Advanced Engineering Mathematics
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Lecture - 08
Cauchy's Theorem - II

Hello friends. Welcome to my second lecture on Cauchy integral theorem. Here we are going to consider the third case where C is any simple closed curve in the complex plane and we will be showing that integral of fz along C is=0. So let us say this is your simple closed curve. Let us take any simple closed curve in the complex z plane okay.

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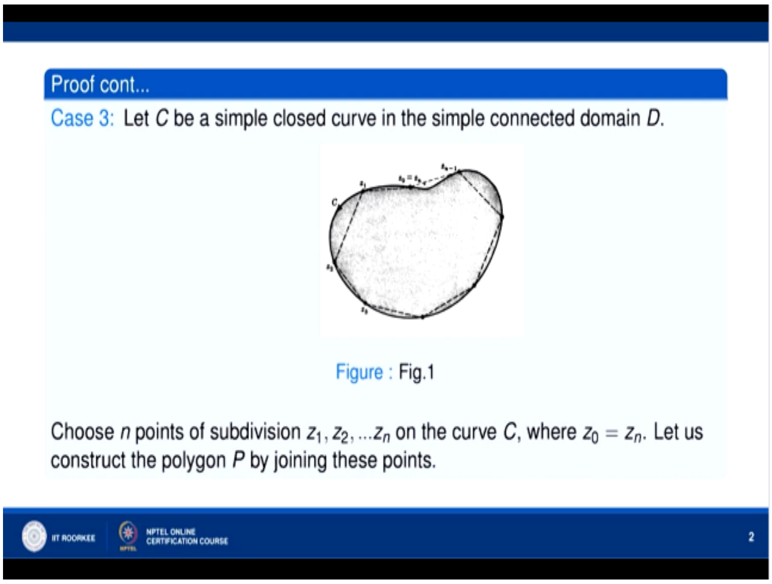

This is your point z0, then we have z1, we divided this curve C into n parts by means of points z0, z1, z2, z3 and so on zn-1 zn, zn coincides with z0. So choose end points of subdivision z1, z2, zn on the curve C where z0 coincides with zn and so that then we construct this polygon by joining z0 to z1, z1 to z2, z2 to z3, z3 to z4 and so on zn-1 to zn. We have already proved that integral over C fz dz=0 if C is a closed polygon okay.

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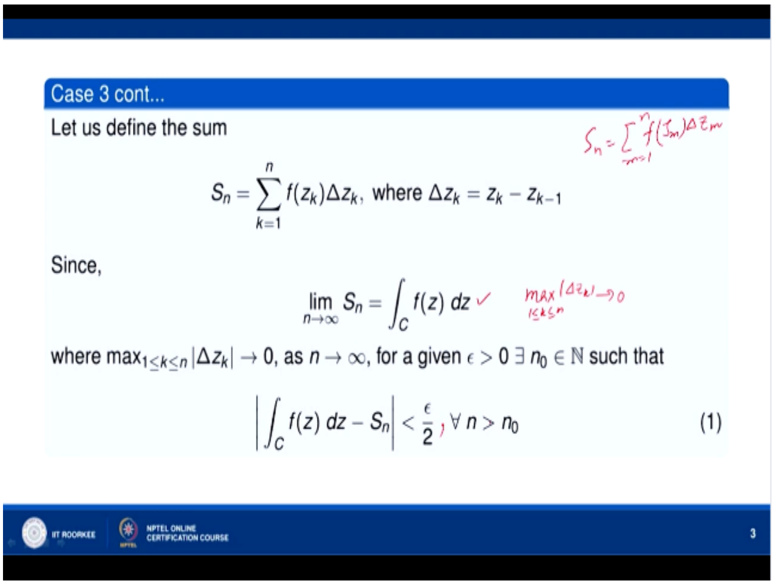

So let us define then the sum Sn to be=sigma k=1 to n f zk delta zk, where delta zk=zk-zk-1. Actually, sum Sn is here k=1 to n f zk delta zk here zeta m or you can say zeta k we have chosen as zk. Then, see we have earlier when we defined the integral over C fz dz as the limit of the sum we had written integral over C Sn=sigma m=1 to n f zeta m delta zm okay. As we had seen earlier zeta m can be chosen arbitrarily on the arc of the curve joining zm-1 to zm.

