Advanced Engineering Mathematics
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Lecture - 07
Cauchy's Theorem - I

Hello friends. Welcome to my lecture on Cauchy's integral theorem. We will have 2 lectures on this theorem. You will see that this theorem is a very important result in complex analysis. We will need 2 results which we have given in the following two examples.

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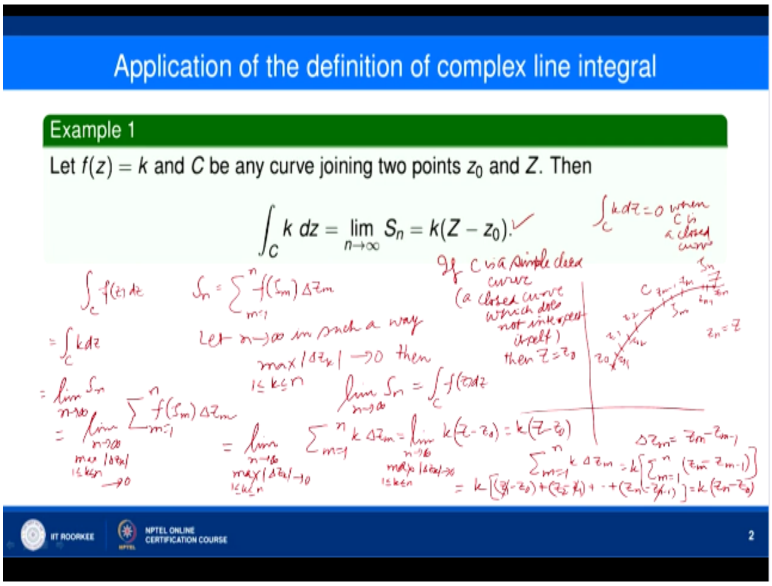

First is that if fz is=a constant k and C be any rectifiable curve joining 2 points z0 and Z in the complex plane. Say suppose we have 2 points, z0 and Z which joins which are the end points of a curve C in the complex plane and this is the direction of increasing values of T. Then, integral over C k times dz which is limit n tends to infinity Sn is=k times Z-z0. Now in our last lecture, we had seen that how to value at integral over C fz dz.

What we had done was we divided the curve C*n parts by means of points z0, z1, z2 and then say zm-1, zm and then zn okay. So zn is equal here is capital Z and then we had chosen arbitrary points zeta 1 here zeta 2 here and say zeta m here between zm-1 and zm and then between zm-1 and zn we had chosen zeta n and then we have formed the sum Sn as sigma m=1 to n f zeta m delta zm.

This was defined as Sn okay. Then, we had said that let n go to infinity in such a way that maximum of mod of delta zk 1<=k<=n tends to 0. Then, limit n tends to infinity, Sn is=integral over C fz dz okay. So here we are given fz=k, so integral over C kdz will be=limit n tends to infinity Sn which is equal to limit n tends to infinity sigma m=1 to n f zeta m delta zm where maximum of mod of delta zk 1<=k<=n goes to 0.

Now this is equal to, now we are given as z=k for all z belonging to C. So f zeta m, zeta m is the point on the curve C. So f zeta m=k for all m=1 to n. So we shall have limit n tends to infinity maximum of mod of delta zk 1<=k<=n goes to 0 sigma m=1 to n k delta zm okay. Now recall that delta zm is=zm-1 okay. So let us find the value of this sum sigma m=1 to n k, k is a constant delta zm.

So k can be written outside the summation and then delta zm is zm-zm-1 sigma m=1 to n okay. When you will expand this what you get is k times z1-z0 then z2-z1 and so on zn-zn-1 okay. So what will happen, z1 will cancel with z1, z2 will cancel with z2 and so on, zn-1 will cancel with zn-1 and what we will get k times zn-z0 okay and zn is=z okay. So what we will get? This will be limit n tends to infinity maximum of mod of delta zk goes to 0.

This is nothing but k times Z-z0. So you can see that summation m=1 to n k times delta zm is k times Z-z0 okay. So whatever be the subdivision of this curve C by taking end points okay this value remains constant. It does not depend on n okay, so this limit is=k times Z-z0 okay. So this is what we get integral over C k dz is=k times Z-z0. Now suppose this curve C is a closed curve, if C is a closed curve then capital Z will be=z0.

So if C is a closed curve, should be closed curve let me say. A simple closed curve means a closed curve which does not intersect itself. So in this curve, if the Z and z0 they are same, they meet then this will be a closed curve. So then what will happen, Z will be=z0 and so integral over C kdz will be=0 okay, so in case of a closed curve when C is a closed curve. We will need this result later on when we prove the Cauchy's theorem okay.

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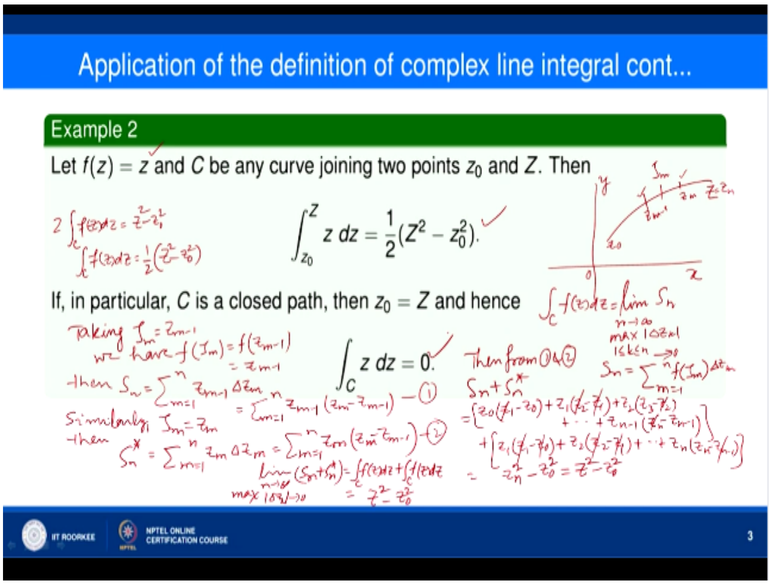

