}{V} + \cdots + a_{mn}\frac{p_m}{V} \geq 1$$

$$\frac{p_1}{V} + \frac{p_2}{V} + \cdots + \frac{p_m}{V} = 1$$

Now let us replace $p_i/V=x_i$ (≥ 0) scenario and in this case all the variables are now replaced by p_i/V. So the 1st variable x1 is replaced by p_1/V, x2 is replaced by p_2/V and like this. So the entire system of inequalities has been changed by making this substitution and this substitution has also to be done for the p1+p2, I mean $p_1/V+p_2/V+\ldots+p_m/V=1$.

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Since the objective of player A is to maximize the value of the game, V which is equivalent to minimizing $1/V$, the resulting linear programming problem can be stated as

Minimize $Z_p \; (= 1/V) = x_1 + x_2 + \ldots + x_m$

subject to the constraints

$$a_{11} x_1 + a_{21} x_2 + \ldots + a_{m1} x_m \geq 1$$
$$a_{12} x_1 + a_{22} x_2 + \ldots + a_{m2} x_m \geq 1$$
$$\vdots \qquad\qquad \vdots$$
$$a_{1n} x_1 + a_{2n} x_2 + \ldots + a_{mn} x_m \geq 1$$

and $\qquad x_1, x_2, \ldots, x_m \geq 0$

where $x_i = p_i / V \geq 0; \; i = 1, 2, \ldots, m$

Now since the objective of the player A is to maximize the value of the game V which is equivalent to minimizing of 1/V therefore the resulting linear programming problem can be stated as follows, that is minimize $1/V = x_1 + x_2 + \ldots + x_m$ subject to the constraints $a_{11} x_1 + a_{21} x_2 + \ldots + a_{m1} x_m \geq 1$ and like this the last equation will be $a_{1n} x_1 + a_{2n} x_2 + \ldots + a_{mn} x_m \geq 1$. All the $x_1, x_2, \ldots, x_m \geq 0$ where $x_i = p_i / V \geq 0$. Now remember that in a linear programming problem we must have all the decision variables ≥ 0.

(Refer Slide Time: 21:12)

Player B minimize the expected loss. Since minimizing of V is equivalent to maximizing $1/V$, so, LPP:

Maximize $Z_q \; (= 1/V) = y_1 + y_2 + \ldots + y_n$

subject to

$$a_{11} y_1 + a_{21} y_2 + \ldots + a_{1n} y_n \leq 1$$
$$a_{21} y_1 + a_{22} y_2 + \ldots + a_{2n} y_n \leq 1$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1} y_1 + a_{m2} y_2 + \ldots + a_{mn} y_n \leq 1$$
$$y_1, y_2, \ldots, y_n \geq 0$$

where $\qquad y_j = q_j / V \geq 0; \; j = 1, 2, \ldots, n$

Now coming to the player B the player B minimizes the expected loss because remember we have said that the player A is the maximizing player and the player B is the minimizing player

since the minimization of V is equivalent to maximization of 1/V. So the LPP can be written as maximize $1/V = y_1 + y_2 + \ldots + y_n$ subject to $a_{11} y_1 + a_{21} y_2 + \ldots + a_{1n} y_n \leq 1$ and similarly $a_{m1} y_1 + a_{m2} y_2 + \ldots + a_{mn} y_n \leq 1$.

Also the conditions $y_1, y_2, \ldots, y_n \geq 0$ where each of the $y_j = q_j / V \geq 0$; $j = 1, 2, \ldots, n$.

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Important:
LP for player *B* is the dual of LP for player *A* and vice-versa.
Therefore, solution of the dual problem can be obtained from the primal simplex table.
Since for both the players $Z_p = Z_q$, the expected gain to player *A* in the game will be exactly equal to expected loss to player *B*.

Now you must have observed a relationship between the LP for the player A and the LP the player B, the LP for the player B is the dual of the LP of the player A and vice-versa. This is a very important result and it shows another very important application of the duality theory which we have studied in the previous lectures. Therefore, the solution of the dual problem can be obtained from the primal simplex table.

