Advanced Engineering Mathematics

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Lecture - 60
Application to Queuing Theory and Reliability Theory

Hello friends. Welcome to my lecture on application of probability theory to queuing theory and reliability theory. Queuing theory is the mathematical study of waiting lines or queues.

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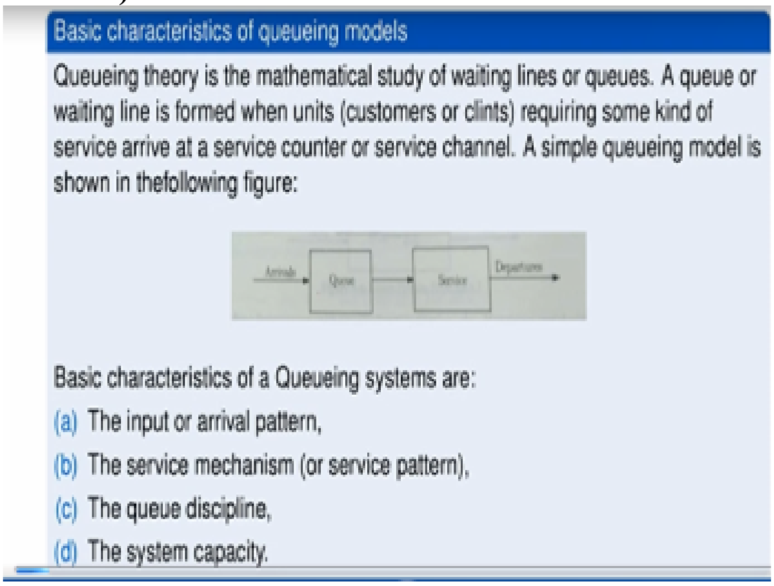

A queue or waiting line is found when units that is customers or clients requiring some kind of service arrive at a service counter or service channel. A simple queueing model is shown in the following figure. There you can see arrivals, queue is formed and then service is done and then they depart. So basic characteristics of queueing system are the input or arrival pattern, the service mechanism or service pattern, the queue discipline, the system capacity.

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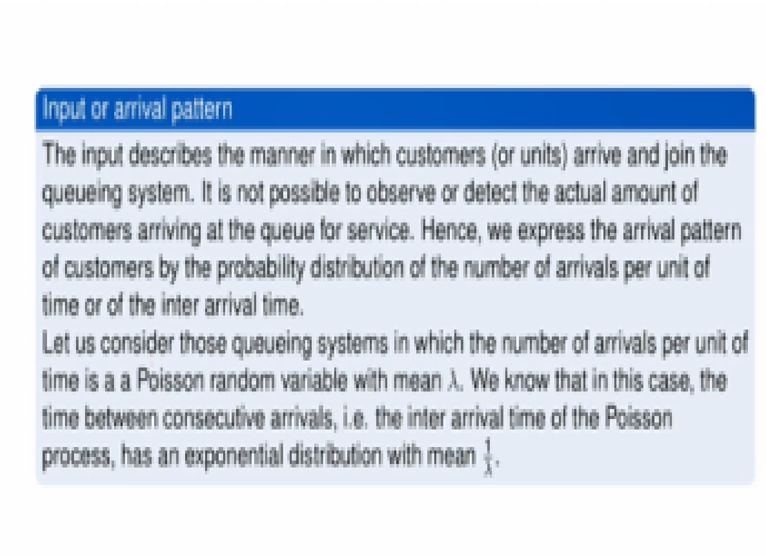

Let us discuss all these characteristics one by one. So input or arrival pattern. The input describes the manner in which customers or units arrive and join the queueing system. It is not possible to observe or detect the actual amount of customers arriving at the queue for service. Hence, we express the arrival pattern of customers by the probability distribution of the number of arrivals per unit of time or of the inter arrival time.

Let us consider those queueing systems in which the number of arrivals per unit of time is a Poisson random variable with mean lambda. We know that in this case, the time between consecutive arrivals that is the inter arrival time of the Poisson process has an exponential distribution with mean 1/lambda.

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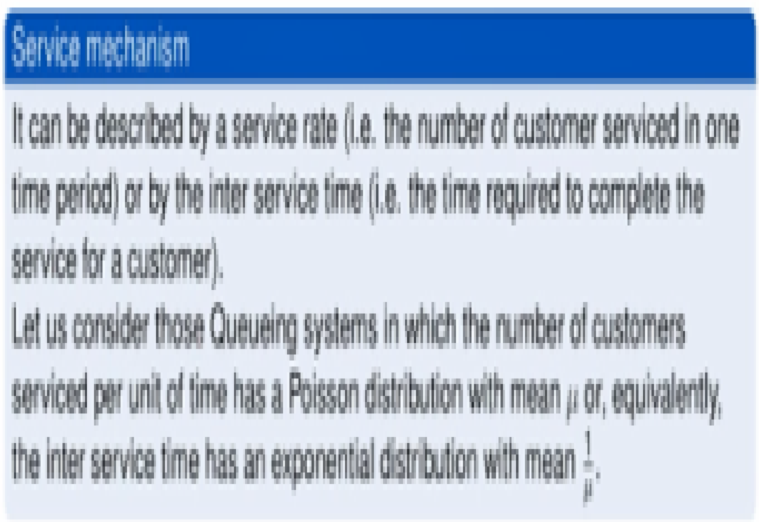

Now let us discuss the service mechanism. It can be described by a service rate that is the number of customer serviced in one time period or by the inter service time that is the time required to complete the service for a customer. Let us consider those queueing systems in which the number of customers serviced per unit of time has a Poisson distribution with mean mu or equivalently the inter service time has an exponential distribution with mean 1/mu.

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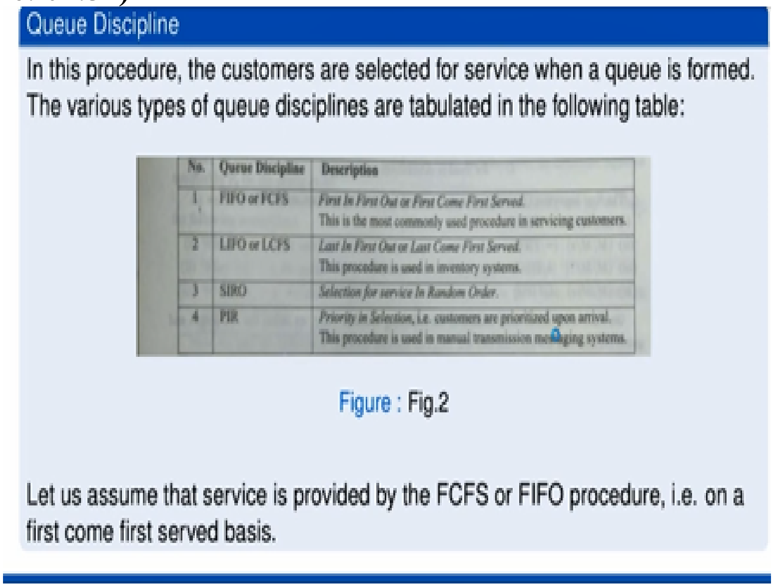

Now next we consider queue discipline. In this procedure, the customers are selected for service when a queue is formed. The various types of queue disciplines are tabulated in the following table. Number one FIFO or FCFS. FIFO means First in First Out or First Come First Served. This is the most commonly used procedure in servicing customers. Number two LIFO or LCFS, Last in First Out or Last Come First Served. This procedure is used in inventory systems.

SIRO, SIRO Selection for Service in Random Order. This is not very usual. PIR, Priority in Selection that is customers are prioritized upon arrival. This procedure is used in manual transmission managing systems. If a message is very important, it is transmitted; first it is given priority over other messages. So it is priority in selection. Now let us assume that service is provided by the FCFS or FIFO procedure. That is on a first come first served basis.

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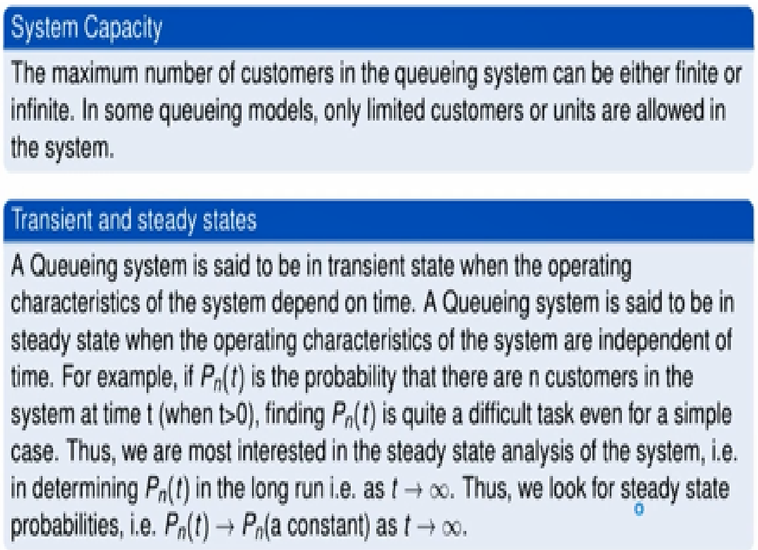

So the maximum number of customers, system capacity. The maximum number of customers in the queueing system can be either finite or infinite. In some queueing models, only limited customers or units are allowed in the system. Now let us talk about transient and steady states. A queueing system is said to be in transient state when the operating characteristics of the system depend on time.

A queueing system is said to be in steady state when the operating characteristics of the system are independent of time. For example, if pn t is the probability that there are n customers in the system at time t. When t>0, finding Pn t is quite a difficult task even for a simple case. Thus, we are most interested in the steady state analysis of the system that is in determining Pn t in the long run.

That is as t goes to infinity. Thus, we look for a steady state probabilities that is Pn t goes to pn, a constant as t goes to infinity.