Now let us go to another example. Now let us consider fz=z okay. So here we take fz=z identity function fz=z and we shall find the value of the integral over z0 to Z, again we are taking the same curve say joining the points z0 and Z in the complex z plane okay. We shall show that it is 1/2 of capital Z square-z0 square. So in particular if C is a closed path, that means z0 becomes=Z then you see that the value of the integral is 0, so what we do here?

Let us again call the definition of the integral of fz integral over C fz dz is=limit n tends to infinity Sn where maximum of mod of delta zk 1<=k<=n goes to 0 and Sn is sigma m=1 to n f zeta m delta zm okay. This is zm-1 let us say and this is zm, zeta m is a point in between zm-1 and zm. Now zeta m is chosen in a completely arbitrary manner in the interval zm-1 on the curve between zm-1 and zm, so we can take zeta m to be zm-1 as well as zeta m to be zm okay.

So taking zeta m to be zm-1 okay, we have f zeta m=f zm-1=zm-1 because we are taking fz=z okay. So then Sn=sigma m=1 to n zm-1*delta zm but delta zm is zm-zm-1. So what we have is this okay. This is what we have. Now similarly let us take zeta m now=zm okay. First, we have taken zeta m=zm-1, now we take zeta m=zm. So then the sum sigma m=1 to n f zeta m delta zm let us call as Sn star.

Sn star=sigma m=1 to n f zeta m=f zm, so we will have zm here and then delta zm. So this will be summation m=1 to n zm*zm-zm-1 okay. Now let us add, let us say call this as equation 1, this as equation 2 then from 1 and 2, Sn+Sn star is=sigma m=1, so let me write Sn first so m=1 to n so that means z0 then we have z1-z0, then we have z1 z2-z1 okay. Then, z2*z3-z2 and so on when you put m=n we get zn-1*zn-zn-1.

This is the value of Sn okay+Sn star, Sn star we got the value of m=1 to n to z1*z1-z0 z2*z2-z0 and so on, zn*zn-zn-1 okay. Now you can see, when you add these two expressions, you see z0 z1 will cancel with z0 z1 okay. Then, z1 z2 here will cancel with this is z2-z1, so z1 z2 will cancel with z1 z2 here and so on and what will happen? So z1 z2 z3 they will all cancel, zn zn-1 will cancel with zn zn-1 here.

And then what we will have here –z0 square, then we will have –z1 square, z1 square will cancel with z1 square here, z2 square will cancel with z2 square here and so on. So this will give you the terms which will be left will be zn*zn that is zn square-z0 square okay. Now let us take the limit as n goes to infinity okay. Then, what will happen? Limit n tends to infinity such that maximum of mod of delta zk goes to 0 okay.

Sn+Sn star by our definition is=integral over C limit of Sn is integral over C fz dz and limit of Sn star is also integral over C fz dz because zeta m can be chosen anywhere on the curve between zm-1 and zm. For one time, we have chosen it as zm-1 and the other time we have chosen it zm. So this will give us the same limit okay. So +integral over C fz dz and this is then=zn square and zn is what? zn coincides with Z, so this is Z square-z0 square.

This quantity is independent of Z, independent of n okay, so Z square-z0 square okay. So twice integral over C fz dz is=Z square-z0 square or we can say integral over C fz dz=1/2 of Z square-z0 square. So this is how we prove this result. When this curve is a closed curve that is z0 n Zn coincide z0 n capital Z coincide then Z square will be=z0 square. So we will have the integral=0. So integral of fz=Z when C is a simple closed curve is 0.

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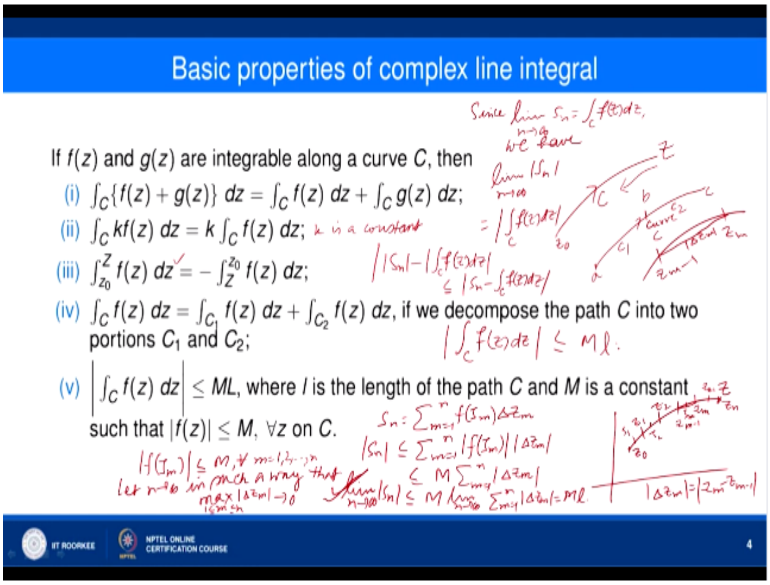

Now let us now list the basic properties of complex line integral. If fz and dz are integrable along a curve C okay, so rectify the curve C then integral over C fz+gz dz is=integral over C fz dz+integral over C gz dz okay and integral over C kfz dz is=k times integral over C fz dz where k is a constant, so k is a constant here, complex constant. Now integral over z0 to Z, if you take two points z0 n, Z and you join them by a curve.