So that zeta m is chosen as zm here okay, so let us define Sn=sigma k=1 to n f zk delta zk where delta zk is zk-zk-1 then we know that Sn when n goes to infinity tends to integral over C fz dz where maximum of mod of delta zk 1<=k<=n goes to 0. So n goes to 0 in such a way that this maximum length goes to 0, then limit of Sn is equal to this.

So from this definition okay definition of limit for a given epsilon>0 we can find n0 belonging to the set of natural numbers N such that mod of integral over C fz dz-Sn is<epsilon/2 for all n>n0 okay.

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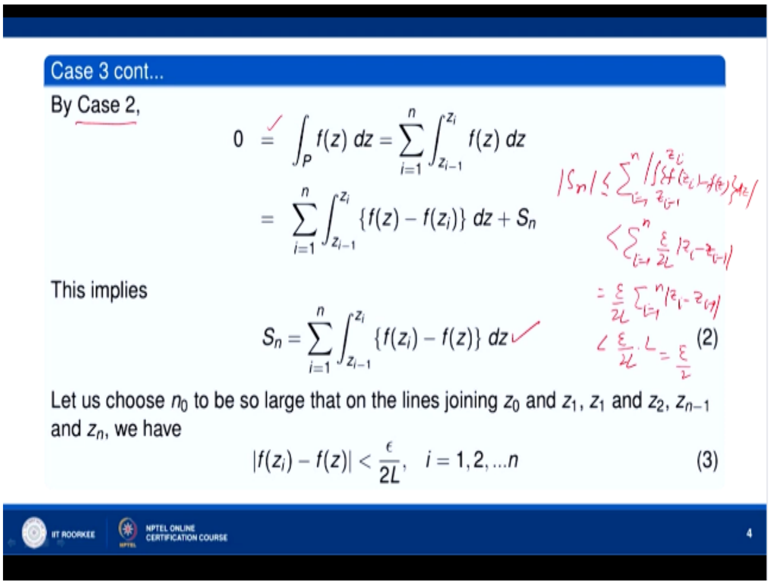

Now P is the closed polygon, you see here this is P, P is this closed polygon joining z0 to z1, z1 to z2 and so on zn-1 to zn. So integral over P fz dz=0, this we have already proved. This is by case 2 okay if 0=integral of P fz dz. Now this integral over P fz dz can be written as sigma i=1 to n integral over zi-1 to zi fz dz okay because P consists of the line segments joining z0 to z1, z1 to z2 and so on.

So we can write it as sum of those n integrals okay and then what I do sigma i=1 to n integral over zi-1 to zi, I subtract f zi here and add f zi okay, so when I add f zi, this Sn is nothing but sigma i=1 to n integral over zi-1 to zi f zi you can say okay. So that I subtract f zi here and add f zi, whatever I am adding that comes inside this Sn okay. Now but then this sum is equal to 0, so Sn will be equal to sigma i=1 to n integral over zi-1 to zi and then f zi-fz dz.

By using the fact that this sum is=0, I get Sn like this okay. Now let us choose n0 to be so large, we have n0 here, this n0 let us choose n0 to be so large that on the lines joining z0 and z1, z1 and z2, zn-1 and zn mod of f zi-fz okay mod of f zi-fz is<epsilon/2. So then what will happen, this follows from the continuity okay. So mod of f zi-fz is<epsilon/2L where L is the length of the curve C the given curve C.

Then, mod of Sn will be<epsilon/2 you can see here mod of Sn will be=mod of Sn will be<=sigma i=1 to n integral over zi-1 to zi okay mod of f zi-fz*dz okay. So mod of f zi-fz is<epsilon/2L, so this is<sigma i=1 to n epsilon/2L*length of the line joining zi to zi-1, so mod of zi-zi-1 okay. Now this is equal to epsilon/2L sigma i=1 to n mod of zi-zi-1 okay and if

you see here this length okay mod of z1-z0+z1-mod of z1-z2 mod of z2-z3 this will be definitely < the length of this one.