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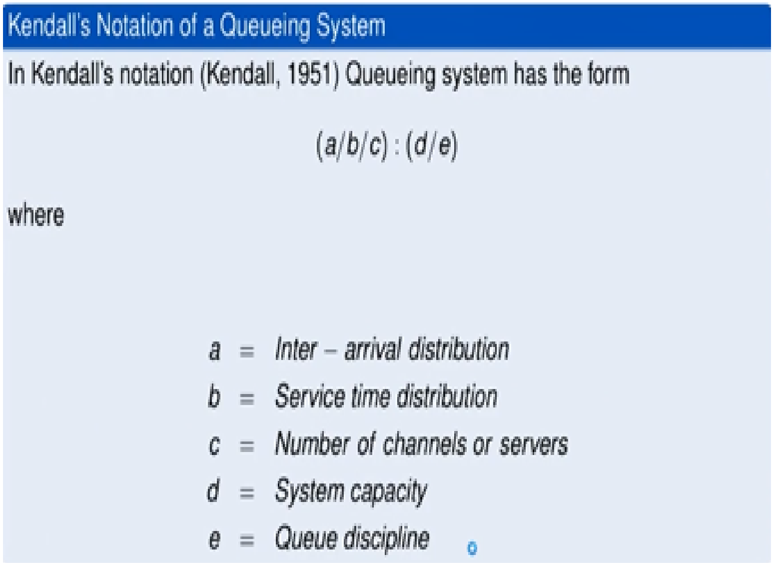

Now let us discuss Kendall's notation of a queueing system. In Kendall's notation, Kendall gave this notation in 1951. So queueing system has this form a oblique b oblique c such that d/e where a is inter arrival distribution, b is service time distribution, c is number of channels or servers, d is system capacity, e is queue discipline.

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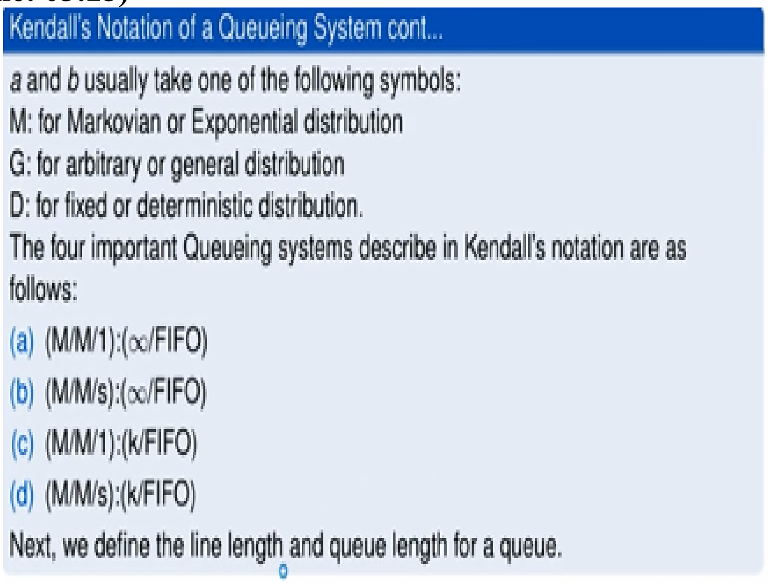

Now a and b usually take one of the following symbols, M for Markovian or Exponential distribution, G for general distribution and D for fixed or deterministic distribution. The 4 important queueing systems described in Kendall's notation are as follows. MM1 infinity oblique FIFO, MMS infinity oblique FIFO, MM1 k oblique FIFO, MMS k oblique FIFO okay.

In this lecture, we will be discussing the first queueing system MM1 infinity oblique FIFO. Next, we define the line length and queue length for a queue.

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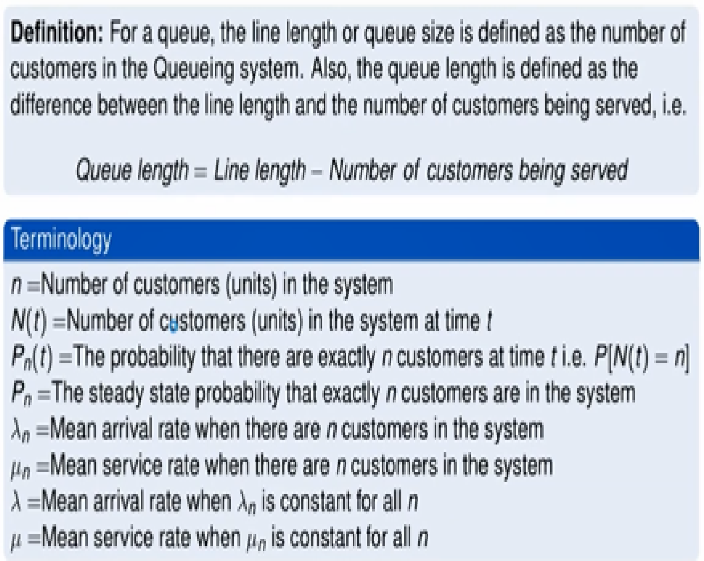

So, for a queue, the line length or queue size is defined as the number of customers in the queuing system. Also, the queue length is defined as the difference between the line length and the number of customers being served. That is queue length=line length-number of customers being served.

Now we will be using the following terminology, n is the number of customers units in the system, N t number of customers in the system at time t, Pn t the probability that there are exactly n customers at time t that is probability that N t=n. Pn, the steady state probability that exactly n customers are there in the system. And then, lambda n is the mean arrival rate okay, mean arrival rate when there are n customers in the system.

Mu n is the mean service rate when there are n customers in the system. Lambda is mean arrival rate when lambda n is constant for all n, and mu is mean service rate when mu n is constant for all n.

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Terminology cont...

$\rho = \frac{\lambda}{\mu}$ =Utilization factor or traffic intensity
$f_s(w)$ =The probability density function of waiting time in the system
$f_q(w)$ =The probability density function of waiting time in the queue
L_s =The expected number of customers in the system or the average line length
L_q =The expected number of customers in the queue or the average queue length
L_w =The expected number of customers in non-empty queues
W_s =The expected waiting time of a customer in the system
W_q =The expected waiting time of a customer in the queue.

Now some more notations are there. Rho is=lambda/mu which is known as utilization factor or traffic intensity, fs w is the probability density function of waiting time in the system, fq w is the probability density function of waiting time in the queue, Ls is the expected number of customers in the system or the average line length. The Lq is the expected number of customers in the queue or the average queue length.

Lw is the expected number of customers in the non-empty queues. Ws is the expected waiting time of a customer in the system, Wq is the expected waiting time of a customer in the queue.

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Transient State Probabilities for Poisson Queue Systems

We now state the differential equations for the transient state probabilities for the Poisson queue systems, which are also knows as birth-death process or immigration-emigration process.
Let $N(t)$ be the number of customers (or units) in the system at time t and $P_n(t)$ be the probability that there are n customers in the system at time t, where $n \geq 1$, i.e.

$$P_n(t) = P[N(t) = n].$$

The differential equations satisfied by $P_n(t)$, are given by

$$P_n'(t) = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t) \quad (1)$$

where $n \geq 1$, and

$$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) \quad (2)$$

Now let us find the differential equations for the transient state probabilities for the Poisson queue systems which are also known as birth-death process or immigration-emigration process. Let N t be the number of customers in the system at time t and pn t be the probability that there are n customers in the system at time t where n>or=1 that is Pn t=probability that N t=n.

Then the differential equations satisfied by Pn t are given by Pn dash t that is derivative of Pn t with respect to t=-lambda n+mu n*Pn t+lambda n-1*pn-1t +mu n+1 Pn+1t where n is>or=1 and when n=0 we have the differential equation P0 dash t=-lambda 0 P0 t+mu 1 P1 t.

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Transient State Probabilities for Poisson Queue Systems cont...

The solution of these differential equations yields the transient state probability, $P_n(t)$,for $n \geq 0$ but this is exceedingly difficult even for simple problems.

Steady State Probabilities for Poisson Queue Systems

The steady state probabilities for poisson queue systems are derived by assuming that $P_n(t) \to P_n$, independent of t, as $t \to \infty$. The equations of steady state probabilities, P_n, can be obtained by putting $P_n'(t) = 0$ and replacing $P_n(t)$ by P_n in equations (1) and (2). Thus we obtain

$$-(\lambda_n + \mu_n)P_n + \lambda_{n-1}P_{n-1} + \mu_{n+1}P_{n+1} = 0 \qquad (3)$$

and

$$-\lambda_0 P_0 + \mu_1 P_1 = 0. \qquad (4)$$

Transient state probabilities for Poisson queue systems. The solution of these differential equations, the solution of the two differential equations that we have got number 1 and 2. Number 1 is for n>or=1 and number 2 is for n=0. So the solution of these differential equations yields the transient state probability, Pn t for n>=0 but this is exceedingly difficult even for simple problems.

So we consider steady state probabilities for Poisson queue systems. The steady state probabilities for Poisson queue systems are derived by assuming that Pn t goes to Pn independent of ts t tends to infinity. The equations of steady state probabilities Pn can be obtained by putting Pn dash t=0 because Pn t is now as t tends to infinity is a constant, so when we take the derivative of Pn t with respect to t, it will be=0.

So Pn dash t=0 and replacing Pn t by Pn in equations 1 and 2 and then we obtain, you can see here if you put Pn dash t=0 and Pn t=Pn then you get –lambda n+mu n*Pn+lambda n-1*Pn-1+mu n+1*Pn+1=0 okay. So we have this equation and similarly the other equation, equation number 2 for n=0 becomes P0 dash t=0 and this is –lambda 0 P0+mu 1 P1 okay of the right side. So we get –lambda 0 P0+mu 1 P1=0, so we get the second equation.

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Steady State Probabilities for Poisson Queue Systems cony...

Equations (3) and (4) are called the balance equations or equilibrium equations of Poisson queue systems.