Remember we have seen a result where from the optimum table of the primal we can actually derive the solution of the dual and vice-versa. Now since for both the players the objective function value that is Zp and Zq they are both same remember the value of the game for the player A and the value of the game for the player B is always same the expected gain to player A in the game will be exactly equal to the expected loss to the player B.

So this is also a very important consequence, that is, the value of the game for both the players will be same.

Note: LP technique requires all variables to be non-negative and therefore to obtain a non-negative of value V of the game, the data to the problem, i.e. a_{ij} in the payoff table should all be non-negative.

If there are some negative elements in the payoff table, add a constant to every element in the payoff table so as to make the smallest element zero.

Solution to this new game will give an optimal mixed strategy for the original game, but the value of the original game equals the value of the new game minus the constant.

It needs to be noted that the LP technique requires all the variables to be non-negative remember they should be ≥ 0 and therefore to obtain a non-negative value of V of the game the data to the problem that is the a_{ij}'s in the payoff table should all be non-negative, remember in the 1st method we had the payoff matrix there was some negative entries. In such type of situations, we need to add a quantity so that all the entries in the payoff table become ≥ 0. If there are some negative elements in the payoff table, we must add a constant to every element in the payoff table. So as to make the smallest element 0, now the solution to this new game will give an optimal mixed strategy for the original game. But the value of the original game equals the value of the new game minus the constant. So we must make sure that whatever we have added has to be subtracted from the results.

So now let us take a numerical example to illustrate how we can use the linear programming
technique to solve this game problem. Transform the following payoff matrix of a 2 person zero-
sum game into an equivalent linear programming problem and solve it using the simplex method.
Now you can see that there are two players A and B both have three strategies each that is
A1,A2,A3 and B1,B2,B3 and the payoff matrix is given by 1,-1,3;3,5,-3;6,2 and-2.

And as I said that moment you find any entry in the payoff matrix negative you must make sure
that it has to be removed by adding a constant such that all the entries they become 0 or >0.

(Refer Slide Time: 26:33)

So in the solution to this problem 1st thing we find is that there is no saddle point in the game and once we have sure that now we can apply the mixed strategies therefore we will add a constant to all the elements of the matrix. Now in the payoff matrix what we find is that we if we add 4 to all the entries then as a result we get the following payoff matrix 5,3,7;7,9,1;10,6 and 2.

Now you could have added 3 also that is not a problem if you had added 3 then 1 of the entries would have become 0 that is also fine but in this case I have added the constant 4. Now all the entries in the payoff matrix have become > 0. Now let us call the probabilities of A1,A2,A3 as p_1, p_2, p_3 and similarly the probabilities corresponding to the strategies B1,B2,B3 as q_1, q_2, q_3.

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> Let p_i $(i = 1, 2, 3)$ and q_j $(j = 1, 2, 3)$ be the probabilities of selecting strategies A_i $(i = 1, 2, 3)$ and B_j $(j = 1, 2, 3)$ by A and B, respectively.
>
> The expected gain for player A is:
>
> $5p_1 + 7p_2 + 10p_3 \geq V$ (if B uses strategy B_1)
>
> $3p_1 + 9p_2 + 6p_3 \geq V$ (if B uses strategy B_2)
>
> $7p_1 + p_2 + 2p_3 \geq V$ (if B uses strategy B_3)
>
> where $p_1 + p_2 + p_3 = 1$ and $p_1, p_2, p_3 \geq 0$

So let p_i and q_j be the probabilities of selecting the strategies A_i, i goes from 1 to 3 and B_j where j goes from 1 to 3, by the two players A and B respectively. Then according to our method the expected gain for the player A can be written as $5p_1 + 7p_2 + 10p_3 \geq V$ this comes if B uses strategy B_1 and similarly the 2nd equation that is $3p_1 + 9p_2 + 6p_3 \geq V$ this happens if B uses the strategy B_2 and the third equation is $7p_1 + p_2 + 2p_3 \geq V$ this comes if B uses the strategy B_3. Of course we must have the conditions $p_1 + p_2 + p_3 = 1$ and all $p_1, p_2, p_3 \geq 0$.