Then, integral along z0 to Z fz dz will be negative of integral from z0 to Z. That is when you reverse the signs of integration along C, suppose this is your curve C okay and this is your z0 point, this is your capital Z point. Suppose you are integrating from z0 to Z okay, that is you get this integral and then you integrate from Z to z0 okay, that means you reverse the signs of integration, then there will be a change of sign.

So integral over C fz dz is integral over C1 fz dz+integral over C2 fz dz if we decompose path C into two portions. Suppose you have this is your path C okay, you decompose it into two parts. Say this is a, this is b and this is c okay, you decompose it into two portions say C1 is the part of the curve from a to b and C2 is the part of the curve from b to c, this is your b here, this is your b this is your curve C, so this is curve C okay.

So integral from a to b+integral from b to c along the curve will be equal to integral over C okay. So now these results can be easily established by using the definition of the line integral as the limit of a sum okay. So they can be taken as an exercise. I will prove that the mod of integral over C fz dz is<=M*L, where L is the length of the path C and M is a constant such that mod of fz is <=M.

Now let us see when we had considered the curve C in the complex z plane. So joining the points z0 and Z, we divide it into n parts by means of points z1, z2, zn-1 and zn okay and then we have taken arbitrary points, arbitrary points in completely arbitrary manner between z0 and z1 as zeta 1. Then, zeta 2 between z1 and z2 and then with zeta m between zm-1 and zm and we form the sum Sn=sigma m=1 to n f zeta m delta zm okay.

And the limit of this Sn as n goes to infinity when in such a way that mod of delta zm maximum value of mod of delta zm goes to 0 when m varies from 1 to n. Then, that gives you the integral over C fz dz okay. So let us see mod of Sn is<=sigma m=1 to n mod of f zeta m*mod of delta zm okay. Now mod of fz is<=m for all z on C, so mod of f zeta m will be<=m for all m=1 to n up to n.

Because zeta m belong to curve C okay, so this is <=m times sigma m=1 to n mod of delta zm. What is mod of delta zm? Let us look at this. You see this is mod of delta zm, let me show you like this. This is zm-1; this is zm okay. Then, mod of delta zm is mod of zm-zm-1 that is the distance between the complex number zm-1 and zm. So this is the mod of delta zm okay. That means the length of the chord joining the points zm-1 and zm okay.

So this is the sum of the lengths of the chords which join the points z0 to z1, z1 to z2 and then so on zn-1 to zn. Now if you let n go to infinity in such a way that let n go to infinity in such a way that maximum of mod of delta zm where m varies from 1 to n. This goes to 0, so you go on increasing the number of points that divide the curve into n parts okay in such a way that the maximum length between any two points goes to 0 okay between the points of the subdivision goes to 0.

Then, what will happen, limit n tends to infinity mod of Sn will be <=m times limit n tends to infinity sigma m=1 to n mod of delta zm. This will be equal to m times n that is the sum of the lengths of the chords will tend to the length of curve C okay. So limit n tends to infinity mod of sigma m=1 to n mod of delta zm will be=M*L and since limit n tends to infinity Sn is=integral over C fz dz.

We have limit n tends to infinity mod of Sn is=mod of integral over C fz dz because this can be easily established because mod of Sn-mod of integral over C fz dz by triangle inequality is

<=mod of Sn-integral over C fz dz okay. So then integral over C fz dz tends to, Sn tends to integral over C fz dz mod of Sn tends to mod of integral over C fz dz.

So this limit is nothing but mod of integral over C fz dz. So this actually this gives us mod of integral over C fz dz<=M*L. So this property is also going to be used in the proof of the Cauchy integral theorem okay.

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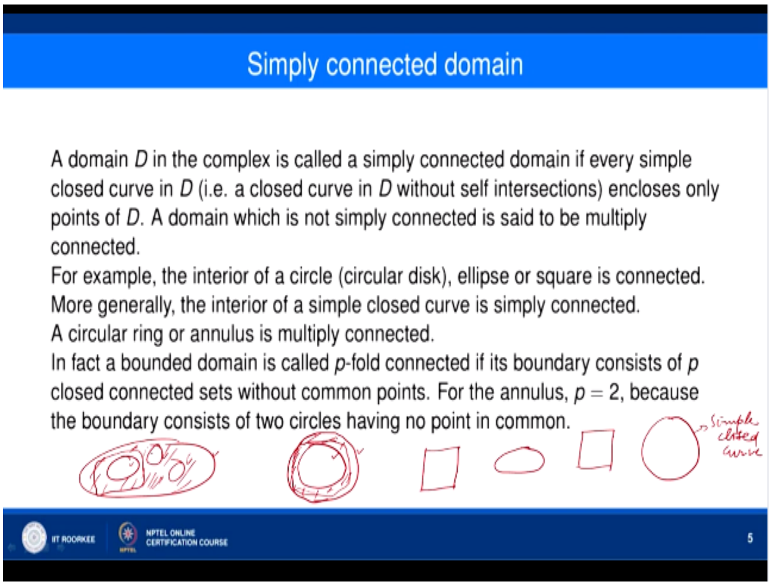

Now let us define what is a simply connected domain. Suppose a domain D in the complex plane is called a simply connected domain if every simple closed curve in D as I defined earlier a simple closed curve with a closed curve which does not intersect itself, so a closed curve in D without self-intersections encloses only points of D okay. A domain which is not simply connected is called a multiply connected domain.