Because this is the length of the polygon of course some part of the polygon lies outside but as n goes to infinity this will give us tend to length of the curve, so this will be < epsilon/2L*L okay. This will be equal to epsilon/2, so when n is sufficiently large, this length sigma i=1 to n mod of zi-zi-1 will tend to L epsilon/2.

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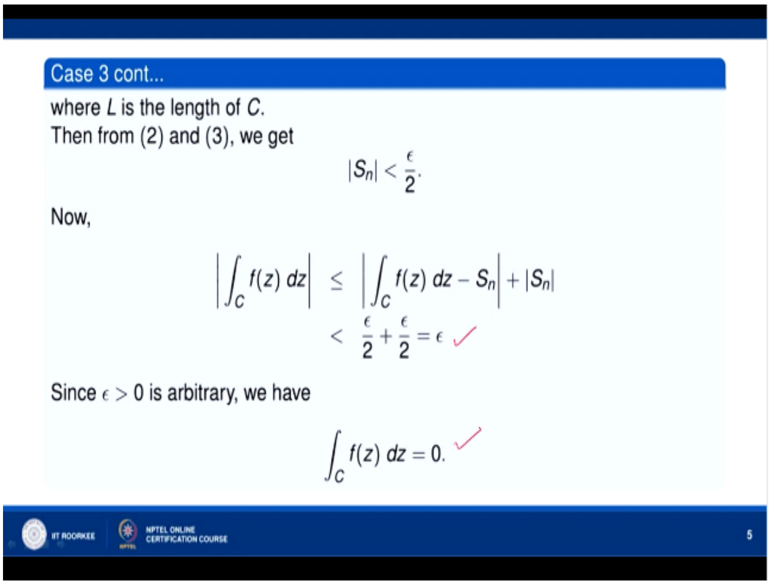

So you can see n0 is sufficiently large so this can be mod of Sn will be<epsilon/2. Now mod of integral over C fz dz is<=mod of integral over C fz dz-Sn+mod of Sn and mod of integral over C fz dz-Sn is<epsilon/2. This we have seen here okay for all n>0 and mod of Sn is also<epsilon/2, so this is<epsilon and epsilon is arbitrary, so we can say integral over C fz dz=0, so this is how we prove that integral of fz along any simple closed path is=0 if fz is analytic in a simply connected domain D and C lies inside D okay.

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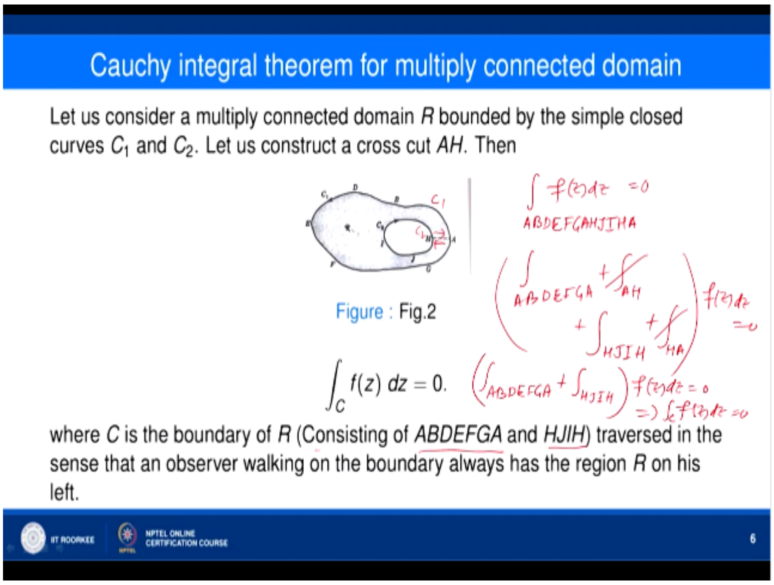

Now let us consider multiply connected domain. We are going to do Cauchy integral theorem for multiply connected domain. So let us consider doubly connected domain here. The boundary of the domain consists of two parts okay. So let us name them as C1 and C2, so this is my C1 and this is C2 okay. Let us construct a cross cut, a cross cut AH then what we do is what do we see?