Using the principle of mathematical induction, it follows that

$$P_n = \frac{\lambda_0 \lambda_1 \lambda_2 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 \ldots \mu_n} P_0 \quad (5)$$

and

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda_0 \lambda_1 \lambda_2 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 \ldots \mu_n} \right)} \quad (6)$$

From these equations, one can determine the values of Pn in terms of P0 and the value of P0. Now let us see equations 3 and 4 are called the balance equations this equation 3 and 4 okay this 3 and this 4 are called balance equations or equilibrium equations of the Poisson queue systems. Using the principle of mathematical induction, it can be shown that this Pn from this equation okay using the principle of mathematical induction one can show that Pn=lambda 0*lambda 1*lambda 2 and so on lambda n-1/mu 1 mu 2 mu 3 mu n*P0.

And P0=1/1+sigma n=1 to infinity lambda 0 lambda 1 lambda 2 lambda n-1/mu 1 mu 2 mu 3 and so on mu n.

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Model-I (M/M/1): (∞/FIFO) Single Server With Infinity Capacity

For this model, we make the following assumptions:

(a) The mean arrival rate is constant, i.e. $\lambda_n = \lambda$ for all n.

(b) The mean service rate is constant i.e. $\mu_n = \mu$ for all n.

(c) The mean arrival rate is less than the mean service rate, i.e. $\lambda < \mu$ or equivalently that $\rho = \frac{\lambda}{\mu} < 1$.

The assumption that the traffic intensity $\rho < 1$ ensures that an infinite queue will not build up. Under assumption (c) and equation (6) simplifies to

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu} \right)^n} = \frac{1}{1 + \frac{\lambda}{\mu} + (\frac{\lambda}{\mu})^2 + \ldots} = \frac{1}{\left(1 - \frac{\lambda}{\mu}\right)^{-1}} = 1 - \frac{\lambda}{\mu} = 1 - \rho \quad (7)$$

Now let us consider the model MM1 infinity oblique FIFO single server with infinite capacity. So here a is m b is also m that means we are considering exponential type distribution number 3, 1 means we are considering single server okay, infinity means the queue can be infinite okay and FIFO means we are considering the discipline as first in first out okay, first in first out FIFO okay.

That is the queue discipline we are considering, a is arrival distribution, b is service time distribution, both we are assuming as that they follow exponential type distribution, c is the number of channels or servers. So we are considering that the system has one server and d is the system capacity, system capacity we are taking as infinite. So we have MM1 infinity FIFO model okay.

For this model, we make following assumptions. The mean arrival rate is constant that is lambda n=lambda for all n. Mean service rate is constant that is mu 1=mu for all n. Mean arrival rate is<the mean service rate. So that means lambda is<mu or we can say rho=lambda/mu is<1, rho is the traffic intensity, so the traffic intensity we are assuming that it is<1 and which will ensure that an infinite queue will not build up.

Because mean service rate is higher than the mean arrival rate, so the infinite queue will not build up. Now under assumption c that is mean arrival rate is<the mean service rate okay, the equation 6 simplifies to let us see the equation 6, this equation 6 becomes 1/1+lambda 0 lambda 1 lambda 2 and lambda n-1 are all=lambda, so we have lambda/mu raised to the power n okay.

So we get the 1/1+sigma n=1 to infinity lambda/mu to the power n which is 1/1+lambda/mu+lambda/mu whole square and so on which is geometric series with ratio lambda/mu. So we have the sum of the series as 1– lambda/mu to the power -1. So 1/1– lambda/mu to the power–1 is 1– lambda/mu and lambda/mu we denote by rho okay. So P0 is=1-rho.

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Substituting equation (7) into (5), we have

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 = \rho^n(1-\rho)$$

Characteristic of Model-I

1.Average or expected number of customers in the system (L_s): If N denotes the number of customers in the Queueing system, then N is a discrete random variable taking values $N = 0, 1, 2, \ldots$ and we know that

$$P_n = P(N = n) = \rho^n(1-\rho), \text{ for } n = 0, 1, 2, \ldots$$

Now let us consider the case substituting equation 7 into 5 okay. So this 7 P0=1-rho, let us put in equation number 5. Equation number 5 now is Pn=lambda/mu to the power n =P0 okay because we are assuming that the mean service rate and the mean arrival rate are constant okay. So we have the Pn=lambda/mu raised to power n*P0 okay and this is rho to the power n and P0 =1-rho, so we have Pn=rho to power n*1–rho.

Now characteristics of model 1 okay so average or expected number of customers in the system. Average or expected numbers or number of customers in the system we denote by Ls. So if N denotes the number of customers in the queueing system, then N is a discrete random variable. It will take values 0, 1, 2, 3 and so on and we know that Pn is the probability that n takes the value N okay. So it is=rho to the power n*1– rho as we have seen just now for n=0, 1, 2, 3 and so on.

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Average or expected number of customers in the system (L_s)

Thus, the expected number of customers in the system is given by

$$L_s = E(N) = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\rho^n(1-\rho) = (1-\rho)\sum_{n=1}^{\infty} n\rho^n$$

We note that

$$L_s = \rho(1-\rho)\sum_{n=1}^{\infty} n\rho^{n-1} = \rho(1-\rho)[1+2\rho+3\rho^2+\ldots+] = \rho(1-\rho)(1-\rho)^{-2} = \rho(1-\rho)^{-1}$$

$$\Rightarrow L_s = \frac{\rho}{1-\rho} = \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda}.$$

Now thus the expected number of customers in the system Ls is given by expected value of n that is sigma n=0 to infinity and Pn which is sigma n=0 to infinity n*rho to the power n 1-rho. Now when n=0, first term is 0, so we can write it as 1–rho is independent of n, so we can write is outside and then sigma n=1 to infinity n rho to the power n. Now Ls can also be written as rho times 1–rho*sigma n=1 to infinity n times rho to the power n-1.

So we have rho*1–rho and then this series sigma n=1 to infinity and rho to the power n–1 becomes 1+2 rho+3 rho square and so on okay and its sum we know it is=1–rho to the power-2 okay. So we have rho*1– rho to the power–1 and therefore Ls is=rho/1–rho. Now rho is lambda/mu, so we can put the value of rho as lambda/mu and we get Ls=lambda/mu–lambda. So that means that average or expected number of customers in the system is=lambda/mu–lambda.

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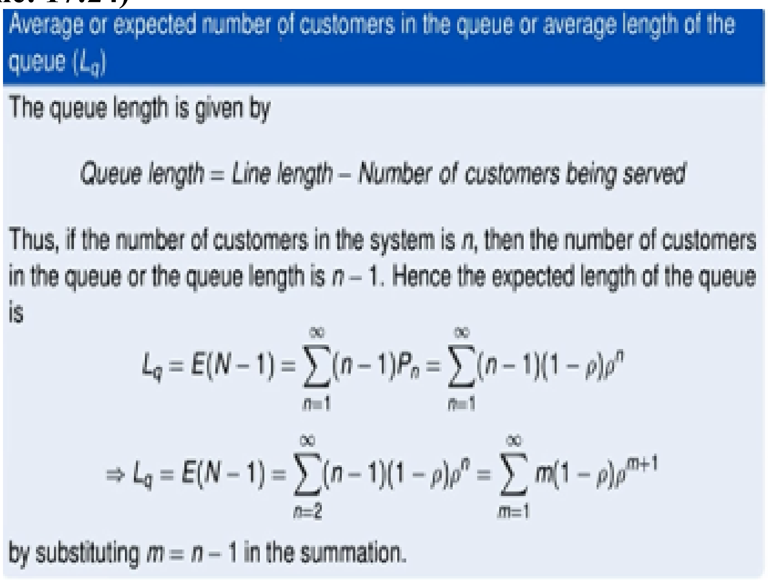

Now average or expected number of customers in the queue or average length of the queue. We know that queue length is given by line length–number of customers being served. Thus, if the number of customers in the system is n, then the number of customers in the queue or the queue length is n-1. Hence the expected length of the queue is Lq= E N-1 and so it is sigma n=1 to infinity n-1*Pn okay

Now when n=1, the first term becomes 0 okay or before that what we can do is we can put the value of Pn, Pn is rho to the power n*1-rho. So this summation becomes sigma n=1 to infinity n-1*1-rho*rho to the power n and therefore using the fact that at n=1 the first term becomes 0 and will start with 2 onwards, so n=2 to infinity n-1 1-rho*rho to the power n and here what we can do is let us put m=n-1 so that m runs from 1 to infinity and we have m times 1–rho*rho to the power m+1.

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Average or expected number of customers in the queue or average length of the queue (L_q) cont...

Hence

$$L_q = \rho^2(1-\rho)\sum_{m=1}^{\infty} m\rho^{m-1} = \rho^2(1-\rho)(1-\rho)^{-2} = \rho^2(1-\rho)^{-1}$$

$$\Rightarrow L_q = \frac{\rho^2}{1-\rho} = \frac{\left(\frac{\lambda}{\mu}\right)^2}{1-\frac{\lambda}{\mu}} = \frac{\lambda^2}{\mu(\mu-\lambda)}.$$

And therefore I can write Lq=rho square times 1-rho*sigma m=1 to infinity m times rho to the power m–1 okay and this we know the sum of this is 1–rho to the power-2, so we have rho square*1–rho to the power-1 and therefore Lq which is the average or expected number of customers in the queue is=rho square/1-rho. Rho is lambda/mu, so lambda/mu whole square upon 1-lambda/mu which gives us Lq as lambda square/mu*mu–lambda.