For example, you can see the interior of a circle okay. A circle is a closed curve and it does not intersect itself. So it is a simple closed curve. If you take a rectangle, it is a closed curve which is a simple closed curve because it does not intersect itself. Let us take an ellipse okay, it is also a simple closed curve. They are all simple closed curves or you can take a square, they are all simple closed curves.

The interior of a simple closed curve is a simply connected domain. In general, the interior of a simply closed curve is a simply connected domain if you take the circular ring okay, this is the domain with boundary consists of two parts okay. The inner circle and outer circle okay.

So this domain shaded region is not a simply connected domain because we said that if you take any curve in this okay it cannot be shrunk to a point without leaving the domain.

Actually, we say that simple closed curve is one which can be shrunk to a point without leaving the domain. So here you can see if you take a closed curve in the shaded and you go on shrinking it, it cannot be shrunk to a point without leaving the domain. So it is not a simply connected domain. There are two boundaries of this region, so this is called a doubly connected domain.

A domain which has p boundaries will be called as p-connected domain. In fact, a bounded domain is called p-fold connected if its boundary consists of p closely connected boundaries without common points for the annulus, so this is annular region okay. The region bounded by two concentric circles is an annular region, so this is called a doubly connected domain. The boundary consists of two circles having no point in common.

So if a domain is not simply connected, it will be called a multiply connected domains. For example, you consider domain like this. You can see here if you take this region, this is not a simply connected domain because if you take a closed curve like this okay, it cannot be shrunk to a point without leaving the domain, so the domain the interior of this closed curve is not simply connected.

And its boundary of this region consists of 1, 2, 3 and 4 parts okay. So it is 4 connected, p connected means 4 connected domain, it is a multiply connected domain.

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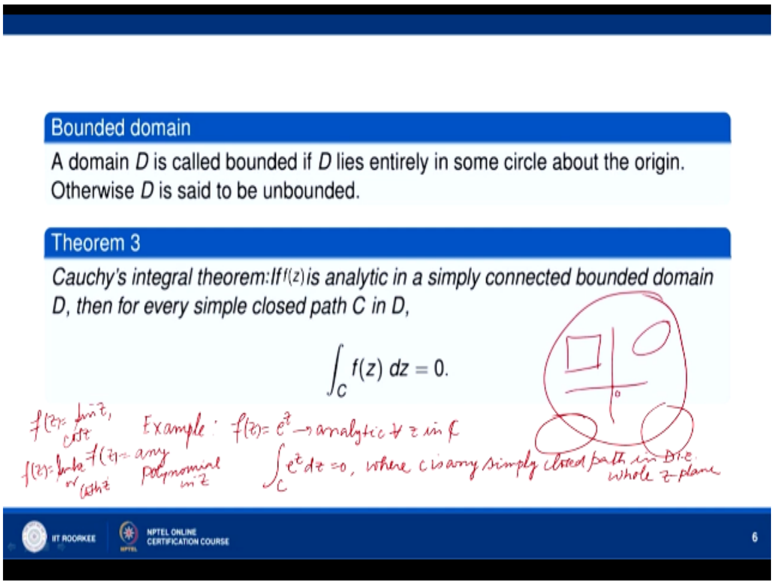

A domain D will be called bounded if it lies entirely in some circle about the origin. That means if you take a domain, it will be called a bounded domain. If you can draw a circle with center at the origin of some radius r such that the given domain lies completely inside it. Say for example you take this as origin, you can draw a circle like this okay in which this lies. So if you can construct a circle, the center at the origin of some radius r such that the domain D lies completely inside it, then the domain D will be called a bounded domain.

Now Cauchy integral theorem if fz is analytic in a simply connected bounded domain D, then for every simple closed path C in D integral over C fz dz is=0. Now you can see how powerful result is this Cauchy integral theorem. It says that you take any simply connected bounded domain, if fz is analytic there then you take any simple closed path in C in D integral over C fz dz=0. For example, let us consider fz=e to the power z.

We know that it is an analytic function. It is analytic for all z in the complex plane for all z in C in the complex z plane. So the curve says that you take any simple closed path in the complex plane okay any simple closed path okay, any simple closed path whether it is circle or it is ellipse or you take a square or rectangle okay, the integral of e to the power z along that path is=0.

So integral over C e to the power z dz=0 where C is a simply closed path, any simply closed path in D. D means the whole complex plane that is the whole z plane. You can take any polynomial fz can be any polynomial in z. The same situation the integral over C fz dz will be

0, you can take fz to be sin z or fz to be cos z or you take fz=sin hyperbolic z or cos hyperbolic z, they are all analytic functions for all z.

And so integral of fz okay will be 0 integral of fz dz will be 0 along any simple closed curve in the complex plane for these functions okay. So this is a very powerful result, let us see how we prove this.