C is the boundary of the region R which consists of ABDEFGA and HJIH okay HJIH traversed in the sense that an observer walking on the boundary of the region R, boundary always has the region R on its left okay. Then, what do we see, you can see integral over ABDEFGA okay and then I move along this direction H okay JI then I come back to H I move in this direction A okay this is equal to 0.

Because now by taking the cross cut we have found a simple closed curve okay and then I break it into parts, so integral over ABDEFGA. Then, integral over AH then integral over HJIH and then integral over HA is=0 okay. Integral over AH and integral over HA cancel okay and then I write integral over ABDEFGA okay ABDEFGA and HJIH okay. So integral over HJIH.

And our C consists of ABDEFGA and HJIH, so we can say that this gives us integral over C fz dz=0. So this is the Cauchy integral theorem for a multiply connected domain.

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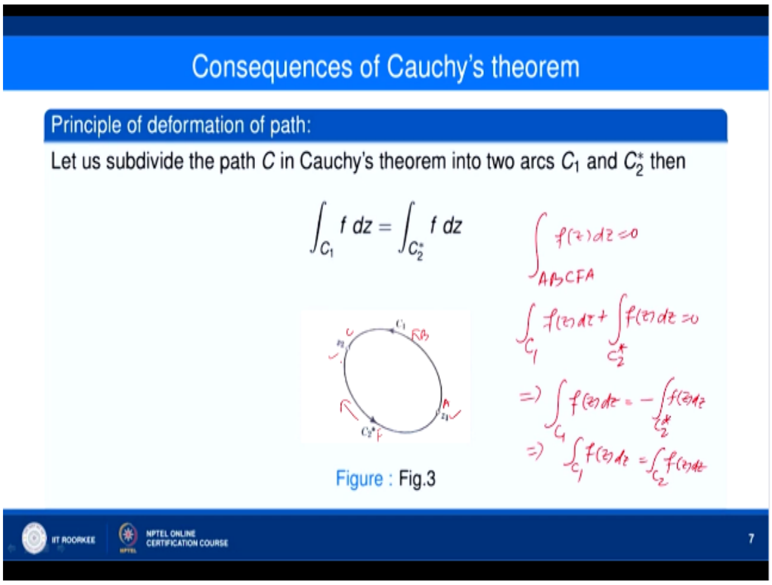

Now let us look at consequences of Cauchy integral theorem. Suppose we have a simply closed curve C in the Cauchy's theorem we can divide it into two arcs okay C1 and C2 star. So this is my curve C okay, this is my curve C, I divide it into 2 parts by from z1 to z2 this is my C1 curve okay and then C2 star is this. So then you can see integral over if this is ABC and here I take a point F then integral over ABCFA fz dz=0.

Now this gives you integral over C1, C1 joins z1 to z2, z1 I have named as A, z2 I have named as C, so integral over C1 fz dz=0 and then integral over this is CFA is C2 star, so integral over C2 star okay. Let us divide the path C in Cauchy two arcs given in C2 star, this is my C1 and this is C2 star. So integral over C2 star will be=0 okay. Now this will give you integral over C1 fz dz=-integral over C2 star fz dz.

And if you reverse the sense of integration along C2 star, then I get C2 curve. So this gives you integral over C1 fz dz=integral over C2. So what it actually tells us that in a simply connected domain if you take any two points z1 and z2 okay and then you join z1 to z2 integral of fz okay along any path that joins z1 to z2 and lies inside the simply connected domain remains same, it does not change.

I mean I joined z1 to z2 by the curve C1, whatever value I get, I get the same value if I join z1 to z2 by other curve C2, so this is the property of the analytic functions.