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Average or expected number of customers in nonempty queue (L_w)

$$L_w = E\{N-1 \mid N-1>0\} = \frac{E(N-1)}{P(N-1>0)} \qquad (8)$$

We know that

$$E(N-1) = L_q = \frac{\rho^2}{1-\rho}$$

and

$$P(N-1>0) = 1 - P_0 - P_1 = 1-(1-\rho)-[\rho(1-\rho)] = \rho - \rho(1-\rho) = \rho^2.$$

Substituting the above in equation (8), we get

$$L_w = \frac{\frac{\rho^2}{1-\rho}}{\rho^2} = \frac{1}{1-\rho} = \frac{1}{1-\frac{\lambda}{\mu}} = \frac{\mu}{\mu-\lambda}.$$

Now let us consider the case average or expected number of customers in non-empty queue. So this means that Lw. Lw is th expectation of N-1, N-1 is the line length okay divided by q N-1 sorry Lw=expectation of N-1 given that N-1>0 that is the queue is non-empty okay, at least one customer is there okay, so N-1 is>0 and therefore expectation of N-1 okay expectation of N-1, N is no, Lw=expectation of N-1 given that N-1 is>0. N is the number of customers, N denotes the number of customers in the queue system.

So number of customers is>1 okay, so expectation of N-1/probability that N-1 is>0 and we know that expectation of N-1 that is Lq okay=rho square/1–rho okay. Lq=rho square/1–rho. So we have P N–1>0 =1–P0–P1. P0 means probability that N=0 and P1 is the probability that N=1, so probability that N>1 can be obtained from 1–P0–P1 so 1–P0 is=1–rho and P1=rho times 1–rho because Pn is=rho to the power n*1–rho for all n>or=1.

So we have on simplifying this we get rho square okay. So Pn–1> 0 is rho square, so Lw=rho square/1–rho/rho square which is 1/1–rho and that is=mu/mu–lambda as rho=lambda/mu.

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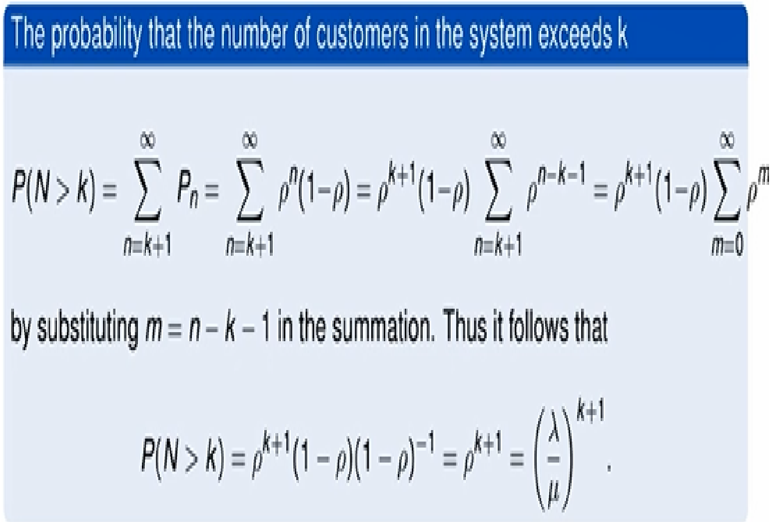

Now let us discuss the probability that number of customers in the system exceeds k okay. So probability that number of customers in the system exceeds k that is N>k. Now this is=sigma n=k+1 to infinity Pn n=k+1 to infinity, the value of Pn is rho to the power n*1-rho. We can write it as rho to the power k+1*1–rho*sigma n=k+1 to infinity rho to the power n–k–1 and this is rho to the power k+1*1– rho and we can put m as n–k–1.

When you take m=n-k–1 then this summation becomes sigma m=0 to infinity rho to the power m and hence probability that n>k is=rho to the power of k+1*1–rho and this is geometric series with geometric ratio rho, it sum is 1/1–rho. So we get 1–rho to the power –1 and this will give you on simplification the probability that N is>k=rho to the power of k+1 that is lambda/mu to the power k+1.

So this is the probability lambda/mu to the power k+1 is the probability that number of customers in the system exceeds k.

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Probability density function of the waiting time of a customer in the system ($f_s(w)$)

Let T_s be the random variable that represents the waiting time of a customer in the system, $f_s(w)$ be the probability density function of T_s and $f_s(w|n)$ be the probability density function of T_s given that there are already n customers in the system when the customer arrives. Then

$$f_s(w) = (\mu - \lambda)e^{-\mu w}\sum_{n=0}^{\infty}\frac{(\lambda w)^n}{n!} = (\mu - \lambda)e^{-\mu w}e^{\lambda w}, \text{ where } w > 0.$$

Thus, the probability density function of the waiting time of the customer in the system is

$$f_s(w) = (\mu - \lambda)e^{-(\mu-\lambda)w}, \text{ where } w > 0 \quad (9)$$

which is the probability density function of the exponential random variable with parameter $\mu - \lambda$.

Now let us discuss probability density function of the waiting time of a customer in the system, we denote it by fs w. So let Ts be the random variable that represents the waiting time of a customer in the system, fs w be the probability density function of Ts and fs w given n be the probability density function of Ts given that there are already n customers in the system when the customer arrives.

Then, fs w is given by mu-lambda*e to the power mu w*sigma n=0 to infinity lambda w raised to the power n/n factorial. Now the sum of the series we are all aware, the sum of the series is e to the power lambda w, so we get mu–lambda e to power -mu w*e to power lambda w where w is>0. Thus, the probability density function of the waiting time of the customer in the system is fs w=mu–lambda*e to the power –mu–lambda*w where w is>0.

This is the probability density function of the exponential random variable with parameter mu–lambda.

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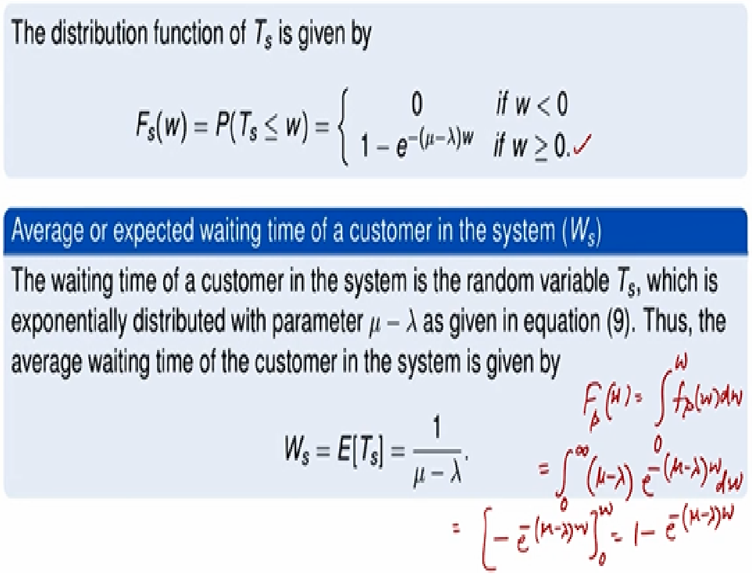

Now the distribution function, we know distribution function fs w is=probability that Ts is<or=w. So when w is<0 okay when w is<0 because we know that fs w, fs w is the probability density function of the random variable, this is=mu–lambda*e to the power -mu–lambda w when w is>0. So when w is<0 okay probability is 0, so we get fs w=0. So when w is>or=0, this fs w is=integral over 0 to w because we want the probability that Ts is <or=w.

So integral over 0 to w fs w dw and this will be=integral 0 to infinity fs w is mu–lambda*e to the power –mu– lambda w dw okay and this is nothing but –e to the power –mu– lambda*w 0 to w. So this is=1–e to the power–mu–lambda w okay, so this is the distribution function of Ts. Now average or expected waiting time of a customer in the system. So the waiting time of a customer in the system is the random variable Ts which is exponentially distributed with parameter mu–lambda as given in equation 9, here okay mu–lambda.

So thus the average waiting time of a customer in the system is 1/mu–lambda because it follows exponential distribution okay.

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The probability that the waiting time of a customer in the system exceeds t

$$P(T_s > t) = \int_{w=t}^{\infty} f_s(w)dw = \int_{w=t}^{\infty} (\mu - \lambda)e^{-(\mu-\lambda)w}dw = \int_{w=t}^{\infty} d[-e^{-(\mu-\lambda)w}]$$

$$\Rightarrow P(T_s > t) = \left[-e^{-(\mu-\lambda)w}\right]_{w=t}^{\infty} = e^{-(\mu-\lambda)t}$$

Probability density function of the waiting time of a customer in the queue ($f_q(w)$)

The probability density function of the waiting time of a customer in the queue is

$$f_q(w) = \begin{cases} \frac{\lambda}{\mu}(\mu - \lambda)e^{-(\mu-\lambda)w} & \text{for } w > 0 \\ 1 - \frac{\lambda}{\mu} & \text{for } w = 0 \\ 0 & \text{for } w < 0 \end{cases}$$

Now the probability that the waiting time of a customer in the system exceeds t, so probability that Ts is>t is integral over w=t to infinity fs w dw, fs w is mu–lambda*e to the power -mu–lambda*w, so on integration we get probability that Ts is>t=e to the power –mu–lambda*t. So this is the probability that waiting time of a customer in the system exceeds t.

Probability density function of the waiting time of a customer in the queue fq w is given by this the probability density function of the waiting time of the customer in the queue is fs w=lambda/mu mu-lambda*e to the power –mu–lambda*w for w>0, 1-lambda/mu okay for w=0 and 0 for w<0.

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The distribution function of the waiting time of the customer in the queue is

$$F_q(w) = P(T_q \le w) = \begin{cases} 0 & \text{for } w < 0 \\ 1 - \frac{\lambda}{\mu} & \text{for } w = 0 \\ \frac{\lambda}{\mu} - \frac{\lambda}{\mu}e^{-(\mu-\lambda)w} & \text{for } w > 0 \end{cases}$$

The probability that the waiting time of a customer in the system exceeds t

$$P(T_q > t) = \int_t^{\infty} f_q(w)dw = \int_t^{\infty} \frac{\lambda}{\mu}(\mu - \lambda)e^{-(\mu-\lambda)w}dw = \frac{\lambda}{\mu}\int_t^{\infty} d[-e^{-(\mu-\lambda)w}]$$

$$\Rightarrow P(T_q > t) = \frac{\lambda}{\mu}\left[-e^{-(\mu-\lambda)w}\right]_t^{\infty} = \frac{\lambda}{\mu}[0 + e^{-(\mu-\lambda)t}] = \frac{\lambda}{\mu}e^{-(\mu-\lambda)t}.$$

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The distribution function of the waiting time of the customer in the queue is fq w which is probability that Tq is<or=w okay probability that Tq is<or=w, so we have defined Tq earlier, fs w is the probability density function of Ts. where Ts is the random variable that represents

the waiting time of a customer in the system, and Tq denotes the random variable that represents the waiting time of a customer in the queue okay.