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Put $x_1 = p_1/V$, $x_2 = p_2/V$ and $x_3 = p_3/V$

The problem for player A, becomes:

Minimise $Z_p (= 1/V) = x_1 + x_2 + x_3$

subject to the constraints

$$5x_1 + 7x_2 + 10x_3 \geq 1$$
$$3x_1 + 9x_2 + 6x_3 \geq 1$$
$$7x_1 + x_2 + 2x_3 \geq 1$$

and $\qquad x_1, x_2, x_3 \geq 0$

Now we will adopt our strategy by making this substitution as $x_1 = p_1/V$, $x_2 = p_2/V$ and $x_3 = p_3/V$ and the problem for the player A will become minimize $Z_p (= 1/V) = x_1 + x_2 + x_3$ subject to the constraints $5x_1 + 7x_2 + 10x_3 \geq 1$, $3x_1 + 9x_2 + 6x_3 \geq 1$, $7x_1 + x_2 + 2x_3 \geq 1$ and all the three variables $x_1, x_2, x_3 \geq 0$.

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Similarly, the problem of player B:

Maximize $Z_q (= 1/V) = y_1 + y_2 + y_3$

subject to $\quad 5y_1 + 3y_2 + 7y_3 \leq 1$
$$7y_1 + 9y_2 + y_3 \leq 1$$
$$10y_1 + 6y_2 + 2y_3 \leq 1$$

and $\qquad y_1, y_2, y_3 \geq 0$

Where $y_1 = q_1/V$; $y_2 = q_2/V$ and $y_3 = q_3/V$.

Now similarly for the player B, we will have the corresponding problem as maximize $Z_q (= 1/V)$ $= y_1 + y_2 + y_3$ which is subject to $5y_1 + 3y_2 + 7y_3 \leq 1$, $7y_1 + 9y_2 + y_3 \leq 1$, $10y_1 + 6y_2 + 2y_3 \leq 1$ all the $y_1, y_2, y_3 \geq 0$ and we have remember made the substitution that $y_1 = q_1/V$; $y_2 = q_2/V$ and $y_3 = q_3/V$.

Similarly, the problem of player B:

Maximize $Z_q \ (= 1/V) = y_1 + y_2 + y_3$

subject to $\quad 5y_1 + 3y_2 + 7y_3 \leq 1$

$$7y_1 + 9y_2 + \ y_3 \leq 1$$

$$10y_1 + 6y_2 + 2y_3 \leq 1$$

and $\quad\quad\quad y_1, y_2, y_3 \geq 0$

Where $y_1 = q_1/V; \ y_2 = q_2/V$ and $y_3 = q_3/V$.

So the problem of player B turns out to be maximize $y_1+y_2+y_3$. Now we are converting this inequalities over here in this LP corresponding to the player B we have less than constraints. So remember we can add the slack variables to make this $\leq$ constraint as equality constraint. Remember that if we do it for the player A we will not be able to do it properly because there we have the $\geq$ constraints. And we will need to subtract the surplus variable and then we will not have a BFS so we will need some artificial variables. So therefore it is much more convenient to look at the $\leq$ constraints corresponding to the player B.

Problem of player B:

Maximize $Z_q = y_1 + y_2 + y_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

$$5y_1 + 3y_2 + 7y_3 + s_1 \quad\quad\quad = 1$$

$$7y_1 + 9y_2 + \ y_3 \quad\quad + s_2 \quad = 1$$

$$10y_1 + 6y_2 + 2y_3 \quad\quad\quad + s_3 = 1$$

And $\quad\quad y_1, y_2, y_3, s_1, s_2, s_3 \geq 0$

And we can simply add the slack variables into each of these equations as shown here in these equations and of course they have 0 coefficients in the objective function and of course as you know that the slack variables should also be ≥ 0. So now we have a resulting six variable problem which we can solve using the simplex procedure.