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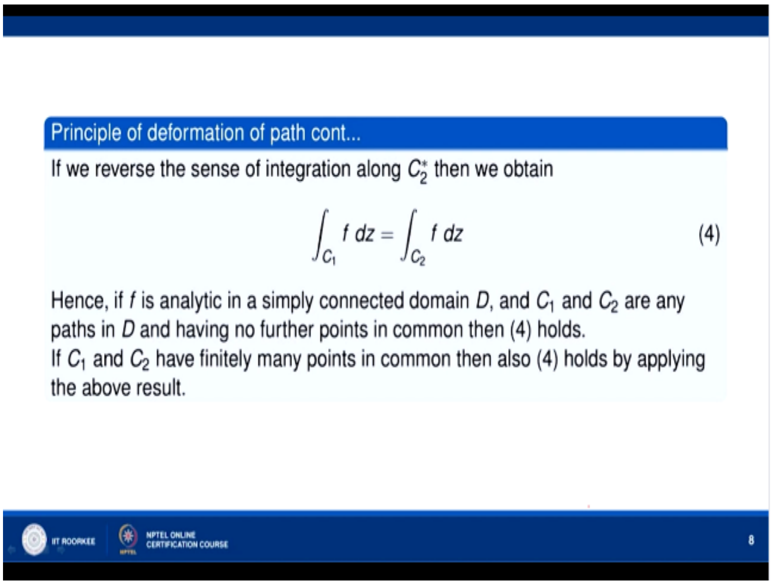

Now so this integral over C1=integral over C2. Now hence if fz is analytic in a simply connected domain D and C1 and C2 are any two paths in D which have no further points in common, then integral over C1 f dz becomes integral of fz dz over C2 okay. Now if C1 and C2 have finitely many points in common, then also this equation holds. Let us see how we get that.

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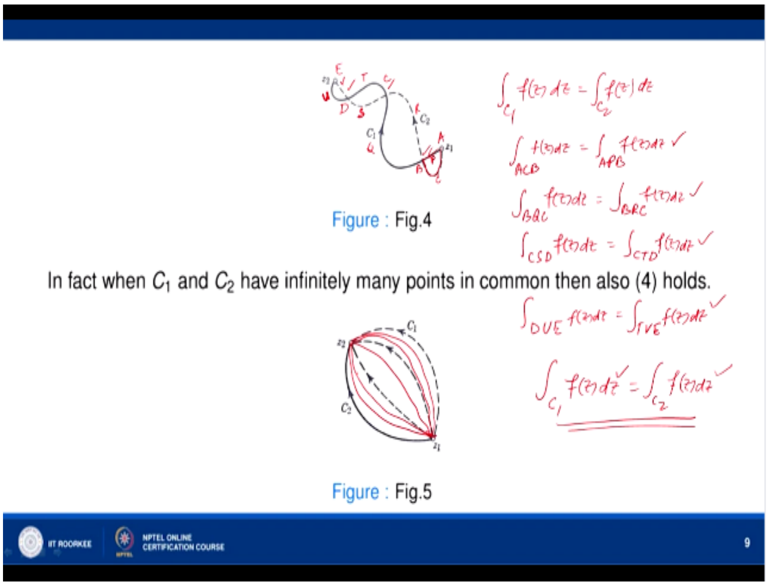

Suppose this curve okay z1 and z2 are joined by D2 curve C1 and C2 okay which have this point, this point and this point common okay. So there are finitely many points that are common then what I do integral along C1 integral along C2 we have to show that integral along C1 fz dz is=integral along C2 fz dz. So in order to prove this, I consider first this part, this part, this part, so in parts will consider.

So let me call it as A, this is B and this is let me say C and this is D and this is E okay. So then integral let me take a point C. So integral along ACB okay, integral along ACB fz dz okay=integral along APB. We can do it in parts by our previous result okay. So integral over ACB is=integral over APB. Similarly, integral over B to this one C okay by taking other points Q and then you can take your point R, integral over BQC is=integral over BRC.

And then integral over CST okay, so let us say this is S point okay CSD is=integral over CTD and lastly we consider this one okay. So let me say U here and V here okay. So DUE is=integral over TVE. So now add all these equations okay, add all these equations we get integral over C1 fz dz=integral over C2 fz dz okay. So if the two curves even if C2 have finitely many points in common then again integral over C1 fz dz=integral over C2 fz dz.

In fact, it can be shown that if C1 and C2 have infinitely many points in common then also this result holds okay. So now let us so integral over C1 is always=integral over C2 if the function fz is analytic in a simply connected domain D and C1 and C2 lie completely inside D which join any two points z1 and z2 inside D okay. Now let us look at this property of analytic functions.