So this is the probability density function of the waiting time of the customer in the queue okay where Tq is the random variable which denotes the waiting time of the customer in the queue. So fs w is given by this and then we have this is not fs w, this is fq w okay. So this is fq w, fq w is given by these values okay, fq w=lambda/mu mu–lambda e to the power-mu–lambda*w for w>0, 1-lambda/mu for w=0 and 0 for w<0 okay.

And then the distribution function of the waiting time of the customer in the queue is given by fq w where probability that Tq is<or=w. So this can be, these are given by 0 when w is<0, 1-lambda/mu when w=0, 1-lambda/mu+lambda/mu e to the power –mu–lambda*w for w>0. We can easily obtain these values because fq w is=integral over –infinity to w fq w dw okay and fq w we have here fq w is 0 when w is<0 okay.

So this will go to 0 to w fq w dw okay, so from here we can easily find these values. Now the probability that the waiting time of a customer in the system exceeds t. Here 0 to w when w is>0 we can put the value of fq w okay fq w for w>0 is given by this expression okay lambda/mu*mu-lambda e to the power–mu–lambda*w. So when you integrate this okay we will get lambda/mu*1-e to the power–mu–lambda*w okay.

So we will get this, if you integrate this, 0 to w fq w dw okay, what we will get 0 to w lambda/mu*mu-lambda e to the power –mu-lambda*w dw and this will be lambda/mu here we will have –e to the power–mu–lambda*w 0 to w. So we will get this as lambda/mu*1-e to the power –mu–lambda*w okay. So that is the formula for probability that Tq is <or= w. Now probability that the waiting time of a customer in the system exceeds t okay.

Probability that Tq is>t, so this is=integral t to infinity fq w dw and fq w dw is lambda/mu mu-lambda e to the power–mu–lambda w dw. So this lambda/mu and this quantity is the derivative of –e–mu–lambda*w okay. So probability that Tq is>t is lambda/mu times –e to the power-mu–lambda*dw*w integral over the limits are t to infinity, so lambda/mu now this goes to 0 when the w goes to infinity.

So this 0 and when you put the lower limit, you get 0+e to the power–mu–lambda*t. So we get mu–lambda*e to the power we get lambda/mu*e to the power–mu–lambda*t, lambda/mu*e to the power–mu–lambda*t okay.

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The average waiting time of a customer in the queue (W_q)

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$$

The average waiting time of a customer in the queue if he/she has to wait

It is given by

$$E[T_q \mid T_q > 0] = \frac{E(T_q)}{P(T_q > 0)} = \frac{W_q}{1 - P(T_q = 0)} = \frac{W_q}{1-\left(1-\frac{\lambda}{\mu}\right)} = \frac{W_q}{\frac{\lambda}{\mu}} = \frac{\frac{\lambda}{\mu(\mu-\lambda)}}{\frac{\lambda}{\mu}} = \frac{1}{\mu-\lambda}$$

$$E[T_q \mid T_q > 0] = \frac{\frac{\lambda}{\mu(\mu-\lambda)}}{\frac{\lambda}{\mu}} = \frac{1}{\mu-\lambda}$$

Now the average waiting time of a customer in the queue Wq is given by lambda/mu*mu–lambda. The average waiting time of a customer in the queue if he, she has to wait is given by E expectation of Tq where Tq is>0. Expectation of Tq is nothing but Wq and Tq>0 is 1-P Tq=0. Now expectation of Tq given Tq>0 is then Wq=lambda/mu times mu–lambda and probability that Tq=0 okay.

Probability that Tq=0 we have seen here probability that Tq=0 okay=1-lambda/mu okay, probability that Tq=0=1-lambda/mu, so 1-this is Wq/1-1-lambda/mu and this gives you Wq/lambda/mu but Wq=lambda/mu*mu-lambda/lambda/mu. So this is=1/mu–lambda, so we get this.

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Theorem (Little's formulae)

For the M/M/1:∞/FIFO model, the relations between L_s, L_q, W_s and W_q are given as follows:

(a) $L_s = \lambda W_s$

(b) $L_q = \lambda W_q$

(c) $W_s = W_q + \frac{1}{\mu}$

(d) $L_s = L_q + \frac{\lambda}{\mu}$.

Now let us discuss Little's formulae for the MM1 infinity FIFO model. The relation between Ls, Lq, Ws and Wq are given as follows. Ls=lambda Ws, Lq=lambda Wq, Ws=Wq+1/mu, Ls=Lq+lambda/mu. So first let us see Ls=lambda Ws okay. We know that Ws is 1/mu–lambda, Ls is what? Ls is lambda/mu–lambda okay.

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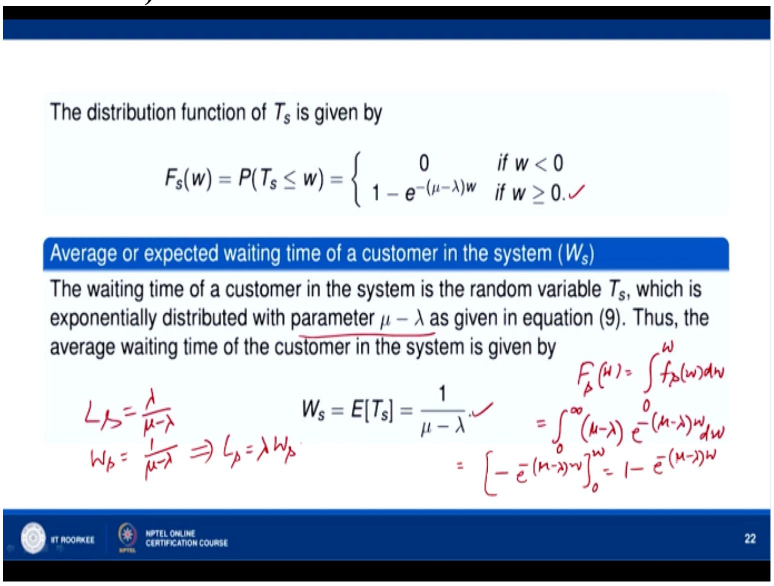

Ls is lambda/mu–lambda, Ws is=1/mu–lambda okay, so these 2 relations imply that Ls=lambda times Ws okay. So this is how we get the first formulae Ls=lambda Ws. Now let us look Lq and Wq. Let us look at Lq. Lq=lambda square/mu*mu–lambda okay.

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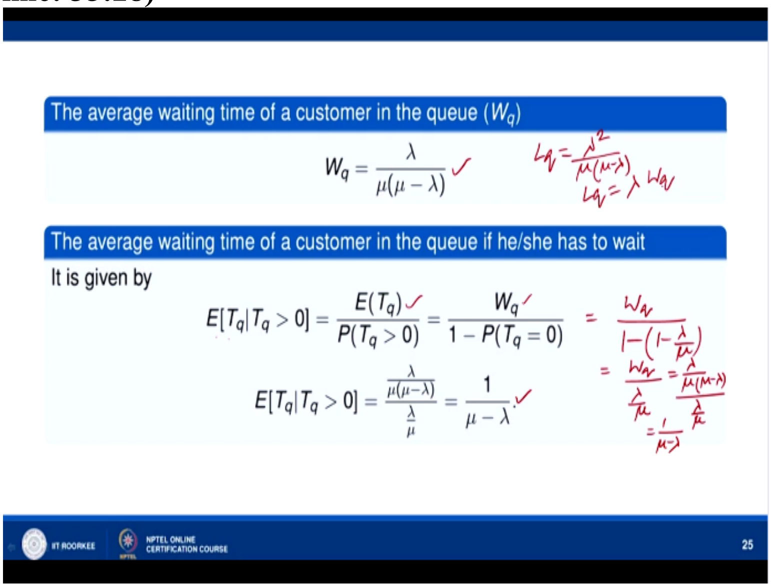

So Lq=lambda square/mu*mu–lambda okay and Wq is lambda/mu times mu–lambda. So, Lq= lambda times Wq okay, so Lq = lambda times Wq.

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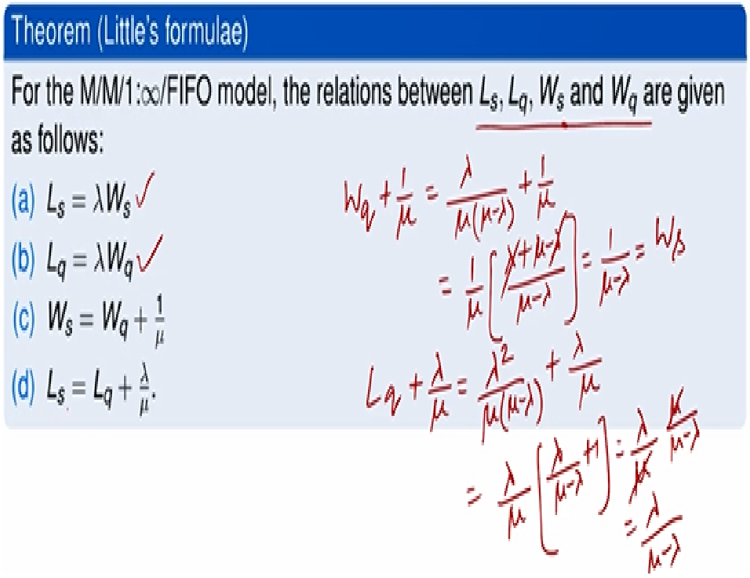

Now let us show that Wq+1/mu, Wq +1/mu=Wq is=lambda/mu times mu–lambda. So this=1/mu lambda+mu-lambda/mu–lambda. So this is 1/mu–lambda and 1/mu–lambda is Ws. Ls is this, Ws is here, Ws is /mu–lambda. So this is Ws okay and then Lq+lambda/mu. Lq is=lambda square/mu times mu–lambda. That is the value of Lq+lambda/mu. So this is lambda/mu times lambda/mu–lambda+1.