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Optimal solution is :

Basis B	Values	y_1	y_2	y_3	s_1	s_2	s_3
y_3	1/10	2/5	0	1	3/20	−1/10	0
y_2	1/10	11/15	1	0	−1/60	7/60	0
s_3	1/5	24/5	0	0	−1/5	−3/5	1
	−1/5	−2/15	0	0	−2/15	−1/15	0

I will omit the proof and the detailed solution but the solution that I obtained for this 6 variable problem is shown here in this table and you can use this table to get the results corresponding to the player B and as I told you that the player LP corresponding to the player A and the LP corresponding to the player B are the dual of each other. So therefore from the optimum table of this problem corresponding to the player B you can get the solution of the dual which is corresponding to the solution of the player A.

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Optimal solution (mixed strategies) for B is:

$y_1 = 0$; $y_2 = 1/10$ and $y_3 = 1/10$

Expected value of the game is:

$Z = 1/V - \text{constant} (= 4) = 5 - 4 = 1$.

Converting back into the original variables:

If $1/V = 1/5$ then $V = 5$. Hence

$y_1 = q_1/V, \Rightarrow q_1 = y_1 \times V = 0$

$y_2 = q_2/V, \Rightarrow q_2 = y_2 \times V = 1/10 \times 5 = 1/2$

$y_3 = q_3/V, \Rightarrow q_3 = y_3 \times V = 1/10 \times 5 = 1/2$

So we find that the optimum solution for the player B is $y_1 = 0$; $y_2 = 1/10$ and $y_3 = 1/10$ and the expected value of the game or rather the objective function is Z=1/V- constant which we actually had used to make all the elements as positive we had added this constant 4 into all the elements of the payoff matrix. So we get the value of Z=5-4 which is 1 and now we will want to convert this into the original variables remember the original variables were the p_i's and the q_i's.

So in this case for the player B we have to convert all the variables that is the y_i variables as q_i's so if $1/V=1/5$ then V=5 and also $y_1 = q_1/V, \Rightarrow q_1 = y_1 \times V = 0$; similarly $y_2 = q_2/V, \Rightarrow q_2 = y_2 \times V = 1/10 \times 5 = 1/2$ and similarly $y_3 = q_3/V, \Rightarrow q_3 = y_3 \times V = 1/10 \times 5 = 1/2$. So what we have done is we have converted the y_1, y_2 variables of the dual which was corresponding to the player B. Back into the q_i's where q_i's were the probabilities corresponding to the strategies of B.

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Optimal strategies for player A

$$x_1 = 2/15, \, x_2 = 1/15 \text{ and } x_3 = 0$$

Thus,

$$p_1 = x_1 \times V = (2/15) \times 5 = 2/3$$
$$p_2 = x_2 \times V = (1/15) \times 5 = 1/3$$
$$p_3 = x_3 \times V = 0$$

Hence, probabilities of using strategies by both players
are: Player A = (2/3, 1/3, 0)
and Player B = (0, 1/2, 1/2)
and the value of the game is $V = 5$.

So the optimum strategies of the player A now let us come to the optimal strategies of the player A this can be derived as I said these are the dual, so the dual variables can also be read the solution from the primal. So the values of $x_1 = 2/15$, $x_2 = 1/15$ and $x_3 = 0$ and then converting back into the probabilities p_1, p_2 we can write $p_1 = x_1 \times V = (2/15) \times 5 = 2/3$. Similarly $p_2 = x_2 \times V = (1/15) \times 5 = 1/3$ and $p_3 = x_3 \times V = 0$. Hence the probabilities of using the strategies by both the players are as follows player A has the strategies (2/3,1/3, 0) and player B has the strategies (0, ½, ½) and the value of the game V=5.

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General Types of Strategic Games

- The Prisoner's Dilemma
- The arms race
- Bach or Stravinsky?
- Matching Pennies
- Stage Hunt

So with this we come to an end of the topic on game theory and before I close I would like to mention some of the general type of strategic games The Prisoners Dilemma, The Arms Race, the Bach and Stravinsky, the Matching Pennies, the Stage Hunt and many more. So those of you are interested in this part of the operation research that is the game theory you can read about these strategic games in details.

So in the next lecture I will give you some exercises and some quiz questions. So with that I will close this lecture. Thank you.