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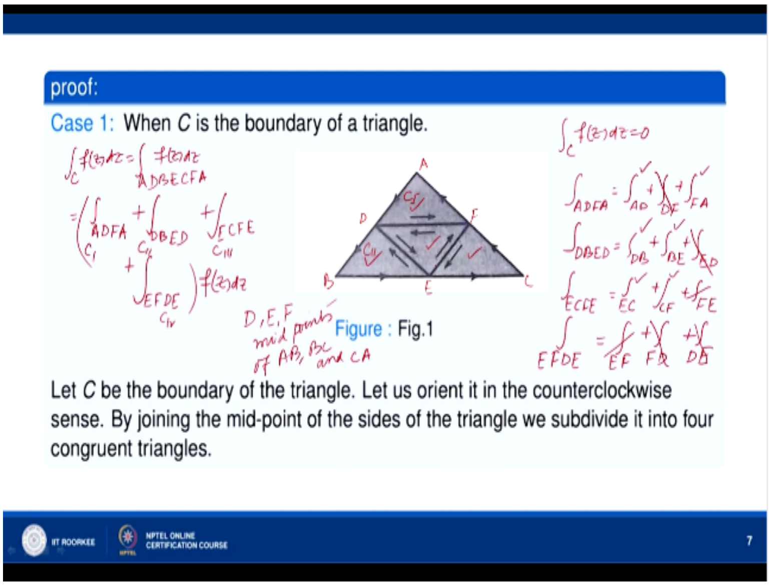

We begin with the case. First, let us consider the case where the simple closed curve C is the boundary of a triangle. You can see, the boundary of the triangle is a simple closed curve, it is a closed curve and it does not intersect itself okay, so C here is the boundary of a triangle. Now what we will do? We will show that integral over C fz dz is=0 okay. We will show that integral over C fz dz=0 where C is the boundary of a triangle.

So integral over C fz dz can be written, so here let me say this is A, this is B, this is C, this is D, E and F okay. Then, integral over C fz dz will be equal to what? Integral C fz dz will be equal to integral over ADBECFA fz okay, ADBECFA okay. Now I can write it as integral over ADFA+integral over DBED+integral over ECFE+integral over EDFE okay, so there are 4 triangles ADFA then DBED, then EDFE, then ECFE okay.

Now I say that integral over C fz dz which is ADBECFA is equal to this. Why? Because when we find the integral over ADFA, ADFA means integral over ADFA means integral over AD+integral over DF+integral over FA and integral over DBED means integral over DB+integral over BE+integral over ED. Integral over ECFE similarly so EC+integral over

CF+integral over FE and then lastly integral over EDFE is equal to integral over ED+integral over DF+integral over FE.

This along the triangle ECFE we are moving the integrals, so EFCE it should be. So here EFCE okay, so we will have here EFCE okay EF then FC then CE okay. Now here we have EDFE, so ED+DF+FE. Now let us add all these integrals. So what will we get? We will write here this way okay. Let us see, what I am doing is AB I am moving along this direction then this direction and then this way.

So integral over EFDE to show ECFE so EC and then integral over CF then integral over FE. So this will be integral of that okay and here in the case of the triangle, so EF FD and then DE. Now whether they cancel, you see so DF cancels with FD because they are in opposite directions. Then, EF cancels with FE and ED cancels with DE okay and what we have AD+DB. This AD+DB means AB okay and then we have BE+EC.

So BE+EC that means that integral over BC and then we have two more CF and FA okay that is integral over CA. So integral over C fz dz which is integral along AB+integral along BC+integral along CA can be written as the sum of 4 triangles and these triangles have been made by taking D, E and F as the mid-points of the sides AB, BC and CA. So these are D, E and F are mid points of AB, BC and CA.

So by taking D, E and F to be the mid points of the sides AB, BC and CA okay and then joining them joining D to E then E to F and F to D, we find 4 triangles okay and we see that integral over the triangle ABCA, integral along the path ABCA is=the sum over the boundaries of the 4 congruent triangles. These are all 4 congruent triangles okay. So let C be the boundary of the triangle.

Let us orient it in the counter clockwise sense okay. By joining the mid points of the sides of the triangle, we subdivide it into 4 congruent triangles okay. So let us name those congruent triangles as C1, C2, C3 and C4 okay. So C1 is ADFA this is C1 ADFA okay this is C1, C2 is DBED this is C2, C3 is EFCE this should be ECFE okay so ECFE this is C3 okay, this is C3 and C4 is this one EFDE okay EFDE this one okay, so this is fourth triangle okay.

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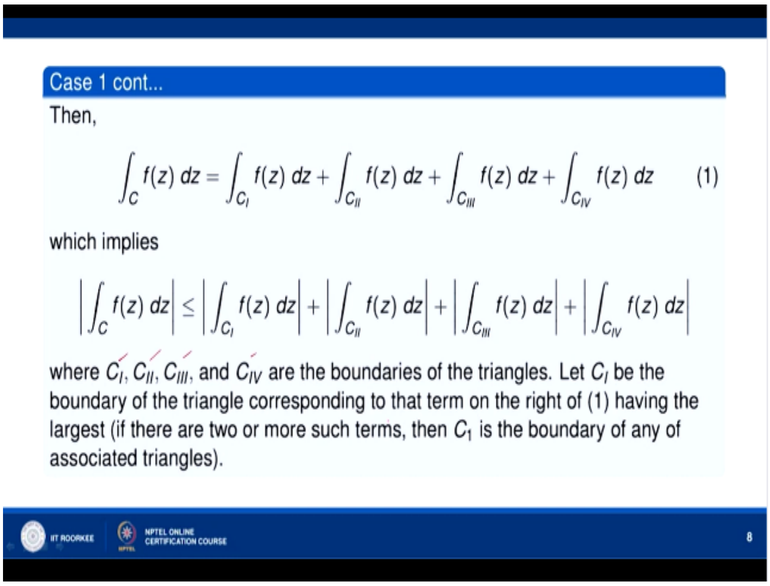

So this is how we write it and then what we do is integral over C fz dz is=integral over C1+integral over C2+integral over C3 and C4 fz dz, where C1, C2, C3, C4 are boundaries of the 4 congruent triangles okay. Mod of integral over C fz dz by triangle inequality is <=mod of integral over C1 fz dz+mod of fz dz over C2+mod of integral over fz dz over C3 and mod of integral over C4 of fz dz.