And we get here lambda/mu lambda+mu-lambda that means mu/mu–lambda. So we can say we get lambda/mu–lambda and lambda/mu–lambda is Ls okay, Ls lambda/mu–lambda. So if one parameter if Ls, Ws or Lq or Wq one is known, the other three can be found from Little's formulae, so they are very useful.

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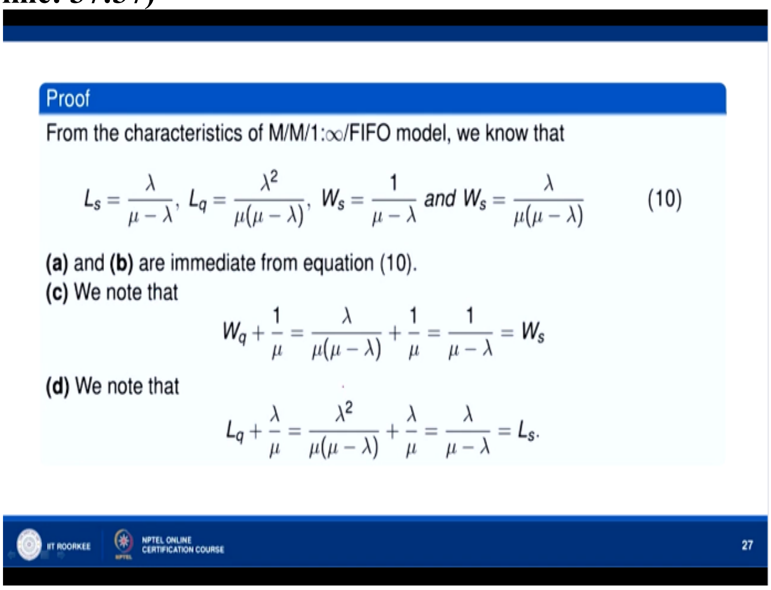

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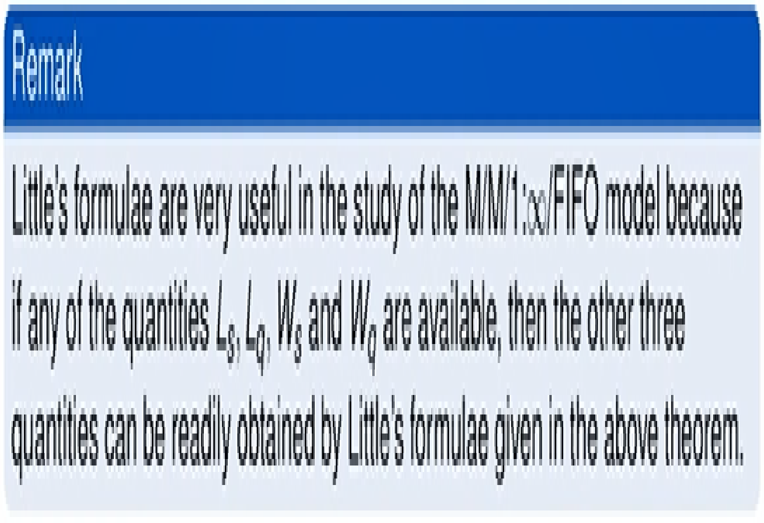

This is the proof. Now so the Little's formulae are very useful in the study of MM1 infinity FIFO model because if any of the quantities Ls, Lq, Ws and Wq are available, then the other three can be readily obtained from Little's formulae which are given in the above theorem.

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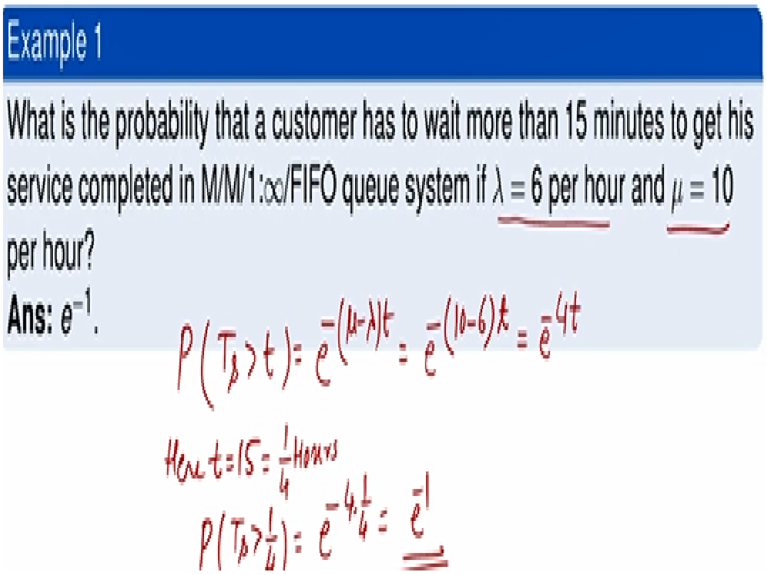

Now let us look at the first question. What is the probability that a customer has to wait more than 15 minutes to get his service completed in MM1 FIFO queue system? If lambda=6 per hour and mu=10 per hour okay. So we are given the mean service rate that is mu=10 per hour and mean arrival rate lambda=6 okay. Now let us see which formula gives us the probability that a customer has to get wait more than 15 minutes to get the service okay.

So let us go to that formula. So the probability that the waiting time of a customer in the system exceeds t okay. So if the customer has to wait more than 15 minutes then probability that Ts is>or=15 we have to find, so we have probability that Ts>t okay. So we have probability that Ts is>t is=e to the power–mu-lambda*t. So this equal to e to the power–mu is 10, lambda is 6*t so we get e to the power–4t.

So if the customer has to wait more than t minutes then probability is Ts>t. Now here we want the probability that the system that the customer has to wait more than 15 minutes okay. So t here t=15.okay so probability that Ts is>15, now 15 we have to convert into hours, so 1/4 hours, so probability that Ts is>1/4=e to the power -4*1/4 which is=e to the power of -1 okay. So the probability that the customer has to wait more than 15 minutes to get the service is e to the power -1.

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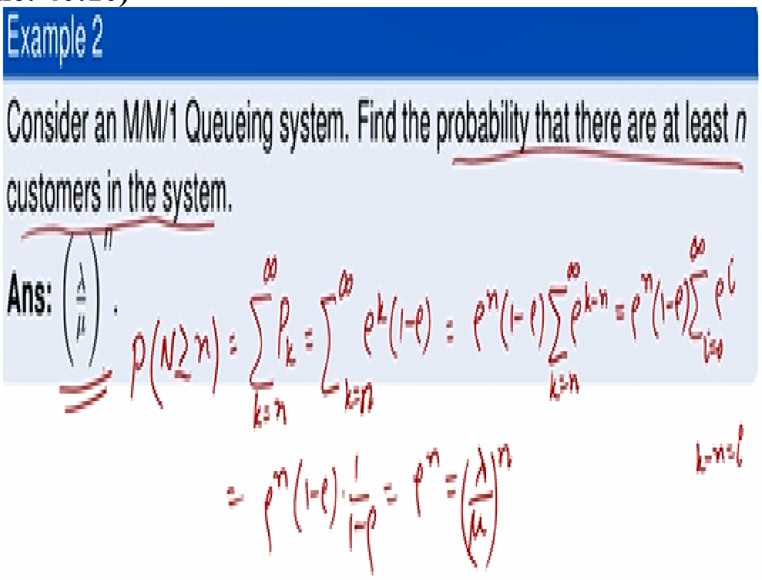

Now find consider MM1 queue system find the probability that there are at least N customers in the system, so number of customers are>or=n. let us go to that formula. Probability that number of the customers in the system exceeds k okay. So probability that the number of customers in the system exceeds k is given by this formula probability that N>k=sigma n=k+1 to infinity Pn okay.

And Pn is rho to the power n*1-rho. So let us go to that formula. So we want the probability that there are at least n customers. So probability that N>or=n. So probability that sigma k=n to infinity Pk okay so this is=sigma k is= n to infinity rho to the power k*1-rho okay so we will have rho to power n*1-rho we can write outside and here we will have k=n to infinity rho to power k-n.

And k-n we can write as=i then this is rho to power n*1-rho*sigma i=0 to infinity rho to the power i. So we will have the rho to the power n*1-rho and this is 1/1-rho. So this will be=rho to the power n and rho is=lambda/mu okay. So lambda/mu raised to the power n. So probability that there are at least n customers in the system is given by lambda/mu raised to the power n.

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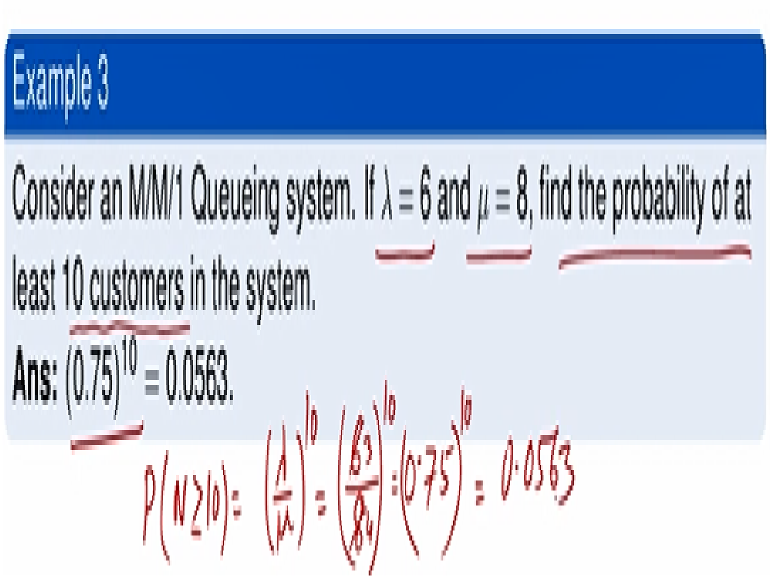

Now consider an MM1 queueing system. If lambda=6, mu=8 find the probability of at least 10 customers in the system. So we have just now found the probability that there are at least n customers in the system. So let us take n=10 here okay. So we have probability that is N>or=10 is=lambda/mu raised to the power 10 and lambda is=6, mu is=8 raised to the power 10, so this is 3/4, so this is 0.75 raised to the power 10 and this comes out to be 0.0563 okay.