See integral over C1 fz dz=integral over C2 fz dz can be imagined as if C2 has been obtained from C1 by a continuous deformation of path. You can see if you continuously deform C1 and keep z1 and z2 fixed okay, continuous deformation if you do you will come to this okay. So by continuous deformation of C1 which joins z1 and z2 okay, we can obtain the curve C2 so if you continuously deform the path which joins any two points z1 and z2 inside a simply connected domain, the integral of the function does not change that is what it says.

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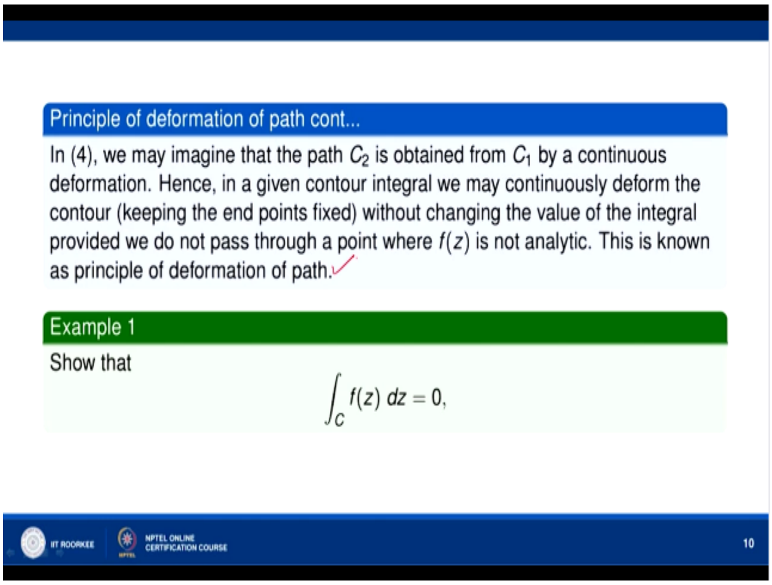

So if in this 4 integral over C1 fz dz=integral over C2 fz dz, we imagine that the path C2 is obtained from C1 by a continuous deformation. Then in a given contour integral we may continuously deform the contour keeping the end points fixed without changing the value of the integral provided we do not pass through a point where fz is not analytic. This is known as principle of deformation of path.

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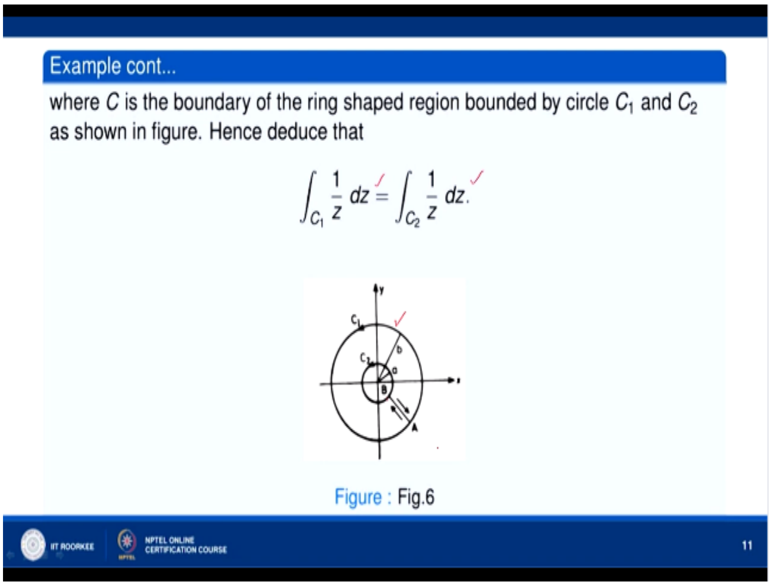

Now let us show that integral over C fz dz=0 in the case where C is the boundary of the ring shaped region bounded by circle C1 and C2 as shown in this figure okay and also deduce that integral over C1 1/z dz=integral over C2 1/z dz. Now here you can see we are having two circles okay. The circle C1 has radius B, center origin. The circle C2 has radius A and center origin okay.