So now what we do is let us say that C1 with the boundary of the triangle which corresponds to the term on the right side of this equation okay having the largest value. If there are two or more such terms, then C1 is the boundary of any of the associated triangles okay.

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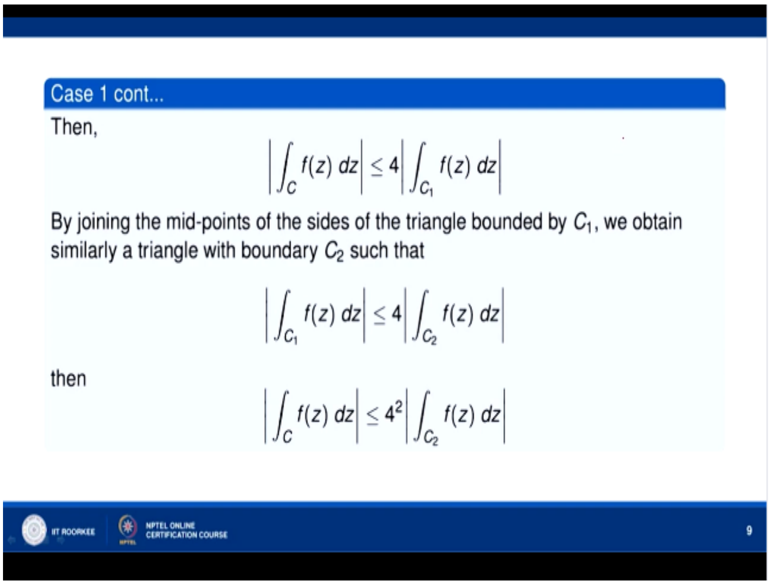

So then what we will do? This will be mod of integral over C fz dz will be<=4 times mod of integral over C1 fz dz because C1 is the boundary of that triangle whose integral over C1 is the largest okay. So each of the terms on the right side of this equation will be <=mod of integral over C1 fz dz okay. So what we have here, you see this is what the situation, out of these 4 congruent triangles we have taken that triangle as C1 okay whose integral of fz along that C1 is having the maximum value okay.

So that C1 is now again subdivided into 4 congruent triangles by taking the mid points of the sides of that triangle C1 and joining them okay. So by similar process by joining the mid points of the sides of the triangle bounded by C1, we obtain similarly a triangle with boundary C2 such that mod of integral over C1 fz dz is <=4 times mod of integral over C2 fz dz.

Now combining this and this equation okay, these two equations we arrive at mod of integral over C fz dz <=4 square mod of integral over C2 fz dz.

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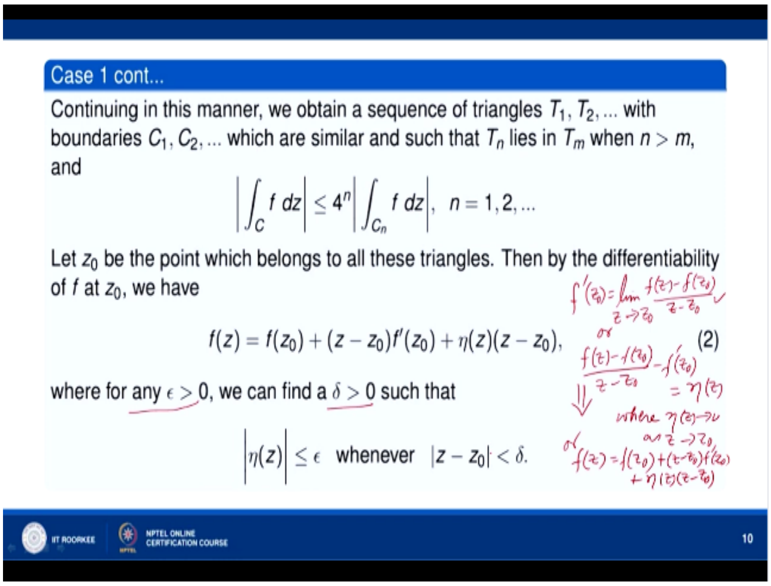

Now continuing this process okay, we obtain a sequence of triangles T1, T2 and T3 and so on with boundaries C1, C2 which are similar triangles and such that Tn lies in Tm when n is>m okay. You can see C2 lies in C1, C3 lies in C2 and so on so when n is>m Tn lies in Tm okay and mod of integral over C f dz will be <=4 to the power n mod of integral over Cn fz dz. Now let z0 be the point which belongs to all these triangles.

Actually, we have here a nested sequence of triangles, so let z0 be the point which belongs to all these triangles. Then, we have assumed that f is analytic okay, so f is differentiable at z0 okay, so fz=f z0+z-z0 f prime z0+eta z z-z0. You see we have f prime z0=limit z tends to z0 fz-f z0/z-z0 or we can write it as fz-f z0/z-z0-f prime z0=eta z okay, we can write like this where eta z goes to 0 as z goes to z0.