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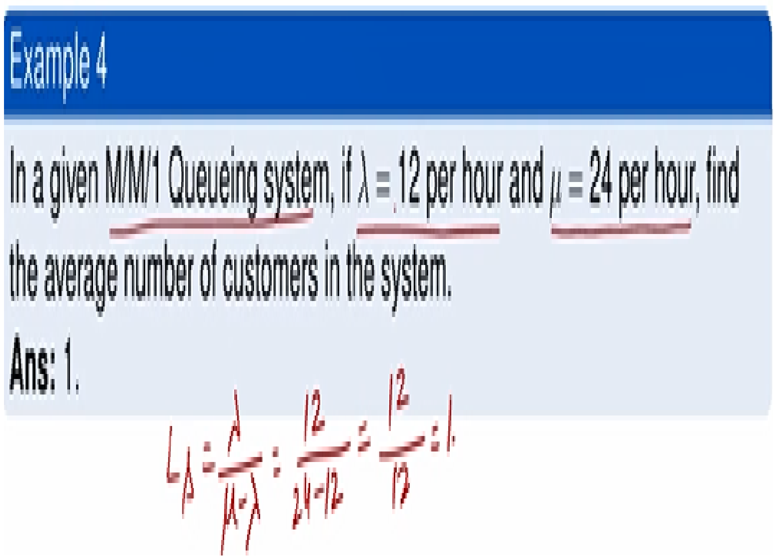

Now let us consider another problem. In a given MM1 queueing system if lambda=12 per hour, mu=24 per hour. So mean service rate is 24 per hour and mean arrival rate is 12 per hour. Find the average number of customers in the system. So let us find the formula for the average number of customers in the system. So average or expected number of customers in the system is given by Ls and Ls=lambda/mu–lambda, so we will apply this formula.

So average number of customers in the system is given by Ls which is lambda/mu-lambda. So Ls is=lambda/mu–lambda and lambda is=12 okay 12 per hour, mu=24 per hour okay. So we have 24-12, so this is 12/12 and we have 1 here okay because this is service rate per hour. In 1 hour, the arrival rate is 12 and in 1 hour the service rate is 24 okay

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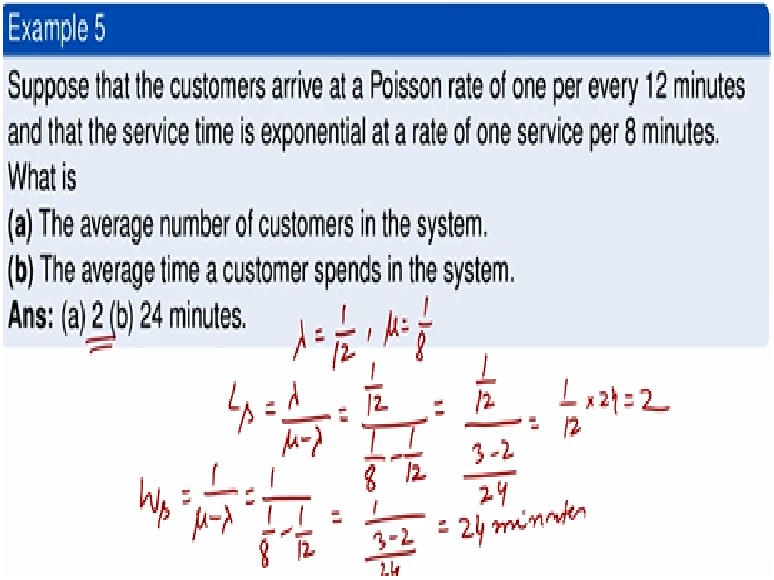

Now suppose that the customers arrive at a Poisson rate of one per every 12 minutes okay and that the service time is exponential at a rate of one service per 8 minutes. So lambda here is=1/12 and mu=1/8 okay. The average number of customers in the system just now we have seen the average number of customers in the system is given by lambda/mu–lambda. So Ls=lambda/mu–lambda and this is 1/12/1/8-1/12.

So we have 1/12 and we have here 24 and we get 3-2, so we get 1/12*24, so we get 2. So average number of customers in the system is 2. The average time a customer spends in the system. Let us find the average time a customer spends in the system. So average or expected time of a customer in the system Ws. Ws is 1/mu–lambda okay. So Ws=1/mu–lambda. Here the average time of customer so Ws=1/mu–lambda and this is 1/1/8-1/12 okay, so 1/24 and we have 3–2, so this is 24 minutes. So 24 minutes is the answer.

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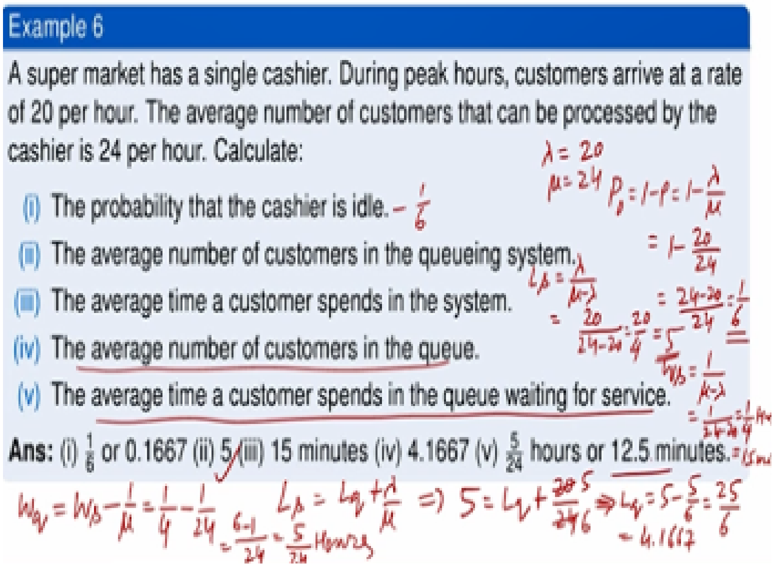

Now we go to this last question on queueing theory. A super market has a single cashier. During peak hours, customers arrive at a rate of 20 per hour okay. So arrival rate is lambda=1/20 and the average number of customers that can be processed by the cashier is 24 per hour. So mu=1/24 okay. Calculate the probability that the cashier is idle. When the cashier is idle, we have to find the P0 okay, the probability that there is no body in the queue okay in the system.

P0=1-rho, P0, we found the formula for P0, P0=1-rho okay and Pn is the probability that there are n customers in the system. So if the cashier is idle, there will be nobody in the system, so P0=1–rho we have to find. So P0 okay the first part is P0=1–rho which is 1–lambda/mu okay and this is 1-1/20/1/24 okay, so 24/20 okay, so not like this. Here lambda= we have to consider per hour okay.

So lambda is=20 and mu=24 okay. So P0=1–rho and therefore it is 1–lambda/mu which is=1-20/24 okay, so 24-20/24. This means that it is 4/24 so 1/6 hours okay. So probability is 1/6. So P0 is this is 1/6 and then the average number of customers in the queueing system. Average number of customers in the system we know, average number of customers in the system is given by Ls.

Ls is=lambda/mu–lambda okay. So Ls=this second part Ls=lambda/mu–lambda okay. So lambda is=20/24-20. So we get 1/4 okay. So average number of customers in the queueing system is given by Ls and Ls is=20/4, so this is=5 okay. So average number of customers in the queueing system is 5, so we have got the answer. Now average time a customer spends in the system okay. Average time a customer spends in the system is given by Ws which is 1/mu–lambda okay.

So Ws=1/mu–lambda which is=1/24–20, so 1/4 that is 15 minutes. This 1/4 hours which is=15 minutes. Then, we have average time a customer spends in the queue waiting for service okay. So average time a customer spends we have got, no we have to first do this, average time of customers in the queue, average number of customers in the queue. We can get from this formula.

Ws we have got=Ws=Wq+1/q okay. Average number of customers in the queue, so Lq we have to find Lq okay. What is the formula for Lq? Lq=Ls, Ls=Lq+lambda/mu okay. Ls=Lq+lambda/mu okay. Ls we have found, Ls=average number of customers in the system okay, average number of customers in the system we have found, Ls=5 okay. So we have 5=Lq+lambda/mu.

Lambda=20, 20/24 okay so we have 5/6, so Lq=5–5/6, so 25/6, 25/6 means 4.1667 okay, so that is Lq. Lq is the average number of customers in the queue. Then, average time a customer spends in the queue waiting for service okay. So that means Wq. Wq is the average time a customer spends in the queue waiting for service okay. So we have Wq. Wq=Ws–1/q, Ws–1/mu.

Ws we found to be=1/4 okay 1/4 hours and 1/mu, 1/mu is 1/24 okay. So this is 24 and we get here 6-1, so 5/24 okay, so 5/24 hours that is to say 12.5 minutes. So that is the average time a customer spends in the queue waiting for service. So this finishes the discussion on queueing theory. Now let us take an example on reliability theory.

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Reliability

Consider an equipment or a system which is subjected to failure. Let $F(t)$ denote the probability of the equipment or system failure within t units of time.The system reliability $R(t)$ is the probability that failure will not occur in t units of time. Thus

$$R(t) = 1 - F(t)$$

Assume that a number of components are involved in the system .Let us consider two configurations: Series and Parallel configuration.