We have joined the two circles by a cross cut, this cross cut AB okay then what do we notice if we use Cauchy integral theorem, the integral over this one okay integral over this is my curve C1, this is my curve C2 okay. So here integral over by this one we have earlier done this Cauchy integral theorem for a multiply connected domain okay. By using Cauchy integral theorem for a multiply connected domain, integral over C1 will be=integral over C2.

Here we are taking the anti-clockwise directions along C1 as well as clockwise direction along C2. We have seen in the Cauchy integral theorem for a multiply connected domain that integral along the two boundaries okay remains equal in the sense also is same. So here the sense is anti-clockwise direction along C1 so along C2 also it is clockwise okay. So integral over C1 1/z dz=integral over C2 1/z dz by using Cauchy integral theorem for a multiply connected domain.

What we will do is we will move along the curve and then the cross cut okay so that it becomes a simple closed curve and then the integral will be 0 by Cauchy integral theorem integral along AB will cancel because once we will move from A to B and once we will move from B to A and we will able to get this.

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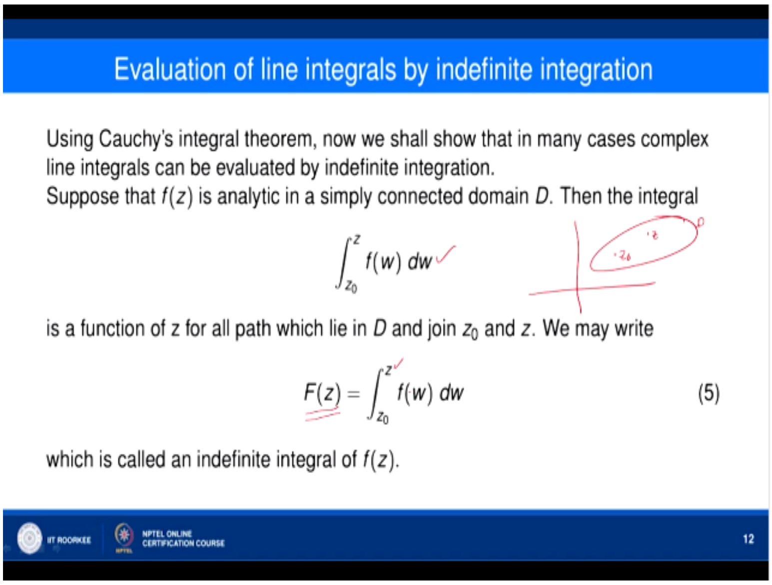

Now let us evaluate line integrals by indefinite integral. Cauchy integral theorem can be used to evaluate integrals by indefinite integral. You can see suppose fz is analytic in a simply connected domain D, then this integral. Let us look at this integral. This is a function of z for all paths which lie in D and join z0 and z. Let us take any domain D okay, so z0 is a point here and z is another point okay.

Then integral of fw with respect to w from z0 to z where z0 and z can be joined by any curve which lies only inside D okay. So then this is a function of z, I denote it by Fz okay and this because z can take any value so this is a function of z and this is called an indefinite integral of fz.

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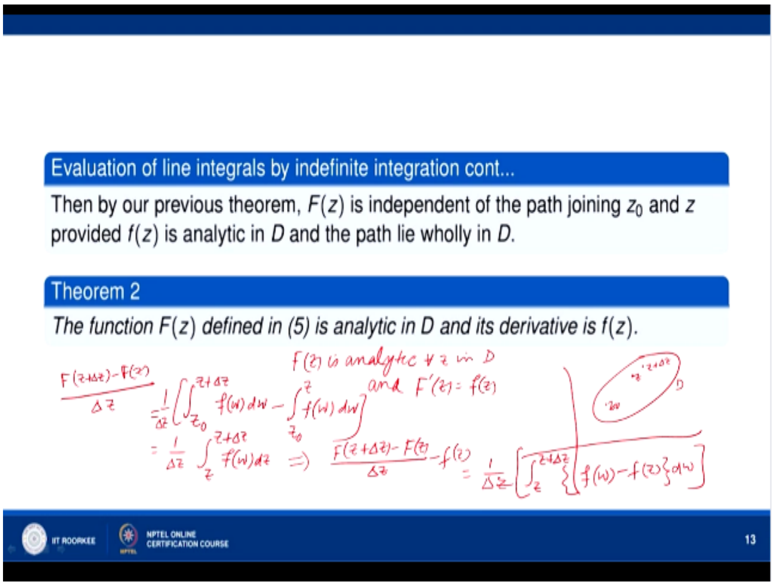

So by our previous theorem Fz is independent of the path we have earlier seen okay that the value of Fz will remain same as we can join z0 to z by any curve okay as long as the curve lies completely inside D. The value of the integral does not change so by our previous theorem Fz is independent of the path joining z0 and z provided fz is analytic in D and the path lies wholly inside D.