So by definition of derivative, we have this equation. This can be alternately written like this and we then can write it as fz equal to so this gives you fz=f z0 by multiplying the equation by z-z0, we have f z0+z-z0*f prime z0+eta z*z-z0 okay. Now eta z goes to 0 as z goes to z0, so what will happen, for any epsilon>0 we can find a delta>0 such that mod of eta z is<=epsilon whenever mod of z-z0 is<delta. This follows from the definition of limit.

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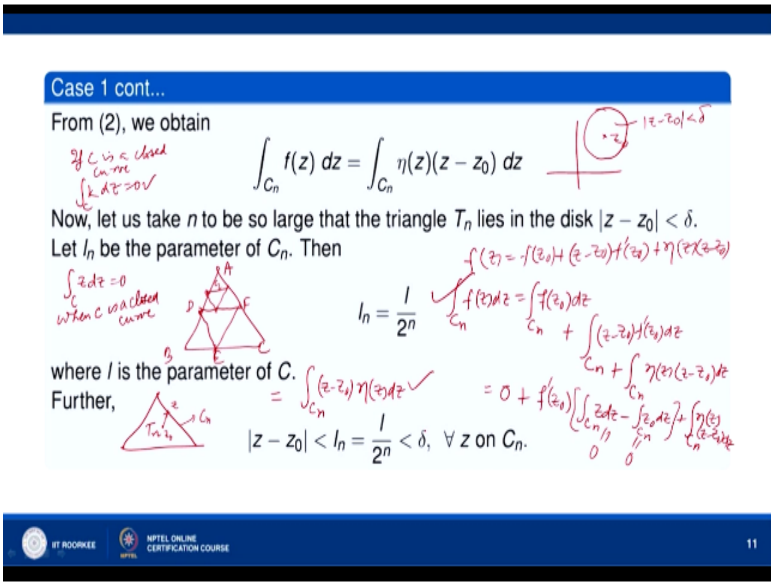

Now from the equation 2 okay let us see how we get this equation okay. Let us integrate this equation 2 okay. Let us integrate this along the curve C okay along the curve Cn okay, Cn is the boundary of the triangle Tn. So what we will have here, when we integrate we have the equation fz=f z0+z-z0 f prime z0+eta z z-z0 okay.

So let us integrate over Cn fz dz. Then, this is equal to integral over Cn f z0 dz+integral over z-z0 f prime z0 integral over Cn dz+integral over Cn eta z z-z0 dz okay. Now we have already shown that if C is a closed path, if C is a closed path, closed curve then integral over integral of k over C okay is=0. This we have already proved. Here f z0 is the constant quantity okay.

So integral over Cn f z0 dz=0 okay and then here f prime z0 is the constant quantity. I can write it outside the integral. Now integral over Cn z-z0 dz can be written as integral over Cn z dz-integral over z0 dz Cn okay. We have also proved that integral over C z dz=0 when C is a closed curve, simple closed curve okay. So integral over Cn z dz is=0 because Cn is the boundary of the triangle Tn okay which is the simple closed curve.

So this is equal to 0 integral over Cn z0 dz is also 0 by using this result. So this is 0 this is 0 okay. So we have integral over Cn eta z z-z0 dz okay. So this is equal to what we have here 0, this is 0, so integral over Cn z-z0*eta z dz, so this integral okay. This integral is equal to this integral. This is 0 and this is 0 okay, so integral over Cn of z dz is equal to this. Now let us take n to be so large that the triangle Tn okay, the triangle Tn whose boundary is given by Cn lies in the disk mod of z-z0<delta.

We have already found a delta corresponding to epsilon because eta z goes to 0 as z goes to z0. So let us take the disk mod of z-z0<delta, so this is your z0 (()) (47:02) this circular disk. This is mod of interior of this okay is mod of z-z0<delta. So take n to be so large that the triangle Tn lies in the because z0 lies in every triangle Tn okay. So when n is very large, Tn will lie inside this disk mod of z-z0<delta. Let us say ln be the parameter of Cn.

If ln is the parameter of Cn then ln will be=l/2 to the power n where l is the parameter of C why because this was your triangle ABC okay. By joining the mid points of the sides of the triangle, we obtained 4 congruent triangles okay. Since we joined the mid points D, E and F okay each of these 4 congruent triangles has 1/2 of the perimeter of the ABC triangle. So the triangle ABC if it has perimeter l okay then each of these 4 congruent triangles will have perimeter l/2, so this is when n=1.

If you take now let us say C1 is that triangle along which the integral of fz is having the largest value. And let us say this is C1 triangle, so again you join the mid points here, you get this okay. So since its length, length of ADF is l/2 then here the C2 triangle will be that triangle along which the integral of fz will be having maximum value. So if C2 is this triangle, then its perimeter will be 1/2 of the perimeter of ADF that means l/2 square. So when n is=2, you will get ln=l/2 square.

So in general, ln will be equal to l/2 to the power n where ln is the parameter of Cn okay. Now mod of z-z0 okay mod of z-z0 is<ln, you see here suppose this is your triangle Tn okay. This is your triangle Tn whose boundary is Cn. This boundary is let us say Cn okay, you take any point z here, z0 is a point which lies inside the triangle Tn. So the length of this line

segment joining z0 to z this length of the line segment z0 to z will always be<the perimeter of Tn okay.

So mod of z-z0 is<the perimeter of Cn that is ln and ln is=l/2 raised to the power n, l/2 raised to the power n is<delta okay because the triangle Tn lies inside the disk okay, so mod of z-z0<delta, so mod of z-z0 is<ln, ln is l/2 raised to power n which is<delta, this will be true for all z on Cn okay.