Consider an equipment or a system which is subjected to failure. Let F t denotes the probability of the equipment or system failure within t units of time. The system reliability R t is the probability that failure will not occur in t units of time and thus we can say that R t=1-F t. Assume that a number of components are involved in the system. Let us consider 2 configurations, series and parallel configuration.

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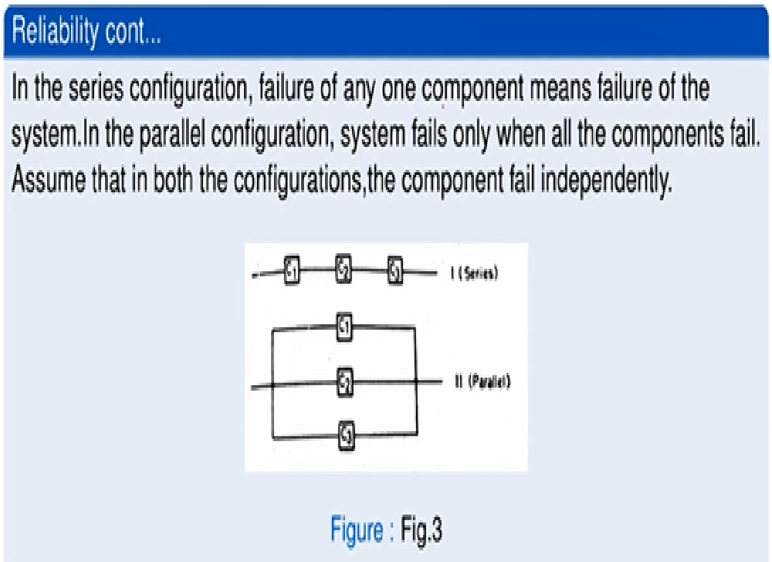

In the series configuration, failure of any one component means failure of the system. In the parallel configuration, system fails only when all the components fail. Assume that in both the configurations the component fail independently. Now this is the system in series configuration, this is the system in parallel configuration. So there are 3 components; C1, C2, C3 which are connected in series and here there are 3 components which are connected in parallel configuration.

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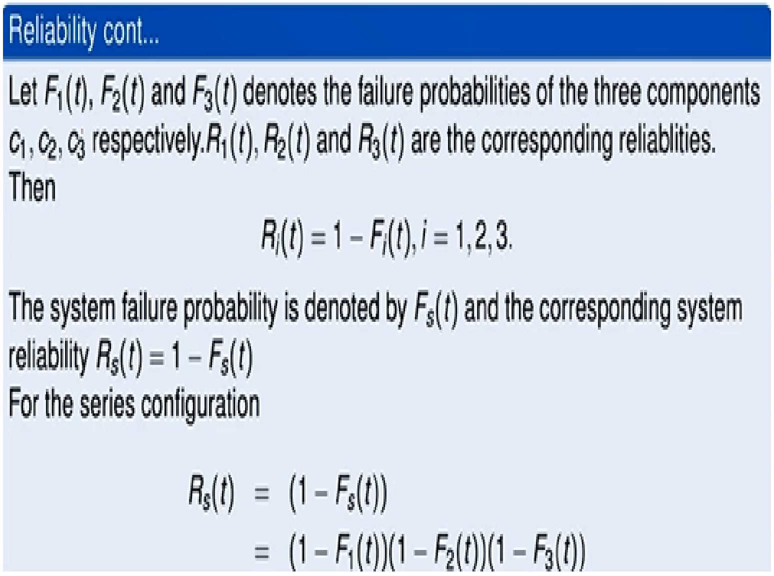

So let F1 t, F2 t and F3 t denote the failure probabilities of the 3 components C1, C2, C3 respectively. R1 t, R2 t, R3 t are the corresponding reliabilities. Then, Ri t=1-Fi t, i=1, 2, 3. The system failure probability will be denoted by Fs t and the corresponding system reliability Rs t will be then=1-Fs t okay. For the series configuration, Rs t is=1-Fs t=1-F1 t*1-F2 t*1-F3 t. Why?

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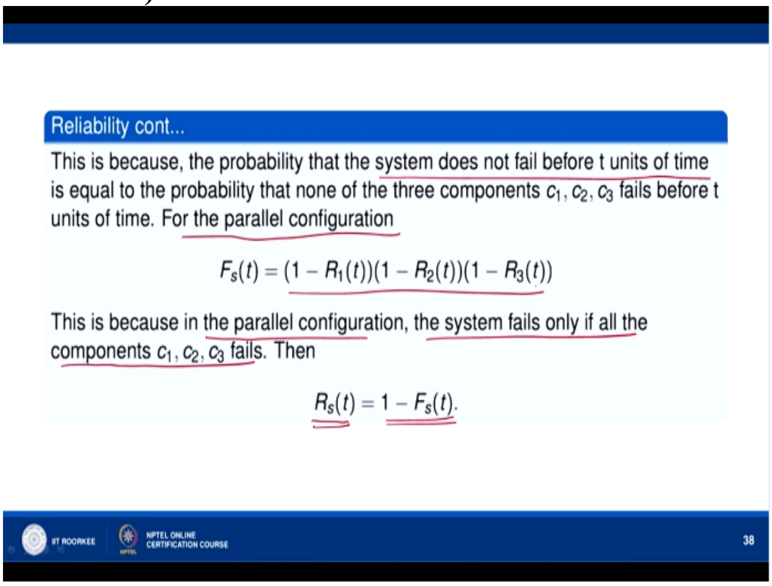

Because the probability that the system does not fail before t units of time okay. We are looking at the probability that the system does not fail. We are finding the probability of the reliability okay. That is the system does not fail before t units of time and we are considering the series configuration. So none of the 3 components must fail and therefore Rs t=1-F1 t*1-F2 t*1-F3 t.

Because 1-F1 t is the probability that the system component C1 does not fail before t units of time and similarly 1-F2 t is the probability that the component C2 does not fail before t units of time and similarly 1-F3 t is the probability that the component C3 does not fail before t units of time. So their product then gives us the probability that the system does not fail before t units of time in the case of series configuration okay.

So for the parallel configuration what will be the probability? Fs t=1-R1 t*1-R2 t*1-R3 t because in the parallel configuration the system fails only if all the 3 components fail okay. So 1-R1 t is the probability that the component C1 fails and 1-R2 t is the probability that the component C2 fails and 1-R3 t is the probability that the component C3 fails. So when all of them fail, the probability will be given by Fs t and then Fs t gives you the system reliability that is Rs t, Rs t=1-Fs t okay.

So in the case of parallel configuration, Rs t will be given by 1-Fs t where Fs t is given by this product okay.

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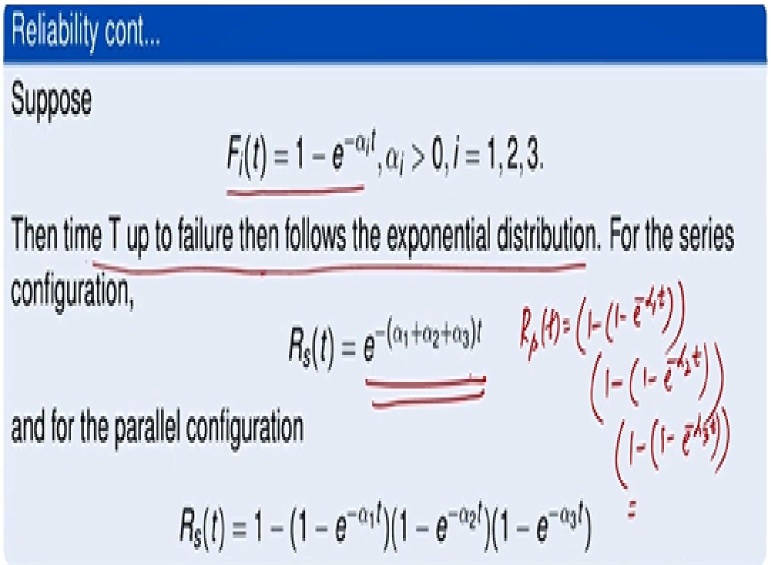

So suppose let us consider an example suppose Fi t is given by 1–e to the power –alpha i t, alpha i is>0, i is=1, 2, 3. So Fi t as we have seen, Fi t is the probability that the component C1 fails okay. This Fi t is the probability of failure of the component okay so Fi t is given by 1–e to the power –alpha i t that means F1 t is 1–e to the power–alpha 1t. This is the probability of the failure of component C1.

And when you take i=2, you get the probability of the failure of component C2 and when you take i=3 you get the probability of the failure of component C3. So Fi t's are given and then the time t up to the failure then follows the exponential distribution for the series configuration. We have seen in the case of series configuration Rs t=Rst is 1–F1 t*1-F2 t*1-F3t, so we have Rs t=1–F1 t.

F1t is 1–e to the power of –alpha 1t okay*1–e to the power –alpha 2t*1–1-e to the power –alpha 3t okay and you get this expression okay and for the parallel configuration Rs t is=parallel configuration Rs t=1-Fs t okay, Fs t is given by 1–R1 t*1-R2 t*1-R3 t okay. So yeah so we get in the case of parallel configuration Rs t=1–Fs t, Fs t=1–R1 t, 1–R1 t is the probability that the first component fails that is F1 t, F1 t*F2 t*F3 t okay.

So we get 1-F1 t*F2 t*F3 t okay so that is the system reliability in the case of parallel configuration. So we have discussed both the cases okay. The system reliability in the case of parallel configuration and the system reliability in the case of series configuration okay and we have taken one example where we have taken particular values of the failure probabilities of the 3 components given by Fi t=1–e to the power–alpha i t where alpha i is>0 and i=1, 2, 3.

And we have found the values of Rs t for both the cases the series configuration as well as parallel configuration. So with that I would like to end this lecture. This is our last lecture on advanced engineering mathematics. Thank you very much for your attention.