The function Fz defined in this equation is analytic in D and its derivative is fz. So we are going to show that this function Fz=z0 to z fw dw is analytic and its derivative is=fz. So we are going to show that fz is analytic for all z in D and F prime z=fz okay. So what you do is let us consider in the domain D we have z0 here, z is any point let us keep it fixed and let us take another point z+delta z which is also inside D okay. the domain D.

Then, Fz+delta z, let us consider Fz+delta z-fz okay divided by let us consider Fz+delta z-fz/delta z. So this will be by definition integral from z0 to z+delta z fw dw-integral from z0 to z fw dw*1/delta z okay. Now I can write it as –z0 to z I can write as z to z0, so z to z0 then z0 to z+delta z will be=1/delta w integral z to z+delta z fw dw okay and then I can this will give me Fz+delta z-Fz/delta z-F prime z. That means subtract F and z both sides Fz both sides.

So then this will be equal to 1/delta z integral z to z+delta z fw- this I can do because z is independent of w, z is fixed okay, this z is fixed. So fz is a constant okay, so integral over z to z+delta z dw will be delta z, delta z delta z will cancel so this Fz I can bring inside the integral okay.

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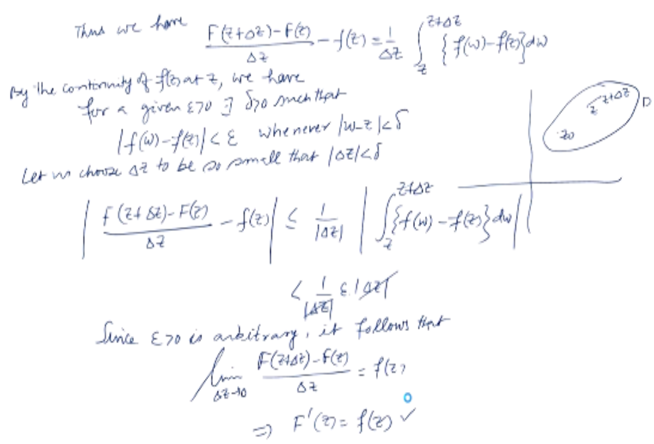

So now what we do? So thus we have okay we have this okay. Now what I do is this is my domain okay, here is z0, here is z and here is z+delta z okay. By the continuity of fz at z okay we have for a given epsilon>0 that gives delta>0 such that mod of fw-fz is<epsilon whenever mod of w-z is <delta. Let us chose delta z to be so small that mod of delta z is<delta. If we do that then what will happen, mod of fz+delta z-Fz/delta z okay–fz will be<=1/mod of delta z*mod of integral over z to z+delta z fw-Fz dw okay.

So we chose delta z to be so small that the line segment joining z to z+delta z lies completely inside the domain okay and then what will happen mod of fw-fz will be<epsilon for all w such that mod of w-z is<delta. So mod of n for any w lying between z to z+delta z okay, this will be<delta okay. So what we will have, this is<1/delta z*epsilon times delta z, we have this okay mod of fw-fz will be<epsilon and will have mod of z+delta z-z which is mod of delta z.

So this will cancel and we will get this is<epsilon okay. So since epsilon>0 is arbitrary, it followed that limit delta z tends to 0 Fz+delta z-Fz/delta z is=fz which implies that F prime z is=fz okay. So fz is differentiable okay, fz is differentiable at any z and therefore fz is analytic and further that F prime z is=fz.

So this is how we prove this theorem where we have shown that the integral of function fz analytic in a simply connected domain can be evaluated easily by indefinite integral. With this I would like to end my lecture. Thank you very much for your attention.