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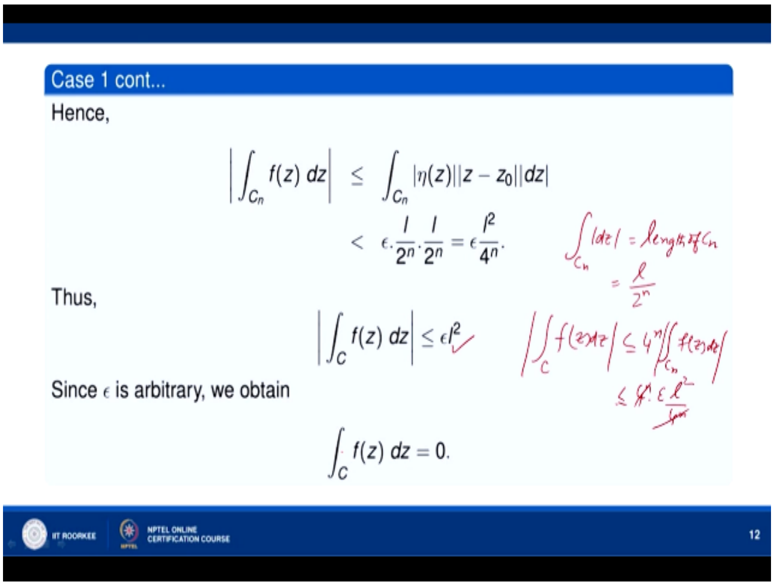

Hence, mod of integral over Cn fz and let us come to this mod of integral over Cn fz dz is equal to this quantity okay this one. Let us come to this okay. So mod of integral over Cn fz dz will be<integral over Cn mod of eta z mod of z-z0 mod of dz okay. Mod of eta z is<epsilon whenever mod of z-z0 is<delta okay, so this is epsilon and mod of z-z0 is<l/2 raised to the power n integral over Cn mod of dz okay is nothing but the length of Cn okay.

So this is equal to l/2 raised to the power n okay. So this is l/2 raised to power n, so epsilon*l square 4 to the power n okay. These are the square over 4 to the power n. Now mod of integral over C fz dz, this we found to be <= 4 to the power n*integral over Cn fz dz mod of this okay and this is<=epsilon*so this is 4*4 to the power n*epsilon l square/4 to the power n, so this will cancel and will get epsilon l square.

Now epsilon can be made arbitrarily small because it is arbitrary. So let us take epsilon to be arbitrary small okay we arrive at integral over C fz dz=0. So when C is a triangle okay, the boundary this curve C is the boundary of a triangle okay, we found that integral over C fz dz=0.

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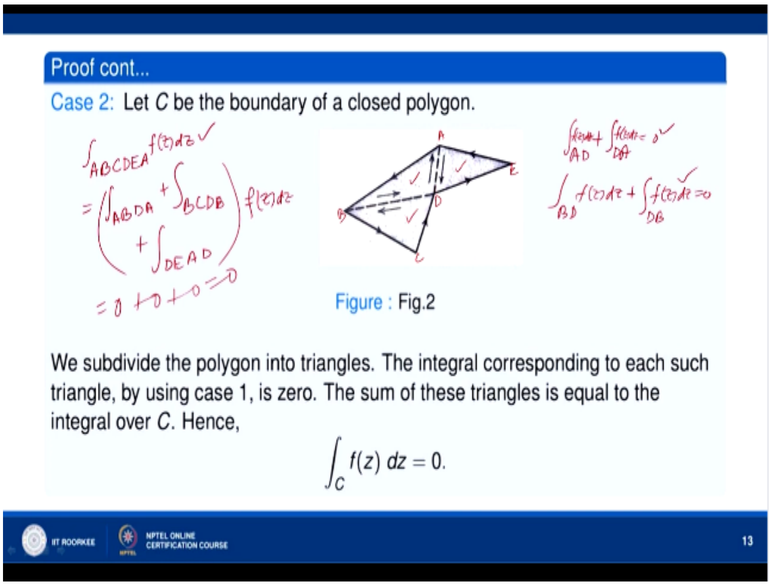

Now let us consider the case 2 where C is the boundary of a closed polygon. You can see suppose this is the polygon, this is A, B, C, D, E this is my polygon ABCDE okay, what I join what I do is I join B to D and A to D okay and then I see that integral over ABCDEA fz dz okay so I convert it into 3 triangles 1, 2, 3 okay and what I notice is that integral over ABDA+integral over BCDB+integral over DEAD.

Sum of these 3 integrals is=integral over ABCDEA while finding integral over ABDA and then BCDB and then DEAD okay we will be moving along BD in the opposite directions, BD and DB and then here again along D in the opposite directions AD and DA. So integral along AD and DA will cancel, this will be equal to 0 and similarly integral along BD so since this is equal to 0 and this is equal to 0 okay.

This sum of these 3 integrals will be equal to this integral okay. Now integral ABDA we have already proved that if C is the boundary of a triangle then integral along C is=0. So integral along ABDA will be 0, integral along BCDB that will also be 0 and then integral along DEAD that will also be 0 and we will get 0. So we subdivide the polygon into triangles, the integral corresponding to each such triangle by using case 1 is 0.

And so the sum of these triangles is=0 and this proves the Cauchy integral theorem for the case where C is the boundary of a closed polygon. Now in the next lecture, we shall consider any simple closed curve C in the complex z plane and show that the integral of fz along that simple closed curve C is=0. With this I would like to end my lecture. Thank you very much for your attention.